Lecture Note 2

Group Constants in Resonance Region

Oct. 2008

Prof. Joo Han-gyu
Department of Nuclear Engineering

Introduction

- Group Constants

  a. average xsec (spectrum-weighted), σ_{xg}

$$\sigma_{xg} = \frac{\int_{E_g}^{E_{g-1}} \sigma(E)\phi(E)dE}{\int_{E_g}^{E_{g-1}} \phi(E)dE}$$

  b. diffusion constant, D_g

  c. resonance integral, RI_g

  d. fission spectrum, χ_g

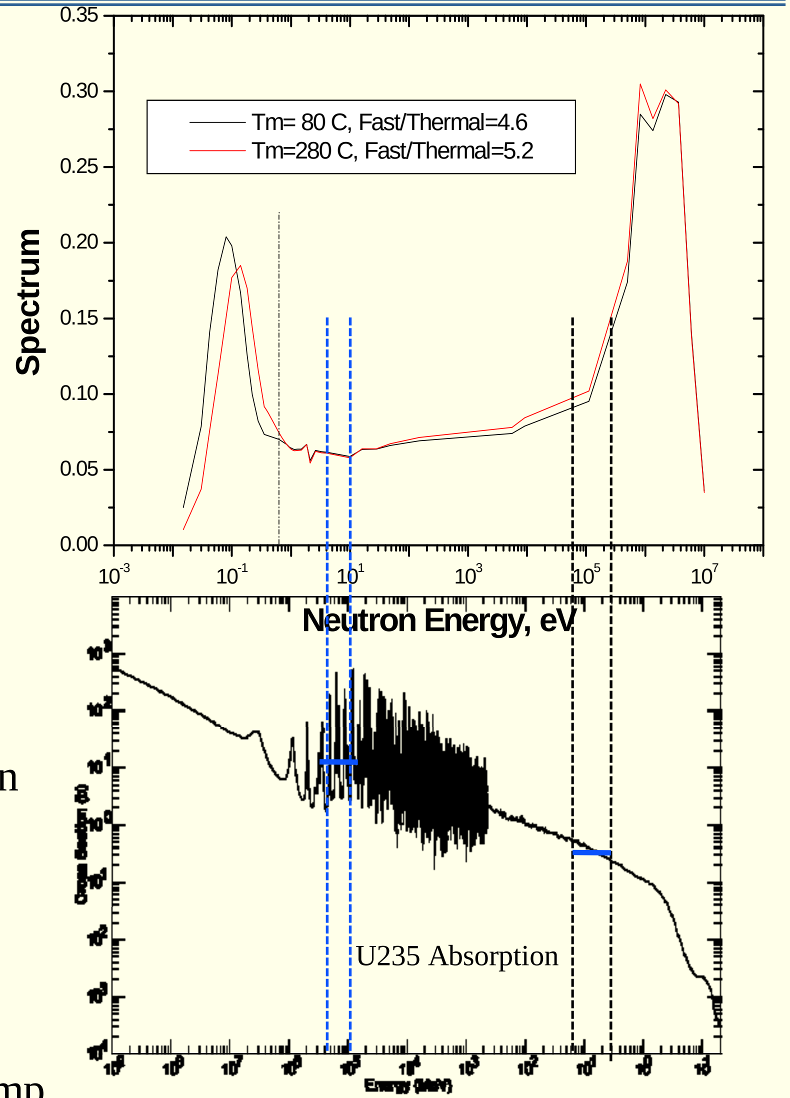
  $\rightarrow$ Mostly averaged xsec is meant.

- Smooth cross cections

  Non-resoance cross sections in fine group (~50G) can be obtained by ultra-fine group slowing down calculations for typical composition as a function of temperature

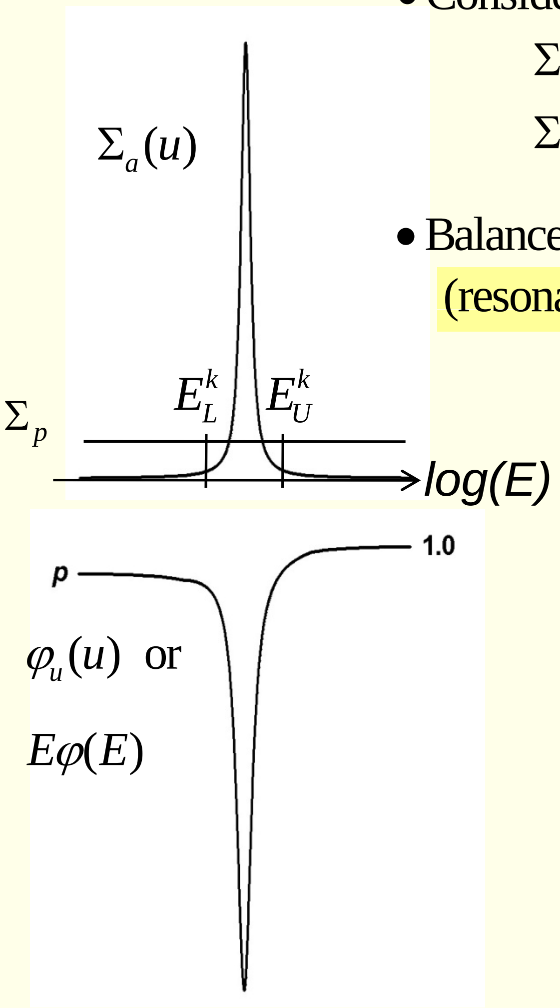
- Cross section change due to deviation from ref. condition
  - for fine group, the change in within group specturm is not large
  - for coarse group (e.g. 2G), the spectrum change is considerable $\rightarrow$ need a reevaluation of spectrum and consequently coarse group xsec by lattice calc.

- Resonance cross cections
  - need to be reevaluated for given composition and temp



$\because$ numerous resonances within a fine group and self-shielding changes depending on composition

Resonance Self-Shielding and Effective Xsec



- Consider a homogeneous mixture of a resonance absorber and moderator

$$\Sigma_p = N_R\sigma_p^R + N_M\sigma_p^M$$
$$\Sigma_t(E) = \Sigma_a^R(E) + \Sigma_s^R(E) + \Sigma_p$$

