Lecture Note 3

P_L Method

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Prof. Joo Han-gyu
Department of Nuclear Engineering

Derivation of Legendre Polynomial

- Laplace Eqn. in Spherical cord. with azimuthal symmetry ($\frac{\partial}{\partial \alpha}=0$)

$$\nabla^2\phi = 0 \rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi}{\partial\theta} = 0$$

- Seperation of variables

$$\phi(\mathrm{r},\theta)=\mathrm{R(r)}\Theta(\theta) \rightarrow \quad \frac{1}{r^2}\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{1}{Y}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) = -\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} = \lambda^2$$

$$\mu = \cos\theta$$

$$d\mu = -\sin\theta d\theta$$

$$\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\underbrace{\sin^2\theta}_{1-\mu^2}\frac{d\Theta}{\sin\theta d\theta} = -\lambda^2$$

$$\frac{d}{d\mu}(1-\mu^2)\frac{d\Theta}{d\mu} + \lambda^2\Theta = 0$$

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \lambda^2\Theta = 0$$

Let $\lambda^2 = l(l+1)$.

Legendre Equation

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + l(l+1)\Theta = 0$$

$$\rightarrow \Theta(\mu)=P_l(\mu)$$

Properties of Legendre Polynomials

- Power Series Solution

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu$$

- Recurrence Relation (HW)

$$(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu)$$

$$P_2(\mu) = \frac{1}{2}(3\mu^2 - 1), P_3(\mu) = \frac{1}{2}\mu(5\mu^2 - 3)\cdots$$

- Orthogonality (HW)

$$\int_{-1}^{1} P_l(\mu)P_m(\mu)d\mu = \begin{cases} 0 & l \neq m \\ \dfrac{2}{2l+1} & l = m \end{cases}$$

Derivation of Spherical Harmonics

- Solution of Laplace Eqn. in Spherical cord. in case of no symmetry

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\alpha^2}$$ added to azimuthally symmetric Laplacian

$$\phi(r,\theta,\alpha)=R(r)\,Y(\theta,\alpha)$$

$$\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y}{\partial\theta})+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}+l(l+1)=0$$

$$Y(\theta,\alpha)=\Theta(\theta)\Phi(\alpha)$$

$$\frac{1}{\Theta}\frac{d}{\sin\theta d\theta}(\sin\theta\frac{d\Theta}{d\theta})+\frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\alpha^2}+l(l+1)=0$$

$$\frac{1}{\Theta}\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})+l(l+1)\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\alpha^2}=m^2$$

$$\frac{1}{\Phi}\frac{d^2\Phi(\alpha)}{d\alpha^2}=-m^2$$

$$\Phi(\alpha)=e^{im\alpha}$$

Associated Legendre Polynomial

- Associated Legendre Equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\ \Theta}{d\theta})+[l(l+1)-\frac{m^2}{\sin^2\theta}]\ \Theta=0$$

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{d\Theta}{d\mu}\right)+\left(l(l+1)-\frac{m^2}{1-\mu^2}\right)\Theta=0$$

- Associated Legendre Polynomial for $m \le l$

$$P_l^m(\mu)=(-1)^m\sqrt{1-\mu^2}^{\,m}\frac{d^m P_l(\mu)}{d\mu^m}=(-1)^m\sin^m\theta\frac{d^m P_l(\mu)}{d\mu^m};m\le l$$

- Rodirgue's formula

$$P_l^m(\mu)=\frac{1}{2^l l!}\sqrt{1-\mu^2}^{\,m}\frac{d^{m+l}}{d\mu^{m+l}}(\mu^2-1)^l,\quad -l\le m\le l$$

- Orthogonality

$$\int_{-1}^{1}P_l^m(\mu)P_{l'}^m(\mu)dx=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$$

$$P_l^{-m}(\mu)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\mu)$$

$$\underbrace{\frac{\sqrt{(l+m)!}}{\sqrt{(l-m)!}}P_l^{-m}(\mu)}_{\tilde{P}_l^{-m}(\mu)}=(-1)^m\underbrace{\frac{\sqrt{(l-m)!}}{\sqrt{(l+m)!}}P_l^m(\mu)}_{\tilde{P}_l^m(\mu)}$$

Spherical Harmonics

- Definition of Spherical Harmonics

$$Y_{lm}(\hat{\Omega}) = Y_{lm}(\theta,\alpha) = \tilde{P}_l^m(\mu)e^{im\alpha} \qquad \rightarrow Y_{l,-m}(\hat{\Omega}) = Y_{l,-m}(\theta,\alpha) = \underbrace{\tilde{P}_l^m(\mu)e^{-im\alpha}}_{Y_{lm}^*(\theta,\alpha)}$$

$$\hat{\Omega} = \hat{\Omega}(\theta,\alpha), \; 0 \le m \le l$$

- Orthogonality

$$\int_{4\pi} Y_{lm}(\hat{\Omega})Y_{lm}^*(\hat{\Omega})d\hat{\Omega}$$

$$= \int_{-1}^{1}\int_0^{2\pi} \tilde{P}_l^m(\mu)e^{im\alpha} \cdot \tilde{P}_{l'}^{m'}(\mu)e^{-im'\alpha} d\alpha d\mu = \frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'}$$

$$\int_0^{2\pi} e^{i(m-m')\alpha}d\alpha = \int_0^{2\pi}\left(\cos(m-m')\alpha + i\sin(m-m')\alpha\right)d\alpha$$

$$= \begin{cases} 0, & m \ne m' \\ 2\pi, & m = m' \end{cases}$$

$$\int_{-1}^{1} \tilde{P}_l^m(\mu)\cdot\tilde{P}_{l'}^m(\mu)d\mu = \frac{\sqrt{(l-m)!}}{\sqrt{(l+m)!}}\frac{\sqrt{(l'-m)!}}{\sqrt{(l'+m)!}}\int_{-1}^{1} P_l(\mu)\cdot P_{l'}(\mu)d\mu = \frac{(l-m)!}{(l'+m)!}\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'} = \frac{2}{2l+1}\delta_{ll'}$$

Spherical Harmonics

$$Y_{00} = 1$$

$$Y_{10} = \tilde{P}_1^0(\mu) = \mu$$

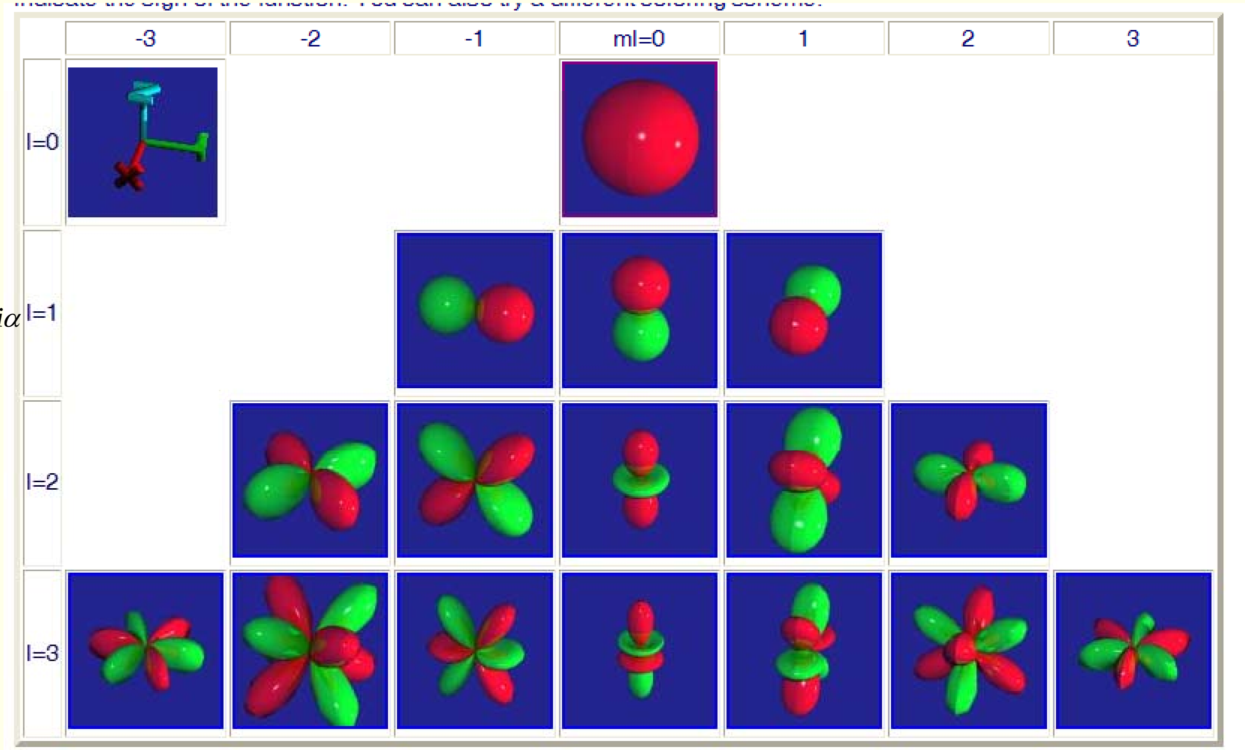
$$P_1^1(\mu) = (-1)^1 \sin\theta \frac{d\mu}{d\mu} = -\sin\theta$$

