Unsteady Problems

Introduction

- Time: 4th coordinate direction
- Direction of influence: forcing at a given instant will affect the flow only in the future – no backward influence → Unsteady flows are parabolic-like in time

IVP in ODEs – 2-Level Methods

- Consider 1st order ODE with an initial condition

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = f(t, \phi(t)) \; ; \quad \phi(t_0) = \phi^0 \; . \tag{6.1}$$

- Basically find the solution ϕ at a short time Δt after the initial point. Then the solution at $t_1 = t_0 + \Delta t$, ϕ^1 can be regarded as a new initial condition and the solution can be advanced.

$$\int_{t_n}^{t_{n+1}} \frac{\mathrm{d}\phi}{\mathrm{d}t}\,\mathrm{d}t = \phi^{n+1} - \phi^n = \int_{t_n}^{t_{n+1}} f(t, \phi(t))\,\mathrm{d}t \; , \tag{6.2}$$

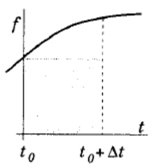
- RHS cannot be evaluated without knowing the solution so some approximation is necessary

SNUTT Seoul National University Towing Tank

2-Level Methods – Cont.

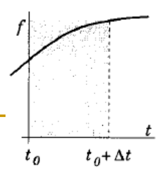
- 4 relatively simple procedures
  - Explicit or forward Euler method

$$\phi^{n+1} = \phi^n + f(t_n, \phi^n)\,\Delta t \; , \tag{6.3}$$



  - Implicit or backward Euler method

$$\phi^{n+1} = \phi^n + f(t_{n+1}, \phi^{n+1})\,\Delta t \; . \tag{6.4}$$

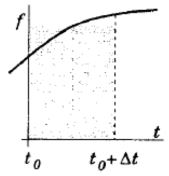


SNUTT Seoul National University Towing Tank

2-Level Methods – Cont.

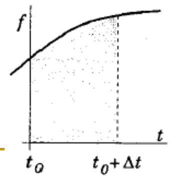
- Midpoint rule – basis of the leapfrog method

$$\phi^{n+1} = \phi^n + f(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}})\,\Delta t\,, \tag{6.5}$$



- Trapezoid rule – basis of the Crank-Nicolson method

$$\phi^{n+1} = \phi^n + \frac{1}{2}\left[f(t_n, \phi^n)) + f(t_{n+1}, \phi^{n+1})\right]\Delta t\,, \tag{6.6}$$



SNUTT Seoul National University Towing Tank

2-Level Methods – Cont.

- 1st method: explicit – conditionally stable
- 2nd – 4th method: implicit – unconditionally stable

SNUTT Seoul National University Towing Tank

IVP in ODEs – Predictor-Corrector Methods

- Predictor-Corrector methods: Combine the best of explicit and implicit methods
  - Explicit methods: easy to program, little memory, little computation time per step, but unstable
  - Implicit methods: require iterative solution, harder to program, more memory, more computation time per time step, but much more stable

SNUTT Seoul National University Towing Tank

Predictor-Corrector Methods – Cont.

- Prediction using the explicit Euler method

$$\phi^{*}_{n+1} = \phi^{n} + f(t_n, \phi^{n})\,\Delta t\,, \tag{6.9}$$

- Correction by the trapezoid rule

$$\phi^{n+1} = \phi^{n} + \frac{1}{2}\left[f(t_n, \phi^{n}) + f(t_{n+1}, \phi^{*}_{n+1})\right]\Delta t\,. \tag{6.10}$$

- 2nd order accurate (trapezoid rule), but stability of the explicit Euler method

SNUTT Seoul National University Towing Tank

IVP in ODEs – Multipoint Methods

- Higher-order approximation – information at more points
  - Multipoint methods: additional points at which data has already been computed
  - Runge-Kutta methods: additional points between t_n and t_{n+1}
  - Adams-Bashforth methods (explicit methods)
    - 1st order: explicit Euler
    - 2nd order:

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}\left[3\,f(t_n,\phi^n) - f(t_{n-1},\phi^{n-1})\right] \qquad (6.11)$$

    - 3rd order:

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{12}\left[23\,f(t_n,\phi^n) - 16\,f(t_{n-1},\phi^{n-1}) + 5\,f(t_{n-2},\phi^{n-2})\right]. \qquad (6.12)$$

SNUTT Seoul National University Towing Tank

Multipoint Methods – Cont.

