II. Interpolation

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Interpolation

☐ Introduction to Interpolation

- **Approximation of Function**
- **Interpolation and Polynomial Approximation**

☐ Polynomial Interpolation

- **Lagrange Interpolation**
- **Newton Interpolation**
- **Hermite Interpolation**

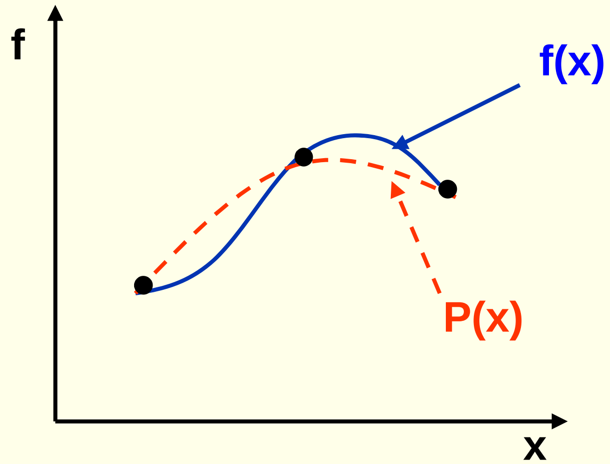
☐ Piecewise Polynomial Interpolation

- **Piecewise Linear Interpolation**
- **Cubic Spline Interpolation**

Approximation of Function

☐ What is approximation of a function?

- **Approximate a true function f(x) by an easily manipulated, lower order func. P(x)**



☐ Two Forms of Approximate Function P(x)

- **Linear Combination**

$$P(x) = a_0 g_0(x) + a_1 g_1(x) + \cdots + a_n g_n(x)$$

- **Rational Form**

$$P(x) = \frac{b_0 g_0(x) + b_1 g_1(x) + \cdots + b_n g_n(x)}{a_0 g_0(x) + a_1 g_1(x) + \cdots + a_m g_m(x)}$$

☐ Types of Approximation Problems

- **Interpolation of tabulated data, passing through all data points given**
- **Curve Fitting of experimental or uncertain data with least squared error**
- **Minimize the maximum error of approximation (minimax)**

Polynomial Interpolation

☐ What is polynomial interpolation?

- Given n+1 base points

x_i	x_0	x_1	…	x_n
f_i	f_0	f_1	…	f_n

- Find a function passing through all given points by a polynomial

$$f(x) \approx P_n(x) = \sum_{i=0}^{n} a_i x^i$$

☐ Needs

- Replace f(x), which would be difficult to evaluate and manipulate, by a simpler, more amenable function P(x)
- Estimate the functional values, derivatives or integrals of f(x) which is known quantitatively for a finite number of arguments called base points

☐ Forms of Polynomial

- Power Form $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
- Shifted Power Form $P(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$
- Newton Form $P(x) = a_0 + a_1(x-c_1) + a_2(x-c_1)(x-c_2) + \cdots + a_n(x-c_1)\cdots(x-c_n)$

Lagrange Interpolation

□ Lagrange Polynomial Theorem

If *f(x)* is a real-valued function whose values are given at the n+1 distinct points, $x_0, x_1, \ldots, x_n$, then

there exists a unique polynomial *P(x)* of degree at most n such that

$$f(x_k) = P(x_k) \quad \forall\ k = 0, 1, \cdots, n$$

where

$$P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

and Lagrange Kernel

$$L_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)} = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

Derivation of Lagrange Interpolation Formula

Let $f(x) \cong P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \Leftrightarrow \mathbf{p}^T\boldsymbol{\alpha}$ $\quad \mathbf{p}^T = \begin{bmatrix} 1 & x & x^2 \cdots x^n \end{bmatrix} \quad \boldsymbol{\alpha} = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$

At n+1 points given, require

$$f_k = f(x_k) = P_n(x_k), \ \forall k$$

$$\begin{aligned} f_0 &= a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^2 \\ &\vdots \\ f_n &= a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^2 \end{aligned} \Leftrightarrow \mathbf{f} = \mathbf{G}\boldsymbol{\alpha} \downarrow \boldsymbol{\alpha} = \mathbf{G}^{-1}\mathbf{f}$$

$$\mathbf{f} = \begin{bmatrix} f_0 \\ \vdots \\ f_n \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 1 & x_0 & x_0^2 \cdots & x_0^{n-1} & x_0^n \\ \vdots & & \ddots & & \vdots \\ 1 & x_n & x_n^2 \cdots & x_n^{n-1} & x_n^n \end{bmatrix}$$

$$P_n(x) = \mathbf{p}^T\boldsymbol{\alpha} = \overbrace{\mathbf{p}^T\mathbf{G}^{-1}}^{\mathbf{l}^T}\mathbf{f} = \mathbf{l}^T\mathbf{f} = \sum_{i=0}^{n} L_i(x) f_i$$

