Engineering Economic Analysis

2019 SPRING

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Chap. 4

UTILITY

Utility Functions

- Utility
  - an indicator of person's overall well-being
  - a numeric measure of a person's happiness
  - a way of describing preferences

- Utility function (for some given preference $\succsim$): any function u: **X**→**R** which satisfies

  $u(\tilde{x}) > u(\tilde{y})$ if and only if $\tilde{x} \succ \tilde{y}$

Utility Functions

- Consider the bundles (4,1), (2,3) and (2,2).
- Suppose (2,3) $\succ$ (4,1) ~ (2,2).
- Assign to these bundles any numbers that preserve the preference ordering; *e.g.* U(2,3) = 6 > U(4,1) = U(2,2) = 4.
- Can call these numbers utility levels.
- If we let $U(x_1,x_2) = x_1x_2$, this function can be a utility function !
- *Given some preferences, can we always find the corresponding utility function?*

Utility Functions

- Existence: Suppose preferences are complete, reflexive, transitive, strongly monotonic and continuous. Then there exists a continuous utility function $u : \mathbf{R}^n_+ \to \mathbf{R}$ which represents those preferences
  - Continuity means that small changes to a consumption bundle cause only small changes to the preference level.
  - Lexicographic preference

- Unique?

Utility Functions

- Consider the bundles (4,1), (2,3) and (2,2).
- Suppose (2,3) $\succ$ (4,1) ~ (2,2).
- $U(x_1,x_2) = x_1x_2$, is a utility function
- Consider $V(x_1,x_2) = U^2 = (x_1x_2)^2$
- $V(2,3) = 36 > V(4,1) = V(2,2) = 16$.
- If we let $V(x_1,x_2) = (x_1x_2)^2$, this function also can be a utility function !

Utility Functions

- (Positive) Monotonic transformation
  - Given a function u, $f(u)$ is monotonic transformation of u if $u_1 > u_2$, then $f(u_1) > f(u_2)$

- (Positive) Monotonic transformation of a utility function is also a utility function that represents the same preferences as the original utility function

- Unique up to (positive) monotonic transformation

Utility Functions

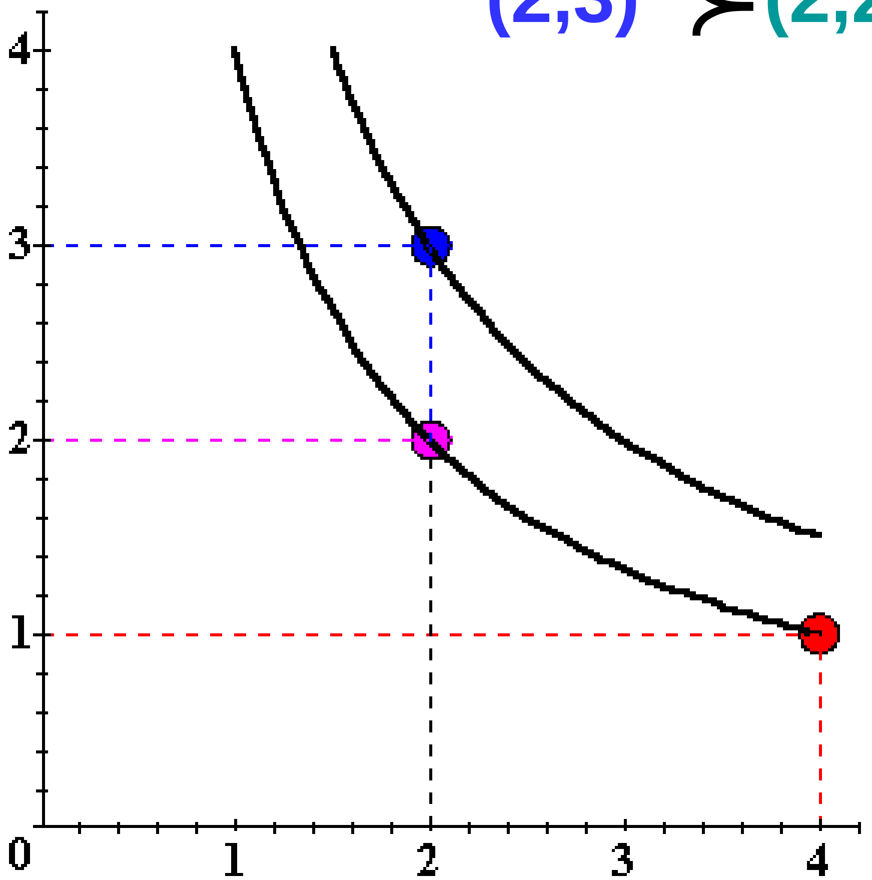
- Ordinal utility: the magnitude of the utility function is only important insofar as it orders(ranks) the different bundles; the size of the utility difference between two bundles does not matter
- Cardinal utility: the size of the utility difference between two bundles is supposed to have some sort of significance
- *For examining consumers' choice behavior, the ordinality is sufficient in analyzing utility functions*

Utility Functions & Indiff. Curves

- Consider the bundles (4,1), (2,3) and (2,2).
- Suppose (2,3) $\succ$ (4,1) $\sim$ (2,2).
- Assign to these bundles any numbers that preserve the preference ordering;
  e.g. U(2,3) = 6 > U(4,1) = U(2,2) = 4.
- Can call these numbers utility levels.
- Note that all bundles in an indifference curve have the same utility level.

