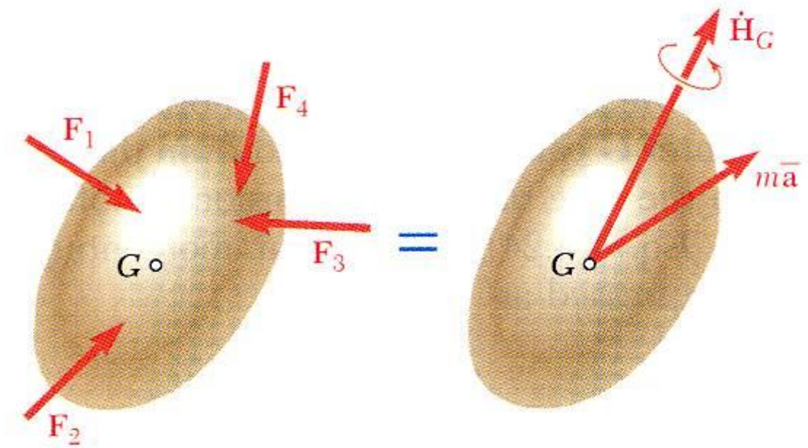
18.1 Introduction



$$\sum \vec{F} = m\vec{\bar{a}}$$

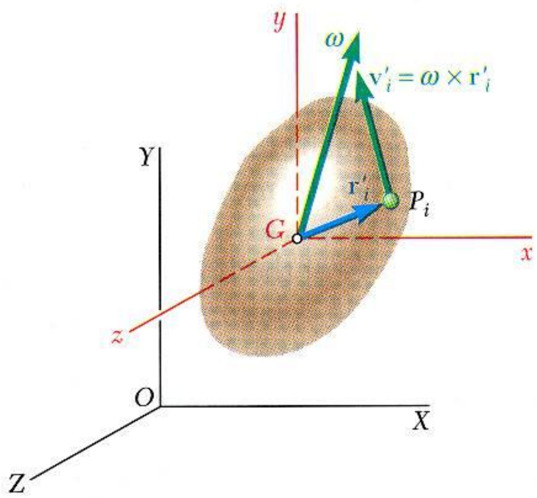
$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

The fundamental relations developed for the plane motion of rigid bodies may also be applied to the general motion of three dimensional bodies.

The relation $\vec{H}_G = \bar{I}\vec{\omega}$ which was used to determine the angular momentum of a rigid slab is not valid for general three dimensional bodies and motion.

The current chapter is concerned with evaluation of the angular momentum and its rate of change for three dimensional motion and application to effective forces, the impulse-momentum and the work-energy principles.

18.2 Rigid Body Angular Momentum in Three Dimensions



Angular momentum of a body about its mass center,

$$\vec{H}_G = \sum_{i=1}^{n} \left(\vec{r}_i' \times \vec{v}_i \Delta m_i \right) = \sum_{i=1}^{n} \left[\vec{r}_i' \times \left(\vec{\omega} \times \vec{r}_i' \right) \Delta m_i \right]$$

The *x* component of the angular momentum,

$$H_x = \sum_{i=1}^{n} \left[\qquad \qquad \right] \Delta m_i$$

$$= \sum_{i=1}^{n} \left[y_i \left(\omega_x y_i - \omega_y x_i \right) - z_i \left(\omega_z x_i - \omega_x z_i \right) \right] \Delta m_i$$

$$= \omega_x \sum_{i=1}^{n} \left(y_i^2 + z_i^2 \right) \Delta m_i - \omega_y \sum_{i=1}^{n} x_i y_i \Delta m_i - \omega_z \sum_{i=1}^{n} z_i x_i \Delta m_i$$

$$H_x = \omega_x \int \left(y^2 + z^2 \right) dm - \omega_y \int xy \, dm - \omega_z \int zx \, dm$$

$$= \qquad \qquad$$

$$H_y = -\bar{I}_{yx} \omega_x + \bar{I}_y \omega_y - \bar{I}_{yz} \omega_z$$

$$H_z = -\bar{I}_{zx} \omega_x - \bar{I}_{zy} \omega_y + \bar{I}_z \omega_z$$

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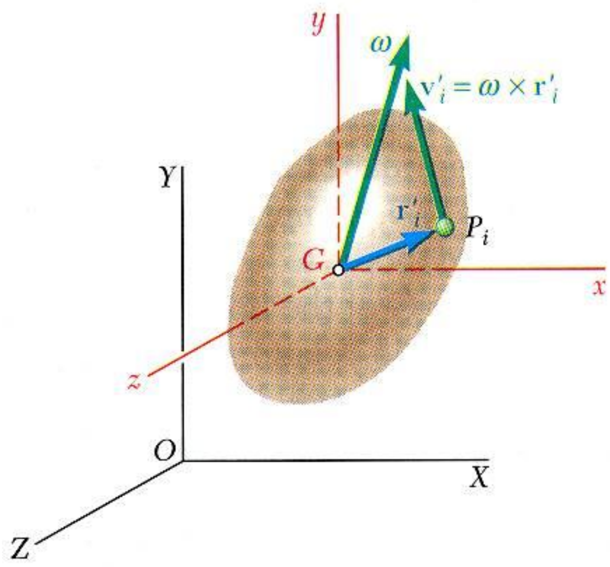