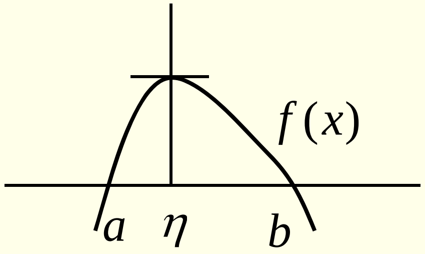
$$\mathbf{p}^T\mathbf{G}^{-1} = \begin{bmatrix} L_0(x) & \cdots & L_n(x) \end{bmatrix}, \ \mathbf{l} = \begin{bmatrix} L_0(x) \\ \vdots \\ L_n(x) \end{bmatrix},$$

Constraint: $f_k = P_n(x_k) = \sum_{i=0}^{n} L_i(x_k) f_i \ \forall k$ $\rightarrow L_i(x_k) = \delta_{ik}$

Let $L_i(x) = C_i \prod_{\substack{k=0 \\ k \neq i}}^{n} (x - x_k)$ $\rightarrow$ $C_i = \dfrac{1}{\prod_{\substack{k=0 \\ k \neq i}}^{n} (x_i - x_k)}$

$$\therefore L_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)}$$

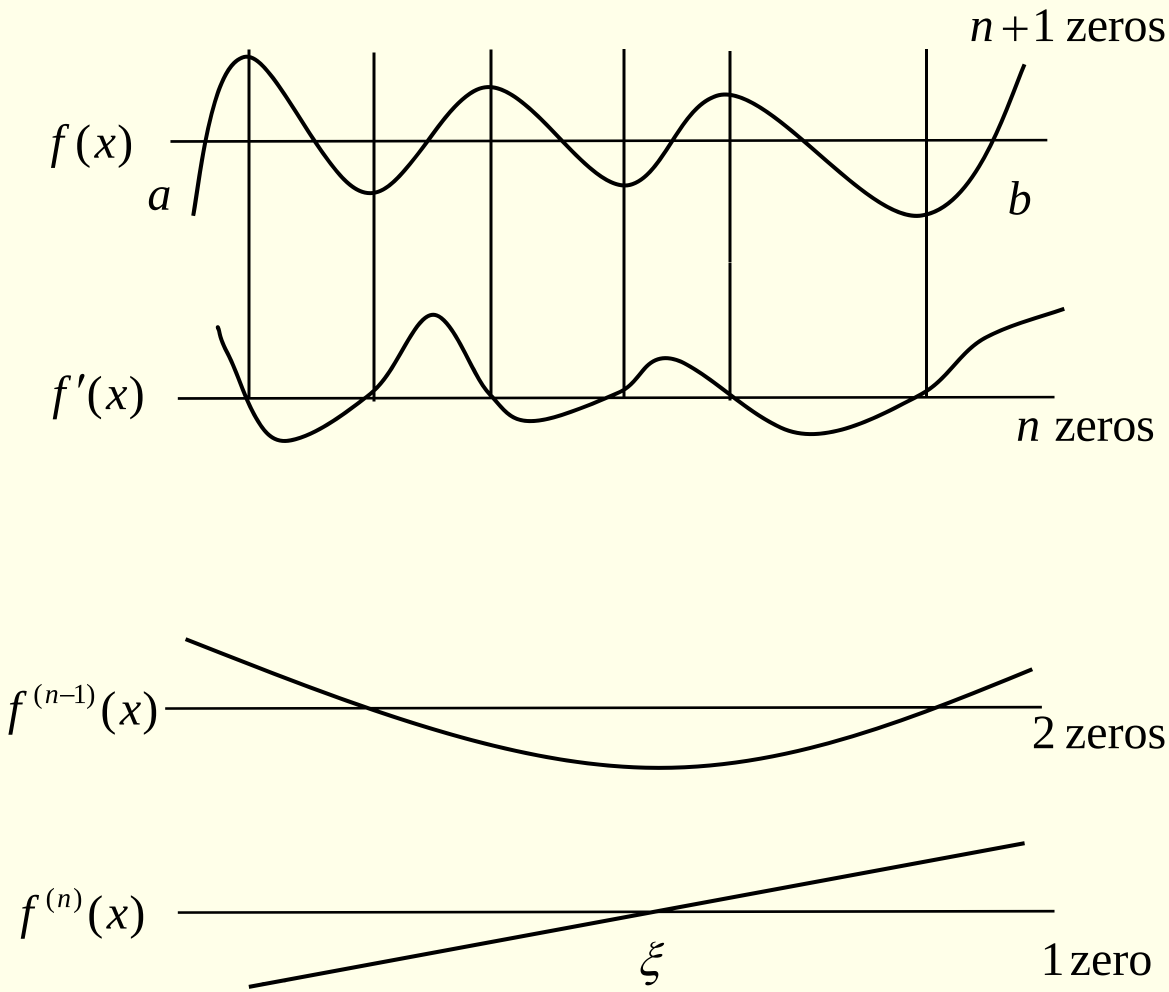
Rolle's Theorem



There exists $\xi \in (a,b)$
for which $f'(\xi) = 0$.

If there are n+1zeros of $f(x)$,
$x_0, \cdots, x_n$, then there is
a point within $[x_0, x_n]$
such that $f^{(n)}(\xi) = 0$

Ex. for n = 6, 7 zeros for g(t)

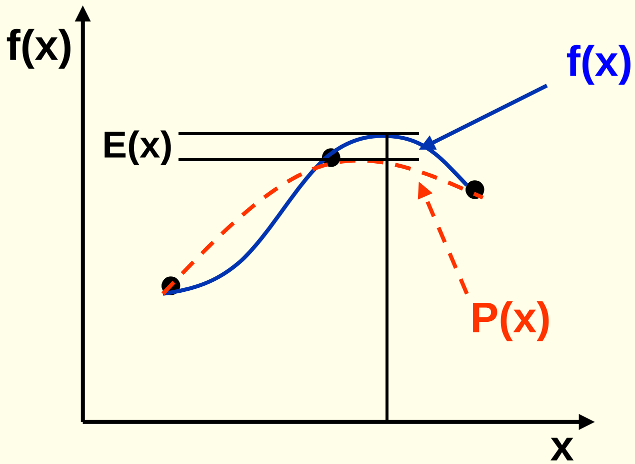


Error of Lagrange Interpolation

Let $f(x) = P_n(x) + E(x)$ $\rightarrow E(x) = f(x) - P_n(x)$

$E(x_k) = 0\ ,\quad k = 0,1,\ldots,n$

$$\rightarrow E(x) = S(x)\cdot\prod_{i=0}^{n}(x - x_i)$$



Define $g(t) = f(t) - P_n(t) - S(x)\cdot\prod_{i=0}^{n}(t - x_i)$ $t \in [a,b]$

$[x_0, x_n] \in [a,b]$

1) $g(x_k) = 0 \quad k = 0,1,\ldots,n$

2) $g(x) = 0$

$\rightarrow n+2$ zeros in $[a,b]$

Rolle's Theorem: $g^{(n+1)}(\xi) = 0 = f^{(n+1)}(\xi) - 0 - (n+1)!\cdot S(x)$ for some $\xi = \xi(x)$

$$\rightarrow S(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \quad \xi \in (a,b)$$

$\therefore E(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\cdot\prod_{i=0}^{n}(x - x_i)$ at least order of n+1 on x

Newton Interpolation

☐ Drawbacks of Lagrange Interpolation

- <u>Excessive amount of calculation</u> is required when many interpolations are to be done using the same data set.
- <u>No estimated error</u> can be made, unless the high order derivatives can be evaluated.
- The addition of a new term requires <u>complete recomputation</u>.
- These are avoided by Divided Difference scheme.