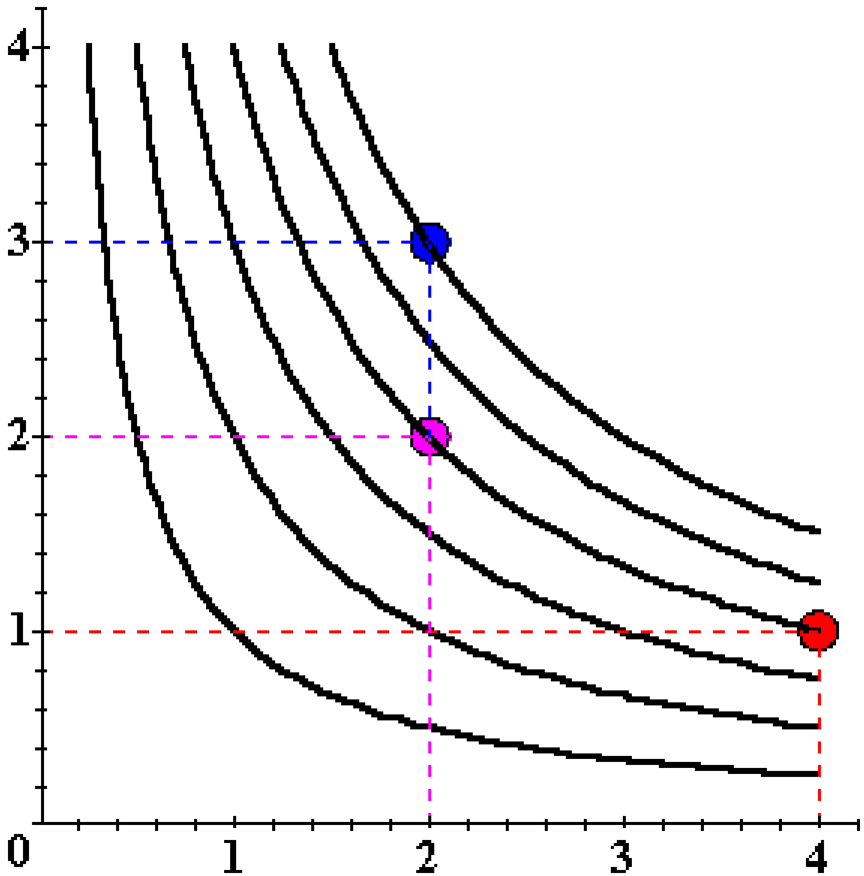
Utility Functions & Indiff. Curves

$(2,3) \succ (2,2) \sim (4,1)$



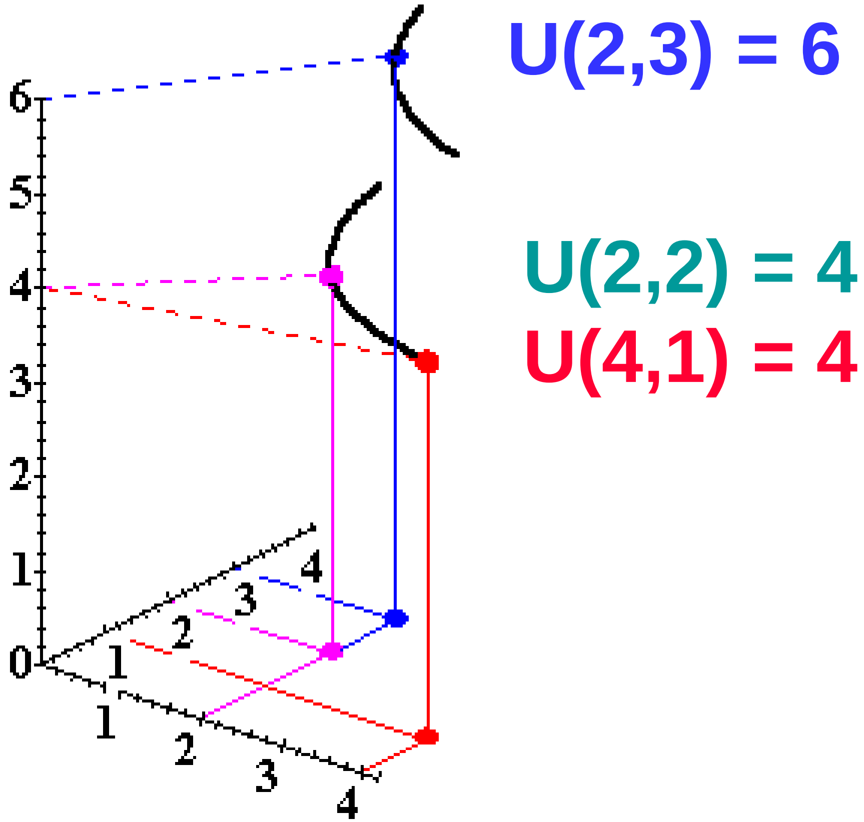
Utility Functions & Indiff. Curves

- Comparing more bundles will create a larger collection of all indifference curves

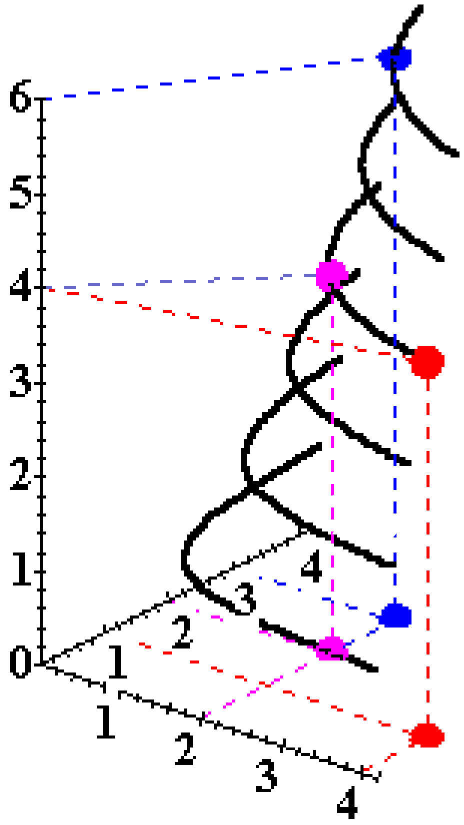


Utility Functions & Indiff. Curves

- Another way to visualize this same information is to plot the utility level on a vertical axis.



