2. Stress and infinitesimal strain

2.1 Problem definition

- Shallow depth – Block movement in jointed rock mass under low-stress environment

  Deep mining – Stress is main concern for the mine stability. Rock mass behaves as continuum.

- Force

  - Body force: Gravity, magnetic, inertia (F/V)

  - Surface force: Conveyed by physical contact (F/A)

2.2 Force and stress

- Force is a vector with magnitude and direction.

  Ex.) F_x, F_y, F_z, F_1, F_2,…

- Stress is a tensor with magnitude, direction and <u>surface orientation.</u>

  Ex.) $\sigma_{xx}\ (= \sigma_x)$, $\sigma_{xy}\ (= \tau_{xy})$, σ_{yz},…

  surface orientation (normal to the surface)

  stress direction

2.2 Force and stress

- Traction is a force per unit area acting on a surface (Fig.2.1(b)).

- "Tensor *is a further extension of ideas we already use when defining quantities like scalars and vectors.*"

  Ex.) Scalar → rank zero

  Vector → rank one

  Stress → rank two

  ⋮

2.2 Force and stress

- Tensor notation

- Introduced in deriving the theory of relativity by Einstein.
- Basic idea is to make summation for the repeatedly used subscripts.

Ex.) $a_i b_i = a_x b_x + a_y b_y + a_z b_z$

$$a_{ii} b_k = a_{xx} b_k + a_{yy} b_k + a_{zz} b_k \quad (i\text{: dummy index, } k\text{: free index})$$

$$a_{ik} b_i c_k = a_{xk} b_x c_k + a_{yk} b_y c_k + a_{zk} b_z c_k$$

$$= a_{xx} b_x c_x + a_{xy} b_x c_y + a_{xz} b_x c_z$$

$$+ a_{yx} b_y c_x + a_{yy} b_y c_y + a_{yz} b_y c_z$$

$$+ a_{zx} b_z c_x + a_{zy} b_z c_y + a_{zz} b_z c_z$$

2.2 Force and stress

- Important symbols in tensor notation

- Kronecker delta

$\delta_{ij} = 1$ if $i=j$.

$= 0$ if $i \neq j$.

Ex.) $\delta_{xx} = 1$, $\delta_{zz} = 1$, $\delta_{xy} = 0$, $\delta_{yz} = 0$

$\delta_{ij}\, a_i b_j = a_x b_x + a_y b_y + a_z b_z = a_i b_i$

- Permutation symbol

$\varepsilon_{ijk} = 0$ if any pair of subscripts are identical.

$= (-1)^p$, where p is the number of subscript transposition.

Ex.) $\varepsilon_{xyz} = 1$, $\varepsilon_{xyx} = 0$, $\varepsilon_{yyz} = 0$, $\varepsilon_{xzz} = 0$

$\varepsilon_{xzy} = -1$, $\varepsilon_{zxy} = 1$, $\varepsilon_{yxz} = -1$, $\varepsilon_{zyx} = -1$

2.2 Force and stress

- Linear algebra in tensor notation

- Matrix operation

$Ax = b \rightarrow a_{ij}x_j = b_i$

$AB = Y \rightarrow a_{ij}b_{jk} = Y_{ik}$

$A^T B = Z \rightarrow a_{ji}b_{jk} = Z_{ik}$

$x^T Bx = c \rightarrow x_i b_{ij} x_j = c$

- Derivatives

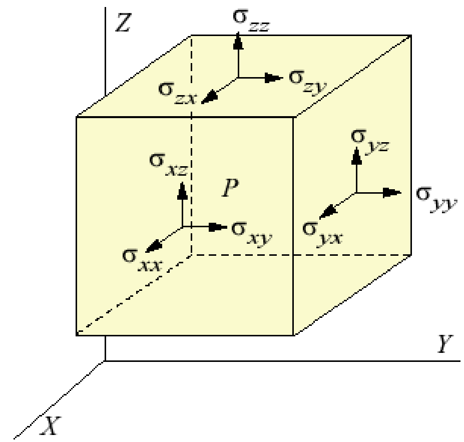
$$Grad\, f = \nabla f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z = f_{,i}$$

$$Div\vec{V} = \nabla \bullet \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = V_{i,i}$$