- Balance during slowing down under Narrow Resonance approximation

  (resonance so narrow that scattering source inside the resonance is negligible)

$$\Sigma_t(E)\varphi(E) = \int_E^{\frac{E}{\alpha}} \Sigma_s(E' \to E)\varphi(E')dE'$$

in case of no absorption

$$\varphi(E') = \frac{s_0}{\xi\Sigma_s E'},\ E' > E_U^k$$

$$= \int_E^{\frac{E}{\alpha}} \frac{\Sigma_s}{(1-\alpha)E'}\varphi(E')dE'$$

$$= \int_E^{\frac{E}{\alpha}} \frac{s_0}{\xi(1-\alpha)E'^2}dE' = -\frac{s_0}{\xi(1-\alpha)E'}\Bigg|_E^{\frac{E}{\alpha}} = \frac{s_0}{\xi(1-\alpha)E}(1-\alpha)$$

$$= \frac{s_0}{\xi E}$$

$$\rightarrow \varphi(E) = \frac{s_0}{\xi\Sigma_t(E)E} \quad or \quad E\varphi(E) = \frac{s_0}{\xi\Sigma_t(E)} \propto \frac{1}{\Sigma_t(E)}$$

Resonance Self-Shielding and Effective Xsec

- Source Normalization for $E\varphi(E)=1$ above Resonance

$$E\varphi(E)=1 \text{ for } E>E_U^k$$

$$E\varphi(E)=\frac{S_0}{\xi\Sigma_s}=\frac{S_0}{\xi\Sigma_p}=1 \quad \rightarrow \frac{S_0}{\xi}=\Sigma_p$$

$$\boxed{\Sigma_t(E)=\underbrace{\Sigma_a^R(E)+\Sigma_s^R(E)}_{\Sigma_t^R(E)}+\Sigma_p=\Sigma_t^R(E)+\Sigma_p}$$

- Self-Shielded Flux

$$E\varphi(E)=\frac{S_0}{\xi\Sigma_t(E)}=\frac{\Sigma_p}{\Sigma_t(E)}=\frac{\Sigma_p}{\Sigma_t^R(E)+\Sigma_p}=\frac{N_R\sigma_p^R+N_M\sigma_p^M}{N_R\sigma_t^R(E)+N_R\sigma_p^R+N_M\sigma_p^M}$$

$$=\frac{\sigma_b}{\sigma_t^R(E)+\sigma_b}$$

$$\boxed{\sigma_b=\frac{\Sigma_p}{N_R}=\sigma_p^R+\frac{N_M}{N_R}\sigma_p^M=f\left(\sigma_p^R,\sigma_p^M,\frac{N_M}{N_R}\right)}$$

dilution parameter

– Unshielded Flux with infinite dilution, with $N_R\rightarrow 0,\ \frac{N_M}{N_R}\rightarrow\infty,\ \sigma_b\rightarrow\infty \quad \rightarrow E\varphi(E)=1$

- Effective Xsec

$$\boxed{du=-\frac{dE}{E}}$$

$$\bar{\sigma}_a^R=\frac{\int\sigma_a^R(E)\varphi(E)dE}{\int\varphi(E)dE}=\frac{\int\sigma_a^R(E)\frac{\sigma_b}{\sigma_t^R(E)+\sigma_b}\frac{dE}{E}}{\int\frac{\sigma_b}{\sigma_t^R(E)+\sigma_b}\frac{dE}{E}}=\frac{\int\sigma_a^R(u)\frac{\sigma_b}{\sigma_t^R(u)+\sigma_b}du}{\int\frac{\sigma_b}{\sigma_t^R(u)+\sigma_b}du}=f(\sigma_b)$$

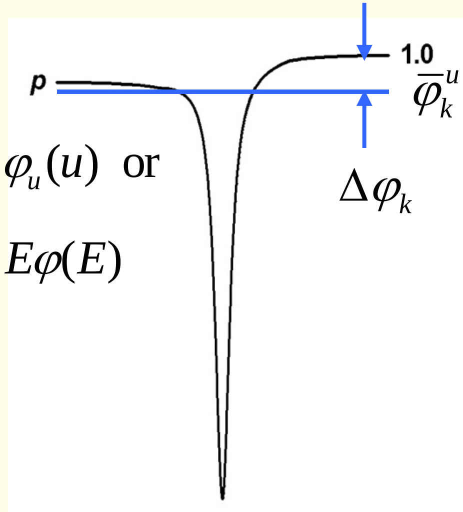
Resonance Integral

- Reaction Rate per Atom with Normalized Flux $E\varphi(E)=1$ above Resonance

$$I_k=\int_{E_L^k}^{E_U^k}\frac{\sigma_a^R(E)\sigma_b}{\sigma_t^R(E)+\sigma_b}\frac{dE}{E}=\int_{u_L^k}^{u_U^k}\frac{\sigma_a^R(u)\sigma_b}{\sigma_t^R(u)+\sigma_b}du=f(\sigma_b,T)$$

given in a form of 2-D table in a Xsec Lib

- Effective Xsec in terms of RI



$$\bar{\sigma}_a^R=\frac{I_k}{\varphi_k}=\frac{I_k/\Delta u}{\varphi_k/\Delta u}=\frac{\bar{I}_k}{\bar{\varphi}_k^u}=\frac{\bar{I}_k}{1-\Delta\varphi_k}$$

$$\bar{\varphi}_k^u=\frac{1}{\Delta u}\int_{u_L^k}^{u_U^k}\frac{\sigma_b}{\sigma_t^R(u)+\sigma_b}du=\frac{1}{\Delta u}\int_{u_L^k}^{u_U^k}\left(1-\frac{\sigma_t^R(u)}{\sigma_t^R(u)+\sigma_b}\right)du$$

$$=1-\frac{1}{\Delta u\sigma_b}\int_{u_L^k}^{u_U^k}\frac{\left(\sigma_a^R(u)+\cancel{\sigma_s^R(u)}\right)\sigma_b}{\sigma_t^R(u)+\sigma_b}=1-\frac{\bar{I}_k}{\sigma_b}\quad\rightarrow\ \Delta\varphi_k=\frac{\bar{I}_k}{\sigma_b}$$

$$\bar{\sigma}_a^R=\frac{\bar{I}_k}{1-\dfrac{\bar{I}_k}{\sigma_b}}$$

If $\sigma_s^R(u)$ is negligible..., really?