☐ Divided Difference (차분상)

- Definition

1. $f[x_i] = f(x_i)$
2. $f[x_i, x_j] = \dfrac{f[x_i] - f[x_j]}{x_i - x_j} = \dfrac{f(x_i)}{x_i - x_j} + \dfrac{f(x_j)}{x_j - x_i}$
3. $f[x_i, x_j, x_k] = \dfrac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$ The order of x_i's in [...] does not matter.
4. …..

$$= \frac{f(x_i)}{(x_i - x_j)(x_i - x_k)} + \frac{f(x_j)}{(x_j - x_i)(x_j - x_k)} + \frac{f(x_k)}{(x_k - x_i)(x_k - x_j)}$$

Newton Interpolation

Let $f(x) = P_n(x) + E(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n \prod_{k=0}^{n-1}(x-x_k) + E(x)$

$f(x_0) = P_n(x_0) = a_0 + E(x_0)$ $\rightarrow$ Require $E(x_0) = 0 \rightarrow a_0 = f[x_0]$

$$f[x] = f[x_0] + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) + E(x)$$

Divide by $x - x_0$ after moving $f[x_0]$ to LHS

$$\frac{f[x]-f[x_0]}{x-x_0} \equiv f[x,x_0] = a_1 + a_2(x-x_1) + \cdots + a_n(x-x_1)\cdots(x-x_{n-1}) + \frac{E(x)}{x-x_0}$$

Insert $x = x_1 \rightarrow a_1 = f[x_1,x_0] = f[x_0,x_1]$

$$\rightarrow f[x,x_0] = f[x_0,x_1] + a_2(x-x_1) + \cdots + a_n(x-x_1)\cdots(x-x_{n-1}) + \frac{E(x)}{(x-x_0)}$$

$$\frac{f[x,x_0]-f[x_0,x_1]}{x-x_1} = f[x,x_0,x_1] = a_2 + \cdots + a_n(x-x_2)\cdots(x-x_{n-1}) + \frac{E(x)}{(x-x_0)(x-x_1)}$$

$\rightarrow x = x_2 \rightarrow a_2 = f[x_2,x_1,x_0] = f[x_0,x_1,x_2]$

$$\rightarrow f[x,x_0,x_1] = f[x_0,x_1,x_2] + a_3(x-x_3) + \cdots + a_n(x-x_2)\cdots(x-x_{n-1}) + \frac{E(x)}{(x-x_0)(x-x_1)}$$

In general, $a_n = f[x_0,x_1,\cdots x_{n-1},x_n]$

$$\rightarrow f[x,x_0,\cdots,x_{n-1}] = f[x_0,x_1,\cdots,x_n] + \frac{E(x)}{\prod_{k=0}^{n-1}(x-x_k)}$$

$$\rightarrow E(x) = \frac{f[x,x_0,\cdots,x_{n-1}] - f[x_0,x_1,\cdots,x_n]}{x_n - x_0}\prod_{k=0}^{n}(x-x_k) = f[x,x_0,\cdots,x_n]\prod_{k=0}^{n}(x-x_k)$$

$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + \cdots + f[x_0,x_1,\cdots,x_{n-1}]\prod_{k=0}^{n}(x-x_k)$$

More About Divided Difference

$$f[x_0] = f(x_0) = f_0$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$\vdots$$

n-th order D.D (n계 차분상)

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}$$: Definition

$$= \sum_{i=0}^{n} f_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j}$$

Proof

$$f[x_0, x_1] = \frac{f_1}{x_1 - x_0} - \frac{f_0}{x_1 - x_0} = \frac{f_1}{x_1 - x_0} + \frac{f_0}{x_0 - x_1}$$

Let

$$f[x_0, \cdots, x_k] = \sum_{i=0}^{k} f_i \prod_{\substack{j=0 \\ j \neq i}}^{k} \frac{1}{x_i - x_j}$$

More About Divided Difference

By Definition

$$f[x_0,\cdots,x_k,x_{k+1}]$$

$$=\frac{f[x_1,\cdots,x_{k+1}]-f[x_0,\cdots,x_k]}{x_{k+1}-x_0}$$

$$=\frac{1}{x_{k+1}-x_0}\left\{\sum_{i=1}^{k+1} f_i\prod_{\substack{j=1\\ j\neq i}}^{k+1}\frac{1}{x_i-x_j}-\sum_{i=0}^{k} f_i\prod_{\substack{j=0\\ j\neq i}}^{k}\frac{1}{x_i-x_j}\right\}$$

$$=\frac{1}{x_{k+1}-x_0}\left\{f_{k+1}\prod_{j=1}^{k}\frac{1}{x_{k+1}-x_j}+\sum_{i=1}^{k} f_i\prod_{\substack{i=1\\ j\neq i}}^{k+1}\frac{1}{x_i-x_j}-\sum_{i=1}^{k} f_i\prod_{\substack{i=0\\ j\neq i}}^{k}\frac{1}{x_i-x_j}-f_0\prod_{j=1}^{k}\frac{1}{x_0-x_j}\right\}$$