2.2 Force and stress

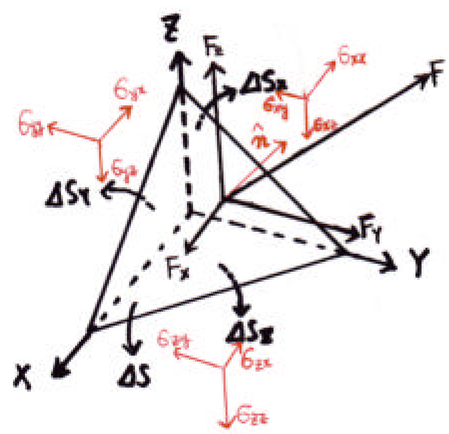
- Stress in rectangular coordinates

$$[s] = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{bmatrix} = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{xy} & s_{yy} & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} \end{bmatrix}$$



- Stress in an arbitrary plane

$$\hat{n} = \left(\frac{\Delta S_x}{\Delta S}, \frac{\Delta S_y}{\Delta S}, \frac{\Delta S_z}{\Delta S} \right)$$



2.2 Force and stress

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$$

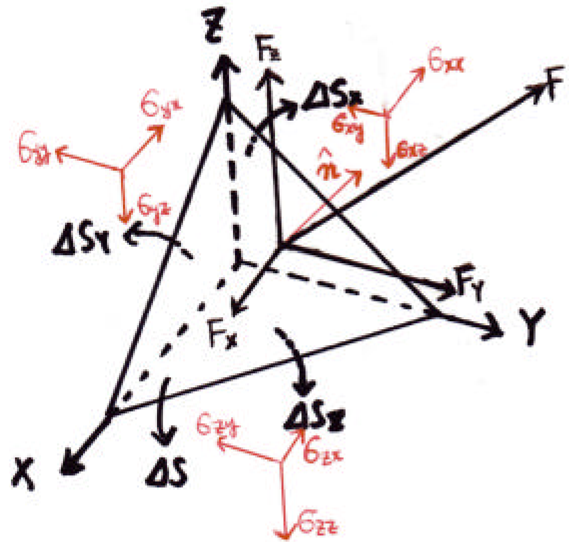
$$\Delta S t^{\vec{n}} = \Delta S t_x^{\vec{n}} + \Delta S t_y^{\vec{n}} + \Delta S t_z^{\vec{n}}$$

$$\Delta S t_x^{\hat{n}} = \Delta S_x s_{xx} + \Delta S_y s_{yx} + \Delta S_z s_{zx}$$

$$\Delta S t_y^{\hat{n}} = \Delta S_x s_{xy} + \Delta S_y s_{yy} + \Delta S_z s_{zy}$$

$$\Delta S t_z^{\hat{n}} = \Delta S_x s_{xz} + \Delta S_y s_{yz} + \Delta S_z s_{zz}$$

$$t_i^{\hat{n}} = n_j s_{ji} \left(= s_{ij} n_j\right)$$

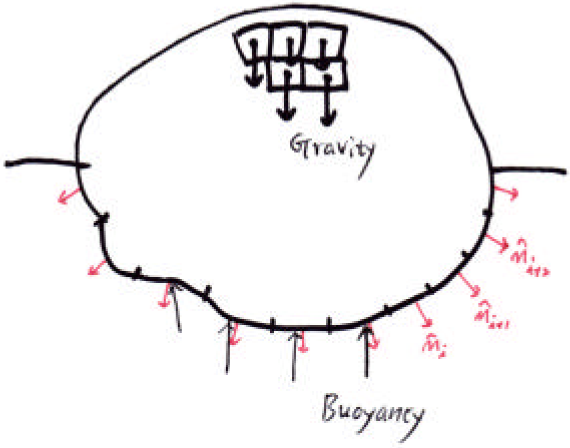


2.2 Force and stress

- Equilibrium conditions

- Force equilibrium & moment equilibrium

$$\sum \vec{F} = 0, \quad \sum \vec{M} = 0$$



- Force equilibrium

$$\int_s t_i^{\hat{n}} ds + \int_v r\, b_i dv = 0$$

$$\int_s n_j s_{ji} ds = \int_v s_{ji,j} dv = 0 \quad (Divergence\ theorem)$$

$$\rightarrow s_{ji,j} + r\, b_i = 0$$

2.2 Force and stress

- Moment equilibrium

$$\vec{M} = \vec{r} \times \vec{F} \; \left(= e_{ijk} r_j F_k \right)$$

$$\int_s \vec{r} \times t^{\hat{n}} ds + \int_v \vec{r} \times r \vec{b} \, dv = 0$$