Resonance Integral

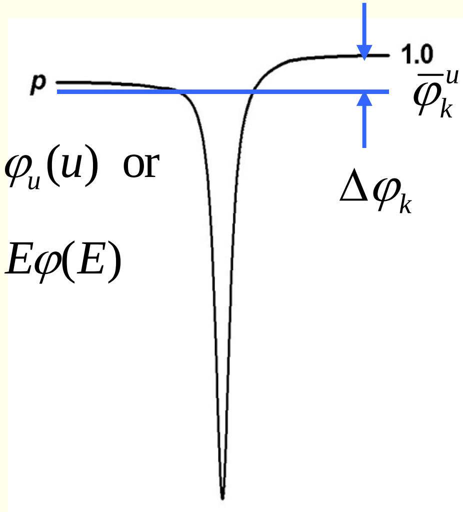
- Reaction Rate per Atom with Normalized Flux $E\varphi(E)=1$ above Resonance

$$I_k=\int_{E_L^k}^{E_U^k}\frac{\sigma_a^R(E)\sigma_b}{\sigma_t^R(E)+\sigma_b}\frac{dE}{E}=\int_{u_L^k}^{u_U^k}\frac{\sigma_a^R(u)\sigma_b}{\sigma_t^R(u)+\sigma_b}du=f(\sigma_b,T)$$

given in a form of 2-D table in a Xsec Lib

- Effective Xsec in terms of RI

$$\bar{\sigma}_a^R=\frac{I_k}{\varphi_k}=\frac{I_k/\Delta u}{\varphi_k/\Delta u}=\frac{\bar{I}_k}{\bar{\varphi}_k^u}=\frac{\bar{I}_k}{1-\Delta\varphi_k}$$



$$\bar{\varphi}_k^u=\frac{1}{\Delta u}\int_{u_L^k}^{u_U^k}\frac{\sigma_b}{\sigma_t^R(u)+\sigma_b}du=\frac{1}{\Delta u}\int_{u_L^k}^{u_U^k}\left(1-\frac{\sigma_t^R(u)}{\sigma_t^R(u)+\sigma_b}\right)du$$

$$=1-\frac{1}{\sigma_b\Delta u}\int_{u_L^k}^{u_U^k}\frac{\left(\sigma_a^R(u)+\cancel{\sigma_s^R(u)}\right)\sigma_b}{\sigma_t^R(u)+\sigma_b}du\quad=1-\frac{\bar{I}_k}{\sigma_b}\quad\rightarrow\ \Delta\varphi_k=\frac{\bar{I}_k}{\sigma_b}$$

$$\bar{\sigma}_a^R(\sigma_b)=\frac{\bar{I}_k(\sigma_b)}{1-\dfrac{\bar{I}_k(\sigma_b)}{\sigma_b}}\cdots(*)$$

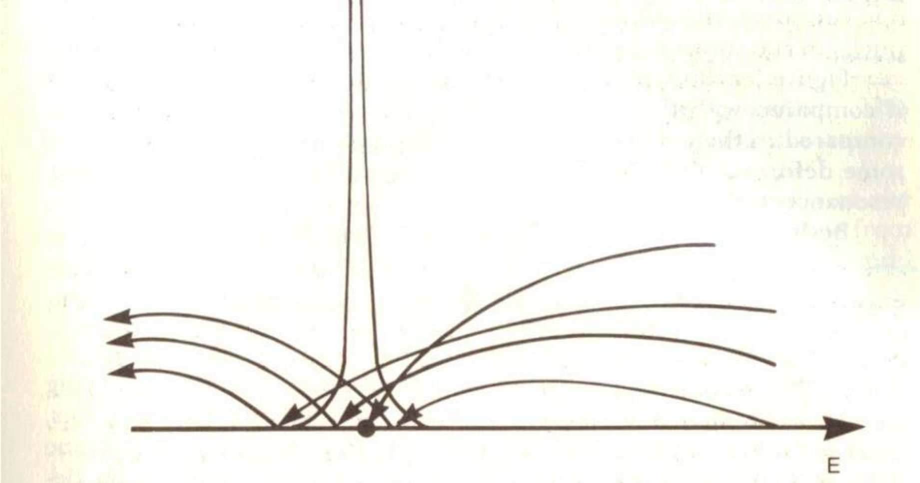
If $\sigma_s^R(u)$ is negligible..., really?

$I_k^a(\sigma_b)$ is adjusted so that (*) gives the correct avg. xsec!

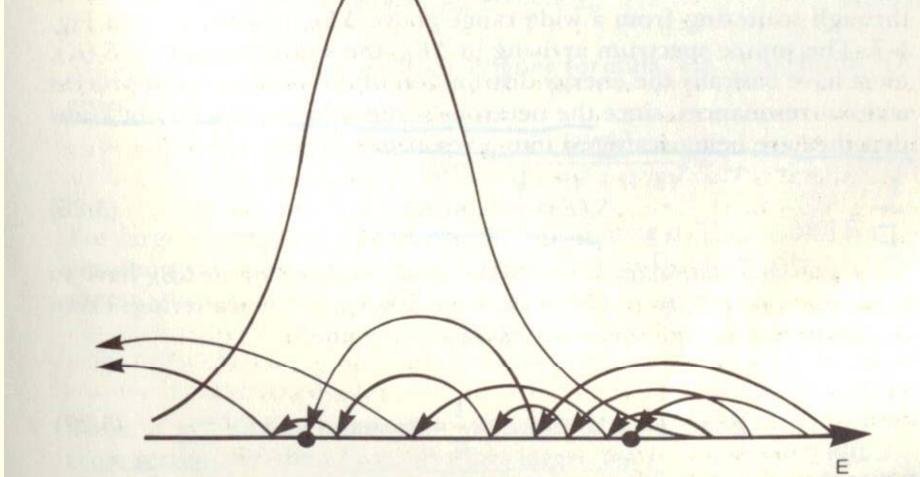
Basic Approximations for Resonance Treatment

- Need : analytical solution for resonance absorption not possible for slowing down in non-hydrogeneous material

Narrow Resonance



Wide Resonance



- Resonance width considered narrow compared with scattering energy loss
- regard all scattering source initiated above the resonance
- scattering inside the resonance bring neutron out of the resonance
- valid more for resonance at high energy

- Resonance considered much wider or scattering loss much smaller than the resonance width
- neutron resides inside the resonance after scattering
- regard in-scattering same as out-scattering
- most likely for scattering with heavy nuclides at low energy

Intermediate Resonance Approximation in Homo. Mixture

by Goldstein and Cohen (1962)

- Scattering by Narrow Resonance

$$R_{SS}^{NR}(E)=\sum_i\int_E^{\frac{E}{\alpha_i}}\frac{\Sigma_{si}\varphi(E')}{(1-\alpha_i)E'}dE' \approx \sum_i\Sigma_{pi}\int_E^{\frac{E}{\alpha_i}}\frac{C}{(1-\alpha_i)E'^2}dE' \approx \sum_i\Sigma_{pi}\frac{C}{E}$$

$$\boxed{\varphi(E')=\frac{C}{E'}}$$

$\boxed{\alpha_i \approx 1,\ \text{Narrow integration range}}$

- Scattering by Wide Resonance

$$R_{SS}^{WR}(E)=\sum_i\int_E^{\frac{E}{\alpha_i}}\frac{\Sigma_{si}\varphi(E')}{(1-\alpha_i)E'}dE' =\sum_i\int_E^{\frac{E}{\alpha_i}}\frac{\Sigma_{si}(E')E'\varphi(E')}{(1-\alpha_i)E'^2}dE' \approx \sum_i\Sigma_{si}(E)E\varphi(E)\int_E^{\frac{E}{\alpha_i}}\frac{dE'}{(1-\alpha_i)E'^2}$$