$$=f_{k+1}\prod_{j=0}^{k}\frac{1}{x_{k+1}-x_j}+\sum_{i=1}^{k} f_i\left[\left(\prod_{\substack{i=1\\ j\neq i}}^{k}\frac{1}{x_i-x_j}\right)\cdot\frac{1}{x_i-x_{k+1}}-\left(\prod_{\substack{i=1\\ j\neq i}}^{k}\frac{1}{x_i-x_j}\right)\cdot\frac{1}{x_i-x_0}\right]\cdot\frac{1}{x_{k+1}-x_0}+f_0\prod_{j=0}^{k+1}\frac{1}{x_0-x_j}$$

$$=f_{k+1}\prod_{\substack{j=0\\ j\neq k+1}}^{k+1}\frac{1}{x_{k+1}-x_j}+\sum_{i=1}^{k} f_i\prod_{\substack{i=1\\ j\neq i}}^{k}\frac{1}{x_i-x_j}\left(\frac{1}{x_i-x_{k+1}}-\frac{1}{x_i-x_0}\right)\cdot\frac{1}{x_{k+1}-x_0}+f_0\prod_{j=1}^{k+1}\frac{1}{x_0-x_j}$$

More About Divided Difference

$$= f_{k+1}\prod_{\substack{j=0\\ j\neq k+1}}^{k+1}\frac{1}{x_{k+1}-x_j}+\sum_{i=1}^{k} f_i\prod_{\substack{i=1\\ j\neq i}}^{k}\frac{1}{x_i-x_j}\cdot\frac{x_{k+1}-x_0}{(x_i-x_0)(x_i-x_{k+1})}\cdot\frac{1}{x_{k+1}-x_0}+f_0\prod_{j=1}^{k+1}\frac{1}{x_0-x_j}$$

$$= f_{k+1}\prod_{\substack{j=0\\ j\neq k+1}}^{k+1}\frac{1}{x_{k+1}-x_j}+\sum_{i=1}^{k} f_i\prod_{i=0}^{k+1}\frac{1}{x_i-x_j}+f_0\prod_{j=1}^{k+1}\frac{1}{x_0-x_j}$$

$$=\sum_{i=0}^{k+1} f_i\prod_{\substack{j=0\\ j\neq i}}^{k+1}\frac{1}{x_i-x_j}\quad Q.E.D.$$

is unchanged if x_i values are given regardless of the order of x_i's.

$$\begin{array}{lllll}
x_0: & y_0=[y_0] & & & \\
 & & [y_0,y_1] & & \\
x_1: & y_1=[y_1] & & [y_0,y_1,y_2] & \\
 & & [y_1,y_2] & & [y_0,y_1,y_2,y_3] \\
x_2: & y_2=[y_2] & & [y_1,y_2,y_3] & \\
 & & [y_2,y_3] & & \\
x_3: & y_3=[y_3] & & &
\end{array}$$

Properties of Divided Difference

Polynomial

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$$

$$= a_0 + a_1(x - x_0) + \cdots + a_n\prod_{i=0}^{n-1}(x - x_i) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i) \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

Divide once by $(x - x_0)$

$$f[x, x_0] = a_1 + a_2(x - x_1) + \cdots + a_n\prod_{i=1}^{n-1}(x - x_i) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=1}^{n}(x - x_i)$$

$$= f[x_0, x_1] + a_2(x - x_1) + \cdots$$

Divide (n-1)times

$$f[x, x_0, \cdots, x_{n-2}] = f[x_0, x_1, \cdots, x_{n-1}] + a_n(x - x_{n-1}) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_{n-1})(x - x_n)$$

$$\frac{f[x, x_0, \cdots, x_{n-2}] - f[x_0, x_1, \cdots, x_{n-1}]}{x - x_{n-1}} = a_n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_n) \qquad \rightarrow a_n = f[x_0, x_1, \cdots, x_n]$$

$$f[x, x_0, \cdots, x_{n-1}] = f[x_0, x_1, \cdots, x_{n-1}, x_n] + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_n)$$

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} = f[x, x_0, \cdots, x_{n-1}, x_n] \rightarrow \text{Exact Error at x}$$