$$\rightarrow e_{ijk} s_{jk} = 0$$

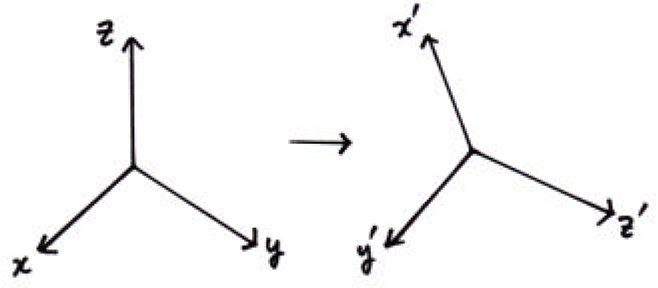
$$i = x : \quad s_{yz} - s_{zy} = 0$$

$$i = y : \quad s_{zx} - s_{xz} = 0 \quad \rightarrow \quad s_{ij} = s_{ji}$$

$$i = z : \quad s_{xy} - s_{yx} = 0$$

2.3 Stress transformation

- Vector transformation



$$\begin{array}{cccc} & x & y & z \\ x' & \cos q_{x'x} & \cos q_{x'y} & \cos q_{x'z} \\ y' & \cos q_{y'x} & \cos q_{y'y} & \cos q_{y'z} \\ z' & \cos q_{z'x} & \cos q_{z'y} & \cos q_{z'z} \end{array} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2.3 Stress transformation

- Stress transformation

$$u_i' = a_{ip} u_p, \quad v_j' = a_{jq} v_q \quad (vector\ transformation)$$

$$u_i' v_j' = s_{ij}' \quad (outer\ \ product)$$

$$u_i' v_j' = a_{ip} u_p a_{jq} v_q = a_{ip} a_{jq} u_p v_q = a_{ip} a_{jq} s_{pq} = a_{ip} s_{pq} a_{jq}$$

$$\rightarrow s_{ij}' = a_{ip} s_{pq} a_{jq} \quad \left(\Sigma' = A \cdot \Sigma \cdot A^T \right)$$

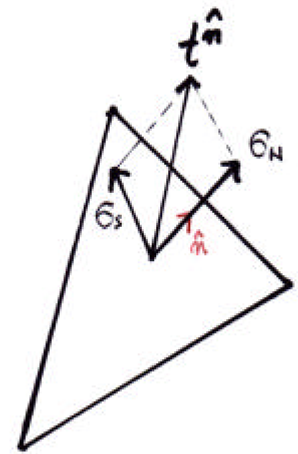
2.4 Principal stress and stress invariants

- Principal stress

- Traction in a surface can be decomposed into normal and shear stresses which are in the same plane with the traction.

$$s_N = \hat{n} \cdot t^{\hat{n}}$$

$$s_S = t^{\hat{n}} - s_N$$



2.4 Principal stress and stress invariants

- The surface on which shear component of the traction is zero is a principal plane and its normal component is called principal stress.

$$t^{\hat{n}} = \Sigma \cdot \hat{n} = s\,\hat{n}$$

$$s_{ij} n_j - s\, n_i = 0$$

$$\left(s_{ij} - s d_{ij} \right) n_j = 0$$

$$\begin{bmatrix} s_{xx} - s & s_{xy} & s_{xz} \\ s_{xy} & s_{yy} - s & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} - s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $s_1 \geq s_2 \geq s_3$

s_1 : major principal stress

s_2 : intermediate principal stress

s_3 : minor principal stress

2.4 Principal stress and stress invariants

- Stress invariants

$$\begin{vmatrix} s_{xx} - s & s_{xy} & s_{xz} \\ s_{xy} & s_{yy} - s & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} - s \end{vmatrix} = 0 \quad for\ \hat{n} \neq (0,0,0)$$

$$\rightarrow s^{3} - I_1 s^{2} + I_2 s - I_3 = 0$$

$$I_1 = s_{ii} \left(= s_{xx} + s_{yy} + s_{zz}\right)$$

$$I_2 = \frac{1}{2}\left[s_{ii} s_{jj} - s_{ij} s_{ij}\right]\left(= s_{xx} s_{yy} + s_{xx} s_{zz} + s_{yy} s_{zz} - s_{xy} s_{xy} - s_{xz} s_{xz} - s_{yz} s_{yz}\right.$$