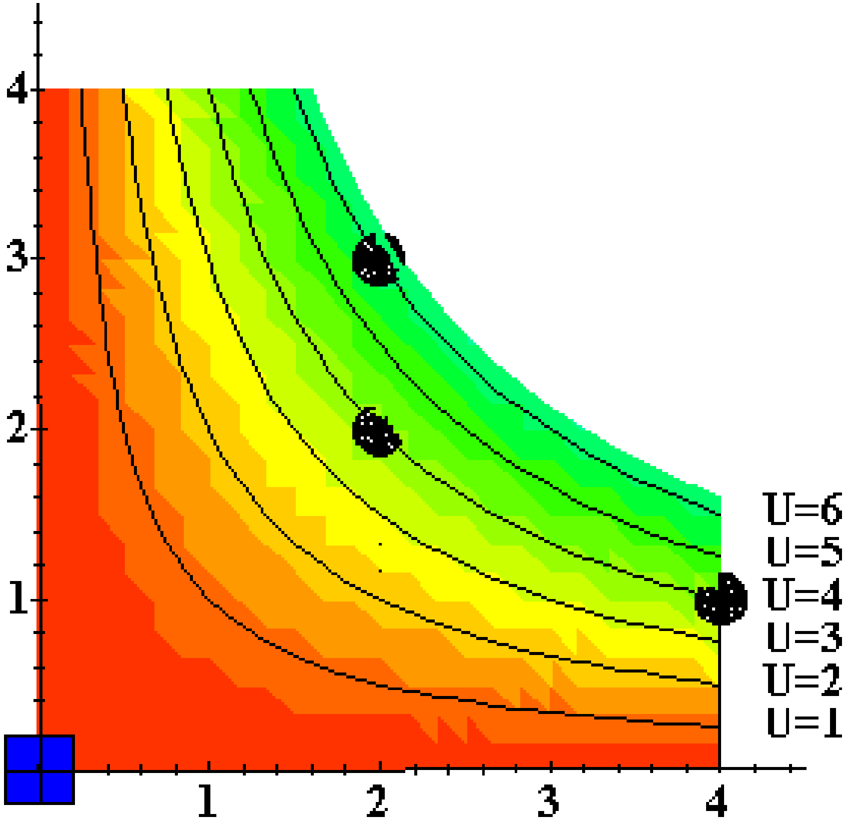
Utility Functions & Indiff. Curves



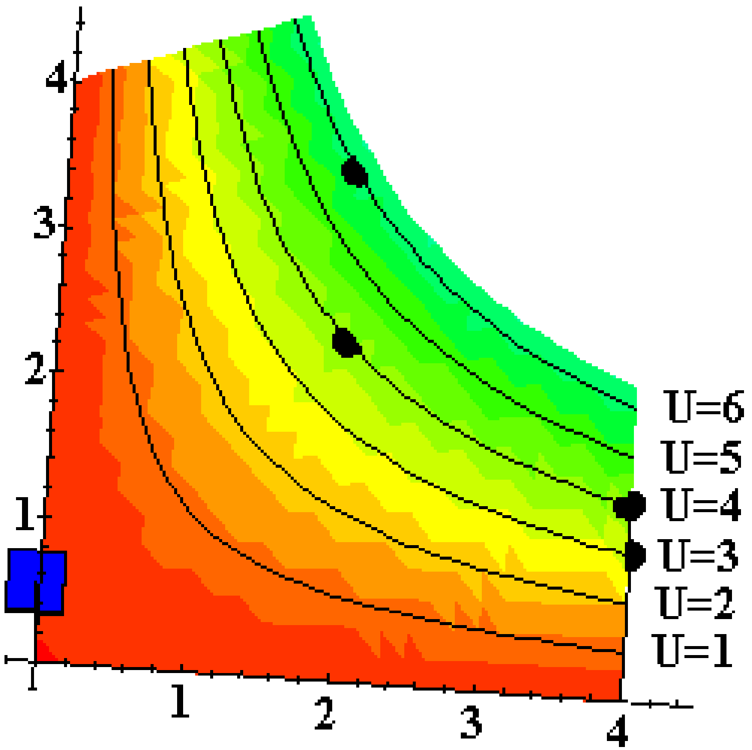
Utility Functions & Indiff. Curves

- Comparing all possible consumption bundles gives the complete collection of the consumer's indifference curves, each with its assigned utility level.
- This complete collection of indifference curves completely represents the consumer's preferences.

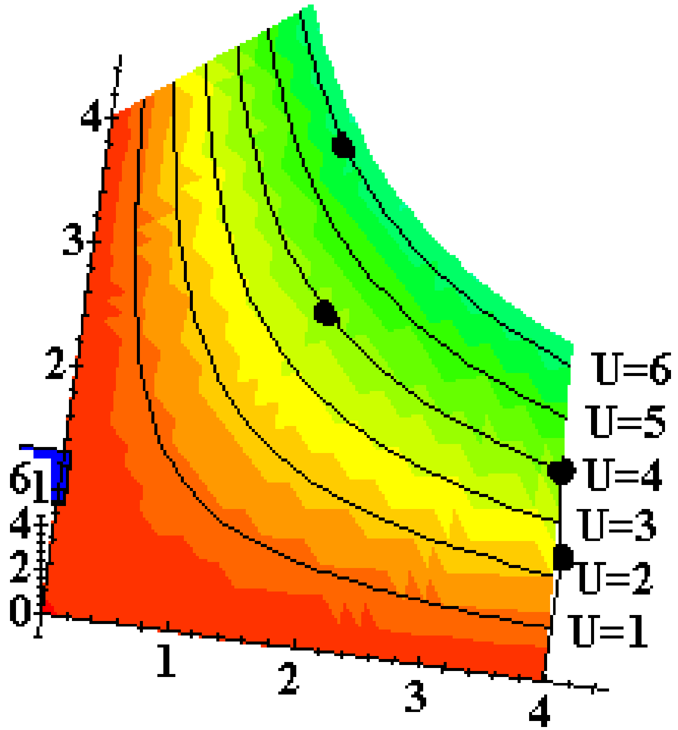
Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves



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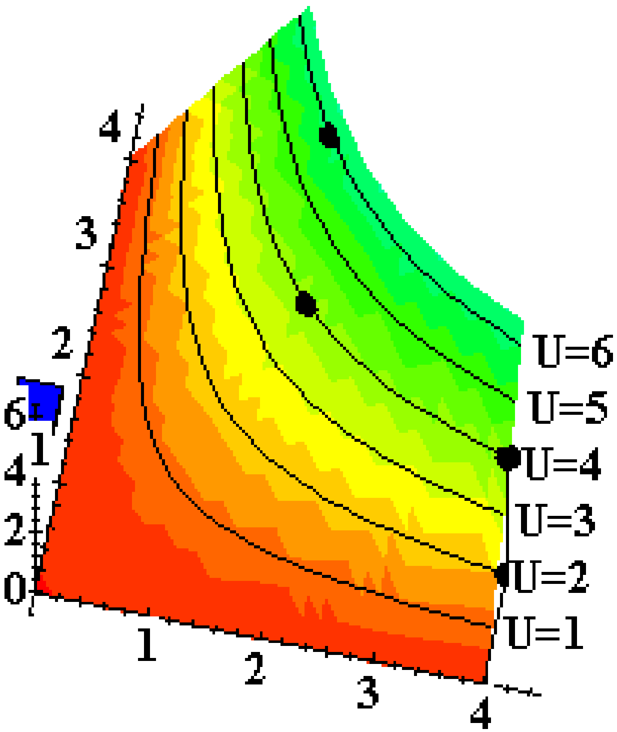

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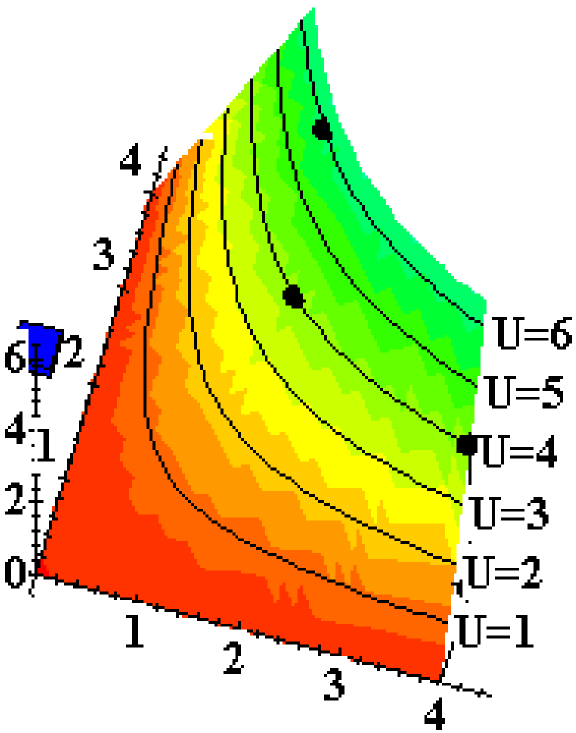