$$Y_{11} = \tilde{P}_1^1(\mu) e^{-i\alpha} = \sqrt{\frac{0!}{2!}} P_1^1(\mu) e^{i\alpha}$$

$$= -\frac{1}{\sqrt{2}} \sin\theta(\cos\alpha + i\sin\alpha)$$

$$Y_{1,-1} = \tilde{P}_1^{-1}(\mu) e^{-i\alpha} = -\tilde{P}_1^{-1}(\mu) e^{-i\alpha}$$

$$= \frac{1}{\sqrt{2}} \sin\theta(\cos\alpha - i\sin\alpha)$$



Spherical Harmonics Expansion of Angular Flux

$$\varphi(\theta,\alpha)=\sum_{l=0}^{L}\sum_{m=-l}^{l}a_{lm}Y_{lm}(\theta,\alpha)$$

$$a_{lm}=\frac{\left\langle\varphi,Y_{lm}^*\right\rangle}{\left\langle Y_{lm},Y_{lm}^*\right\rangle}=\frac{\left\langle\varphi,Y_{lm}^*\right\rangle}{\frac{4\pi}{2l+1}}=\frac{2l+1}{4\pi}\underbrace{\left\langle\varphi,Y_{lm}^*\right\rangle}_{\phi_{lm}}=\frac{2l+1}{4\pi}\phi_{lm}$$

$$=\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{2l+1}{4\pi}\phi_{lm}Y_{lm}(\theta,\alpha)$$

- L=1, P_1 Expansion

$$\varphi(\theta,\alpha)=\frac{1}{4\pi}\phi_{00}+\frac{3}{4\pi}(\phi_{11}Y_{11}+\phi_{10}Y_{10}+\phi_{1,-1}Y_{1,-1})$$

$$=\frac{1}{4\pi}\phi_{00}+\frac{3}{4\pi}[\phi_{11}(-\frac{1}{\sqrt{2}}\sin\theta(\cos\alpha+i\sin\alpha))+\phi_{1,-1}(\frac{1}{\sqrt{2}}\sin\theta(\cos\alpha-i\sin\alpha))+\phi_{10}\cos\theta]$$

$$=\frac{1}{4\pi}\phi_{00}+\frac{3}{4\pi}[\sin\theta\cos\alpha(-\frac{1}{\sqrt{2}}(\phi_{11}-\phi_{1,-1}))+\sin\theta\sin\alpha(-\frac{i}{\sqrt{2}}(\phi_{11}+\phi_{1,-1}))+\phi_{10}\cos\theta]$$

$$=\frac{1}{4\pi}\phi_{00}+\frac{3}{4\pi}(\Omega_x\phi_{1x}+\Omega_y\phi_{1y}+\Omega_z\phi_{1z})$$

$$=\frac{1}{4\pi}\phi_{00}+\frac{3}{4\pi}\hat{\Omega}\cdot\vec{J}$$

For 1-D $\rightarrow \frac{1}{4\pi}\phi_{00}+\frac{3}{4\pi}\mu J$

1-D Boltzmann Transport Equation in Plane Geometry

- 1-D Boltzmann Transport Eqn

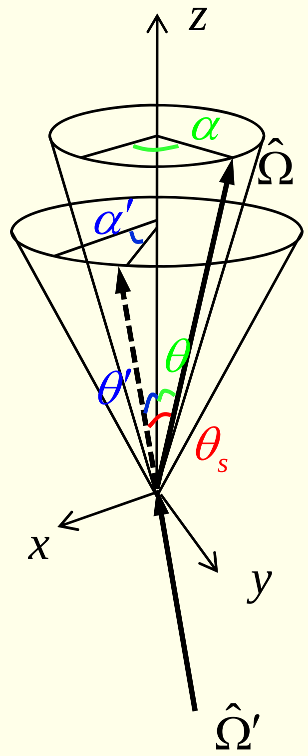
$$\mu\frac{\partial\varphi}{\partial z}+\Sigma_t\varphi=\frac{\chi}{4\pi}\psi+\int_{E'}\int_{\Omega'}\Sigma(z,E'\rightarrow E,\hat{\Omega}'\rightarrow\hat{\Omega})\varphi(z,E',\hat{\Omega}')d\hat{\Omega}'dE'$$

- Legendre Expansion of Angular Flux

$$\varphi(z,E,\mu)=\sum_{l=0}^{L}\frac{2l+1}{4\pi}\phi_l(z,E)P_l(\mu)$$

- Legendre Expansion of Differential Scattering Xsec

$$\Sigma(z,E'\rightarrow E,\mu_s)=\sum_{l=0}^{L}\Sigma_l(z,E'\rightarrow E)\frac{2l+1}{4\pi}P_l(\mu_s) \qquad \mu_s=\mu_s(\mu,\mu',\alpha,\alpha')$$



- Insert the expansion into the 1-D BTE.

$$\mu\frac{\partial}{\partial z}\sum_{l=0}^{L}\frac{2l+1}{4\pi}\phi_l(z,E)P_l(\mu)+\Sigma_t\left(\sum_{l=0}^{L}\frac{2l+1}{4\pi}\phi_l(z,E)P_l(\mu)\right)$$

$$=\frac{\chi}{4\pi}\psi+\int_0^\infty\int_{-1}^{1}\int_0^{2\pi}\left(\sum_{l=0}^{L}\Sigma_l(z,E'\rightarrow E)\frac{2l+1}{4\pi}P_l(\mu_s)\right)\cdot\left(\sum_{l'=0}^{L}\frac{2l'+1}{4\pi}\phi_{l'}(z,E')P_{l'}(\mu')\right)d\alpha' d\mu' dE'$$

Addition Theorem

$$\mu_s = \hat{\Omega}\cdot\hat{\Omega}' = P_1(\mu_s)$$
$$= (\sin\theta\cos\alpha\hat{x} + \sin\theta\sin\alpha\hat{y} + \cos\theta\hat{z})\cdot(\sin\theta'\cos\alpha'\hat{x} + \sin\theta'\sin\alpha'\hat{y} + \cos\theta'\hat{z})$$
$$= \sin\theta\sin\theta'\cos\alpha\cos\alpha' + \sin\theta\sin\theta'\sin\alpha\sin\alpha' + \cos\theta\cos\theta'$$
$$= \sin\theta\sin\theta'(\cos\alpha\cos\alpha' + \sin\alpha\sin\alpha') + \cos\theta\cos\theta'$$
$$= \sin\theta\sin\theta'\cos(\alpha-\alpha') + \cos\theta\cos\theta'$$
$$= (-P_1^1(\mu)\cdot -P_1^1(\mu'))\cos(\alpha-\alpha') + P_1(\mu)P_1(\mu')$$
$$= P_1(\mu)P_1(\mu') + P_1^1(\mu)P_1^1(\mu')\cos(\alpha-\alpha')$$

$$\boxed{\tilde{P}_l(\mu) = (-1)^m \sin^m\theta \frac{d^m P_l(\mu)}{d\mu^m}; m \le l}$$