$$I_3 = \left|s_{ij}\right|$$

CUBIC EQUATION: $x^3 + a_1x^2 + a_2x + a_3 = 0$

Let

$$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54},$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Solutions:

$$\begin{cases} x_1 = S + T - \frac{1}{3}a_1 \\ x_2 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 + \frac{1}{2}i\sqrt{3}\,(S-T) \\ x_3 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 - \frac{1}{2}i\sqrt{3}\,(S - \end{cases}$$

If a_1, a_2, a_3 are real and if $D = Q^3 + R^2$ is the *discriminant*, then

(i) one root is real and two complex conjugate if $D > 0$

(ii) all roots are real and at least two are equal if $D = 0$

(iii) all roots are real and unequal if $D < 0$.

If $D < 0$, computation is simplified by use of trigonometry.

Solutions if $D < 0$:

$$\begin{cases} x_1 = 2\sqrt{-Q}\cos\left(\frac{1}{3}\theta\right) - \frac{1}{3}a_1 \\ x_2 = 2\sqrt{-Q}\cos\left(\frac{1}{3}\theta + 120^\circ\right) - \frac{1}{3}a_1 \\ x_3 = 2\sqrt{-Q}\cos\left(\frac{1}{3}\theta + 240^\circ\right) - \frac{1}{3}a_1 \end{cases} \quad \text{where } \cos\theta = R/\sqrt{-Q^3}$$

$$x_1 + x_2 + x_3 = -a_1, \quad x_1x_2 + x_2x_3 + x_3x_1 = a_2, \quad x_1x_2x_3 = -a_3$$

where x_1, x_2, x_3 are the three roots.

2.5 Differential equations of static equilibrium

Refer to section 2.2

$$s_{ji,j} + r b_i = 0$$

$$\frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} + \frac{\partial s_{xz}}{\partial z} + r b_x = 0$$

$$\frac{\partial s_{yx}}{\partial x} + \frac{\partial s_{yy}}{\partial y} + \frac{\partial s_{yz}}{\partial z} + r b_y = 0$$

$$\frac{\partial s_{zx}}{\partial x} + \frac{\partial s_{zy}}{\partial y} + \frac{\partial s_{zz}}{\partial z} + r b_z = 0$$

2.6 Plane problems and biaxial stress

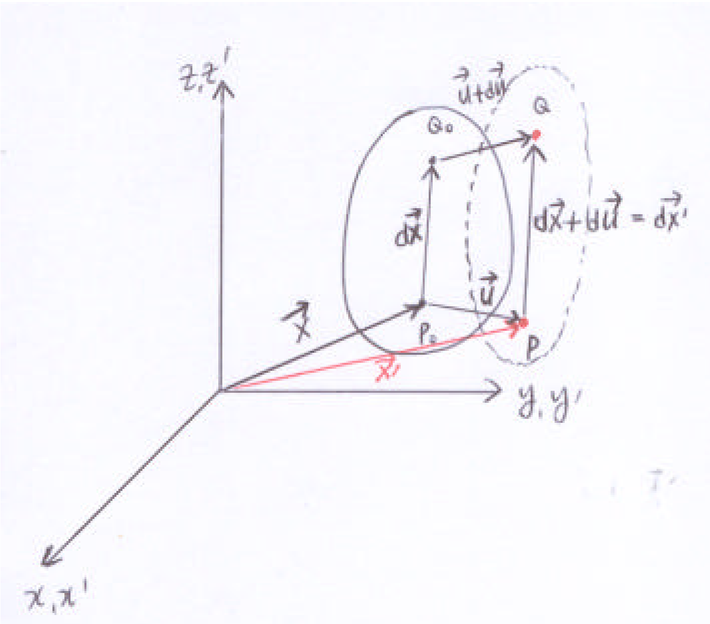
Apply the principles in 3D to 2D problems

Problem

- Calculate traction, normal stress and shear stress in a surface whose normal vector is as follow (Σ is stress on the object).

$$\hat{n} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, \right) \quad \Sigma = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