$$=\sum_i\Sigma_{si}(E)E\varphi(E)\frac{1}{E} =\sum_i\Sigma_{si}(E)\varphi(E)$$

- Intermediate Resonance (IR) with Normalization, $C=1$

$$R_{SS}(E)=\sum_i\left(\lambda_i\frac{\Sigma_{pi}}{E}+(1-\lambda_i)\Sigma_{si}(E)\varphi(E)\right) \qquad R_{SS}(u)=\sum_i\left(\lambda_i\Sigma_{pi}+(1-\lambda_i)\Sigma_{si}(u)\varphi(u)\right)$$

$$R_{SS}(E)dE=\sum_i\left(\lambda_i\Sigma_{pi}\frac{dE}{E}+(1-\lambda_i)\Sigma_{si}(E)\varphi(E)dE\right) \qquad R_{SS}(u)du=\sum_i\left(\lambda_i\Sigma_{pi}du+(1-\lambda_i)\Sigma_{si}(u)\varphi(u)du\right)$$

Slowing-Down Equation with IR Source

- Balance Equation for Fuel and Moderator Mixture

$$\Sigma_t(E)\varphi(E) = \sum_i \left(\lambda_i \frac{\Sigma_{pi}}{E} + (1-\lambda_i)\Sigma_{si}(E)\varphi(E) \right) = \lambda \frac{\Sigma_p}{E} + (1-\lambda_F)\Sigma_s^F(E)\varphi(E)$$

$$\boxed{\lambda_i = 1 \text{ for moderator}}$$

$$\text{where } \Sigma_t(E) = \Sigma_a^F(E) + \Sigma_s^F(E) + \Sigma_p^M$$

$$*\Sigma_s^F(E) = \Sigma_s^{F,res}(E) + \Sigma_p^F$$

$$\sum_i \lambda_i \Sigma_{pi} = \Sigma_p^M + \lambda_F \Sigma_p^F = \lambda \Sigma_p \quad \text{where } \Sigma_p = \Sigma_p^M + \Sigma_p^F \ (\lambda \text{ properly defined})$$

- Move flux dependent term of RHS to LHS

$$\left(\Sigma_a^F(E) + \lambda_F \Sigma_s^F(E) + \Sigma_p^M\right)\varphi(E) = \left(\Sigma_a^F(E) + \lambda_F \Sigma_s^{F,res}(E) + \lambda\Sigma_p\right)\varphi(E) = \frac{\lambda\Sigma_p}{E}$$

- Flux with IR Approximation

$$\therefore E\varphi(E) = \frac{\lambda\Sigma_p}{\Sigma_a^F(E) + \lambda_F \Sigma_s^{F,res}(E) + \lambda\Sigma_p}$$

$$or\ \varphi(u) = \frac{\sigma_b}{\sigma_a^F(u) + \sigma_b}$$

$$or\ E\varphi(E) = \frac{\sigma_b}{\sigma_a^F(E) + \lambda_F \sigma_s^{F,res}(E) + \sigma_b} \quad \text{where } \sigma_b = \frac{\lambda\Sigma_p}{N}$$

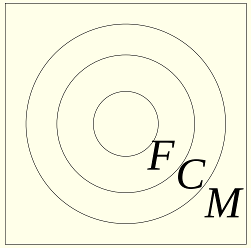
with neglect of $\lambda_F \sigma_s^{F,res}(u)$ for $\lambda_F << 1$.

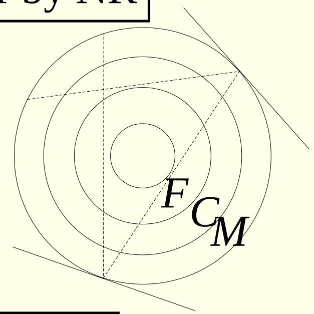
Slowing Down Equation in a Single Pin Cell

- Balance Equation for Fuel in terms of Collision Probabilities

$$\varphi_M(u) = 1\ ,\ u < u_L^k$$
in moderator by NR

$$V_F \Sigma_F(u)\varphi_F(u) = \sum_{J \neq F} V_J \Sigma_J \cdot 1 \cdot \tilde{P}_{JF} + V_F \left(\lambda_F \Sigma_{pF} + (1-\lambda_F)\Sigma_{sF}(E)\varphi_F(u) \right) \tilde{P}_{FF}$$





– Reciprocity in CP
$$V_J \Sigma_J \tilde{P}_{JF} = V_F \Sigma_F(u) \tilde{P}_{FJ}$$

$$\tilde{P}_{ji} = \frac{1}{V_j} \int_{V_i} \int_{V_j} \Sigma_i n(\vec{r}_j \to \vec{r}_i) dV_j dV_i$$

$$\Sigma_J = \Sigma_{sJ} = const.$$
$$\Sigma_F = \Sigma_{tF}(u)$$

$$\sum_{J \neq F} V_J \Sigma_J \tilde{P}_{JF} = V_F \Sigma_F(u) \sum_{J \neq F} \tilde{P}_{FJ} = V_F \Sigma_F(u) (\underbrace{1 - \tilde{P}_{FF}}_{P_{esc}^F})$$

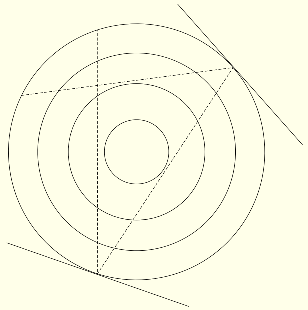
$$\cancel{V_F} \left(\Sigma_F(u) - (1-\lambda_F)\Sigma_{sF}(u)\tilde{P}_{FF} \right) \varphi_F(u) = \cancel{V_F} \Sigma_F(u) P_{esc}^F + \cancel{V_F} \lambda \Sigma_{pF} \tilde{P}_{FF}$$

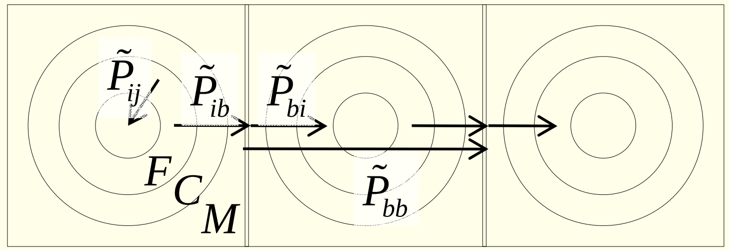
- IR Flux

$$\varphi_F(u) = \frac{\lambda_F \Sigma_{pF} \tilde{P}_{FF} + \Sigma_F(u) P_{esc}^F}{\Sigma_F(u) - (1-\lambda_F)\Sigma_{sF}(u)\tilde{P}_{FF}}$$