$$P_1^1(\mu) = \left(-1\right)^1 \sin\theta \frac{d\mu}{d\mu} = -\sin\theta$$

- Addition Theorem

$$P_l(\mu_s) = P_l(\mu)P_l(\mu') + 2\sum_{m=1}^{l} \tilde{P}_l^m(\mu)\tilde{P}_l^m(\mu')\cos m(\alpha-\alpha')$$

$$= \sum_{m=-l}^{l} Y_l^m(\mu,\alpha)Y_l^{m*}(\mu',\alpha')$$

Application of Addition Theorem

$$\int_{-1}^{1}\int_{0}^{2\pi}\left(\sum_{l=0}^{L}\Sigma_l(z,E'\to E)\frac{2l+1}{4\pi}P_l(\mu_s)\right)\cdot\left(\sum_{l'=0}^{L}\frac{2l'+1}{4\pi}P_{l'}(\mu')\right)d\alpha' d\mu'$$

$$=\int_{-1}^{1}\int_{0}^{2\pi}\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)\left(P_l(\mu)P_l(\mu')+2\sum_{m=-l}^{l}\tilde{P}_l^m(\mu)\tilde{P}_l^m(\mu')\cos m(\alpha-\alpha')\right)$$

$$\cdot\left(\sum_{l'=0}^{L}\frac{2l'+1}{4\pi}\phi_{l'}(z,E')P_{l'}(\mu')\right)d\alpha' d\mu'$$

$$\boxed{\int_0^{2\pi}\cos m(\underbrace{\alpha-\alpha'}_{t})d\alpha'=\int_{\alpha}^{\alpha-2\pi}-\cos mt dt=\int_{\alpha-2\pi}^{\alpha}\cos mt dt=\frac{1}{m}\left(\sin m\alpha-\sin m(\alpha-2\pi)\right)=0}$$

$$=2\pi\int_{-1}^{1}\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)P_l(\mu)P_l(\mu')\cdot\frac{2l+1}{4\pi}\phi_l(z,E')P_l(\mu')d\mu'$$

$$=\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)\phi_l(z,E')P_l(\mu)\cdot\underbrace{\int_{-1}^{1}P_l(\mu')\frac{2l+1}{2}P_l(\mu')d\mu'}_{=1}$$

$$=\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)\phi_l(z,E')P_l(\mu)$$

B.T.E. by Legendre Expansion

- B.T.E.

$$\mu\frac{\partial}{\partial z}\sum_{l=0}^{L}\frac{2l+1}{4\pi}\phi_l(z,E)P_l(\mu)+\Sigma_t\left(\sum_{l=0}^{L}\frac{2l+1}{4\pi}\phi_l(z,E)P_l(\mu)\right)$$
$$=\frac{\chi}{4\pi}\psi+\int_0^\infty\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)P_l(\mu)\phi_l(z,E')dE'$$

- Apply $2\pi\int_{-1}^{1}P_n(\mu)d\mu$ to B.T.E.

  - Total Reaction Term

$$2\pi\int_{-1}^{1}\Sigma_t(z,E)\left(\sum_{l=0}^{L}\frac{2l+1}{4\pi}\phi_l(z,E)P_l(\mu)\right)P_n(\mu)d\mu=\Sigma_t(z,E)\phi_n(z,E)$$

  - Fission Source Term

$$2\pi\int_{-1}^{1}\frac{\chi}{4\pi}P_n(\mu)d\mu=\delta_{0n}\chi(E)\psi(z)$$

  - Scattering Source Term

$$2\pi\int_{-1}^{1}\int_0^\infty\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)P_l(\mu)\phi_l(z,E')dE'P_n(\mu)d\mu=\int_0^\infty\Sigma_n(z,E'\to E)\phi_n(z,E')dE'$$

Legendre Expansion of B.T.E.

- Leakage Term

$$2\pi\int_{-1}^{1}\left(\mu\frac{\partial}{\partial z}\sum_{l=0}^{L}\frac{2l+1}{4\pi}\phi_l(z,E)P_l(\mu)\right)\cdot P_n(\mu)d\mu \qquad \boxed{(2l+1)\mu P_l(\mu)=(l+1)P_{l+1}(\mu)+lP_{l-1}(\mu)}$$

$$=2\pi\frac{\partial}{\partial z}\int_{-1}^{1}\left(\sum_{l=0}^{L}\frac{1}{4\pi}\left(\underline{(l+1)P_{l+1}(\mu)}+\underline{lP_{l-1}(\mu)}\right)\phi_l(z,E)P_l(\mu)\right)\cdot P_n(\mu)d\mu$$

$$i)\ \left.\begin{matrix} l+1=n \\ l=n-1\end{matrix}\right\} \rightarrow nP_n(\mu)\phi_{n-1}(z,E) \qquad ii)\ \left.\begin{matrix} l-1=n \\ l=n+1\end{matrix}\right\} \rightarrow (n+1)P_n(\mu)\phi_{n+1}(z,E)$$

$$=\frac{1}{2}\frac{\partial}{\partial z}\left[\int_{-1}^{1}nP_n^2(\mu)\phi_{n-1}(z,E)d\mu+\int_{-1}^{1}(n+1)P_n^2(\mu)\phi_{n+1}(z,E)d\mu\right]$$

$$=\frac{1}{2}\frac{\partial}{\partial z}\left[\frac{2n}{2n+1}\phi_{n-1}(z,E)+\frac{2(n+1)}{2n+1}\phi_{n+1}(z,E)\right]=\frac{n}{2n+1}\frac{\partial}{\partial z}\phi_{n-1}(z,E)+\frac{n+1}{2n+1}\frac{\partial}{\partial z}\phi_{n+1}(z,E)$$

- Resulting Equation

$$\frac{n}{2n+1}\frac{\partial\phi_{n-1}}{\partial z}+\frac{n+1}{2n+1}\frac{\partial\phi_{n+1}}{\partial z}+\Sigma_t\phi_n=\delta_{0n}\chi\psi+\int_0^{\infty}\Sigma_n(z,E'\rightarrow E)\phi_n(z,E')dE'$$

Multi-group Formation

- Total Reaction Term

$$\int_{E_g}^{E_{g-1}} \Sigma_t(z,E)\phi_n(z,E)dE = \bar{\Sigma}_t(z)\cdot \underbrace{\int_{E_g}^{E_{g-1}} \phi_n(z,E)dE}_{\phi_{ng}} = \Sigma_{tng}\cdot\phi_{ng}$$

$$where \;\; \bar{\Sigma}_t(z) = \frac{1}{\phi_{ng}}\int_{E_g}^{E_{g-1}} \Sigma_t(z,E)\phi_n(z,E)dE = \Sigma_{tng}$$

- Leakage Term

$$\phi_{ng}(z) = \int_{E_g}^{E_{g-1}} \phi_n(z,E)dE \quad \longrightarrow \quad \frac{n}{2n+1}\frac{\partial\phi_{n-1,g}}{\partial z} + \frac{n+1}{2n+1}\frac{\partial\phi_{n+1,g}}{\partial z}$$

- Scattering Source Term

$$\begin{aligned}
\int_{E_g}^{E_{g-1}}\int_0^{\infty} \Sigma_n(z,E'\to E)\phi_n(z,E')dE'dE &= \int_0^{\infty}\int_{E_g}^{E_{g-1}} \Sigma_n(z,E'\to E)dE\phi_n(z,E')dE' \\
&= \sum_{g'=1}^{G}\int_{E_{g'}}^{E_{g'-1}}\int_{E_g}^{E_{g-1}} \Sigma_n(z,E'\to E)dE\phi_n(z,E')dE' \\
&= \sum_{g'=1}^{G}\Sigma_{ng'g}\phi_{ng'}
\end{aligned}$$

$$where \;\; \Sigma_{ng'g} = \frac{1}{\phi_{ng'}}\int_{E_{g'}}^{E_{g'-1}}\int_{E_g}^{E_{g-1}} \Sigma_n(z,E'\to E)dE\phi_n(z,E')dE'$$

Multi-group Formation

- Multigroup P_L Equation

$$\frac{n}{2n+1}\frac{\partial \phi_{n-1,g}}{\partial z}+\frac{n+1}{2n+1}\frac{\partial \phi_{n+1,g}}{\partial z}+\Sigma_{tng}\phi_{ng}=\delta_{0ng}\chi_g\psi+\sum_{g'=1}^{G}\Sigma_{ng'g}\phi_{ng'} \qquad \begin{array}{l} n=0,\cdots,L \\ g=1,\cdots,G \end{array}$$