2.7 Displacement and strain



$$\vec{X} = (x, y, z) \quad \vec{X}' = (x', y', z')$$

$$\vec{u} = \vec{X}' - \vec{X} \quad d\vec{X}' = d\vec{X} + d\vec{u}$$

$$(dX')^2 = d\vec{X}' \cdot d\vec{X}' \rightarrow dX_k^{'} dX_k^{'}$$

$$dX_k^{'} = \frac{\partial X_k^{'}}{\partial X_j} dX_j$$

$$(dX')^2 - (dX)^2 = \frac{\partial X_k^{'}}{\partial X_i} \frac{\partial X_k^{'}}{\partial X_j} dX_i dX_j - d_{ij} dX_i dX_j$$

$$= \left(\frac{\partial X_k^{'}}{\partial X_i} \frac{\partial X_k^{'}}{\partial X_j} - d_{ij} \right) dX_i dX_j$$

2.7 Displacement and strain

$$L_{ij} = \frac{1}{2}\left(\frac{\partial X'_k}{\partial X_i} \frac{\partial X'_k}{\partial X_j} - d_{ij} \right)$$

$$X'_k = u_k + X_k$$

$$L_{ij} = \frac{1}{2}\left(\left(\frac{\partial u_k}{\partial X_i} + d_{ki} \right)\left(\frac{\partial u_k}{\partial X_j} + d_{kj} \right) - d_{ij} \right)$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} + d_{ij} - d_{ij} \right)$$

$$\approx \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \quad \text{for infinitesimal displacement}$$

$$\therefore (dX')^2 - (dX)^2 = 2L_{ij} dX_i dX_j$$

2.7 Displacement and strain

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

$$\left[L_{ij}\right] = \begin{bmatrix} \frac{\partial u_x}{\partial X_x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial X_y} + \frac{\partial u_y}{\partial X_x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial X_z} + \frac{\partial u_z}{\partial X_x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial X_x} + \frac{\partial u_x}{\partial X_y}\right) & \frac{\partial u_y}{\partial X_y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial X_z} + \frac{\partial u_z}{\partial X_y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial X_x} + \frac{\partial u_x}{\partial X_z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial X_y} + \frac{\partial u_y}{\partial X_z}\right) & \frac{\partial u_z}{\partial X_z} \end{bmatrix}$$

…..Linear (Infinitesimal) strain tensor

2.7 Displacement and strain

- Normal strain

$$\left(dX'\right)^2-\left(dX\right)^2=\left(dX'-dX\right)\left(dX'+dX\right)$$

$$\approx 2\left(dX'-dX\right)dX=2L_{ij}dX_i dX_j$$

$$dX'-dX=\frac{L_{ij}dX_i dX_j}{dX}$$

$$\frac{dX'-dX}{dX}=\frac{dX_i}{dX}L_{ij}\frac{dX_j}{dX}=\hat{n}\cdot L\cdot\hat{n}\quad\left(\text{normal strain}\right)$$

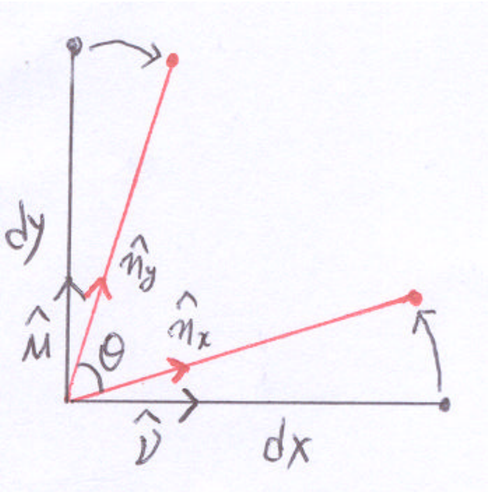
$$\hat{e}_x\cdot L\cdot\hat{e}_x=L_{xx}$$

$$\hat{e}_y\cdot L\cdot\hat{e}_y=L_{yy}$$

$$\hat{e}_z\cdot L\cdot\hat{e}_z=L_{zz}$$

2.7 Displacement and strain

- Shear strain



$$\hat{n}_x \cdot \hat{n}_y = \cos q = \sin\left(\frac{p}{2} - q\right) = \sin g \approx g \quad (\text{shear strain})$$