Transformation of $\vec{\omega}$ into $\vec{H}_G$ is characterized by the inertia tensor for the body,

$$\begin{pmatrix} +\bar{I}_x & -\bar{I}_{xy} & -\bar{I}_{xz} \\ -\bar{I}_{yx} & +\bar{I}_y & -\bar{I}_{yz} \\ -\bar{I}_{zx} & -\bar{I}_{zy} & +\bar{I}_z \end{pmatrix}$$

With respect to the principal axes of inertia,

$$\begin{pmatrix} \bar{I}_{x'} & 0 & 0 \\ 0 & \bar{I}_{y'} & 0 \\ 0 & 0 & \bar{I}_{z'} \end{pmatrix}$$

$$H_x = +\bar{I}_x\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$

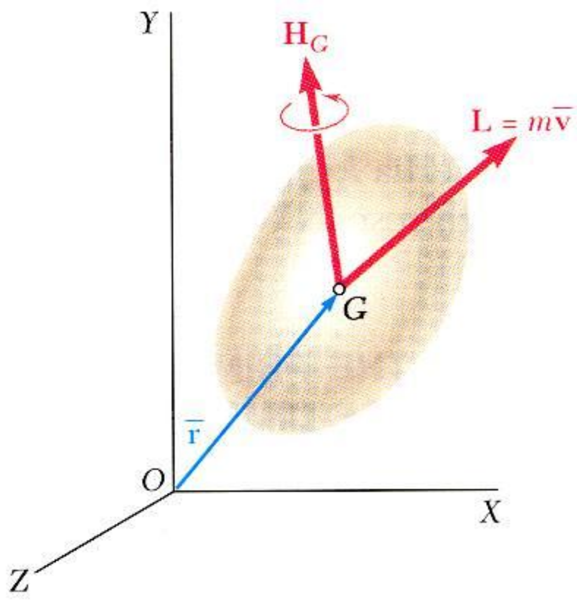
$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{I}_{yz}\omega_z$$

$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\omega_z$$

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \quad H_{y'} = \bar{I}_{y'}\omega_{y'} \quad H_{z'} = \bar{I}_{z'}\omega_{z'}$$

The angular momentum $\vec{H}_G$ of a rigid body and its angular velocity $\vec{\omega}$ have the same direction if, and only if, $\vec{\omega}$ is directed along a principal axis of inertia.

18.2 Rigid Body Angular Momentum in Three Dimensions



The momenta of the particles of a rigid body can be reduced to:

$$\vec{L} = \text{linear momentum}$$
$$= m\vec{\bar{v}}$$

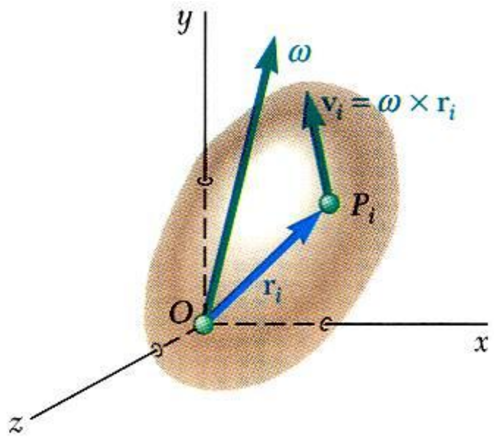
$$\vec{H}_G = \text{angular momentum about } G$$

$$H_x = +\bar{I}_x\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$
$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{I}_{yz}\omega_z$$
$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\omega_z$$

The angular momentum about any other given point *O* is

$$\vec{H}_O = \vec{\bar{r}} \times m\vec{\bar{v}} + \vec{H}_G$$

18.2 Rigid Body Angular Momentum in Three Dimensions



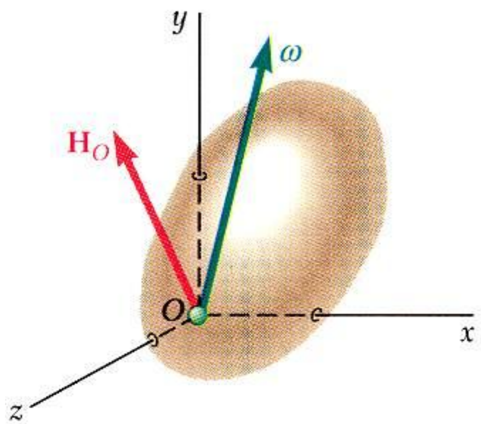
The angular momentum of a body constrained to rotate about a fixed point may be calculated from

$$\vec{H}_O = \vec{r} \times m\vec{v} + \vec{H}_G$$

Or, the angular momentum may be computed directly from the moments and products of inertia with respect to the *Oxyz* frame.