- Matrix form for L=3

$$\begin{bmatrix} \Sigma_{t0g} & \frac{\partial}{\partial z} & & \\ \frac{1}{3}\frac{\partial}{\partial z} & \Sigma_{t1g} & \frac{2}{3}\frac{\partial}{\partial z} & \\ & \frac{2}{5}\frac{\partial}{\partial z} & \Sigma_{t2g} & \frac{3}{5}\frac{\partial}{\partial z} \\ & & \frac{3}{7}\frac{\partial}{\partial z} & \Sigma_{t3g} \end{bmatrix} \begin{bmatrix} \phi_{0g} \\ \phi_{1g} \\ \phi_{2g} \\ \phi_{3g} \end{bmatrix} = \begin{bmatrix} S_{0g} \\ S_{1g} \\ S_{2g} \\ S_{3g} \end{bmatrix}$$

Multi-group P_L Method

- Reduction of odd moments

$$
\begin{array}{c:c:c:c:c}
\frac{n-1}{2n-1}\frac{\partial}{\partial z} & \Sigma_{tn-1g} & \frac{n}{2n-1}\frac{\partial}{\partial z} & & \\
\hdashline
X & \frac{n}{2n+1}\frac{\partial}{\partial z} & \Sigma_{tng} & \frac{n+1}{2n+1}\frac{\partial}{\partial z} & Y \\
\hdashline
 & & \frac{n+1}{2n+3}\frac{\partial}{\partial z} & \Sigma_{tn+1g} & \frac{n+2}{2n+3}\frac{\partial}{\partial z}
\end{array}
\quad
\begin{array}{l}
\cdots\cdots(1) \\
\cdots\cdots(2) \\
\cdots\cdots(3)
\end{array}
\qquad \text{for even } n
$$

$$(2)-(1)\times\frac{1}{\Sigma_{tn-1g}}\frac{n}{2n+1}\frac{\partial}{\partial z}$$

$$X=-\frac{1}{\Sigma_{tn-1g}}\frac{n}{2n+1}\frac{\partial}{\partial z}\frac{n-1}{2n-1}\frac{\partial}{\partial z}=-\frac{1}{\Sigma_{tn-1g}}\frac{n-1}{2n-1}\frac{\partial}{\partial z}\frac{n}{2n+1}\frac{\partial}{\partial z}=-\frac{n}{2n+1}\frac{\partial}{\partial z}D_{n-1g}$$

$$\boxed{D_{n-1g}\equiv\frac{1}{\Sigma_{tn-1g}}\frac{n-1}{2n-1}\to D_{ng}\equiv\frac{1}{\Sigma_{tng}}\frac{n}{2n+1}}$$

$$\tilde{\Sigma}_{tng}=\Sigma_{tng}-\frac{1}{\Sigma_{tn-1g}}\frac{n}{2n+1}\frac{\partial}{\partial z}\frac{n}{2n-1}\frac{\partial}{\partial z}$$

$$=\Sigma_{tng}-\frac{1}{\Sigma_{tn-1g}}\frac{n^2}{(2n+1)(n-1)}\frac{\partial}{\partial z}\frac{n-1}{2n-1}\frac{\partial}{\partial z}=\Sigma_{tng}-\frac{n^2}{(2n+1)(n-1)}\frac{\partial}{\partial z}D_{n-1g}\frac{\partial}{\partial z}$$

Multi-group P_L Method

$$(2)-(3)\times\frac{1}{\Sigma_{tn+1g}}\frac{n+1}{2n+1}\frac{\partial}{\partial z}$$

$$Y=-\frac{1}{\Sigma_{tn+1g}}\frac{n+1}{2n+1}\frac{\partial}{\partial z}\frac{n+2}{2n+3}\frac{\partial}{\partial z}=-\frac{n+2}{2n+1}\frac{\partial}{\partial z}D_{n+1g}\frac{\partial}{\partial z}$$

$$\boxed{\begin{array}{ccccc} \frac{n-1}{2n-1}\frac{\partial}{\partial z} & \Sigma_{tn-1g} & \frac{n}{2n-1}\frac{\partial}{\partial z} & & \\ X & \frac{n}{2n+1}\frac{\partial}{\partial z} & \Sigma_{tng} & \frac{n+1}{2n+1}\frac{\partial}{\partial z} & Y \\ & & \frac{n+1}{2n+3}\frac{\partial}{\partial z} & \Sigma_{tn+1g} & \frac{n+2}{2n+3}\frac{\partial}{\partial z} \end{array}}$$

$$\tilde{\tilde{\Sigma}}_{tn-1g}=\tilde{\Sigma}_{tng}-\frac{1}{\Sigma_{tn+1g}}\frac{n+1}{2n+1}\frac{\partial}{\partial z}\frac{n+1}{2n+3}\frac{\partial}{\partial z}$$

$$\boxed{D_{n+1g}\equiv\frac{1}{\Sigma_{tng}}\frac{n+1}{2n+3}\frac{\partial}{\partial z}}$$

$$=\tilde{\Sigma}_{tng}-\frac{n+1}{2n+1}\frac{\partial}{\partial z}D_{n+1}\frac{\partial}{\partial z}\quad=\Sigma_{tng}-\frac{n^2}{(2n+1)(n-1)}\frac{\partial}{\partial z}D_{n-1g}\frac{\partial}{\partial z}-\frac{n+1}{2n+1}\frac{\partial}{\partial z}D_{n+1g}\frac{\partial}{\partial z}$$

- Matrix form

$$\begin{bmatrix} d_0 & -u_0 & & & \\ -l_2 & d_2 & -u_2 & & \\ & -l_n & d_n & -u_n & \\ & & -l_{L-1} & d_{L-1} & \end{bmatrix}\begin{bmatrix}\phi_{0g}\\ \phi_{2g}\\ \phi_{ng}\\ \phi_{L-1g}\end{bmatrix}=S$$

for an odd L

$$d_{ng}=\Sigma_{tng}-\frac{n^2}{(2n+1)(n-1)}\frac{\partial}{\partial z}D_{n-1}\frac{\partial}{\partial z}-\frac{n+1}{2n+1}\frac{\partial}{\partial z}D_{n+1}\frac{\partial}{\partial z}$$

$$l_{ng}=\frac{n}{2n+1}\frac{\partial}{\partial z}D_{n-1g}$$

$$u_{ng}=\frac{n+2}{2n+1}\frac{\partial}{\partial z}D_{n+1g}$$

P_1 Approximation and Diffusion Theory

- P_1 Expansion of Angular Flux in 1-D

$$\varphi(z,E,\mu)=\frac{1}{4\pi}\phi+\frac{3}{4\pi}\mu J \quad (\phi_0=\phi,\ \phi_1=J)$$

- P_1 Equation

$$① \ \Sigma_t(z,E)\phi(z,E)+\frac{\partial J(z,E)}{\partial z}=\chi(E)\psi+\int_0^\infty \Sigma_0(z,E'\to E)\phi(z,E')dE'$$

$$② \ \frac{1}{3}\frac{\partial \phi}{\partial z}+\Sigma_t J=0+\int_0^\infty \Sigma_1(z,E'\to E)J(E')dE'$$