→ Need to find $\tilde{P}_{FF}(u)$ to determine resonance flux

Fuel to Fuel Collision Probability in a Cell and in a Lattice





- Cell first-flight probabilities
  - $\tilde{P}_{ij}$: First Flight (FF) collision probability for source i and destination j
  - $\tilde{P}_{jb}$: Probability for a neutron in i to reach boundary
  - $\tilde{P}_{bi}$: Probability for a neutron istropically entering boudary (cosine current) to have first collision in i
  - $\tilde{P}_{bb}$: Probability for a neutron istropically entering boudary (cosine current) to pass through the cell without collision

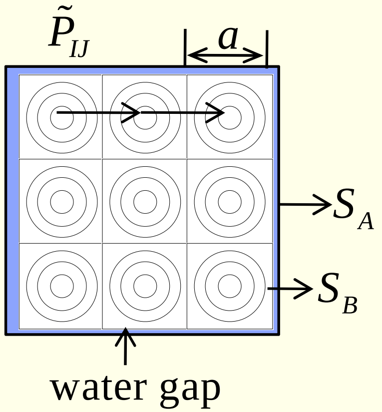
- Reflection (reentrance) Ratio, $R(=\alpha)$: Return ratio of neutrons escaping from the cell
  –1.0 for infinite lattice

- Lattice CP (Upper case suffixes)

$$\tilde{P}_{IJ} = \tilde{P}_{ij} + \tilde{P}_{ib} R \tilde{P}_{bi} + \underbrace{\tilde{P}_{ib} R \tilde{P}_{bb} R \tilde{P}_{bi}}_{\text{Coll. in 3rd cell}} + \tilde{P}_{ib} R (\tilde{P}_{bb} R)^2 \tilde{P}_{bi} + \cdots = \tilde{P}_{ij} + \frac{\tilde{P}_{ib} R \tilde{P}_{bi}}{\underbrace{1 - R\tilde{P}_{bb}}_{X}}$$

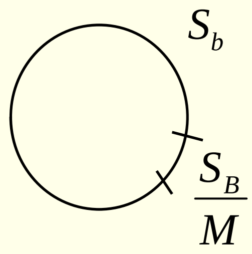
Lattice FF CP increased by this much

Reflection Probability for an Assembly



- Assembly area (S_B) fraction to total cell surface area

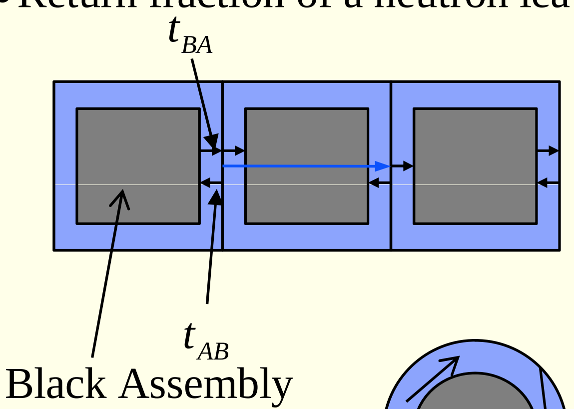
$$f = \frac{4 \cdot na}{n^2 a} = \frac{1}{n} = \frac{1}{\sqrt{M}} \qquad \leftarrow M = n^2$$



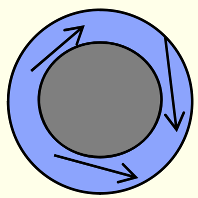
$\rightarrow$ fraction of neutrons exiting through assembly surface per unit neutron exiting a cell

$\rightarrow$ these neutrons can encounter the assembly gap

- Return fraction of a neutron leaving FA avoiding loss in the FA gap



$$g = t_{BA}t_{AB} + t_{BA}\underbrace{(1-P_{AA}^{\infty})}_{\text{assembly bypass fraction}} t_{AB} + \underbrace{t_{BA}(1-P_{AA}^{\infty})^2 t_{AB}}_{\text{absorbed in the 4th FA}} + \cdots = \frac{t_{BA}t_{AB}}{1-P_{AA}^{\infty}}$$



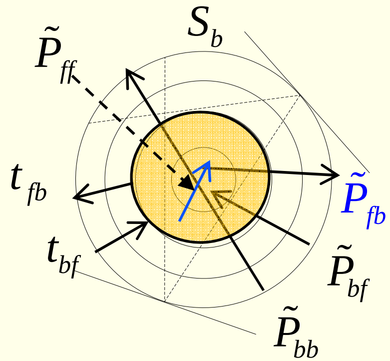
- Reflection probability for a neutron leaving a cell to return in a lattice

$$R = (1-f) + f \cdot g$$

with no gap, $g = 1 \rightarrow R = 1$.

First Flight Probabilities for a Pin Cell

- Lattice Enhanced Fuel-to-Fuel Collision Prob.



vol. to surface prob. which can't be found from v-v prob.

$$X_{FF} = \frac{\tilde{P}_{fb} R \tilde{P}_{bf}}{1 - R\tilde{P}_{bb}} = \frac{\tilde{P}_{fb}\tilde{P}_{bf}}{\frac{1}{R} - \tilde{P}_{bb}} = \frac{(1-\tilde{P}_{ff})t_{fb} \cdot t_{bf}\gamma_f}{\frac{1}{R} - \tilde{P}_{bb}}$$

- First−flight probabilities for a cell

$\tilde{P}_{ff}$: first collision in fuel $\rightarrow P^f_{esc} = 1 - \tilde{P}_{ff}$

$\tilde{P}_{fb}$: fuel vol. to boundary

$\tilde{P}_{bf}$: boundary surface to fuel vol. with cosine incoming current (isotropic)

$\tilde{P}_{bb}$: boundary-to-boundary transmission

t_{fb} : fuel surface to boundary for neutrons exiting fuel with cosine current

t_{bf} : cell surface to fuel surface for neutrons incoming with cosine current

γ_f :F.F. blackness of fuel

Σ_f : total Xsec of fuel

- Reciprocity Relations

$$\gamma_f = \frac{4V_f}{S_f}\Sigma_f P^f_{esc} = \underbrace{\bar{l}_f \Sigma_f}_{x} P^f_{esc} = x\left(1 - \tilde{P}_{ff}\right)$$

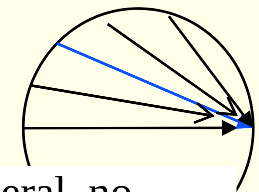
$$\tilde{P}_{bf} = \frac{4V_f}{S_b}\Sigma_f \tilde{P}_{fb}$$

$S_f t_{fb} = S_b t_{bf}$

$\tilde{P}_{fb} \approx \tilde{P}^f_{esc} t_{fb}$

$\tilde{P}_{bf} \approx t_{bf}\gamma_f$

isotropic exiting current at fuel surface?