Hermite Polynomial

- Objective: Find a polynomial satisfying the derivative as well as function value

x_i	x_0	x_1	…	x_n
f_i	f_0	f_1	…	f_n
f'_i	f'_0	f'_1	…	f'_n

(2n+2 constraints)

Define a (2n+1)-th order polynomial as: $P(x)=\sum_{i=0}^{n} f_i H_i(x)+\sum_{i=1}^{n} f_i' \hat{H}_i(x)$

Conditions, $\forall$ j(=0,1,$\cdots$,n)

① $H_i(x_j)=\delta_{ij}$

② $\hat{H}_i(x_j)=0$

③ $\hat{H}_i'(x_j)=\delta_{ij}$

④ $H_i'(x_j)=0$

null after differentiation $\forall j$ except i

②, ③

$$\rightarrow \hat{H}_i(x)=c(x-x_0)^2(x-x_1)^2\cdots(x-x_{i-1})^2(x-x_i)(x-x_{i+1})^2\cdots(x-x_n)^2 = c(x-x_i)\prod_{\substack{k=0\\k\neq i}}^{n}(x-x_k)^2$$

$$\hat{H}'(x)=c\prod_{\substack{k=0\\k\neq i}}^{n}(x-x_k)^2+c(x-x_i)\left(\prod_{\substack{i=0\\j\neq i}}^{n}(x-x_j)^2\right)'$$

Hermite Polynomial

$$\hat{H}'(x_i) = c\prod_{\substack{j=0\\j\neq i}}^{n}(x-x_j)^2 + c(x_i - x_i)\left(\prod_{\substack{i=0\\j\neq i}}^{n}(x-x_j)\right)' = 1 \qquad \rightarrow c = \frac{1}{\prod_{\substack{j=0\\j\neq i}}^{n}(x_i-x_j)^2}$$

$$\therefore \hat{H}_i(x) = \frac{1}{\prod_{\substack{j=0\\j\neq i}}^{n}(x_i-x_j)^2}(x-x_i)\prod_{\substack{k=0\\k\neq i}}^{n}(x-x_k)^2 = (x-x_i)\prod_{\substack{k=0\\k\neq i}}^{n}\frac{(x-x_k)^2}{(x_i-x_j)^2} = (x-x_i)L_i^2(x)$$

$Note: L_i(x_j) = \delta_{ij}$

Let $H_i(x) = (ax+b)L_i^2(x)$

$$H_i(x_j) = (ax_j+b)\delta_{ij} \quad \forall j \qquad j=i:\ ax_i + b = 1$$

$$H_i'(x) = aL_i^2(x) + (ax+b)\cdot 2L_i(x)L_i'(x) \qquad H_i'(x_j) = a\delta_{ij} + (ax_j+b)\cdot 2\delta_{ij}L_i'(x_j)$$

$\underline{j=i}$ $\quad a + 2\underbrace{(ax_i+b)}_{1}L_i'(x_i) = 0$

$$a = -2L_i'(x_i) \qquad b = 1 - ax_i$$

$$\therefore H_i(x) = \left(1-2L_i'(x_i)(x-x_i)\right)L_i^2(x)$$

Error $E(x) = \dfrac{f^{(2n+2)}(\xi)}{(2n+2)!}\prod_{j=0}^{n}(x-x_j)^2$

Piecewise Polynomial Interpolation

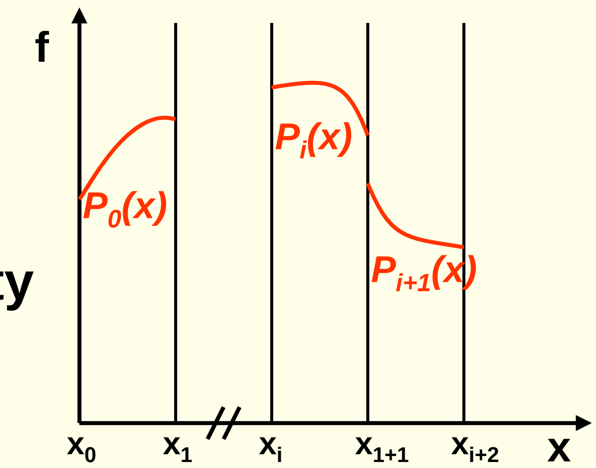
□Why piecewise polynomial pnterpolation?

