Basic Mathematics for Quantum Mechanics

(Sakurai, Modern Quantum Mechanics)

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Kets and Bras

$$\langle \alpha | \beta \rangle$$

bra c ket

$\langle \alpha |$ $\quad$ $| \beta \rangle$

P. A. M. Dirac

- Hilbert space

Ket Space

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

$$c|\alpha\rangle = |\alpha\rangle c$$

Operator and Eigenkets

$$A \cdot \left(|\alpha\rangle \right) = A|\alpha\rangle$$

- Eigenvalues and eigenkets

$$|a'\rangle, |a''\rangle, |a'''\rangle, \ldots$$

$$A|a'\rangle = a'|a'\rangle, \quad A|a''\rangle = a''|a''\rangle$$

Bra Space

- Dual Correspondence (DC)

$$|\alpha\rangle \overset{\text{DC}}{\leftrightarrow} \langle\alpha|$$

$$|a'\rangle, |a''\rangle, \ldots \overset{\text{DC}}{\leftrightarrow} \langle a'|, \langle a''|, \ldots$$

$$|\alpha\rangle + |\beta\rangle \overset{\text{DC}}{\leftrightarrow} \langle\alpha| + \langle\beta|$$

$$c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \leftrightarrow c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|$$

Inner Product

$$\langle \beta | \alpha \rangle = \left(\langle \beta | \right) \cdot \left(| \alpha \rangle \right)$$

Bra (c) ket

- Postulates

$$\langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^*$$

$$\langle \alpha | \alpha \rangle \geq 0$$

where the equaility sign holds only if $|\alpha\rangle$ is a null ket.

Orthogonality & Normalization

- Orthogonality

$$\langle \alpha | \beta \rangle = 0$$

- Normalization

$$|\tilde{\alpha}\rangle = \left(\frac{1}{\sqrt{\langle \alpha | \alpha \rangle}} \right) |\alpha\rangle$$

$$\langle \tilde{\alpha} | \tilde{\alpha} \rangle = 1$$

Operators

$$X \cdot \left(|\alpha\rangle \right) = X|\alpha\rangle$$

$$X + Y = Y + X$$

$$X + \left(Y + Z \right) = \left(X + Y \right) + Z$$

$$\left(\langle\alpha| \right) \cdot X = \langle\alpha|X$$

Hermitian Adjoint & Hermitian Operator

- $X^\dagger$ is called Hermitian adjoint of X if

$$X|\alpha\rangle \overset{\text{DC}}{\leftrightarrow} \langle\alpha|X^\dagger$$

- An operator X is said to be Hermitian if

$$X = X^\dagger$$

Operator Multiplication I

$$XY \neq YX$$

$$X(YZ) = (XY)Z = XYZ$$

$$X\left(Y|\alpha\rangle\right) = (XY)|\alpha\rangle = XY|\alpha\rangle, \qquad \left(\langle\beta|X\right)Y = \langle\beta|(XY) = \langle\beta|XY$$

Operator Multiplication II

$$(XY)^{\dagger} = Y^{\dagger} X^{\dagger}$$

$$XY|\alpha\rangle = X(Y|\alpha\rangle) \leftrightarrow (\langle\alpha|Y^{\dagger})X^{\dagger} = \langle\alpha|Y^{\dagger}X^{\dagger}$$

- Outer product

$$(|\beta\rangle)\cdot(\langle\alpha|) = |\beta\rangle\langle\alpha|$$

Associative Axiom

- Important Examples

$$\left(|\beta\rangle\langle\alpha|\right)\cdot|\gamma\rangle = |\beta\rangle\cdot\left(\langle\alpha|\gamma\rangle\right)$$

예: If $X = |\beta\rangle\langle\alpha|$, then $X^{\dagger} = |\alpha\rangle\langle\beta|$

$$\left(\langle\beta|\right)\cdot\left(X|\alpha\rangle\right) = \left(\langle\beta|X\right)\cdot\left(|\alpha\rangle\right) = \langle\alpha|X^{\dagger}|\beta\rangle^{*}$$

bra ket bra ket

$$\langle\beta|X|\alpha\rangle = \langle\beta|\cdot\left(X|\alpha\rangle\right) = \left\{\left(\langle\alpha|X^{\dagger}\right)\cdot|\beta\rangle\right\}^{*} = \langle\alpha|X^{\dagger}|\beta\rangle^{*}$$

- For a Hermitian X, we have

$$\langle\beta|X|\alpha\rangle = \langle\alpha|X|\beta\rangle^{*}$$

Eigenvalues of Hermitian Operator

Theorem. *The eigenvalues of a Hermitian operator A are real; the eigenkets of A corresponding to different eigenvalues are orthogonal.*