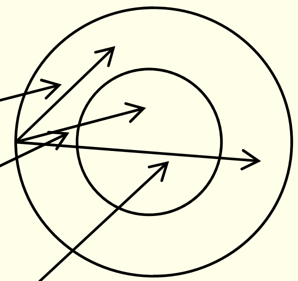
in general, no.
But practically OK for large Σ_f

Determination of FF Probabilities for a Cell

- How to determine t_{bf} or t_{fb} ?
  - Assume black fuel $(\Sigma_f > 5cm^{-1}) \to \gamma_f^\infty = 1$
  - Perform CP calcultion to obtain volume CP kernel and then γ_f^b (fuel blackness for boundary)
  - $t_{bf} = \gamma_f^b$
  - $t_{fb} = \dfrac{S_b}{S_f} t_{bf}$

$$\boxed{\gamma_f^b = \frac{4V_f}{S_b}\Sigma_f P_{esc}^f = \frac{4}{S_b}\Sigma_f V_f (1-\sum_{j=1}^n \tilde{P}_{fj}) = \frac{4}{S_b}(\Sigma_f V_f - \sum_{j=1}^n P_{fj}) \cdots (1)}$$

- $\tilde{P}_{bb}$? FF cell blackness $\to \tilde{P}_{bb} = 1-\gamma_b$



$\gamma_b = 1 - \tilde{P}_{bb}$

$= \sum_{j \neq f} \tilde{P}_{bj}^\infty \leftarrow$ for neutrons never reached fuel

$+ t_{bf}\gamma_f \leftarrow$ for neutrons entering fuel sureface then reacting within fuel

$+ t_{bf}(1-\gamma_f)(1-t_{fb}) \leftarrow$ for neutrons passing fuel then reacting within non fuel

$$= t_{bf}t_{fb}\gamma_f + \underbrace{\sum_{j\neq f}\tilde{P}_{bj}^\infty + t_{bf}(1-t_{fb})}_{\gamma_b^0} = \frac{S_f}{S_b}t_{fb}^2 \underbrace{\bar{l}_f \Sigma_f}_{x} \tilde{P}_{esc}^f + \gamma_b^0 = \frac{S_f}{S_b}t_{fb}^2 x(1-\tilde{P}_{ff}) + \gamma_b^0$$

$\to$ cell blackness with $\Sigma_f = 0$ can be calculated in a way similar to (1)

- Cell Transmission Prob.

$$\tilde{P}_{bb} = 1 - \frac{S_f}{S_b}t_{fb}^2 x(1-\tilde{P}_{ff}) - \gamma_b^0$$

Fuel Collision Probability for Assembly

- Lattice Enhanced Fuel-to-Fuel Collision Prob.

$$X_{FF}=\frac{\tilde{P}_{fb}R\tilde{P}_{bf}}{1-R\tilde{P}_{bb}}=\frac{(1-\tilde{P}_{ff})t_{fb}\cdot t_{bf}\gamma_f}{\frac{1}{R}-\tilde{P}_{bb}}=\frac{(1-\tilde{P}_{ff})t_{fb}\cdot\frac{S_f}{S_b}t_{fb}x(1-\tilde{P}_{ff})}{\frac{1}{R}-\left(1-\frac{S_f}{S_b}t_{fb}^2x(1-\tilde{P}_{ff})-\gamma_b^0\right)}=\frac{x(1-\tilde{P}_{ff})^2\frac{S_f}{S_b}t_{fb}^2}{\frac{1-R}{R}+\frac{S_f}{S_b}t_{fb}^2x(1-\tilde{P}_{ff})+\gamma_b^0}$$

$$=\frac{x(1-\tilde{P}_{ff})^2}{x(1-\tilde{P}_{ff})+\underbrace{\frac{S_b}{S_ft_{fb}^2}\gamma_b^0}_{A}+\underbrace{\frac{S_b}{S_ft_{fb}^2}\frac{f(1-g)}{1-f+f\cdot g}}_{B}}=\frac{x(1-\tilde{P}_{ff})^2}{x(1-\tilde{P}_{ff})+A+B}=\frac{\gamma_f(P_{esc}^f)^2}{\gamma_f+A+B}$$

If no gap, g=1 $\rightarrow$ B=0.

$$\tilde{P}_{FF}>\tilde{P}_{ff}\quad\rightarrow\quad 1-\tilde{P}_{FF}<1-\tilde{P}_{ff}$$

- Ratio of assembly fuel escape prob. to cell fuel escape prob. in the limit of black fuel

$$D=\frac{1-\tilde{P}_{FF}}{1-\tilde{P}_{ff}}=\frac{1-\tilde{P}_{ff}-X_{FF}}{1-\tilde{P}_{ff}}=1-\frac{x(1-\tilde{P}_{ff})}{x(1-\tilde{P}_{ff})+A+B}\qquad \lim_{\Sigma_f\to\infty}D=?$$

$$\lim_{\Sigma_f\to\infty}x(1-\tilde{P}_{ff})=\lim_{\Sigma_f\to\infty}\gamma_f=1\qquad \lim_{\Sigma_f\to\infty}D=1-\frac{1}{1+\underbrace{A+B}_{C}}=1-\frac{1}{1+C}=\frac{C}{1+C}$$

Dancoff Factor and Wigner Approximation

- Various Definitions of Dancoff Factors
  - Ratio of fuel escape probability of FA to that of cell for black fuel
  - FF Blackness of all other materials other than fuel
  - Probability of a neutron leaving a fuel pin (black fuel pin) to have first collision in other materials than fuel

$$D_\infty = 1 - \frac{1}{1 + A + \cancel{B}} = \frac{A}{1+A} \text{ for infinite medium}$$

- Single Term Rational Approximation of P_{ff}

$$P_{ff} = \frac{x}{x+a} \quad \text{with } a = \begin{cases} 1.00 & \text{for Wigner Approx.} \\ 1.16 & \text{for Bell Factor} \end{cases}$$

$$= \frac{\bar{l}\Sigma_f}{\bar{l}\Sigma_f + a} = \frac{\Sigma_f}{\Sigma_f + a\frac{1}{\bar{l}}} = \frac{\Sigma_f}{\Sigma_f + a\Sigma_l}$$

$\Sigma_l = \frac{1}{\bar{l}} = \frac{S}{4V}$: leakage cross section or surface area density

Assembly fuel to fuel CP is also given as a rational form!