$$\vec{H}_O = \sum_{i=1}^{n} \left(\vec{r}_i \times \vec{v}_i \Delta m \right)$$

$$= \sum_{i=1}^{n} \left[\vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i \right) \Delta m_i \right]$$

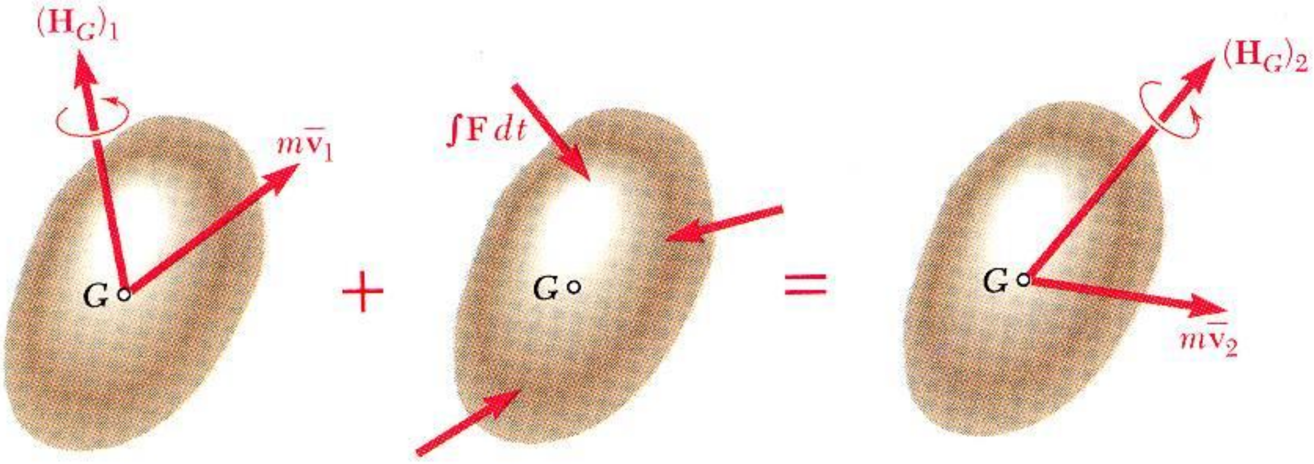


$$H_x = +I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z$$

$$H_y = -I_{yx} \omega_x + I_y \omega_y - I_{yz} \omega_z$$

$$H_z = -I_{zx} \omega_x - I_{zy} \omega_y + I_z \omega_z$$

18.3 Principle of Impulse and Momentum



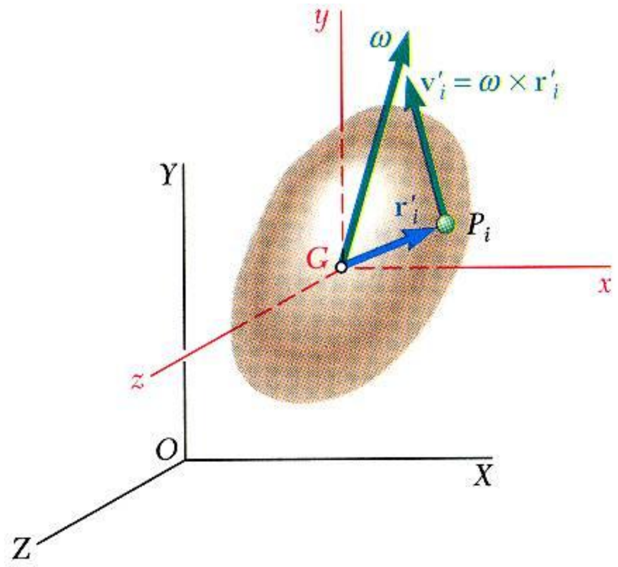
The principle of impulse and momentum can be applied directly to the three-dimensional motion of a rigid body,

$$\textit{Syst Momenta}_1 + \textit{Syst Ext Imp}_{1\text{-}2} = \textit{Syst Momenta}_2$$

The free-body diagram equation is used to develop component and moment equations.

For bodies rotating about a fixed point, eliminate the impulse of the reactions at *O* by writing equation for moments of momenta and impulses about *O*.