Proof.

$$A|a'\rangle = a'|a'\rangle$$

$$\langle a''|A = a''^{*}\langle a''|$$

$$(a' - a''^{*})\langle a''|a'\rangle = 0$$

Eigenkets as base Kets

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$$

$$c_{a'} = \langle a'|\alpha\rangle$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

Completeness Relation

$$\sum_{a'} |a'\rangle\langle a'| = I$$

$$\langle\alpha|\alpha\rangle = \langle\alpha|\cdot\left(\sum_{a'}|a'\rangle\langle a'|\cdot|\alpha\rangle\right) = \sum_{a'}\left|\langle a'|\alpha\rangle\right|^2$$

$$\sum_{a'}|c_{a'}|^2 = \sum_{a'}\left|\langle a'|\alpha\rangle\right|^2 = 1$$

Projection Operator

$$A_{a'} \equiv |a'\rangle\langle a'|$$

$$\left(|a'\rangle\langle a'|\right)\cdot|\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle = c_{a'}|a'\rangle$$

$$\sum_{a'} \Lambda_{a'} = I$$

• For an observable, there corresponds a Hermitian operator. If we measure the observable, only eigenvalues of the Hermitian operator can be measured. A measurement always causes the system to jump into an eigenstate of the Hermitian operator.

Measurement II

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$|\alpha\rangle \xrightarrow{A \text{ measurement}} |a'\rangle$$

$$\text{Probability for } a' = \left|\langle a'|\alpha\rangle\right|^2$$

$$|a'\rangle \xrightarrow{A \text{ measurement}} |a'\rangle$$

Expectation value

$$\langle A \rangle = \langle \alpha | A | \alpha \rangle$$

$$\langle A \rangle = \sum_{a'} \sum_{a''} \langle \alpha | a'' \rangle \langle a'' | A | a' \rangle \langle a' | \alpha \rangle = \sum_{a'} \underset{\text{measured value } a'}{a'} \underbrace{\left| \langle a' | \alpha \rangle \right|^2}_{\text{probability for obtaining } a'}$$

measured value a'

probability for obtaining a'

Compatibility & Incompatibility

- Observables *A* and *B* are compatible when the corresponding operators commute, i.e.,

$$[A, B] = 0$$

- Observables *A* and *B* are incompatible when

$$[A, B] \neq 0$$

Uncertainty Principle

$$\left\langle (\Delta A)^2 \right\rangle \left\langle (\Delta B)^2 \right\rangle \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2$$

where

$$\Delta A = A - \langle A \rangle$$

$$\left\langle (\Delta A)^2 \right\rangle = \left\langle \left(A^2 - 2A\langle A \rangle + \langle A \rangle^2 \right) \right\rangle = \left\langle A^2 \right\rangle - \langle A \rangle^2$$

- The uncertainly relation can be obtained from the Schwarz inequality in a straight forward manner.