- Multi-group form

$$① \to \Sigma_{tg}(z)\phi_g(z)+\frac{\partial J_g(z)}{\partial z}=\chi_g\psi+\sum_{g'=1}^{G}\Sigma_{g'g}\phi_{g'}(z)$$

$$② \to \frac{1}{3}\frac{\partial \phi_g}{\partial z}+\Sigma_{tg}^{(1)}J_g=\sum_{g'=1}^{G}\Sigma_{g'g}^{(1)}J_{g'}$$

coupled on J_g

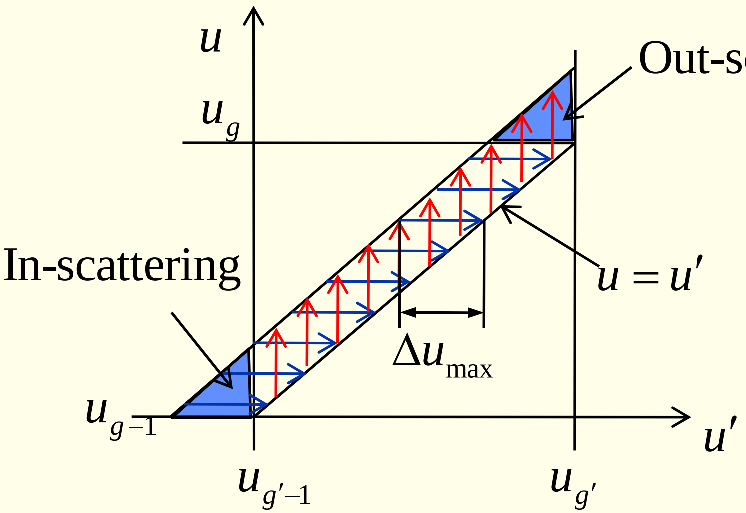
$$\Sigma_{tg}^{(1)}(z)=\frac{1}{J_g}\int_{E_g}^{E_{g-1}}\Sigma_t(z,E)J(z,E)dE \qquad \Sigma_{g'g}^{(1)}(z)=\frac{1}{J_g}\int_{E_{g'}}^{E_{g'-1}}\int_{E_{g-1}}^{E_g}\Sigma_1(z,E'\to E)J(z,E')dEdE'$$

P_1 Approximation and Diffusion Theory

- Approximation for inscattering

$$\underbrace{\sum_{g'=1}^{G}\Sigma^{(1)}_{g'g}J_{g'}}_{\text{In-scattering}} \cong \underbrace{\overbrace{\boxed{\sum_{g'=1}^{G}\Sigma^{(1)}_{gg'}}}^{\Sigma^{(1)}_{sg}}}_{\text{Out-scattering}} J_g$$

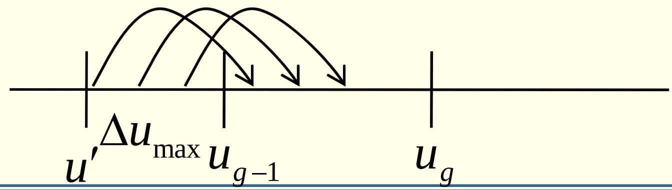
- Inscattering vs. Outscattering



Consider downscattering only to the adjacent group for broad energy group

Contributions from the difference btwn two triangles would be small

for
① Broad group
② Small scattering power
③ Weak absorption (constant flux over u)



$$\int_{E_g}^{E_{g-1}}\int_{E_{g'-1}}^{E_{g'}}\Sigma_1(z,E'\to E)J(z,E')dE'dE$$

P_1 Approximation and Diffusion Theory

- Approximated Multi-group Second P_1

$$\bar{\mu}\Sigma_{sg}^{(0)}$$

$$② \rightarrow \frac{1}{3}\frac{d\phi_g}{dz} + \Sigma_{tg}^{(1)} J_g = \Sigma_{sg}^{(1)} J_g$$

$$J_g(\Sigma_{tg}^{(1)} - \Sigma_{sg}^{(1)}) = -\frac{1}{3}\frac{d\phi_g}{dz} \longrightarrow J_g = -\boxed{\frac{1}{3(\Sigma_{tg}^{(1)} - \Sigma_{sg}^{(1)})}}\frac{d\phi_g}{dz} \quad \text{: Fick's Law}$$

$$D_g$$

– Further Approximation

$\Sigma_{tg}^{(1)} \rightarrow \Sigma_{tg}$: Neglect the difference between current and flux weighting

$\Sigma_{trg} = \Sigma_{tg} - \Sigma_{sg}^{(1)}$: Transport Xsec.

$$D_g = \frac{1}{3\Sigma_{trg}}$$

P_1 Approximation and Diffusion Theory

- Fick's Law in First P_1

$$-\frac{d}{dz}D_g\frac{d\phi_g}{dz}+\Sigma_{rg}\phi_g=\chi_g\psi+\sum_{\substack{g'=1\\g'\neq g}}^{G}\Sigma_{g'g}\phi_g$$

$$\Sigma_{r0g}=\Sigma_{t0g}-\Sigma_{0gg}$$

- Boundary Condition for Diffusion Equation

$$J_{in}(z_b,t)=0 \longrightarrow J_{in}=\frac{1}{4}\phi-\frac{1}{2}J=0$$

$$\phi=2J=-2D\frac{d\phi}{dz}$$: Marshak B.C.

$$\frac{1}{\phi}\frac{d\phi}{dz}=-\frac{1}{2D}=\tilde{\alpha}$$: Robin B.C.

Multi-group Formation

- Self Scattering Correction

  - Move self scattering term from RHS

$$\Sigma_{rng} = \Sigma_{tng} - \Sigma_{ngg}$$

$$\Sigma_{rng} = \frac{1}{\phi_{ng}} \int_{E_g}^{E_{g-1}} \Sigma_t(E)\phi_n(E)dE - \frac{1}{\phi_{ng}} \int_{E_g}^{E_{g-1}} \int_{E_g}^{E_{g-1}} \Sigma_n(E' \to E)dE \underline{\phi_n(E')} dE' \cong \bar{\Sigma}_{t0g} - \bar{\Sigma}_{ngg}$$

ϕ_n required $\longrightarrow$ use ϕ_0 instead

Simplified 1-D, P3 Equations

- General 1-D P_3 Equation

$$
\begin{bmatrix}
\Sigma_{t0g} & \frac{d}{dx} & & \\
\frac{1}{3}\frac{d}{dx} & \Sigma_{t1g} & \frac{2}{3}\frac{d}{dx} & \\
 & \frac{2}{5}\frac{d}{dx} & \Sigma_{t2g} & \frac{3}{5}\frac{d}{dx} \\
 & & \frac{3}{7}\frac{d}{dx} & \Sigma_{t3g}
\end{bmatrix}
\begin{bmatrix}
\phi_{0g} \\ \phi_{1g} \\ \phi_{2g} \\ \phi_{3g}
\end{bmatrix}
=
\begin{bmatrix}
\frac{1}{4\pi}\chi_g\psi + \sum_{g'=1}^{G}\Sigma_{g'g}^{(0)}\phi_{0g} \\
\sum_{g'=1}^{G}\Sigma_{g'g}^{(1)}\phi_{1g} \\
\sum_{g'=1}^{G}\Sigma_{g'g}^{(2)}\phi_{1g} \\
\sum_{g'=1}^{G}\Sigma_{g'g}^{(3)}\phi_{1g}
\end{bmatrix}
$$

- Assume:
  - $\Sigma_{tng} = \Sigma_{t0g} = \Sigma_{tg}$
  - P_1 Scattering

    $$\Sigma_{g'g}^{(n)} = 0 \quad \forall n \geq 2$$
  - Inconsistent P_1

    $$\sum_{g'=1}^{G}\Sigma_{1g'g}\phi_{g'} \approx \sum_{g'=1}^{G}\Sigma_{1gg'}\phi_g = \Sigma_{sg}^{(1)}\phi_g$$
  - Transport correction

    $$\Sigma_{trg} = \Sigma_{tg} - \Sigma_{sg}^{(1)}, \quad D_{0g} = \frac{1}{3\Sigma_{trg}}$$

$$\frac{1}{3}\frac{d\phi_{0g}}{dx} + \Sigma_{tg}\phi_{1g} + \frac{2}{3}\frac{d\phi_{2g}}{dx} = \Sigma_{sg}^{(1)}\phi_{1g}$$

$$\frac{1}{3}\frac{d\phi_{0g}}{dx} + \Sigma_{trg}\phi_{1g} + \frac{2}{3}\frac{d\phi_{2g}}{dx} = 0$$