- Adams-Moulton methods
  - 1st order: implicit Euler
  - 2nd order: trapezoid rule
  - 3rd order:

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{12}\left[5\,f(t_{n+1},\phi^{n+1}) + 8\,f(t_n,\phi^n) - f(t_{n-1},\phi^{n-1})\right]. \qquad (6.13)$$

- Common method: Adams-Bashforth as a predictor & Adams-Moulton as a corrector
- Advantage: easy to use and program, require only one evaluation of the derivative per time step
- Disadvantage: cannot be started using only data at the initial time point

SNUTT Seoul National University Towing Tank

IVP in ODEs – Runge-Kutta Methods

- 2nd order Runge-Kutta:
  - 1st step: ½ step predictor based on explicit Euler
  - 2nd step: midpoint rule corrector

$$\phi^*_{n+\frac{1}{2}} = \phi^n + \frac{\Delta t}{2} f(t_n, \phi^n) , \qquad (6.14)$$

$$\phi^{n+1} = \phi^n + \Delta t \, f(t_{n+\frac{1}{2}}, \phi^*_{n+\frac{1}{2}}) . \qquad (6.15)$$

SNUTT Seoul National University Towing Tank

Runge-Kutta Methods – Cont.

- 4th order Runge-Kutta – most popular
  - 1st step: explicit Euler predictor
  - 2nd step: implicit Euler corrector at $t_{n+1/2}$
  - 3rd step: midpoint rule predictor for full step
  - 4th step: Simpson's rule final corrector (coeffs. 1, 2, 2, 1)

$$\phi^*_{n+\frac{1}{2}} = \phi^n + \frac{\Delta t}{2} f(t_n, \phi^n) , \qquad (6.16)$$

$$\phi^{**}_{n+\frac{1}{2}} = \phi^n + \frac{\Delta t}{2} f(t_{n+\frac{1}{2}}, \phi^*_{n+\frac{1}{2}}) , \qquad (6.17)$$

$$\phi^*_{n+1} = \phi^n + \Delta t \, f(t_{n+\frac{1}{2}}, \phi^{**}_{n+\frac{1}{2}}) , \qquad (6.18)$$

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{6} \Big[f(t_n, \phi^n) + 2 f(t_{n+\frac{1}{2}}, \phi^*_{n+\frac{1}{2}}) + 2 f(t_{n+\frac{1}{2}}, \phi^{**}_{n+\frac{1}{2}}) + f(t_{n+1}, \phi^*_{n+1}) \Big] . \qquad (6.19)$$

TT niversity Towing Tank

IVP in ODEs – Other Methods

- Implicit 3-level 2nd order scheme

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)_{n+1} \approx \frac{3\,\phi^{n+1} - 4\,\phi^n + \phi^{n-1}}{2\,\Delta t} \,. \qquad (6.20)$$

- This leads to

Implicit

$$\phi^{n+1} = \frac{4}{3}\phi^n - \frac{1}{3}\phi^{n-1} + \frac{2}{3} f(t_{n+1}, \boxed{\phi^{n+1}})\,\Delta t \,. \qquad (6.21)$$

SNUTT Seoul National University Towing Tank

Application to Generic Transport Eqn

- Discretization of convective and diffusive fluxes and source terms are same as for steady problems. However, the question of the time at which the fluxes and sources are to be evaluated must be answered.

$$\frac{\partial(\rho\phi)}{\partial t} = -\mathrm{div}\,(\rho\phi\boldsymbol{v}) + \mathrm{div}\,(\Gamma\,\mathrm{grad}\,\phi) + q_\phi = f(t, \phi(t)) \,, \qquad (6.22)$$

SNUTT Seoul National University Towing Tank

Transport Eqn – Explicit Methods

- Explicit Euler methods
  - All fluxes and sources are evaluated using known values at t_n.
  - Consider 1D version of Eq. (6.22) – often model eqn for the N-S eqns

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} + \frac{\Gamma}{\rho}\frac{\partial^2 \phi}{\partial x^2} . \qquad (6.23)$$