- The oscillatory nature of high-degree polynomials and the property that a fluctuation over a small portion of interval can induce large fluctuations over the entire range restricts their use.
- This form is more useful for seeking the numerical approximation for the solution of the system equations.

□What is piecewise interpolation polynomial?

- Let

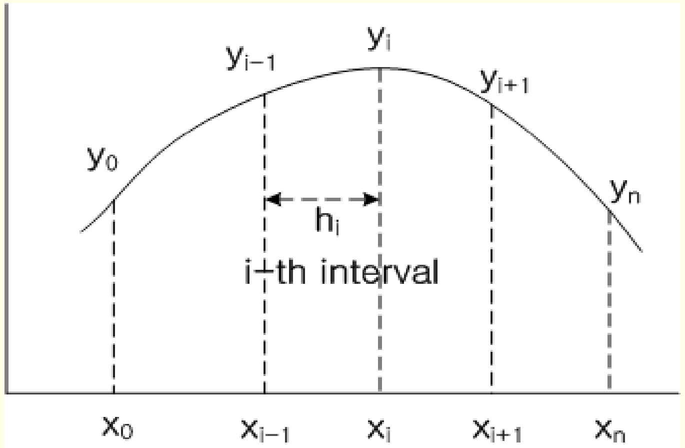
$$f(x) \cong pp(x) = \begin{cases} P_1(x) & x \in [x_0, x_1] \\ P_2(x) & x \in [x_1, x_2] \\ \vdots & \\ P_n(x) & x \in [x_{n-1}, x_n] \end{cases} \quad \Leftrightarrow \quad pp(x) = P_i(x) \quad x \in [x_{i-1}, x_i]$$



- polynomial order depends on continuity requirements

$$f(x_i) = pp(x_i) \qquad i = 0, 1, 2, \cdots, n$$
$$f'(x_i) = pp'(x_i)$$
$$f''(x_i) = pp''(x_i)$$

Cubic Spline



Let

$$P_i(x) = a_i + b_i(x - x_{i-1}) + c_i(x - x_{i-1})^2 + d_i(x - x_{i-1})^3$$

i) $P_i(x_{i-1}) = a_i = y_{i-1}$

$$P_i(x_i) = y_{i-1} + b_i h_i + c_i h_i^2 + d_i h_i^3 = y_i \quad (i = 1, \cdots, n)$$

(function value on the right end)

$$h_i b_i + h_i^2 c_i + h_i^3 d_i = y_i - y_{i-1} \cdots (1)$$

ii) $P_i'(x) = b_i + 2c_i(x - x_{i-1}) + 3d_i(x - x_{i-1})^2$

- Continuity of slope

$$P_i'(x_i) = P_{i+1}'(x_i) : \qquad b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1} \qquad b_i + 2c_i h_i + 3d_i h_i^2 - b_{i+1} = 0 \cdots (2)$$

iii) $P_i''(x) = 2c_i + 6d_i(x - x_{i-1})$

- Continuity of second derivative

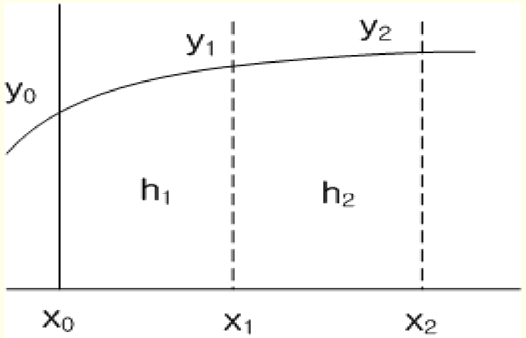
$$P_i''(x_i) = P_{i+1}''(x_i)$$

$$2c_i + 6d_i h_i = 2c_{i+1} \qquad \rightarrow c_i - 3h_i d_i - c_{i+1} = 0 \cdots (3)$$

Cubic Spline

unknowns	4n			
Function values	n+1	4n-2		$\rightarrow$ 2 constraints missing
continuity	3(n-1)			

Use two slopes at the ends



$$f(x) = y_o \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= y_o \frac{(x-x_1)(x-x_2)}{h_1(h_1+h_2)} - y_1 \frac{(x-x_0)(x-x_2)}{h_1 h_2} + y_2 \frac{(x-x_0)(x-x_1)}{h_2(h_1+h_2)}$$

$$f'(x) = y_o \frac{x-x_2+x-x_1}{h_1(h_1+h_2)} - y_1 \frac{x-x_2+x-x_0}{h_1 h_2} + y_2 \frac{x-x_1+x-x_0}{h_2(h_1+h_2)}$$