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}\left(J + J_c\right)$$

$$d\vec{u} = \frac{\partial u_i}{\partial X_j} dX_j = J \cdot d\vec{X}$$

$$\hat{n}_x \approx \hat{u} + J \cdot \hat{u}, \quad \hat{n}_y \approx \hat{m} + J \cdot \hat{m}$$

$$g = \hat{n}_x \cdot \hat{n}_y = \left(\hat{u} + J \cdot \hat{u}\right) \cdot \left(\hat{m} + J \cdot \hat{m}\right)$$

$$= \hat{u} \cdot \hat{m} + \hat{u} \cdot \left(J + J_c\right) \cdot \hat{m} + \hat{u} \cdot J_c \cdot J \cdot \hat{m}$$

$$= \hat{u} \cdot 2L \cdot \hat{m}$$

$$\hat{e}_x \cdot L \cdot \hat{e}_y = L_{xy} = \frac{1}{2} g_{xy} \quad \hat{e}_y \cdot L \cdot \hat{e}_z = L_{yz} = \frac{1}{2} g_{yz}$$

$$\hat{e}_z \cdot L \cdot \hat{e}_x = L_{zx} = \frac{1}{2} g_{zx}$$

2.7 Displacement and strain

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right)$$

$$\left[L_{ij} \right] = \begin{bmatrix} e_x & \frac{1}{2} g_{xy} & \frac{1}{2} g_{xz} \\ \frac{1}{2} g_{xy} & e_y & \frac{1}{2} g_{yz} \\ \frac{1}{2} g_{xz} & \frac{1}{2} g_{yz} & e_z \end{bmatrix}$$

2.7 Displacement and strain

- Decomposition of relative displacement (du)

$$du_i = \frac{\partial u_i}{\partial X_j} dX_j$$

$$= \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i} \right) \right] dX_j$$

$$= \left[L_{ij} + \Omega_{ij} \right] dX_j$$

$$\left[\Omega_{ij} \right] = \begin{bmatrix} 0 & w_{xy} & w_{xz} \\ -w_{xy} & 0 & w_{yz} \\ -w_{xz} & -w_{yz} & 0 \end{bmatrix} \quad \text{.... rotation tensor}$$

$$= \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \quad \left(\vec{\Omega} = \frac{1}{2} \vec{\nabla} \times \vec{u} = \frac{1}{2} \begin{bmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{bmatrix} \right)$$

2.8 Principal strains, strain transformation, volumetric strain and deviator strain

Principal strains: Eigen values/vectors of strain tensor

Strain transformation: analogous to the stress transformation

$$\text{Volumetric strain} = \frac{V - V_0}{V_0} = \frac{(1+e_1)(1+e_2)(1+e_3)\,dX_1 dX_2 dX_3}{dX_1 dX_2 dX_3} - 1$$

$$= e_1 + e_2 + e_3 + e_1 e_2 + e_2 e_3 + e_3 e_1 + e_1 e_2 e_3$$

$$\approx e_1 + e_2 + e_3$$

$$\text{Deviator strain} = \begin{bmatrix} e_x - \Delta/3 & g_{xy} & g_{xz} \\ g_{xy} & e_y - \Delta/3 & g_{yz} \\ g_{xz} & g_{yz} & e_z - \Delta/3 \end{bmatrix} \quad \Delta : \text{volumetric strain}$$

2.9 Strain compatibility equations

Independent strain components: 6 $(e_x, e_y, e_z, g_{xy}, g_{yz}, g_{zx})$

Independent displacement components : 3 (u_x, u_y, u_z)

Random $u_x, u_y, u_z \rightarrow e_x, e_y, e_z, g_{xy}, g_{yz}, g_{zx}$ (o)

Random $e_x, e_y, e_z, g_{xy}, g_{yz}, g_{zx} \rightarrow u_x, u_y, u_z$ (×)