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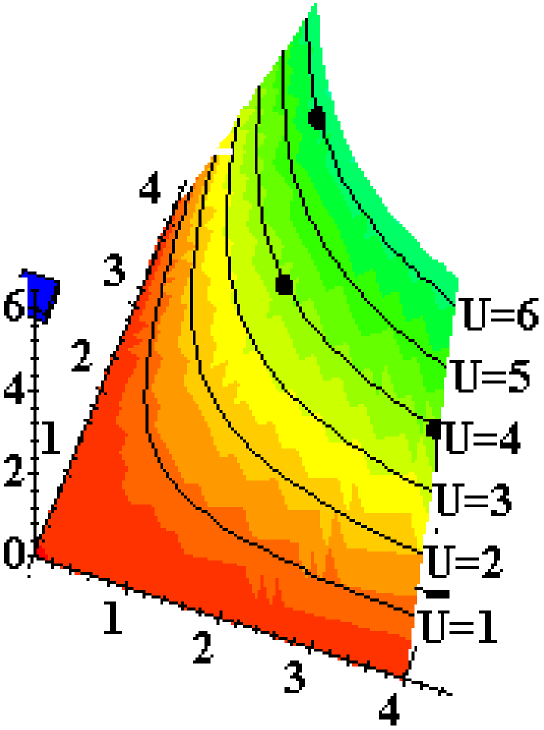

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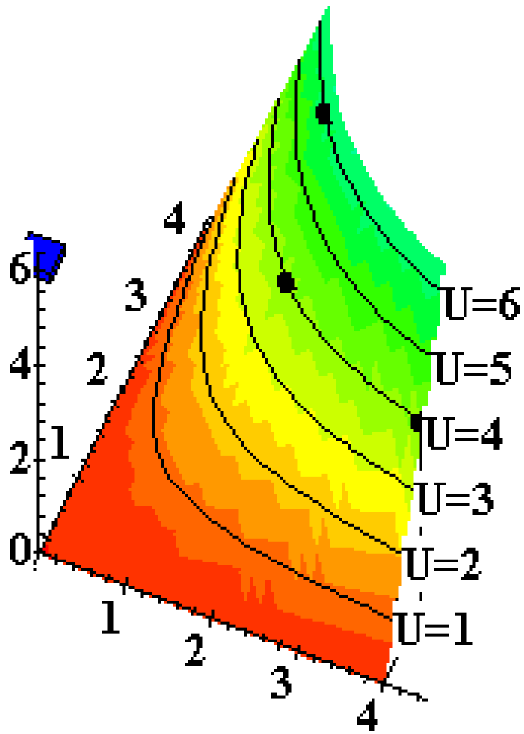

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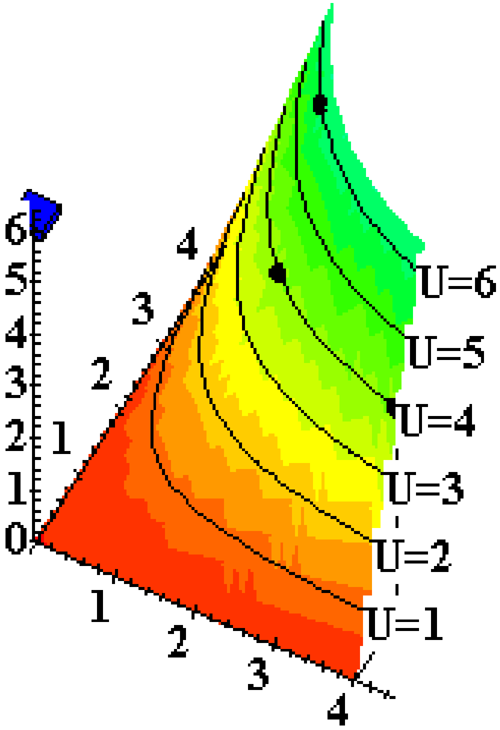

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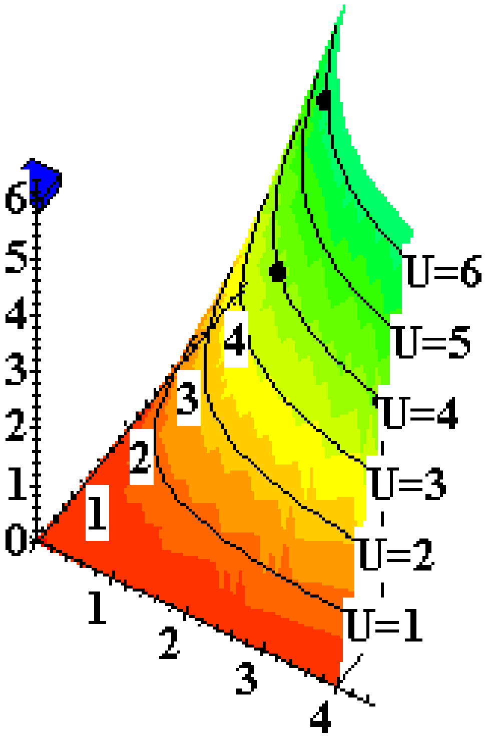

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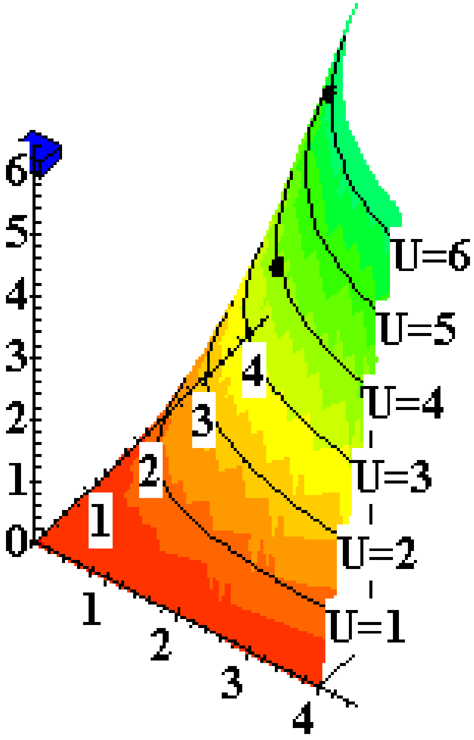

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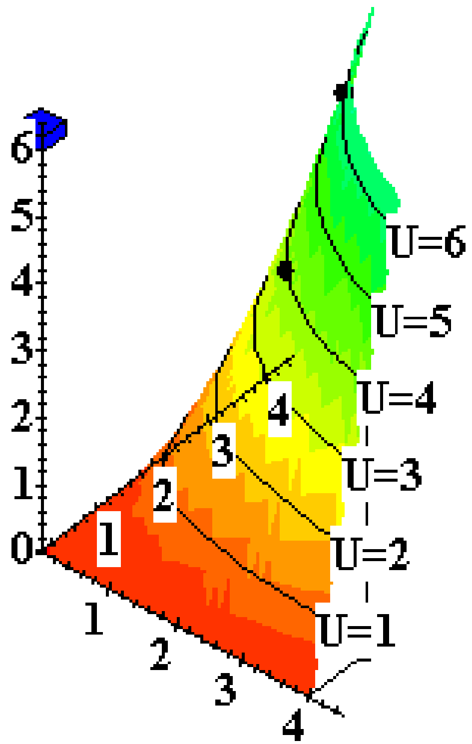