  - With CDS and uniform grid

$$\phi_i^{n+1} = \phi_i^n + \left[-u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\,\Delta x} + \frac{\Gamma}{\rho}\frac{\phi_{i+1}^n + \phi_{i-1}^n - 2\,\phi_i^n}{(\Delta x)^2}\right]\Delta t , \qquad (6.24)$$

  - Can be rewritten

$$\phi_i^{n+1} = (1-2d)\,\phi_i^n + \left(d - \frac{c}{2}\right)\phi_{i+1}^n + \left(d + \frac{c}{2}\right)\phi_{i-1}^n , \qquad (6.25)$$

$$d = \frac{\Gamma\,\Delta t}{\rho(\Delta x)^2} \quad \text{and} \quad \boxed{c = \frac{u\,\Delta t}{\Delta x}} . \qquad (6.26)$$

Courant number

Explicit Methods – Cont.

- Explicit Euler methods – Cont.
  - To improve instability, UDS for convection

$$\phi_i^{n+1} = \phi_i^n + \left[-u\frac{\phi_i^n - \phi_{i-1}^n}{\Delta x} + \frac{\Gamma}{\rho}\frac{\phi_{i+1}^n + \phi_{i-1}^n - 2\,\phi_i^n}{(\Delta x)^2}\right]\Delta t . \qquad (6.34)$$

  - which leads to

$$\phi_i^{n+1} = (1 - 2d - c)\phi_i^n + d\phi_{i+1}^n + (d+c)\phi_{i-1}^n . \qquad (6.35)$$

  - Courant number should be smaller than unity → a fluid particle cannot move more than one grid length in a single time step

Explicit Methods – Cont.

- Leapfrog method
  - 3-level scheme – midpoint rule integration to a time interval of size $2\Delta t$

$$\phi_i^{n+1} = \phi_i^{n-1} + f(t_n, \phi^n)\, 2\Delta t \ . \qquad (6.38)$$

  - With CDS

$$\phi_i^{n+1} = \phi_i^{n-1} + \left[-u \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\,\Delta x} + \frac{\Gamma}{\rho} \frac{\phi_{i+1}^n + \phi_{i-1}^n - 2\,\phi_i^n}{(\Delta x)^2} \right] 2\Delta t \ . \qquad (6.39)$$

  - Or

$$\phi_i^{n+1} = \phi_i^{n-1} - 4d\phi_i^n + (2d - c)\phi_{i+1}^n + (2d + c)\phi_{i-1}^n \ . \qquad (6.40)$$

  - Unconditionally unstable method, but the instability is very weak.

SNUTT
Seoul National University
Towing Tank

Explicit Methods – Cont.

- One way to stabilize the scheme

$$\phi_i^n \approx \frac{1}{2}\left(\phi_i^{n-1} + \phi_i^{n+1}\right) \ , \qquad (6.41)$$

  - DuFort-Frankel method

$$(1 + 2d)\phi_i^{n+1} = (1 - 2d)\phi_i^{n-1} + (2d - c)\phi_{i+1}^n + (2d + c)\phi_{i-1}^n \ . \qquad (6.42)$$

SNUTT
Seoul National University
Towing Tank

Transport Eqn – Implicit Methods

- Implicit Euler method
  - If stability is a prime requirement

$$\phi_i^{n+1} = \phi_i^n + \left[-u \frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\,\Delta x} + \frac{\Gamma}{\rho} \frac{\phi_{i+1}^{n+1} + \phi_{i-1}^{n+1} - 2\,\phi_i^{n+1}}{(\Delta x)^2} \right] \Delta t \quad ,(6.43)$$

  - Or

$$(1+2d)\,\phi_i^{n+1} + \left(\frac{c}{2} - d\right)\phi_{i+1}^{n+1} + \left(-\frac{c}{2} - d\right)\phi_{i-1}^{n+1} = \phi_i^n \,. \qquad (6.44)$$

  - The resulting system of algebraic eqns is very similar to the one obtained for steady problems, only difference in A_P and Q_P
  -

Implicit Methods – Cont.