$$D_{0g}\frac{d\phi_{0g}}{dx} + \phi_{1g} + 2D_{0g}\frac{d\phi_{2g}}{dx} = 0$$

Simplified 1-D, P3 Equations

$$\begin{bmatrix} \Sigma_{tg} & \frac{d}{dx} & & \\ D_{0g}\frac{d}{dx} & 1 & 2D_{0g}\frac{d}{dx} & \\ & \frac{2}{5}\frac{d}{dx} & \Sigma_{tg} & \frac{3}{5}\frac{d}{dx} \\ & & \frac{3}{7}\frac{d}{dx} & \Sigma_{tg} \end{bmatrix} \begin{bmatrix} \phi_{0g} \\ \phi_{1g} \\ \phi_{2g} \\ \phi_{3g} \end{bmatrix} = \begin{bmatrix} \frac{1}{4\pi}\chi_g\psi + \sum_{g'=1}^{G}\Sigma_{g'g}\phi_{0g} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\phi_{1g} \rightarrow J_{0g}$$
$$\phi_{3g} \rightarrow J_{2g}$$
$$D_{2g} = \frac{3}{7\Sigma_{tg}}$$

$$\begin{bmatrix} \Sigma_{rg} & \frac{d}{dx} & & \\ D_{0g}\frac{d}{dx} & 1 & 2D_{0g}\frac{d}{dx} & \\ & \frac{2}{5}\frac{d}{dx} & \Sigma_{tg} & \frac{3}{5}\frac{d}{dx} \\ & & D_{2g}\frac{d}{dx} & 1 \end{bmatrix} \begin{bmatrix} \phi_{0g} \\ J_{0g} \\ \phi_{2g} \\ J_{2g} \end{bmatrix} = \begin{bmatrix} q_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longleftarrow$$

$$q_0 = \frac{1}{4\pi}\chi_g\psi + \sum_{\substack{g'=1 \\ g'\neq g}}^{G}\Sigma_{g'g}\phi_{0g}$$

$$\Sigma_{rg} = \Sigma_{tg} - \Sigma_{gg}$$

$$J_{0g} = -D_{0g}\frac{d\phi_{0g}}{dx} - 2D_{0g}\frac{d\phi_{2g}}{dx}$$
$$J_{2g} = -D_{2g}\frac{d\phi_{2g}}{dx}$$

Coupled P3 Equation

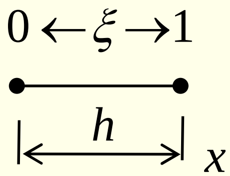
- Definitions of normalized variables

$$\mathrm{X} = \frac{1}{h}\, x;\ D_{\mathrm{X}} = \frac{d}{d\mathrm{X}};\ \frac{d}{d\,x} = \frac{1}{h} D_{\mathrm{X}};\ \frac{d^2}{d\,x^2} = \frac{1}{h^2} D_{\mathrm{X}}^2; \quad \text{* Normalized Coordinate Variable *}$$

$$D_0 = \frac{1}{3\,\mathrm{S_{tr}}};\ D_2 = \frac{3}{7\,\mathrm{S_t}}; \quad \text{* Diffusion Coefficients *}$$

$$\mathrm{b}_0 = \frac{D_0}{h};\ \mathrm{b}_2 = \frac{D_2}{h}; \quad \text{* Relative Diffusivity *}$$

$$\mathrm{S_{D0}} = \frac{D_0}{h^2};\ \mathrm{S_{D2}} = \frac{D_2}{h^2}; \quad \text{* Diffusion Xsec *}$$



- P3 equation in terms of normalized variables

$$\hat{\phi}_0(\xi) = \phi_0(\xi) + 2\phi_2(\xi)$$

$$\begin{pmatrix} \mathrm{S_r} & \frac{\mathrm{D_X}}{\mathrm{h}} & 0 & 0 \\ \mathrm{D_X\, b_0} & \boxed{1} & 2\,\mathrm{D_X\, b_0} & 0 \\ 0 & \frac{2\,\mathrm{D_X}}{5\,\mathrm{h}} & \mathrm{S_t} & \frac{3\,\mathrm{D_X}}{5\,\mathrm{h}} \\ 0 & 0 & \mathrm{D_X\, b_2} & 1 \end{pmatrix}$$

$$J_0(\xi) = -\beta_0 D_\xi \left(\phi_0(\xi) + 2\phi_2(\xi)\right) = -\beta_0 D_\xi \hat{\phi}_0(\xi)$$

$$J_2(\xi) = -\beta_2 D_\xi \phi_2(\xi)$$

- Reduced form after eliminating odd moments

$$\boxed{\Sigma_{Dm} = \frac{\beta_m}{h}}$$

$$\begin{pmatrix} -\frac{\mathrm{D_X^2\, b_0}}{\mathrm{h}} + \mathrm{S_r} & -\frac{2\,\mathrm{D_X^2\, b_0}}{\mathrm{h}} \\ -\frac{2\,\mathrm{D_X^2\, b_0}}{5\,\mathrm{h}} & -\frac{4\,\mathrm{D_X^2\, b_0}}{5\,\mathrm{h}} - \frac{3\,\mathrm{D_X^2\, b_2}}{5\,\mathrm{h}} + \mathrm{S_t} \end{pmatrix} \longrightarrow \begin{pmatrix} -\mathrm{D_X^2\, S_{D0}} + \mathrm{S_r} & -2\,\mathrm{D_X^2\, S_{D0}} \\ -\frac{2}{5}\,\mathrm{D_X^2\, S_{D0}} & -\frac{4}{5}\,\mathrm{D_X^2\, S_{D0}} - \frac{3}{5}\,\mathrm{D_X^2\, S_{D2}} + \mathrm{S_t} \end{pmatrix}$$

Removal of Derivative from Off-diagonal

- Introduction of Summed Flux

$$K \begin{pmatrix} -D_X^2 S_{D0} + S_r & -2 D_X^2 S_{D0} \\ -\frac{2}{5} D_X^2 S_{D0} & -\frac{4}{5} D_X^2 S_{D0} - \frac{3}{5} D_X^2 S_{D2} + S_t \end{pmatrix}$$

$$\Sigma_r \phi_0(\xi) - \Sigma_{D0} D_\xi^2 \left(\phi_0(\xi) + 2\phi_2(\xi)\right) = q_0(\xi)$$

$$-\frac{2}{5}\Sigma_{D0} D_\xi^2 \left(\phi_0(\xi) + 2\phi_2(\xi)\right) + \left(-\frac{3}{5}\Sigma_{D2} D_\xi^2 + \Sigma_t\right)\phi_2(\xi) = 0$$

$$\boxed{\hat{\phi}_0(\xi) = \phi_0(\xi) + 2\phi_2(\xi), \ \ \phi_0(\xi) = \hat{\phi}_0(\xi) - 2\phi_2(\xi)}$$

$$\begin{bmatrix} -\Sigma_{D0} D_\xi^2 + \Sigma_r & -2\Sigma_r \\ -\frac{2}{5}\Sigma_{D0} D_\xi^2 & -\frac{3}{5}\Sigma_{D2} D_\xi^2 + \Sigma_t \end{bmatrix} \begin{bmatrix} \hat{\phi}_0(\xi) \\ \phi_2(\xi) \end{bmatrix} = \begin{bmatrix} q_0(\xi) \\ 0 \end{bmatrix}$$

-remove D_ξ^2 from off-diagonal by $(2) - \frac{2}{5}(1)$

$$\begin{bmatrix} -\Sigma_{D0} D_\xi^2 + \Sigma_r & -2\Sigma_r \\ -\frac{2}{5}\Sigma_r & -\frac{3}{5}\Sigma_{D2} D_\xi^2 + \frac{4}{5}\Sigma_r + \Sigma_t \end{bmatrix} \begin{bmatrix} \hat{\phi}_0(\xi) \\ \phi_2(\xi) \end{bmatrix} = \begin{bmatrix} q_0(\xi) \\ -\frac{2}{5} q_0(\xi) \end{bmatrix}$$

- Final equation after making the coeff. of $\Sigma_{D2} D_\xi^2$ unity by mutiplying $\frac{5}{3}$

$$\begin{bmatrix} -\Sigma_{D0} D_\xi^2 + \Sigma_r & -2\Sigma_r \\ -\frac{2}{3}\Sigma_r & -\Sigma_{D2} D_\xi^2 + \frac{4}{3}\Sigma_r + \frac{5}{3}\Sigma_t \end{bmatrix} \begin{bmatrix} \hat{\phi}_0(\xi) \\ \phi_2(\xi) \end{bmatrix} = \begin{bmatrix} q_0(\xi) \\ -\frac{2}{3} q_0(\xi) \end{bmatrix}$$