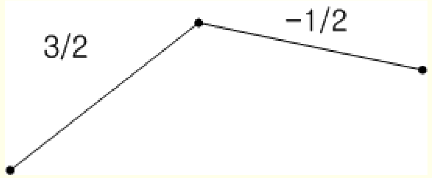
$$f'(x_0) = y_o \frac{-(h_1+h_2)-h_1}{h_1(h_1+h_2)} + y_1 \frac{h_1+h_2}{h_1 h_2} + y_2 \frac{-h_1}{h_2(h_1+h_2)}$$

$$= -y_o\left(\frac{1}{h_1+h_2}+\frac{1}{h_1}\right) + y_1\left(\frac{1}{h_1}+\frac{1}{h_2}\right) + y_2\left(\frac{1}{h_1+h_2}-\frac{1}{h_2}\right)$$

$$= \frac{1}{h_1+h_2}\left(-y_0(2+\gamma) + y_1(2+\gamma+\frac{1}{\gamma}) - y_2\frac{1}{\gamma}\right) \quad \leftarrow \gamma = \frac{h_2}{h_1}$$

Cubic Spline

$$f'(x_0)=\frac{1}{h_1+h_2}\left(-y_0(2+\gamma)+y_1(2+\gamma+\frac{1}{\gamma})-y_2\frac{1}{\gamma}\right)$$



$$if \quad h_1=h_2=h\rightarrow\gamma=1$$

$$y_0'=f'(x_0)=\frac{1}{h}\left(-y_0\frac{3}{2}+y_1\cdot 2-y_2\cdot\frac{1}{2}\right)$$

$$=\frac{1}{h}\left(\frac{3}{2}(y_1-y_0)-\frac{1}{2}(y_2-y_1)\right) \quad \text{extrapolation of slopes}$$

$$=b_1$$

- At the right end

$$f'(x_2)=y_0\frac{h_2}{h_1(h_1+h_2)}+y_1\frac{h_1+h_2}{h_1h_2}+y_2\frac{h_2+h_1+h_2}{h_2(h_1+h_2)}$$

$$=\frac{1}{h_1+h_2}(\frac{1}{\gamma}y_0-(2+\gamma+\frac{1}{\gamma})y_1+y_2(2+\gamma)) \qquad \leftarrow \gamma=\frac{h_1}{h_2}$$

$$y_n'=f'(x_n)=\frac{1}{h}\left(\frac{1}{2}y_{n-2}-y_{n-1}\cdot 2+y_n\cdot\frac{3}{2}\right)$$

$$=\frac{1}{h}\left(\frac{3}{2}(y_n-y_{n-1})-\frac{1}{2}(y_{n-1}-y_{n-2})\right) \quad =b_n+2c_nh_n+3d_nh_n^2$$

Linear System for Cubic Spline

$$h_i b_i + h_i^2 c_i + h_i^3 d_i = y_i - y_{i-1} \cdots (1)$$
$$b_i + 2c_i h_i + 3d_i h_i^2 - b_{i+1} = 0 \cdots (2)$$
$$c_i - 3h_i d_i - c_{i+1} = 0 \cdots (3)$$

$$\begin{bmatrix} h_1 & h_1^2 & h_1^3 & & & & & & & & \\ 1 & & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & h_i & h_i^2 & h_i^3 & & & & & \\ & & & 1 & 2h_i & 3h_i^2 & -1 & 0 & & & \\ & & & & 1 & 3h_i & 0 & -1 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & h_n & h_n^2 & h_n^3 \\ & & & & & & & & 1 & 2h_n & 3h_n^2 \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ d_1 \\ \vdots \\ b_i \\ c_i \\ d_i \\ \vdots \\ b_n \\ c_n \\ d_n \end{bmatrix} = \begin{bmatrix} y_1 - y_0 \\ y_0' \\ y_2 - y_1 \\ \\ y_{i+1} - y_i \\ 0 \\ 0 \\ \vdots \\ 0 \\ y_n - y_{n-1} \\ y_n' \end{bmatrix}$$

n f given function values at the right end

2(n-1) f', f'' continuity at the intermediate points

(3n-1) constraints

+2 slopes at both ends $\rightarrow$ for 3n unknowns (except a_i)