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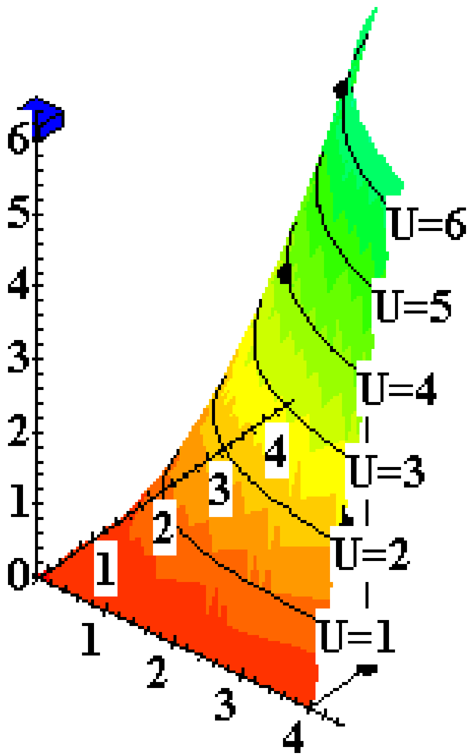

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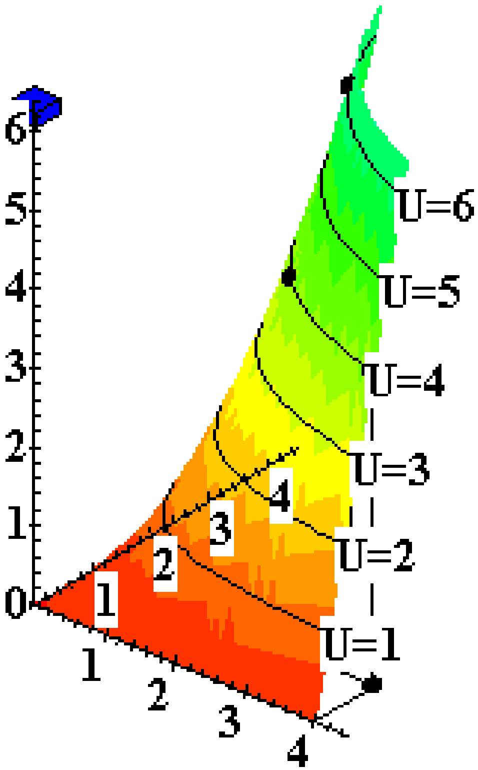

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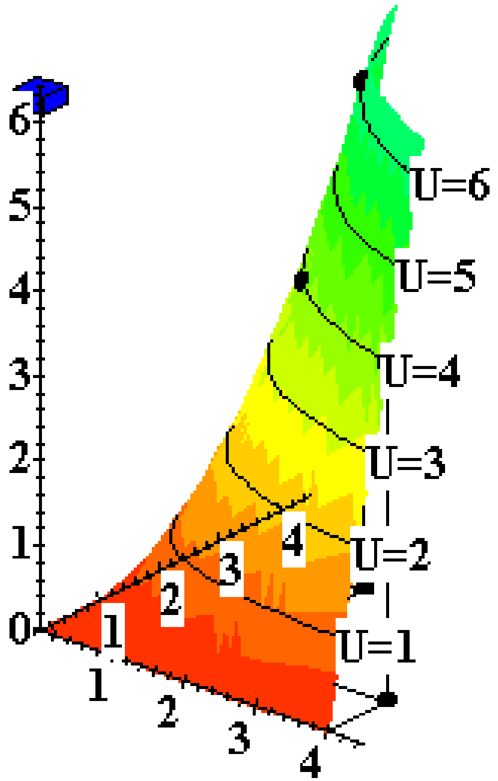

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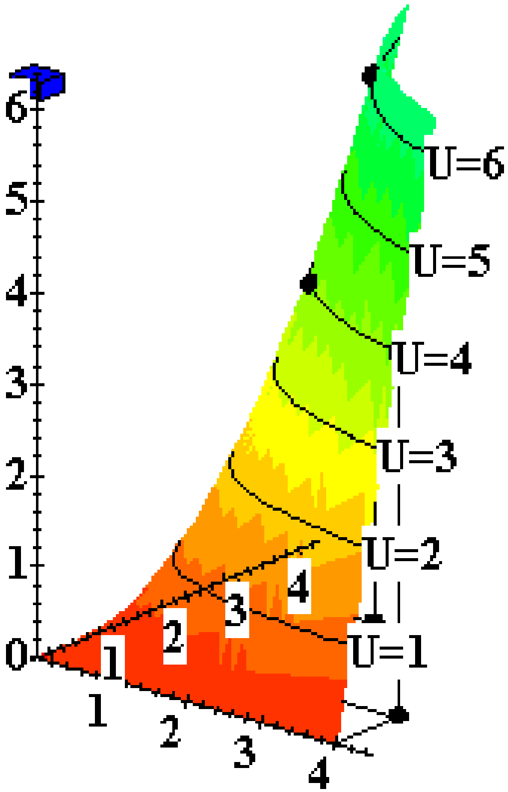

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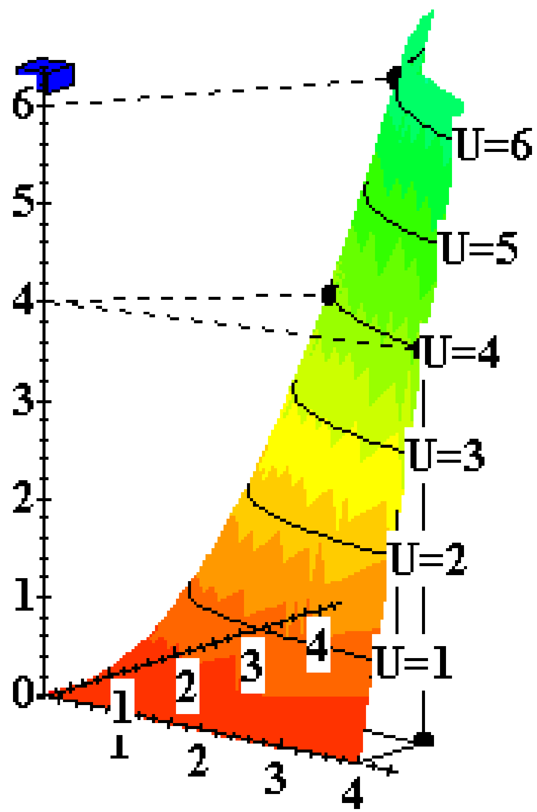
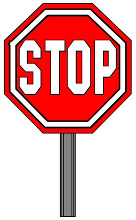