- Implicit Euler method

$$A_{\mathrm{P}}\phi_i^{n+1} + A_{\mathrm{E}}\phi_{i+1}^{n+1} + A_{\mathrm{W}}\phi_{i-1}^{n+1} = Q_{\mathrm{P}} \,, \qquad (6.45)$$

where

$$A_{\mathrm{E}} = \frac{\rho u}{2\,\Delta x} - \frac{\Gamma}{(\Delta x)^2} \,; \quad A_{\mathrm{W}} = -\frac{\rho u}{2\,\Delta x} - \frac{\Gamma}{(\Delta x)^2} \,;$$
$$A_{\mathrm{P}} = -(A_{\mathrm{E}} + A_{\mathrm{W}}) + \frac{\rho}{\Delta t} \,; \quad Q_{\mathrm{P}} = \frac{\rho}{\Delta t}\phi_i^n \qquad (6.46)$$

  - With CDS on coarse grids, oscillatory solutions are produced.
  - Disadvantages: 1st order in time, solve a large coupled set of eqns at each time step, more storage
  - Advantage: possible to use large time step

Implicit Methods – Cont.

- Crank-Nicolson method
  - Utilize 2nd order accuracy and relative simplicity of the trapezoid rule

n+1

$$\phi_i^{n+1} = \phi_i^n + \frac{\Delta t}{2}\boxed{\left[-u\frac{\phi_{i+1}^{n+1}-\phi_{i-1}^{n+1}}{2\Delta x} + \frac{\Gamma}{\rho}\frac{\phi_{i+1}^{n+1}+\phi_{i-1}^{n+1}-2\phi_i^{n+1}}{(\Delta x)^2}\right]} + \frac{\Delta t}{2}\boxed{\left[-u\frac{\phi_{i+1}^{n}-\phi_{i-1}^{n}}{2\Delta x} + \frac{\Gamma}{\rho}\frac{\phi_{i+1}^{n}+\phi_{i-1}^{n}-2\phi_i^{n}}{(\Delta x)^2}\right]} . \quad (6.48)$$

n

Implicit Methods – Cont.

- Crank-Nicolson method – Cont.

$$A_P\phi_i^{n+1} + A_E\phi_{i+1}^{n+1} + A_W\phi_{i-1}^{n+1} = Q_i^t , \quad (6.49)$$

where:

$$A_E = \frac{\rho u}{4\,\Delta x} - \frac{\Gamma}{2(\Delta x)^2} ; \quad A_W = -\frac{\rho u}{4\,\Delta x} - \frac{\Gamma}{2(\Delta x)^2} ,$$
$$A_P = \frac{\rho}{\Delta t} - (A_E + A_W) , \quad (6.50)$$
$$Q_i^t = \left(A_W + A_E + \frac{\rho}{\Delta t}\right)\phi_i^n - A_E\phi_{i+1}^n - A_W\phi_{i-1}^n .$$

  - Q_i^t represents an additional source term, which contains the contribution from the previous time level.
  - Can be regarded as an equal blend of 1st order explicit Euler and implicit Euler schemes.

Implicit Methods – Cont.

- 3 Time Level Method

$$\rho\,\frac{3\,\phi_i^{n+1}-4\,\phi_i^{n}+\phi_i^{n-1}}{2\Delta t}\Delta t = \left[-\rho u\,\frac{\phi_{i+1}^{n+1}-\phi_{i-1}^{n+1}}{2\Delta x}+\Gamma\,\frac{\phi_{i+1}^{n+1}+\phi_{i-1}^{n+1}-2\phi_i^{n+1}}{(\Delta x)^2}\right]\Delta t\,. \quad (6.51)$$

  - Resulting algebraic equation

$$A_{\mathrm{P}}\phi_i^{n+1}+A_{\mathrm{E}}\phi_{i+1}^{n+1}+A_{\mathrm{W}}\phi_{i-1}^{n+1}=\frac{2\rho}{\Delta t}\phi_i^{n}-\frac{\rho}{2\,\Delta t}\phi_i^{n-1}\,. \quad (6.52)$$

  - The central coeff has a stronger contribution

$$A_{\mathrm{P}}=-(A_{\mathrm{E}}+A_{\mathrm{W}})+\frac{3\rho}{2\,\Delta t}\,, \quad (6.53)$$

SNUTT Seoul National University Towing Tank

Examples

- Read through!

SNUTT Seoul National University Towing Tank