3. Solution of Consolidation Equation

① Governing Eq. : $\frac{\partial^2 u}{\partial z^2} = \frac{1}{C_v} \frac{\partial u}{\partial t}$

② Seek solution in non-dimensional terms

[0-1] $W = \frac{u}{u_i}$, u_i = initial pore water pressure

[0-1] $Z = \frac{z}{H}$, H = max. drainage path

[?] $T = \frac{t}{\tau}$, τ = characteristic time

- Sub. the non-dim. terms into the Gov. Eq.

$$\frac{C_v \cdot \tau}{H^2} \frac{\partial^2 W}{\partial Z^2} = \frac{\partial W}{\partial T}$$

If $\tau = \frac{H^2}{C_v}$, $\frac{C_v \cdot \tau}{H^2} = 1$

$$\rightarrow \frac{\partial^2 W}{\partial Z^2} = \frac{\partial W}{\partial T}$$

- Let $\frac{\partial}{\partial Z} = {}'$, $\rightarrow \frac{\partial^2 W}{\partial Z^2} = W''$

$\frac{\partial}{\partial T} = \dot{}$, $\rightarrow \frac{\partial W}{\partial T} = \dot{W}$

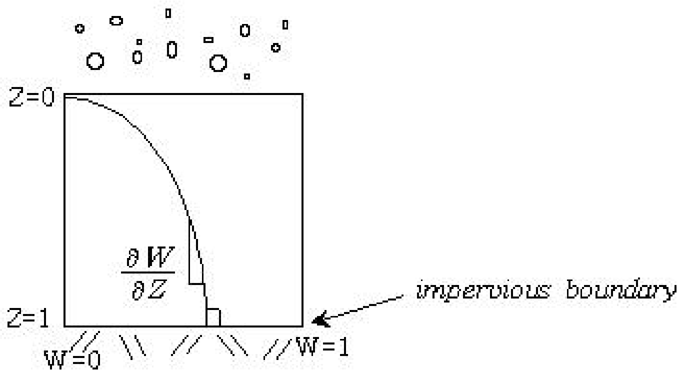
- Then, $W'' = \dot{W}$ [Parabolic 2nd order p.d.e]

- Boundary conditions & Initial conditions

B.C : i) $W(Z=0,\ T)=0$ [top drainage]

ii) $\frac{\partial W}{\partial Z}(Z=1,\ T)=0$ [impervious bottom]

I.C. : i) $W(Z, T=0) = 1$ [initially uniform p.w.p. distribution]



③ Solve the equation by separation of variables

$$W(Z,\ T) \overset{?}{=} W_Z(Z) \cdot W_T(T)$$

$$W'' = W_Z'' W_T$$

$$\dot{W} = W_Z \dot{W}_T$$

$$W_Z'' W_T = W_Z \dot{W}_T$$

$$\frac{\dot{W}_T}{W_T} = \frac{W_Z''}{W_Z} = -k$$

two ordinary d. e.

- $\dot{W}_T = -k\,W_T$

$$W_T = Ae^{-kT}$$

- $W_Z'' = -k\,W_Z$

$$W_Z = C\sin\sqrt{k}Z + D\cos\sqrt{k}Z$$

A, C, D : arbitrary constants

$$W(Z,T) = \frac{4}{\pi}\sum_{m=0}^{\infty}\left[\frac{1}{2m+1}e^{-\frac{\pi^2}{4}(2m+1)^2T}\cdot\sin\frac{\pi}{2}(2m+1)z\right]$$

$$= \frac{4}{\pi}\left[e^{-\frac{\pi^2}{4}T}\cdot\sin\frac{\pi}{2}z + \frac{1}{3}e^{-\frac{9\pi^2}{4}T}\cdot\sin\frac{3\pi}{2}z + \ldots\right]$$

Approximate solutions :

- For large T (>0.2)

$$W(Z,T) \fallingdotseq \frac{4}{\pi}e^{-\frac{\pi^2}{4}T}\sin\frac{\pi}{2}Z$$

[first term]

- For small T (<0.2)

$$W(Z,T) \fallingdotseq \frac{2}{\sqrt{\pi}}\int_0^{\frac{z}{2\sqrt{T}}} e^{-\tau^2}d\tau$$

$$= E_{rf}\left(\frac{z}{2\sqrt{T}}\right)$$

[solution of infinite layer]