Example of Uncertainty Relation

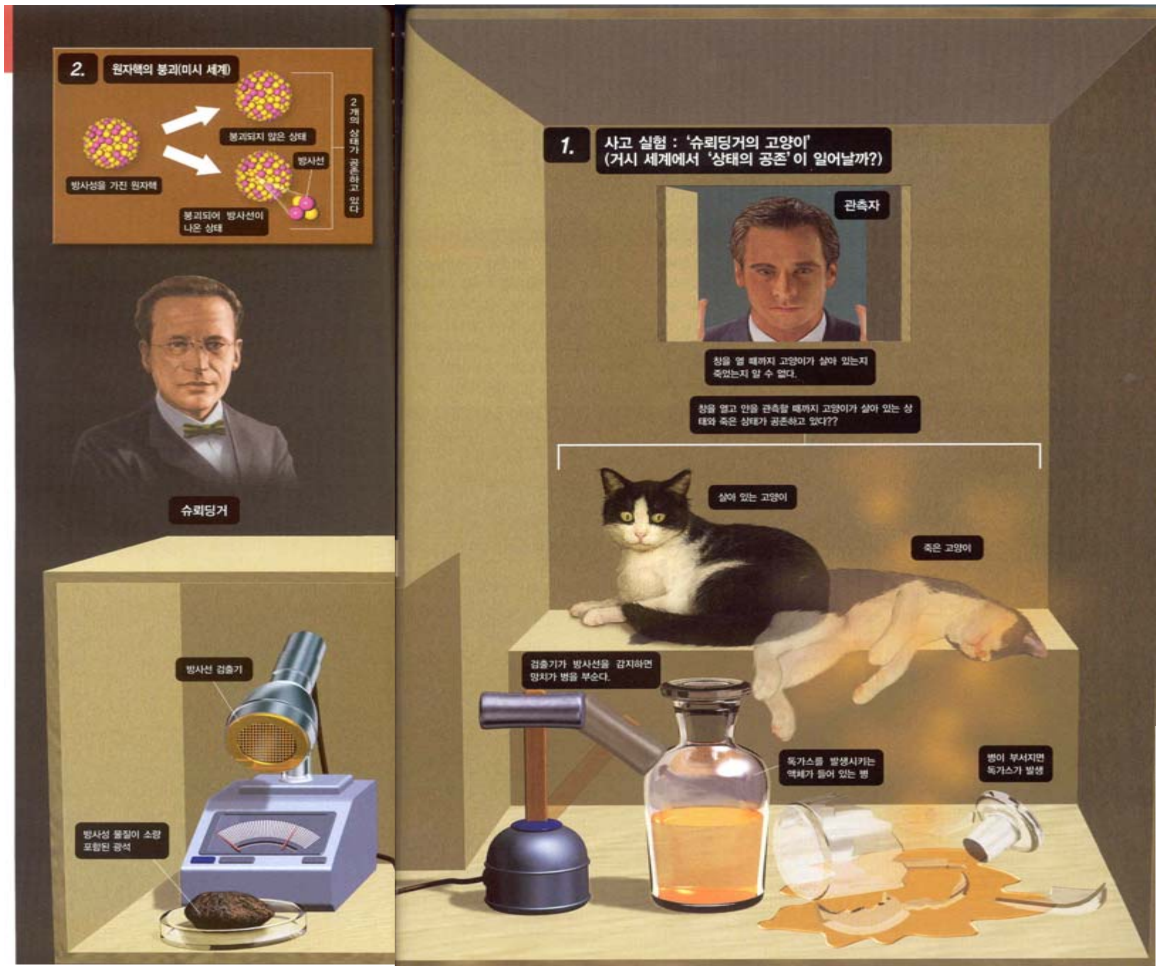
$$X, \quad P = -i\hbar\nabla$$

$$[X, P] = i\hbar$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{1}{4}|i\hbar|^2$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Schrödinger’s Cat



Newton Highlight – 양자론, 뉴턴코리아, 2006.

Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right), \qquad p = i\sqrt{\frac{m\hbar\omega}{2}}\left(-a + a^\dagger\right)$$

a : Annihilation operator

$a^\dagger$: Creation operator

$$\left[a, a^\dagger\right] = \left(\frac{1}{2\hbar}\right)\left(-i[x, p] + i[p, x]\right) = 1$$

Number Operator I

- Number Operator

$$N = a^{\dagger} a$$

$$H = \hbar\omega\left(N + \tfrac{1}{2}\right)$$

$$N\left|n\right\rangle = n\left|n\right\rangle$$

$$H\left|n\right\rangle = \left(n + \tfrac{1}{2}\right)\hbar\omega\left|n\right\rangle$$

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

Number Operator II

$$\left[N,a\right]=\left[a^{\dagger}a,a\right]=a^{\dagger}\left[a,a\right]+\left[a^{\dagger},a\right]a=-a$$

$$\left[N,a^{\dagger}\right]=a^{\dagger}$$

$$Na^{\dagger}\left|n\right\rangle=\left(\left[N,a^{\dagger}\right]+a^{\dagger}N\right)\left|n\right\rangle=\left(n+1\right)a^{\dagger}\left|n\right\rangle$$

$$Na\left|n\right\rangle=\left(\left[N,a\right]+aN\right)\left|n\right\rangle=\left(n-1\right)a\left|n\right\rangle$$

Annihilation and Creation Operators

$$a\left|n\right\rangle = c\left|n-1\right\rangle$$

$$\left\langle n\right|a^{\dagger}a\left|n\right\rangle = \left|c\right|^{2}$$

$$n = \left|c\right|^{2}$$

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$$

$$a^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

$$n = \left\langle n\right|N\left|n\right\rangle = \left(\left\langle n\right|a^{\dagger}\right)\cdot\left(a\left|n\right\rangle\right) \geq 0$$

$$E_0 = \tfrac{1}{2}\hbar\omega$$

Energies of a Harmonic Oscillator

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega, \qquad \left(n = 0, 1, 2, 3, \ldots\right)$$

Who Made the Law?
Is There God?

Richard Dawkins, *The God Delusion* (2006)

Francis S. Collins, *The Language of God* (2006)

Alister McGrath and Joanna C. McGrath, *The Dawkins Delusion?* (2007)

Timothy Keller, *The Reason for God* (2008)

Ian G. Barbour, *When Science Meets Religion* (2000)