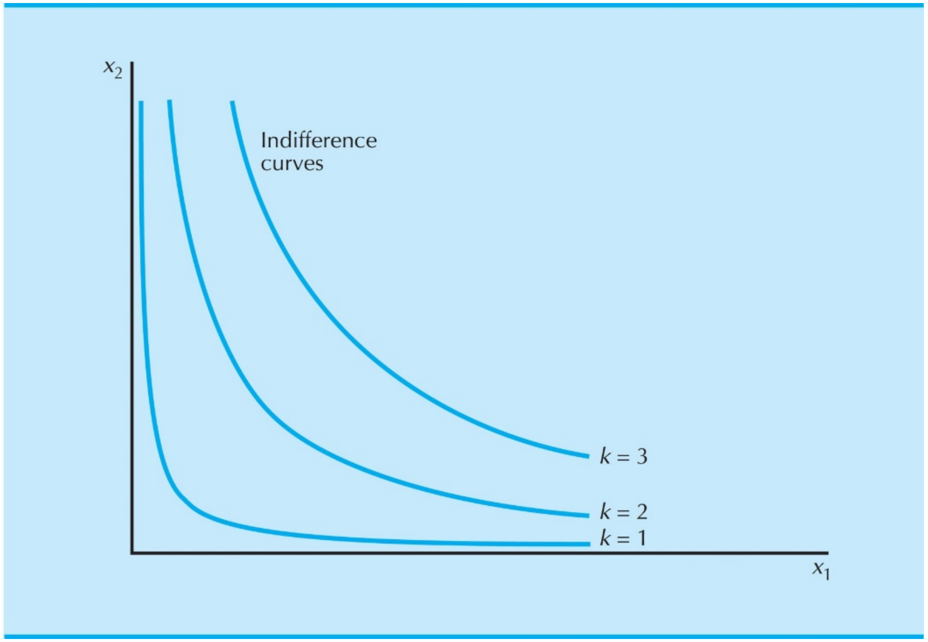


Indifference Curves from Utility Functions

- $U(x_1,x_2) = x_1x_2$
- Note that $U(2,3) = 6 > U(4,1) = U(2,2) = 4$
- To find a corresponding I.C.,

  let $U(x_1,x_2) = x_1x_2 = k$, that is, $x_2 = k/x_1$

Figure 4.3



Some Examples of Utility Functions

- Perfect substitutes
  - Suppose that consumer would require 2 units of good 2 to give up 1 unit of good 1 (MRS = -2)
  - Good 1 is twice as valuable as good 2
  - $U(x_1,x_2) = 2x_1 + x_2$

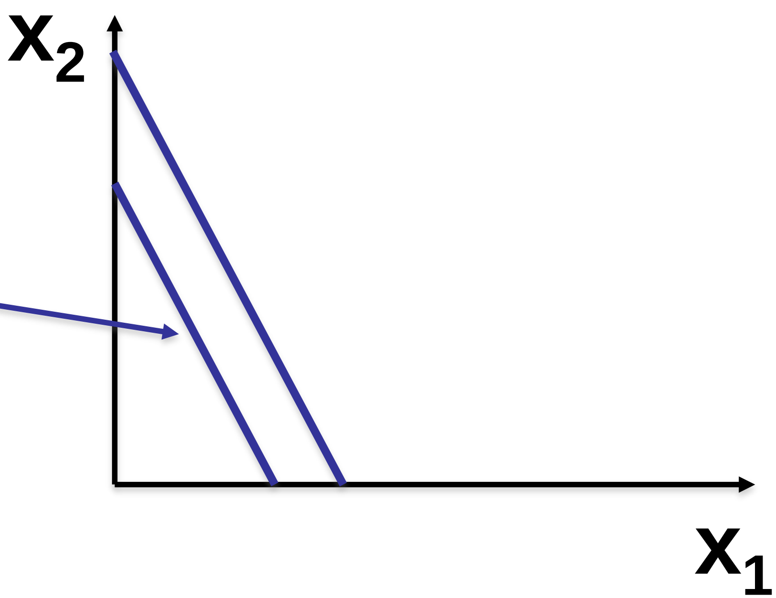
  - Suppose that consumer would require ‘a’ units of good 2 to give up ‘b’ unit of good 1 (MRS = -a/b)
  - Good 1 is (a/b) times as valuable as good 2
  - $U(x_1,x_2) = (a/b)x_1 + x_2$ (Equivalently $ax_1 + bx_2$)

Some Examples of Utility Functions

- Perfect substitutes
  - In case that ‘b’ number of good 1 can be substituted for ‘a’ number of good 2
  - MRS = -a/b

$$u(x_1, x_2) = ax_1 + bx_2$$

$$x_2 = -\frac{a}{b}x_1 + \frac{k}{b}$$



Some Examples of Utility Functions

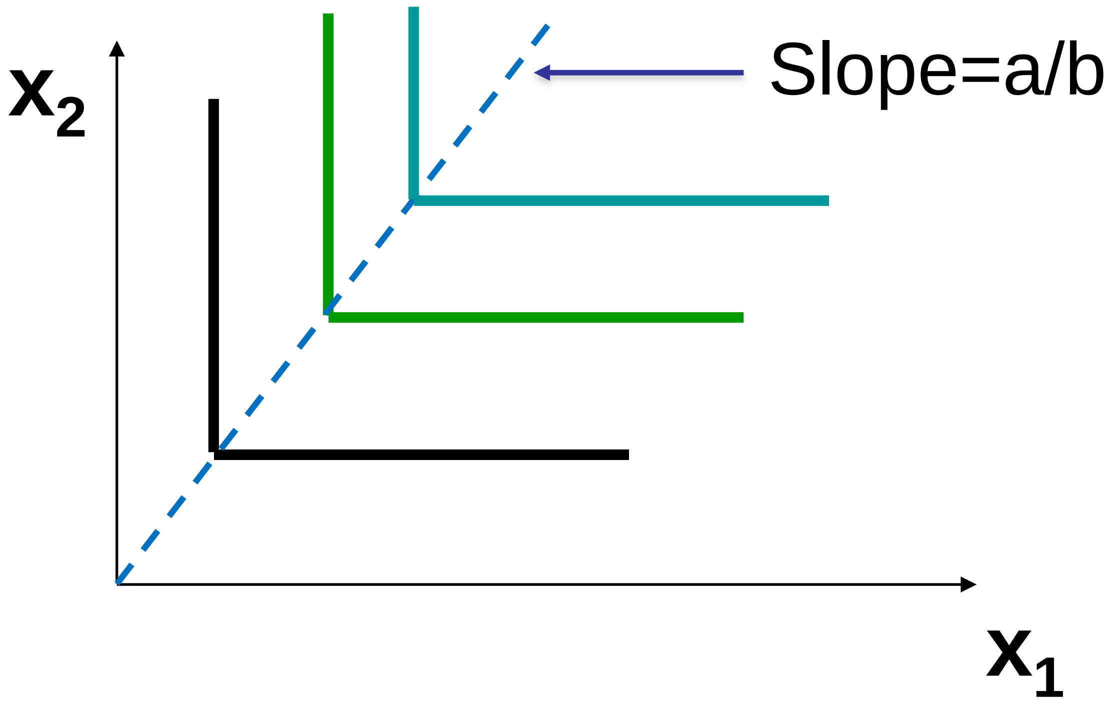
- Perfect complements
  - In case of right shoes & left shoes case, consumer only cares about the number of *pairs*
  - The # of pairs = min $\{x_1, x_2\}$
  - This can be a utility function