Partial Current Relations

- Angular flux in terms of expansion

$$\varphi(\mu,\xi)=\phi_0(\xi)+\frac{3}{4\pi}J_0(\xi)P_1(\mu)+\frac{5}{4\pi}\phi_2(\xi)P_2(\mu)+\frac{7}{4\pi}J_2(\xi)P_3(\mu)$$

- Patial current relations

$$J_{l-1}(\xi)=\int_{-1}^{1}\varphi(\xi,\mu)P_l(\mu)d\mu : \text{only } l\text{-th moment is relevant } \because \text{ orthogonality}$$

$$J_{pl-1}(\xi)=\int_{0}^{1}\varphi(\xi,\mu)P_l(\mu)d\mu : \text{ out-going current}$$

$$J_{ml-1}(\xi)=-\int_{-1}^{0}\varphi(\xi,\mu)P_l(\mu)d\mu \text{ (to make inward normal positive)}$$

$$J_{p0}=\frac{\phi_0}{4}+\frac{J_0}{2}+\frac{5}{16}\phi_2$$

$$J_{m0}=\frac{\phi_0}{4}-\frac{J_0}{2}+\frac{5}{16}\phi_2$$

$$J_{p2}=-\frac{\phi_0}{16}+\frac{J_2}{2}+\frac{5}{16}\phi_2$$

$$J_{m2}=-\frac{\phi_0}{16}-\frac{J_2}{2}+\frac{5}{16}\phi_2$$

$$\longrightarrow$$

$$\phi_0=\frac{8}{5}\left(J_{p0}+J_{m0}-J_{p2}-J_{m2}\right)$$

$$\phi_2=\frac{8}{25}\left(J_{p0}+J_{m0}+4J_{p2}+4J_{m2}\right)$$

$$\longleftarrow \quad \boxed{\phi_0=\hat{\phi}_0-2\phi_2}$$

Albedo Matrix

- Patial current relations in terms of summed flux

J_{p0} ... $-\frac{3}{16}$ ... 0 ... J_{p2} ... $\frac{7}{16}$ ... $\frac{1}{2}$ ...

J_{m0} ... $-\frac{3}{16}$ ... 0 ... J_{m2} ... $\frac{7}{16}$ ... $\frac{1}{2}$ ...

$$\longrightarrow \quad \begin{aligned} \mathbf{J}_{out} &= \mathbf{M}_{\phi}\Phi + \mathbf{M}_{J}\mathbf{J}_{net} \\ \mathbf{J}_{in} &= \mathbf{M}_{\phi}\Phi - \mathbf{M}_{J}\mathbf{J}_{net} \\ \mathbf{J}_{net} &= \mathbf{J}_{out} - \mathbf{J}_{in} \end{aligned}$$

- Summed flux in terms of partial currents

$f_0 = \frac{8}{25}$ ... J_0^m ... $+ J_0$ ... $J_2^m + J_2$ ...

$f_2 = \frac{8}{25}$ ... $J_0^m + J_0$ ... $+ 4$ ... $J_2^m + J_2$ ...

- Albedo matrix for zero-incoming currents

$$\begin{bmatrix} J_{m0} \\ J_{m2} \end{bmatrix} = 0 \rightarrow$$ J_0 ... $-\frac{3}{8}$ ... J_2 ... $\frac{7}{8}$ ...

$\mathbf{M}_{\alpha}$

$\mathbf{J} = \mathbf{M}_{\alpha}\Phi$ at periphery

where $\Phi = [\hat{\phi}_0, \phi_2]^T$

while $\mathbf{J} = -\begin{bmatrix} \beta_0 D_{\xi} & 0 \\ 0 & \beta_2 D_{\xi} \end{bmatrix} \Phi$ for both interior and boundary

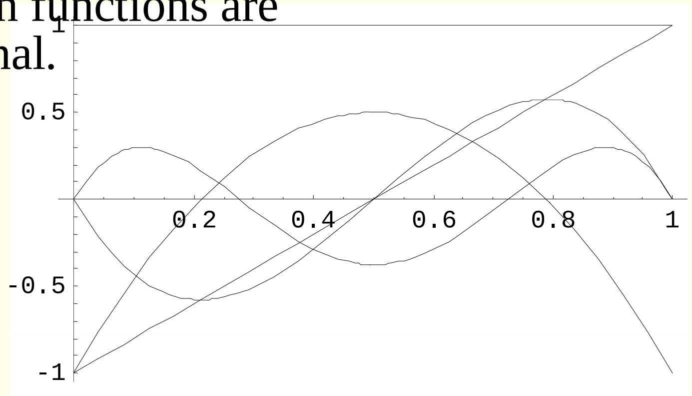
FDM with 2x2 block can solve the P_3 problem!

One-Node Nodal Expansion Method

- NEM Polynomials

Only odd and even functions are mutually orthogonal.

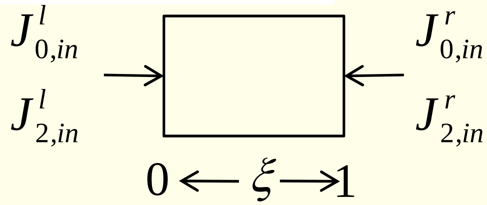
```
P0
P1 ... 1;
P2
P3
P4 x ... 5 x
Plot ... , 1
```



- NEM Flux Expansion

From now on, the summed flux $\hat{\phi}_0$ is represented plainly by ϕ_0.

The actual flux thus should be $\phi = \phi_0 - 2\phi_2$.



$$\Phi(\xi) = \begin{bmatrix} \phi_0(\xi) \\ \phi_2(\xi) \end{bmatrix},\ \phi_m(\xi) = \bar{\phi}_m + \sum_{i=1}^{4} a_{m,i} P_i(\xi) \quad m = 0 \text{ or } 2 \text{ (m for moment).}$$

- Surface fluxes and currents (ξ=0 for left, ξ=1 for right)

$f_l = \ldots - a_{0,1} - a_{0,2} \ldots;\ \ldots - a_{2,1} - a_{2,2} \ldots \qquad f_r = \ldots + a_{0,1} - a_{0,2} \ldots;\ \ldots + a_{2,1} - a_{2,2} \ldots$

$$\boxed{\mathbf{J}(\xi) = -\begin{bmatrix} \beta_0 & 0 \\ 0 & \beta_2 \end{bmatrix} \Phi'(\xi)}$$

$J_l = -b_0 \ldots a_{0,1} + 6\, a_{0,2} - 6\, a_{0,3} + 6\, a_{0,4};\ -b_2 \ldots a_{2,1} + 6\, a_{2,2} - 6\, a_{2,3} + 6\, a_{2,4} \qquad J_r = -b_0 \ldots a_{0,1} - 6\, a_{0,2} - 6\, a_{0,3} - 6\, a_{0,4};\ -b_2 \ldots a_{2,1} - 6\, a_{2,2} - 6\, a_{2,3} - 6\, a_{2,4}$

Lower Order Coeff. in terms of Surface Flux and Partial Currents

- $a_{m,1}, a_{m,2}$ in terms of surface fluxes

$$a_{0,1} \rightarrow -\frac{1}{2}\left(\phi_{0,l} - \phi_{0,r}\right)$$
$$a_{0,2} \rightarrow -\frac{1}{2}\left(-2\bar{\phi}_0 + \phi_{0,l} + \phi_{0,r}\right)$$
$$a_{2,1} \rightarrow -\frac{1}{2}\left(\phi_{2,l} - \phi_{2,r}\right)$$
$$a_{2,2} \rightarrow -\frac{1}{2}\left(-2\bar{\phi}_2 + \phi_{2,l} + \phi_{2,r}\right)$$