$$\tilde{P}_{FF} = \tilde{P}_{ff} + \frac{x(1-\tilde{P}_{ff})^2}{x(1-\tilde{P}_{ff}) + C} = \frac{x}{x+a} + \frac{x\left(\frac{a}{x+a}\right)^2}{x\frac{a}{x+a} + C} = \frac{x}{x + \frac{aC}{\underbrace{a+C}_{\alpha}}} = \frac{x}{x+\alpha}$$

Escape Cross Section

- Dancoff Factor with Rational Approximation

$$D = \lim_{\Sigma_f \to \infty} \frac{1-\tilde{P}_{FF}}{1-\tilde{P}_{ff}} = \lim_{\Sigma_f \to \infty} \frac{\dfrac{\alpha}{x+\alpha}}{\dfrac{a}{x+a}} = \lim_{\Sigma_f \to \infty} \frac{\dfrac{\alpha}{x}}{\dfrac{a}{x}} = \frac{\alpha}{a} = \frac{C}{a+C} \equiv \frac{C}{1+C}$$

$$\therefore \lim_{\Sigma_f \to \infty} a = 1 \rightarrow \text{ Winger approximation is valid for black fuel } \quad \text{or } \alpha = D, \text{ otherwise } \alpha = aD.$$

- Fuel Flux for Assembly with Rational Approximation

Equivalent to homogeneous system with additional escape cross section!

$$\tilde{P}_{FF} = \frac{x}{x+\alpha} = \frac{\bar{l}\,\Sigma_f}{\bar{l}\,\Sigma_f + \alpha} = \frac{\Sigma_f}{\Sigma_f + \underbrace{\alpha\Sigma_l}_{\Sigma_e}} = \frac{\Sigma_f}{\Sigma_f + \Sigma_e}$$

$$P^F_{esc} = 1 - \tilde{P}_{FF} = \frac{\Sigma_e}{\Sigma_f + \Sigma_e}$$

escape cross section

$$\sigma_b = \frac{1}{N_F}\left(\lambda_F \Sigma_{pF} + \Sigma_e\right)$$

$$\varphi_F(u) = \frac{\lambda_F \Sigma_{pF} \tilde{P}_{FF} + \Sigma_F(u) P^F_{esc}}{\Sigma_F(u) - (1-\lambda_F)\Sigma_{sF}(u)\tilde{P}_{FF}} = \frac{\lambda_F \Sigma_{pF} \dfrac{\cancel{\Sigma_f}}{\Sigma_f + \Sigma_e} + \cancel{\Sigma_F(u)} \dfrac{\Sigma_e}{\Sigma_f + \Sigma_e}}{\cancel{\Sigma_F(u)} - (1-\lambda_F)\Sigma_{sF}(E) \dfrac{\cancel{\Sigma_f}}{\Sigma_f + \Sigma_e}} = \frac{\lambda_F \Sigma_{pF} + \Sigma_e}{\Sigma_F(u) + \Sigma_e - (1-\lambda_F)\Sigma_{sF}(E)}$$

$$= \frac{\lambda_F \Sigma_{pF} + \Sigma_e}{\Sigma_{aF}(u) + \lambda_F \Sigma_{sF}(u) + \Sigma_e} = \frac{\lambda_F \Sigma_{pF} + \Sigma_e}{\Sigma_{aF}(u) + \lambda_F \Sigma^{Res}_{sF}(u) + \lambda_F \Sigma_{pF} + \Sigma_e} = \frac{\sigma_b}{\sigma_{aF}(u) + \lambda_F \sigma^{Res}_{sF}(u) + \sigma_b}$$

Two-Term Rational Approximation by Calvik (1962)

- General Form

$$\tilde{P}_{ff} = \sum_{n=1}^{N} \frac{b_i x}{x + a_i} \qquad \text{where } \sum_{n=1}^{N} b_i = 1, \ \because \tilde{p}_{ff} = 1 \text{ as } x \to \infty$$

- Two-Term Form

$$\tilde{P}_{ff} = x\left(\frac{b_1}{x+a_1} + \frac{b_2}{x+a_2}\right) \qquad \boxed{= x\left(\frac{2}{x+2} - \frac{1}{x+3}\right)}$$

where $b_1 + b_2 = 1$

$$\lim_{x\to\infty} x(1-\tilde{P}_{ff}) = \lim_{\Sigma_f\to\infty} \gamma_f = 1$$

$$1 - \tilde{P}_{ff} = \frac{1}{x} \to \tilde{P}_{ff} = 1 - \frac{1}{x} \quad \to \lim_{x\to\infty} \tilde{P}'_{ff} = \frac{1}{x^2}$$

$$\boxed{\lim_{x\to 0} \tilde{P}'_{ff} = \frac{b_1}{a_1} + \frac{b_2}{a_2} = \frac{2}{3} \leftarrow \text{white boundary condition for weak absorption}}$$

$$\tilde{P}'_{ff} = \left(\frac{b_1 x}{x+a_1} + \frac{b_2 x}{x+a_2}\right)' = \frac{a_1 b_1}{(x+a_1)^2} + \frac{a_2 b_2}{(x+a_2)^2} \qquad b_1 = 2, b_2 = -1, a_1 = 2, a_2 = 3$$

$$\lim_{x\to\infty} \tilde{P}'_{ff} = \frac{1}{x^2}\left(a_1 b_1 + a_2 b_2\right) \equiv \frac{1}{x^2} \quad \to a_1 b_1 + a_2 b_2 = 1$$

N-term Rational Approximation

- For assembly

$$\tilde{P}_{FF} = \tilde{P}_{ff} + \frac{x(1-\tilde{P}_{ff})^2}{x(1-\tilde{P}_{ff})+C} = \frac{x(1-\tilde{P}_{ff})\tilde{P}_{ff} + x(1-\tilde{P}_{ff})^2 + C\tilde{P}_{ff}}{x(1-\tilde{P}_{ff})+C} = \frac{x(1-\tilde{P}_{ff})+C\tilde{P}_{ff}}{x(1-\tilde{P}_{ff})+C}$$

$$\boxed{1-\tilde{P}_{ff} = \underbrace{\sum_{n=1}^{N} b_n}_{=1} - x\sum_{n=1}^{N}\frac{b_n}{x+a_n} = \sum_{n=1}^{N}\frac{a_n b_n}{x+a_n}}$$

$$= \frac{x\left(\sum_{n=1}^{N}\frac{a_n b_n}{x+a_n} + C\sum_{n=1}^{N}\frac{b_n}{x+a_n}\right)}{x\sum_{n=1}^{N}\frac{a_n b_n}{x+a_n} + C} = \frac{x\left(\sum_{n=1}^{N} a_n b_n \prod_{\substack{j=1\\ j\neq n}}^{N}(x+a_j) + C\sum_{n=1}^{N} b_n \prod_{\substack{j=1\\ j\neq n}}^{N}(x+a_j)\right)}{x\sum_{n=1}^{N} a_n b_n \prod_{\substack{j=1\\ j\neq n}}^{N}(x+a_j) + C\prod_{n=1}^{N}(x+a_n)}$$