  - Suppose that *one pair* requires '1' unit of good 1 and '2' units of good 2
  - The # of pairs = min $\{x_1, 1/2x_2\}$
  - Equivalently, $U(x_1,x_2)= 2\min\{x_1, 1/2x_2\}= \min\{2x_1, x_2\}$

Some Examples of Utility Functions

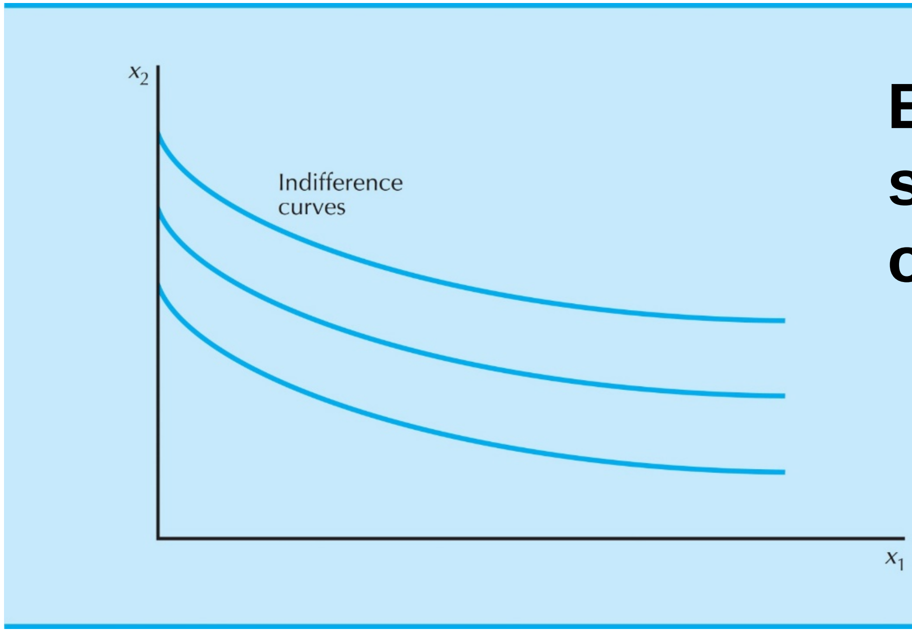
- Perfect complements
  - Consumer wants to consume the goods in proportion of b-to-a; thus (b, a) makes a pair

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$



Some Examples of Utility Functions

- A utility function of the form
$$U(x_1,x_2) = f(x_1) + x_2$$
is linear in just x_2 and is called quasi-linear.
- *E.g.* $U(x_1,x_2) = 2x_1^{1/2} + x_2$.



Each curve is a vertically shifted copy of the others.

Figure 4.4

Some Examples of Utility Functions

- Any utility function of the form

$$U(x_1,x_2) = x_1^a x_2^b$$

  with a > 0 and b > 0 is called a Cobb-Douglas utility function.

- *E.g.* $U(x_1,x_2) = x_1^{1/2} x_2^{1/2}$ (a = b = 1/2)
  $V(x_1,x_2) = x_1 x_2^3$ (a = 1, b = 3)

- Well-behaved (nicely monotonic and convex) preference

Some Examples of Utility Functions

- Cobb-Douglas Indifference curves

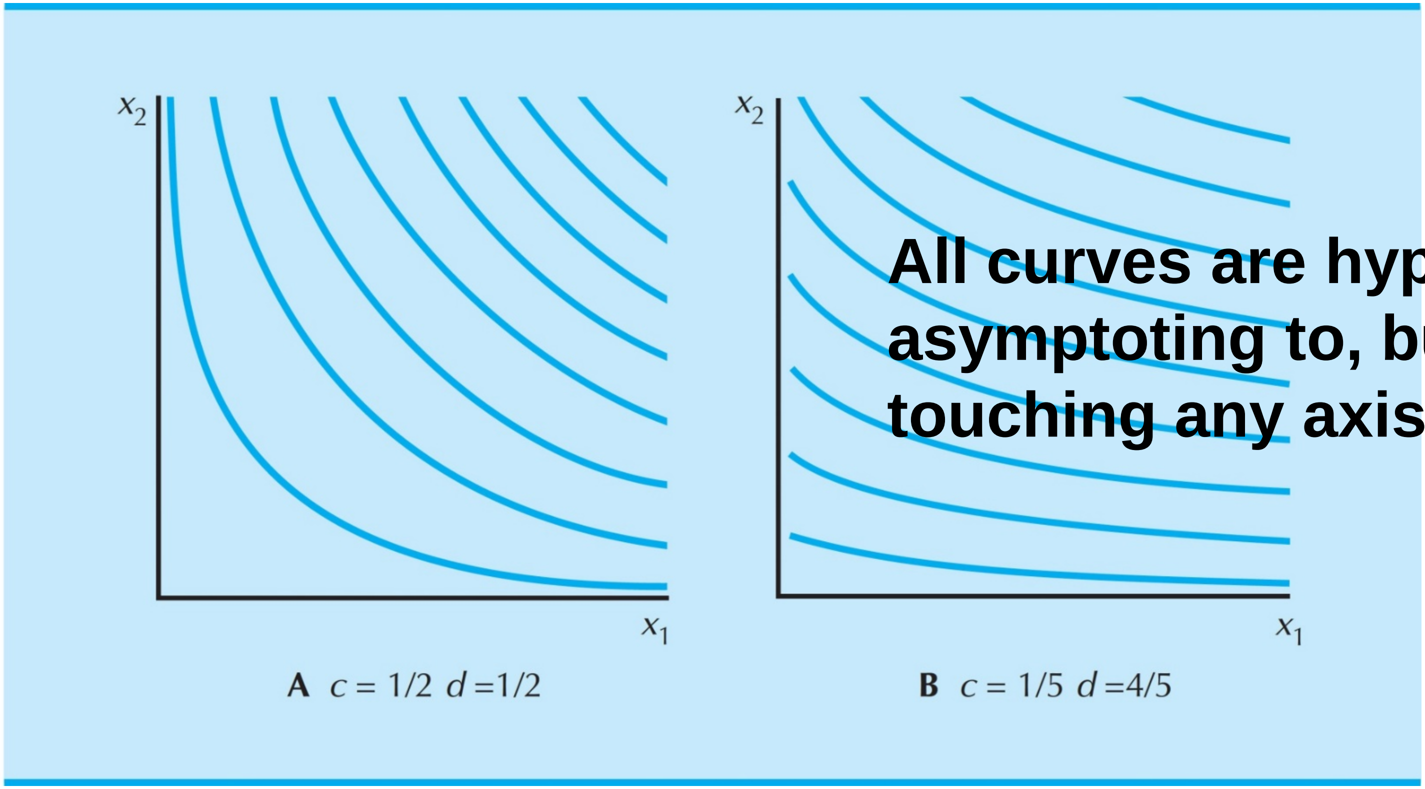


Figure 4.5

All curves are hyperbolic, asymptoting to, but never touching any axis.