18.4 Kinetic Energy



Kinetic energy of particles forming rigid body,

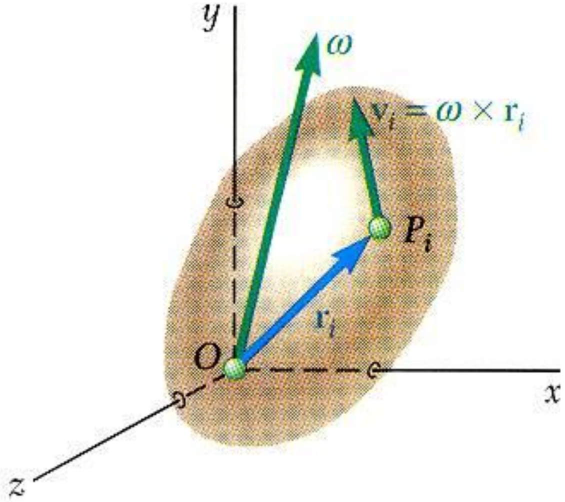
$$
\begin{aligned}
T &= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum_{i=1}^{n} \Delta m_i \bar{v}_i'^2 \\
&= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum_{i=1}^{n} \left|\vec{\omega}\times\vec{r}_i'\right|^2 \Delta m_i \\
&= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2 - 2\bar{I}_{xy}\omega_x\omega_y \\
&\qquad - 2\bar{I}_{yz}\omega_y\omega_z - 2\bar{I}_{zx}\omega_z\omega_x)
\end{aligned}
$$

If the axes correspond instantaneously with the principle axes,

$$
T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2)
$$

With these results, the principles of work and energy and conservation of energy may be applied to the three-dimensional motion of a rigid body.

18.4 Kinetic Energy



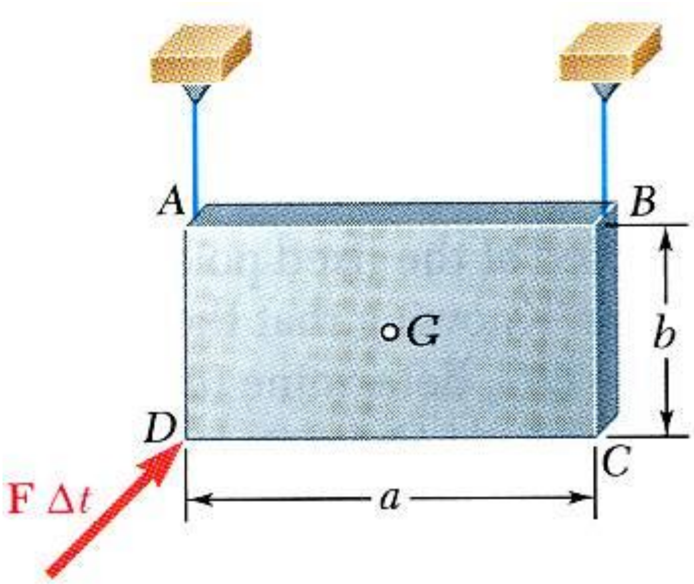
Kinetic energy of a rigid body with a fixed point,

$$T = \tfrac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{zx}\omega_z\omega_x\right)$$

If the axes *Oxyz* correspond instantaneously with the principle axes *Ox'y'z'*,

$$T = \tfrac{1}{2}\left(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2\right)$$

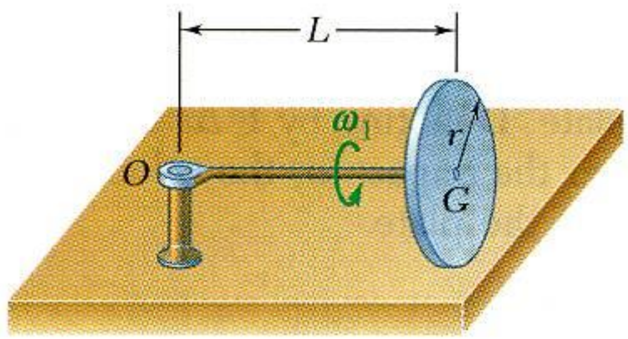
Sample Problem 18.1



Rectangular plate of mass *m* that is suspended from two wires is hit at *D* in a direction perpendicular to the plate. Immediately after the impact, determine *a*) the velocity of the mass center *G*, and *b*) the angular velocity of the plate.

Sample Problem 18.1

Sample Problem 18.2



A homogeneous disk of mass m is mounted on an axle OG of negligible mass. The disk rotates counter-clockwise at the rate ω_1 about OG. Determine: a) the angular velocity of the disk, b) its angular momentum about O, c) its kinetic energy, and d) the vector and couple at G equivalent to the momenta of the particles of the disk.

Sample Problem 18.2