- Surface flux in terms of partial current

$$\phi_{0,l} \rightarrow \frac{8}{25}\left(7 J_{0,li} + J_{0,lo} + 2 J_{2,li} + J_{2,lo}\right)$$
$$\phi_{2,l} \rightarrow \frac{8}{25}\left(J_{0,li} + J_{0,lo} + 4 J_{2,li} + J_{2,lo}\right)$$
$$\phi_{2,r} \rightarrow \frac{8}{25}\left(J_{0,ri} + J_{0,ro} + 4 J_{2,ri} + J_{2,ro}\right)$$
$$\phi_{0,r} \rightarrow \frac{8}{25}\left(7 J_{0,ri} + J_{0,ro} + 2 J_{2,ri} + J_{2,ro}\right)$$

$$\mathbf{J}_{out} = \mathbf{M}_\phi \Phi + \mathbf{M}_J \mathbf{J}_{net}$$
$$= \mathbf{M}_\phi \Phi + \mathbf{M}_J \left(\mathbf{J}_{out} - \mathbf{J}_{in}\right)$$
$$\Phi = \mathbf{M}_\phi^{-1}\left(\left(\mathbf{I} - \mathbf{M}_J\right)\mathbf{J}_{out} + \mathbf{M}_J \mathbf{J}_{in}\right) = \mathbf{M}_\phi^{-1}\mathbf{M}_J\left(\mathbf{J}_{out} + \mathbf{J}_{in}\right)$$

$$\boxed{\leftarrow \mathbf{I} - \mathbf{M}_J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \mathbf{M}_J}$$

- $a_{m,1}, a_{m,2}$ in terms of partial currents

$$a_{0,1} \rightarrow -\frac{4}{25}\left(7 J_{0,li} + 7 J_{0,lo} - 7 J_{0,ri} - 7 J_{0,ro} + 3 J_{2,li} + 3 J_{2,lo} - 3 J_{2,ri} - 3 J_{2,ro}\right)$$
$$a_{0,2} \rightarrow \frac{1}{25}\left(25 \bar{\phi}_0 - 4\left(7 J_{0,li} + 7 J_{0,lo} + 7 J_{0,ri} + 7 J_{0,ro} + 3 J_{2,li} + 3 J_{2,lo} + 3 J_{2,ri} + 3 J_{2,ro}\right)\right)$$
$$a_{2,1} \rightarrow -\frac{4}{25}\left(J_{0,li} + J_{0,lo} - J_{0,ri} - J_{0,ro} + 4 J_{2,li} + 4 J_{2,lo} - 4 J_{2,ri} - 4 J_{2,ro}\right)$$
$$a_{2,2} \rightarrow \frac{1}{25}\left(25 \bar{\phi}_2 - 4\left(J_{0,li} + J_{0,lo} + J_{0,ri} + J_{0,ro} + 4 J_{2,li} + 4 J_{2,lo} + 4 J_{2,ri} + 4 J_{2,ro}\right)\right)$$

But no need if surface fluxes are used

Higher Order Coeff. in terms of Surface Flux and Partial Currents

- Need for spatial moments (1-st and 2-nd)
  - Five expansion coefficients
  - Three physical unknowns per moment flux: $\bar{\phi}_m, \phi_m^l, \phi_m^r$ or $\bar{\phi}_m, J_{m,o}^l, J_{m,o}^r$
  - Three physical constraints: nodal balance, current continuity at both surfaces
  - Lacking two constraints $\rightarrow$ first and second moment balance

fm = [illegible]

sm = [illegible]

$$\longrightarrow \quad \tilde{\phi}_1 = \begin{pmatrix} \frac{a_{0,1}}{3} + \frac{a_{0,3}}{5} \\ \frac{a_{2,1}}{3} + \frac{a_{2,3}}{5} \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} \frac{a_{0,2}}{5} - \frac{3\,a_{0,4}}{35} \\ \frac{a_{2,2}}{5} - \frac{3\,a_{2,4}}{35} \end{pmatrix}$$

- $a_{m,3}, a_{m,4}$ in terms of moments

$$\begin{aligned}
a_{0,3} &\rightarrow -\frac{5\,a_{0,1}}{3} + 5\,\tilde{\phi}_{0,1} \\
a_{2,3} &\rightarrow -\frac{5\,a_{2,1}}{3} + 5\,\tilde{\phi}_{2,1} \\
a_{0,4} &\rightarrow \frac{7}{3}\left(a_{0,2} - 5\,\tilde{\phi}_{0,2}\right) \\
a_{2,4} &\rightarrow \frac{7}{3}\left(a_{2,2} - 5\,\tilde{\phi}_{2,2}\right)
\end{aligned}$$

$\rightarrow$ All the coefficients $a_{m,i}$ are obtained in terms of $\bar{\phi}_m, \tilde{\phi}_{m,1}, \tilde{\phi}_{m,2}, J_{m,o}^l, J_{m,o}^r$

Outgoing Current Relation

- Outgoing current relations in terms of flux expansion coefficients

$$\mathbf{J}_{out} = \mathbf{M}_{\phi}\Phi + \mathbf{M}_{J}\mathbf{J}_{net}$$

$$\mathbf{J}_{in} = \mathbf{M}_{\phi}\Phi - \mathbf{M}_{J}\mathbf{J}_{net}$$

$$J_{lo} = \frac{1}{4}\left(\text{[illegible]}_0 - a_{0,1} - a_{0,2} \right) + \frac{1}{2} b_0 \left(\text{[illegible]}\, a_{0,1} + 6\, a_{0,2} + 6\, a_{0,3} + \text{[illegible]}\, a_{0,4} \right) - \frac{3}{16}\left(\text{[illegible]}_2 - a_{2,1} - a_{2,2} \right)$$
$$+ \frac{1}{16}\left(f_0 + a_{0,1} + a_{0,2} \right) + \frac{7}{16}\left(\text{[illegible]}_2 - a_{2,1} - a_{2,2} \right) + \frac{1}{2} b_2 \left(\text{[illegible]}\, a_{2,1} + 6\, a_{2,2} - 6\, a_{2,3} + 6\, a_{2,4} \right)$$

$$J_{ro} = \frac{1}{4}\left(\text{[illegible]}_0 + a_{0,1} - a_{0,2} \right) + \frac{1}{2} b_0 \left(\text{[illegible]}\, a_{0,1} - 6\, a_{0,2} + 6\, a_{0,3} - \text{[illegible]}\, a_{0,4} \right) - \frac{3}{16}\left(\text{[illegible]}_2 + a_{2,1} - a_{2,2} \right)$$
$$+ \frac{1}{16}\left(f_0 - a_{0,1} + a_{0,2} \right) + \frac{7}{16}\left(\text{[illegible]}_2 + a_{2,1} - a_{2,2} \right) - \frac{1}{2} b_2 \left(\text{[illegible]}\, a_{2,1} - 6\, a_{2,2} - 6\, a_{2,3} - 6\, a_{2,4} \right)$$

- Insert $a_{m,i}$ $(i = 1 \cdots 4)$ given in terms of $\bar{\phi}_m, \tilde{\phi}_{m,1}, \tilde{\phi}_{m,2}, J^l_{m,o}, J^r_{m,o}$ and $J^l_{m,i}, J^r_{m,i}$ which are known. then rearrange for $J^l_{m,o}$ and $J^r_{m,o}$.

$\rightarrow$ $J^l_{m,o}$ and $J^r_{m,o}$ are obtained in terms of unkown $\bar{\phi}_m, \tilde{\phi}_{m,1}, \tilde{\phi}_{m,2}$, and known $J^l_{m,i}, J^r_{m,i}$

Moment Balance Equations

- First moment balance equation

$$\int_0^1 \begin{bmatrix} -\Sigma_{D0} D_\xi^2 + \Sigma_r & -2\Sigma_r \\ -\dfrac{2}{3}\Sigma_r & -\Sigma_{D2} D_\xi^2 + \dfrac{4}{3}\Sigma_r + \dfrac{5}{3}\Sigma_t \end{bmatrix} \begin{bmatrix} \hat{\phi}_0(\xi) \\ \phi_2(\xi) \end{bmatrix} P_1(\xi) d\xi = \int_0^1 \begin{bmatrix} q_0(\xi) \\ -\dfrac{2}{3} q_0(\xi) \end{bmatrix} P_1(\xi) d\xi$$

- Nodal balance equation

- Second moment balance equation

Calculation Sequence

1. Assume incoming currents
2. Solve for odd moments
3. Solve for coupled equation for even moments (0th, 2nd)
4. Update outgoing current
5. Move to the next node