$$\boxed{\sum_{n=1}^{N}\beta_n = 1 \quad \because \text{unity coeff. of } x^{n-1}}$$

$$= x\frac{\left(\sum_{n=1}^{N} a_n b_n + C\sum_{n=1}^{N} b_n\right)x^{n-1} + \cdots}{\left(\sum_{n=1}^{N} a_n b_n + C\right)x^n + \cdots} = x\frac{x^{N-1} + d_{N-2}x^{N-2} + \cdots}{x^N + c_{N-1}x^{N-1} + \cdots} = x\frac{P_{N-1}(x)}{\prod_{n=1}^{N}(x+\alpha_n)} = x\sum_{n=1}^{N}\frac{\beta_n}{x+\alpha_n}$$

by method of partial fraction

$$= \sum_{n=1}^{N}\frac{\beta_n \Sigma_f}{\Sigma_f + \alpha_n \Sigma_l}$$

Flux with Rational Approximation

$$\tilde{P}_{FF} = \sum_{n=1}^{N} \frac{\beta_n \Sigma_f}{\Sigma_f + \alpha_n \Sigma_l} \qquad \tilde{P}_{ESC} = 1 - \tilde{P}_{FF} = \sum_{n=1}^{N} \left(\beta_n - \frac{\beta_n \Sigma_f}{\Sigma_f + \alpha_n \Sigma_l} \right) = \sum_{n=1}^{N} \frac{\alpha_n \beta_n \Sigma_l}{\Sigma_f + \alpha_n \Sigma_l}$$

$$\varphi_F(u) = \frac{\lambda_F \Sigma_{pF} \tilde{P}_{FF} + \Sigma_F(u) P_{esc}^F}{\Sigma_F(u) - (1-\lambda_F)\Sigma_{sF}(u)\tilde{P}_{FF}}$$

$$= \frac{\lambda_F \Sigma_{pF} \sum_{n=1}^{N} \frac{\beta_n \cancel{\Sigma_f}}{\Sigma_f + \alpha_n \Sigma_l} + \cancel{\Sigma_F}(u) \sum_{n=1}^{N} \frac{\alpha_n \beta_n \Sigma_l}{\Sigma_f + \alpha_n \Sigma_l}}{\cancel{\Sigma_F}(u) - (1-\lambda_F)\Sigma_{sF}(u) \sum_{n=1}^{N} \frac{\beta_n \cancel{\Sigma_f}}{\Sigma_f + \alpha_n \Sigma_l}} = \frac{\sum_{n=1}^{N} \left(\lambda_F \Sigma_{pF} + \alpha_n \Sigma_l \right) \frac{\beta_n}{\Sigma_f + \alpha_n \Sigma_l}}{1 - (1-\lambda_F)\Sigma_{sF}(u) \sum_{n=1}^{N} \frac{\beta_n}{\Sigma_f + \alpha_n \Sigma_l}}$$

If $\lambda_F = 1$

$$\varphi_F(u) = \sum_{n=1}^{N} \beta_n \frac{\Sigma_{pF} + \alpha_n \Sigma_l}{\Sigma_f + \alpha_n \Sigma_l} \approx \sum_{n=1}^{N} \beta_n \frac{\lambda_F \Sigma_{pF} + \alpha_n \Sigma_l}{\Sigma_a(u) + \lambda_F \Sigma_{sF}(u) + \alpha_n \Sigma_l}$$

appromimated separate equivalence with IR

$$= \beta \frac{\Sigma_p + \alpha_1 \Sigma_l}{\Sigma_f + \alpha_1 \Sigma_l} + (1-\beta) \frac{\Sigma_{pF} + \alpha_2 \Sigma_l}{\Sigma_f + \alpha_2 \Sigma_l} \quad \text{for two term}$$

Effective Cross Section with Calvik's Rational Approx.

$$\varphi_F(u) = \beta \frac{\Sigma_p + \alpha_1 \Sigma_l}{\Sigma_t(u) + \alpha_1 \Sigma_l} + (1-\beta)\frac{\Sigma_p + \alpha_2 \Sigma_l}{\Sigma_t(u) + \alpha_2 \Sigma_l}$$

$$= \beta \frac{\sigma_p + \alpha_1 \sigma_l}{\sigma_t(u) + \alpha_1 \sigma_l} + (1-\beta)\frac{\sigma_p + \alpha_2 \sigma_l}{\sigma_t(u) + \alpha_2 \sigma_l} \text{ where } \sigma_l = \frac{\Sigma_l}{N_F} = \frac{S}{4VN_F}$$

$$= \beta \frac{\sigma_p + \alpha_1 \sigma_l}{\sigma_a(u) + \sigma_s^{res}(u) + \underbrace{\sigma_p + \alpha_1 \sigma_l}_{\sigma_{b1}}} + (1-\beta)\frac{\sigma_p + \alpha_2 \sigma_l}{\sigma_a(u) + \sigma_s^{res}(u) + \underbrace{\sigma_p + \alpha_2 \sigma_l}_{\sigma_{b2}}}$$

$$= \beta \frac{\sigma_{b1}}{\sigma_a(u) + \sigma_{b1}} + (1-\beta)\frac{\sigma_{b2}}{\sigma_a(u) + \sigma_{b2}} \text{ with neglect of } \sigma_s^{res} \text{ with adjusted RI's}$$

$$\bar{\sigma}_g = \frac{\int_g \left(\beta \frac{\sigma_a(u)\sigma_{b1}}{\sigma_a(u) + \sigma_{b1}} + (1-\beta)\frac{\sigma_a(u)\sigma_{b2}}{\sigma_a(u) + \sigma_{b2}} \right) du}{\int_g \left(\beta \frac{\sigma_{b1}}{\sigma_a(u) + \sigma_{b1}} + (1-\beta)\frac{\sigma_{b2}}{\sigma_a(u) + \sigma_{b2}} \right) du}$$

$$= \frac{\beta \bar{I}_g(\sigma_{b1}) + (1-\beta)\bar{I}_g(\sigma_{b2})}{1 - \left(\beta \frac{\bar{I}_g(\sigma_{b1})}{\sigma_{b1}} + (1-\beta)\frac{\bar{I}_g(\sigma_{b2})}{\sigma_{b2}} \right)}$$

← Equivalence Theorem used in CASMO

$$\bar{\sigma}_{i,g} = \bar{\sigma}_g \frac{\bar{I}_g(\sigma_p + D_i \sigma_l)}{\bar{I}_g(\sigma_p + D_\infty \sigma_l)}$$

with position-dependent Dancoff