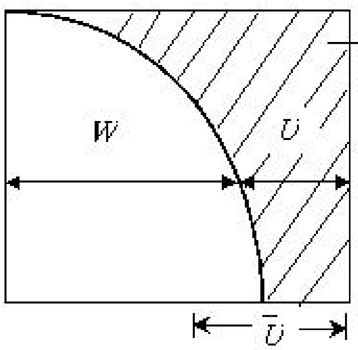
$$E_{rf\,\theta} = \frac{2}{\sqrt{\pi}}\int_0^{\theta} e^{-x^2}dx\,,\quad e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \ldots$$

$$E_{rf\,\theta} = \frac{2}{\sqrt{\pi}}\left(\theta - \frac{1}{3}\theta^3 + \frac{1}{10}\theta^5 - \frac{1}{42}\theta^7 + \ldots\right)$$

④ Degree of Consolidation (($U(Z,T)$))

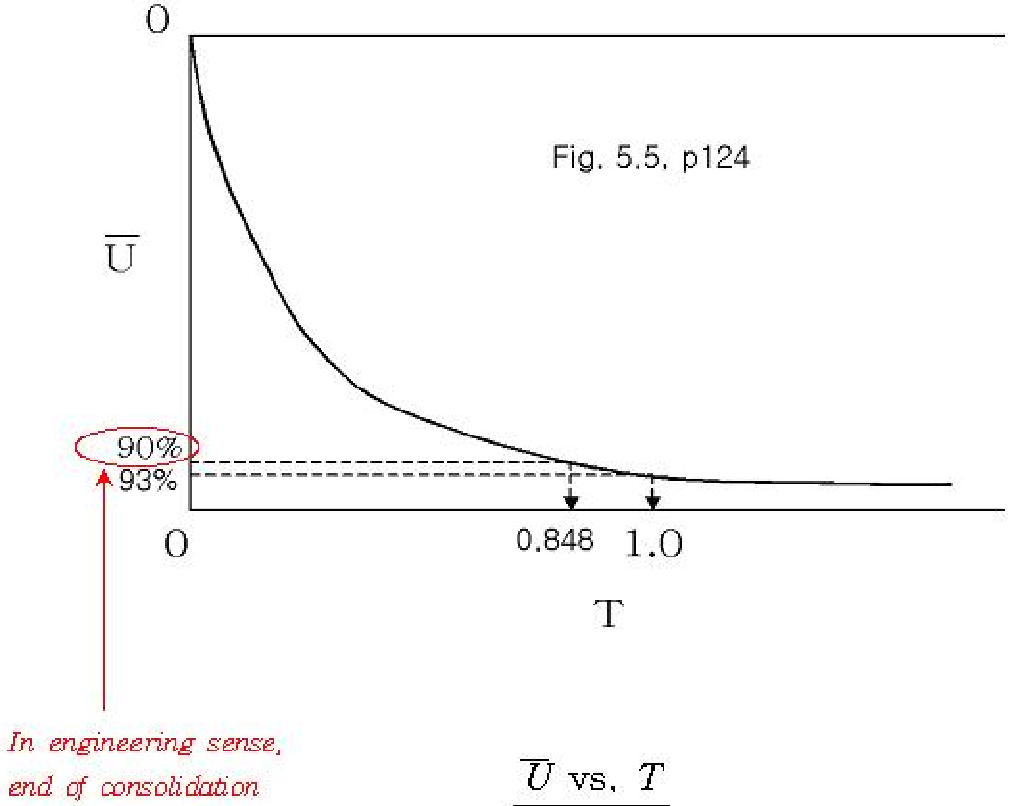
$$U(Z,T) = 1 - W(Z,T)$$

In average,



$$\bar{U} = \frac{1 - \int_0^1 W(Z,T)\,dZ}{\int_0^1 W(Z,0)\,dZ}$$

$$= 1 - \sum_{m=0}^{\infty}\frac{8}{(2m+1)^2\pi^2}e^{-\frac{\pi^2(2m+1)^2}{4}T}$$



$\overline{U}$ vs. T