Some Examples of Utility Functions

- Some examples of monotonic transformation of Cobb-Douglas utility function, $U(x_1,x_2) = x_1^{a} x_2^{b}$
  - $v(x_1,x_2) = \ln[u(x_1,x_2)] = a\ln x_1 + b\ln x_2$
  - $v(x_1,x_2) = [u(x_1,x_2)]^{\frac{1}{a+b}} = x_1^{\frac{a}{a+b}} x_2^{\frac{b}{a+b}} = x_1^{\frac{a}{a+b}} x_2^{1-\frac{a}{a+b}}$

$$= x_1^{c} x_2^{1-c} \quad where\ \mathrm{c} = \frac{a}{a+b}$$

Marginal Utility

- The marginal utility of commodity *i* (say, good 1) is the rate-of-change in utility associated with a small change in the amount of commodity *i*

$$MU_1 = \frac{\Delta U}{\Delta x_1} = \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1}$$

$$MU_1 = \lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1}$$

Note that the amount of good 2 is held fixed!

- The change in utility: $dU = MU_i dx_i$

Marginal Utility and MRS

- Marginal Rate of Substitution
  - The rate at which a consumer is just willing to substitute a small amount of good j for good i
  - To keep utility constant
  - To stay on the same indifference curve
- MRS = - MU_1/MU_2

$$\frac{dx_2}{dx_1} = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

Marginal Utility and MRS

- Given a utility function, $U(x_1, x_2)$
- By total differentiation,

$$dU = \frac{\partial U(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial U(x_1, x_2)}{\partial x_2} dx_2$$

- By the condition of MRS,

$$dU = \frac{\partial U(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial U(x_1, x_2)}{\partial x_2} dx_2 = 0$$

- Rearrangement gives

$$\frac{dx_2}{dx_1} = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

Marginal Utility and MRS

- Consider a function describing I.C., $x_2(x_1)$
- Then this function satisfies the following identity

$$U(x_1, x_2(x_1)) \equiv k$$

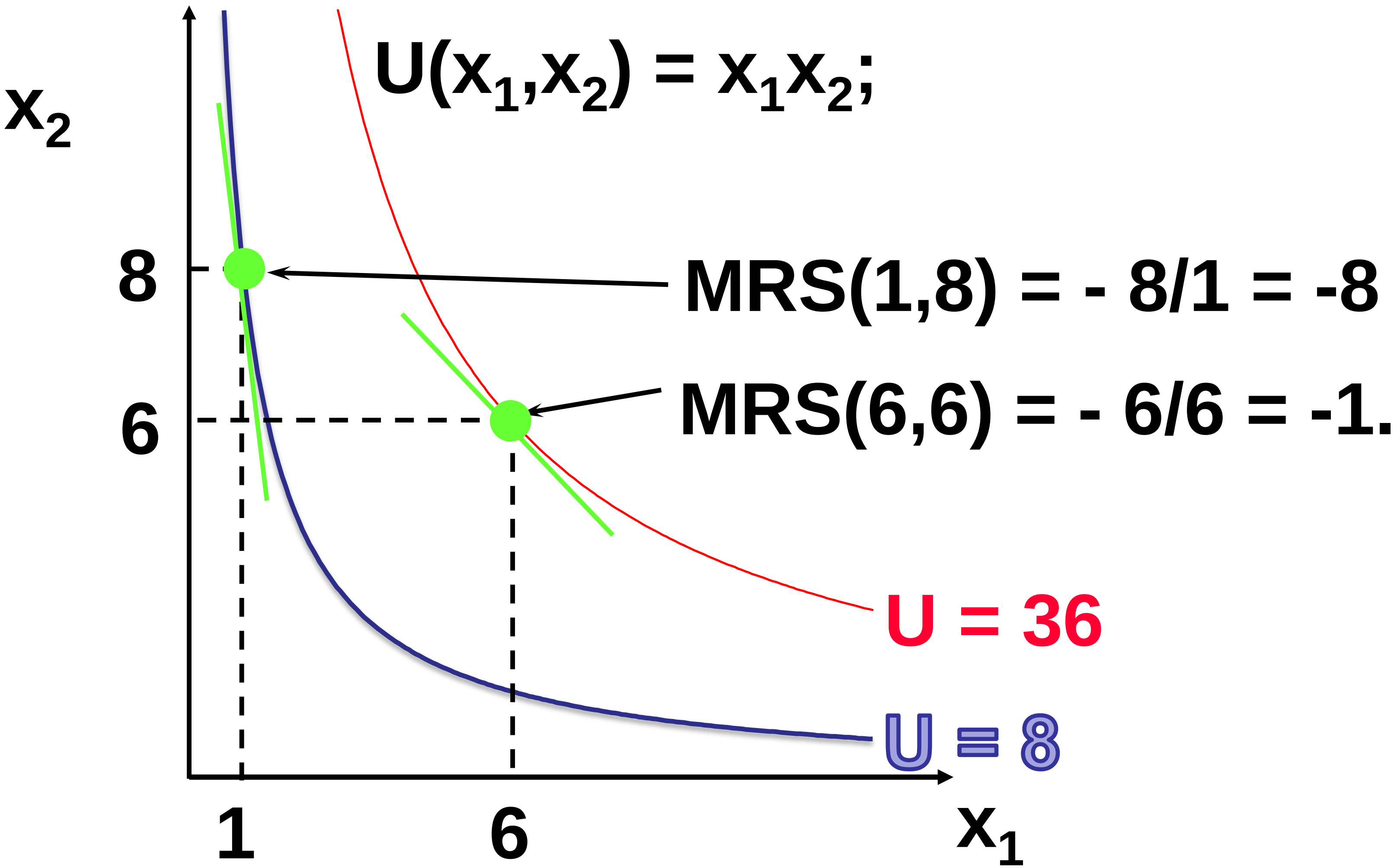
- By differentiating both sides w.r.t. x_1

$$\frac{\partial U(x_1, x_2)}{\partial x_1} + \frac{\partial U(x_1, x_2)}{\partial x_2}\frac{\partial x_2(x_1)}{\partial x_1} = 0$$

- Rearrangement gives

$$\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

Example: Marg. Rates-of-Substitution



Monotonic Transformations & MRS

- Applying a monotonic transformation to a utility function representing a preference relation simply creates another utility function representing the same preference relation.
- What happens to marginal rates-of-substitution when a monotonic transformation is applied?

Monotonic Transformations & MRS: Cobb-Douglas Example

So MRS is unchanged by a positive monotonic transformation.