Finding restricting conditions:

$$e_{xx} = \frac{\partial u_x}{\partial X_x}, \quad e_{yy} = \frac{\partial u_y}{\partial X_y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial X_y} + \frac{\partial u_y}{\partial X_x}\right)$$

$$\frac{\partial^2 e_{xx}}{\partial X_y{}^2} = \frac{\partial^3 u_x}{\partial X_x \partial X_y{}^2}, \quad \frac{\partial^2 e_{yy}}{\partial X_x{}^2} = \frac{\partial^3 u_y}{\partial X_x{}^2 \partial X_y}, \quad \frac{\partial^2 e_{xy}}{\partial X_x \partial X_y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial X_x \partial X_y{}^2} + \frac{\partial^3 u_y}{\partial X_x{}^2 \partial X_y}\right)$$

$$\therefore \frac{\partial^2 e_{xx}}{\partial X_y{}^2} + \frac{\partial^2 e_{yy}}{\partial X_x{}^2} = 2\frac{\partial^2 e_{xy}}{\partial X_x \partial X_y}$$

2.9 Strain compatibility equations

$$\frac{\partial^2 e_{xx}}{\partial X_y^{\;2}} + \frac{\partial^2 e_{yy}}{\partial X_x^{\;2}} = 2\frac{\partial^2 e_{xy}}{\partial X_x \partial X_y}$$

$$\frac{\partial^2 e_{yy}}{\partial X_z^{\;2}} + \frac{\partial^2 e_{zz}}{\partial X_x^{\;2}} = 2\frac{\partial^2 e_{yz}}{\partial X_y \partial X_z}$$

$$\frac{\partial^2 e_{zz}}{\partial X_x^{\;2}} + \frac{\partial^2 e_{xx}}{\partial X_z^{\;2}} = 2\frac{\partial^2 e_{zx}}{\partial X_x \partial X_z}$$

$$\frac{\partial}{\partial X_x}\left(-\frac{\partial e_{yz}}{\partial X_x} + \frac{\partial e_{zx}}{\partial X_y} + \frac{\partial e_{xy}}{\partial X_z}\right) = \frac{\partial^2 e_{xx}}{\partial X_y \partial X_z}$$

$$\frac{\partial}{\partial X_y}\left(\frac{\partial e_{yz}}{\partial X_x} - \frac{\partial e_{zx}}{\partial X_y} + \frac{\partial e_{xy}}{\partial X_z}\right) = \frac{\partial^2 e_{yy}}{\partial X_z \partial X_x}$$

$$\frac{\partial}{\partial X_z}\left(\frac{\partial e_{yz}}{\partial X_x} + \frac{\partial e_{zx}}{\partial X_y} - \frac{\partial e_{xy}}{\partial X_z}\right) = \frac{\partial^2 e_{zz}}{\partial X_x \partial X_y}$$

2.10 Stress-strain relations

$$e_{xx} = \frac{1}{E}\left[s_{xx} - n\left(s_{yy} + s_{zz}\right)\right], \quad e_{yy} = \frac{1}{E}\left[s_{yy} - n\left(s_{xx} + s_{zz}\right)\right], \quad e_{zz} = \frac{1}{E}\left[s_{zz} - n\left(s_{xx} + s_{yy}\right)\right]$$

$$g_{xy} = \frac{1}{G}t_{xy}, \quad g_{yz} = \frac{1}{G}t_{yz}, \quad g_{zx} = \frac{1}{G}t_{zx} \quad \left(G = \frac{E}{2(1+n)}\right)$$

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ g_{xy} \\ g_{yz} \\ g_{zx} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -n & -n & 0 & 0 & 0 \\ -n & 1 & -n & 0 & 0 & 0 \\ -n & -n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+n) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+n) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+n) \end{bmatrix}\begin{bmatrix} s_{xx} \\ s_{yy} \\ s_{zz} \\ t_{xy} \\ t_{yz} \\ t_{zx} \end{bmatrix}$$

$$\begin{bmatrix} s_{xx} \\ s_{yy} \\ s_{zz} \\ t_{xy} \\ t_{yz} \\ t_{zx} \end{bmatrix} = \frac{E(1-n)}{(1+n)(1-2n)}\begin{bmatrix} 1 & n/(1-n) & n/(1-n) & 0 & 0 & 0 \\ n/(1-n) & 1 & n/(1-n) & 0 & 0 & 0 \\ n/(1-n) & n/(1-n) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2n)}{2(1-n)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2n)}{2(1-n)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2n)}{2(1-n)} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ g_{xy} \\ g_{yz} \\ g_{zx} \end{bmatrix}$$

2.11 Cylindrical polar co-ordinates

Refer to p.37, textbook

2.12 Geomechanics conventions for displacement, strain and stress

Compressive/contractile force/displacement are positive.
Refer to Figure 2.12.