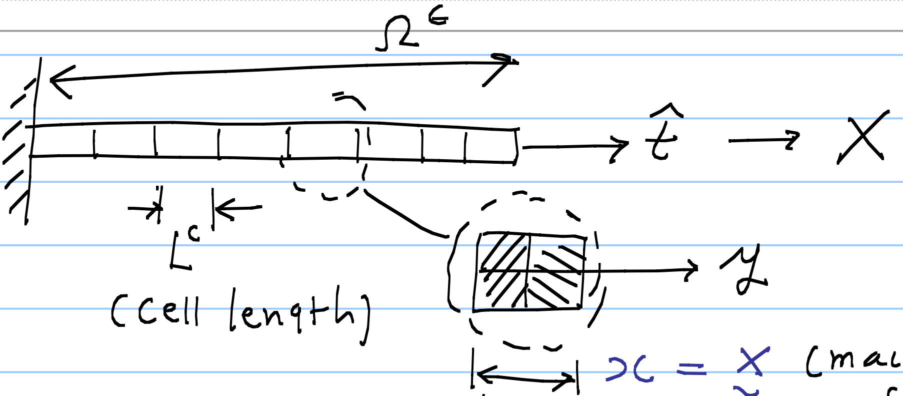
Homogenization: 1-D case



$x = \underset{\sim}{X}$ (macroscopic coord.) (a)

$Y \in [Y_L, Y_R] = [0, l]$

$y = X/\epsilon$ (microscopic coord.) (b)

$\epsilon = ? \sim \dfrac{\text{Cell length}}{\text{Bar length}}$

☒ <u>Find homogenized equations (as $\epsilon \to 0$) for periodic unit cells</u>

Field equation

(S) $\left\{ \begin{array}{l} \qquad\qquad\qquad\qquad (c) \\ \qquad\qquad\qquad\qquad (d) \end{array} \right.$

Easier to work with weak form:

(e)

To go from X to (x, y), we need

i) $dX =$ (f)

ii) Length (or <u>volume</u>) of unit cell

$$L_X^c = \qquad (g)$$

iii) $$\frac{d\phi}{dX} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial X}$$

(ϕ: some ftn) $\quad = \qquad$ (f)

iv) $$\lim_{\epsilon \to 0} \int_{\Omega^\epsilon} \phi(X)\, dX = \lim_{\epsilon \to 0} \int_{\Omega^\epsilon} \phi(x, y)\, dX$$

$$= \int_\Omega \qquad]\, dX$$

$$= \int_\Omega \qquad dx \qquad (h)$$

With (f–h), go back to the weak-form of the govern. eq:

Let

$$\begin{cases} u^{\varepsilon}(X) = u^{\varepsilon}(x,y) \\ \qquad\quad = u^{0}(x,y) + \epsilon u^{1}(x,y) + \epsilon^{2} u^{2}(x,y) + \cdots \\ \sigma^{\varepsilon}(X) = \sigma^{0}(x,y) + \epsilon \cdot \sigma^{1}(x,y) + \cdots \end{cases} \qquad (1)$$

• $v(X) =$

$$\begin{cases} v \in H^{1}(\Omega^{\epsilon}) \text{ with } v\big|_{X=0} = 0, \\ u \in H^{1}(\Omega^{\epsilon}) \text{ with } u\big|_{X=0} = 0 \end{cases}$$

$\left(H^{1} = \{ w \mid w \in L_2,\ W_{,x} \in L_2 \},\ L_2 = \{ w \mid \int_0^1 w^2 dx < \infty \} \right.$

▪ (1) (f) → (e)

$$\int_{\Omega^{\epsilon}} E \Big[\qquad\qquad \Big] \Big[\qquad\qquad \Big] dX$$

$$= \int_{\Omega^{e}} \gamma(X)\, v\, dX + \hat{t}\, v(L) \qquad (2)$$

$$\rightarrow \int_{\Omega^\epsilon} E \Big\{ \frac{1}{\epsilon^2} \frac{\partial u^0}{\partial y} \frac{\partial v}{\partial y}$$

$$+ \frac{1}{\epsilon} \left[\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) \frac{\partial v}{\partial y} + \frac{\partial u^0}{\partial y} \frac{\partial v}{\partial x} \right]$$

$$+ \left[\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) \frac{\partial v}{\partial x} + \left(\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y} \right) \frac{\partial v}{\partial y} \right]$$

$$+ \epsilon (\cdots) \Big\} dX$$

$$= \int r(X) v \, dX + \hat{t} \, v(L) \qquad (3)$$

Integrate term by term of (3) by using (4) as $\epsilon \to 0$

$$\frac{1}{\epsilon^2} \int_\Omega \frac{1}{L_y^c} \left[\int_Y E \frac{\partial u^0}{\partial y} \frac{\partial v}{\partial y} dy \right] dx$$

$$+ \frac{1}{\epsilon} \int_\Omega \frac{1}{L_y^c} \left\{ \int_Y E \left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) \frac{\partial v}{\partial y} + \frac{\partial u^0}{\partial y} \frac{\partial v}{\partial x} \right] dy \right\} dx$$

$$+\int_{\Omega}\frac{1}{L_y^c}\left\{\int_{Y}E\left[\left(\frac{\partial u^0}{\partial x}+\frac{\partial u^1}{\partial y}\right)\frac{\partial v}{\partial x}+\left(\frac{\partial u^1}{\partial x}+\frac{\partial u^2}{\partial y}\right)\frac{\partial v}{\partial y}\right]dy\right\}dx$$

$$+\ O(\epsilon)$$

$$=\int_{\Omega}\left(\frac{1}{L_y^c}\int_{Y}\gamma\, dy\right)dx+\hat{t}\,v(L) \qquad (4)$$

$\gamma(x,y)$

Next Step: Equate terms of the same order of ϵ

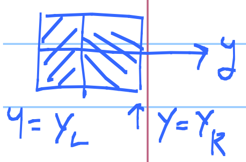
[1] $O\left(\frac{1}{\epsilon^2}\right)$ term

(5)

∵ Because v is arbitrary, let us convert [ ] into something like $\int(\ )\,v\,dy$

By Integration-by-Part (or Divergence)

$$\int_\Omega \frac{1}{L_y c}\left\{ \qquad \qquad \qquad \right\} dx = 0 \qquad (6)$$



Because v is arbitrary,

$$-\frac{\partial}{\partial y}\left(E\,\frac{\partial u^0}{\partial y}\right) = 0 \quad \text{in } Y$$

$$\sigma \longrightarrow E\,\frac{\partial u^0}{\partial y} \equiv 0 \quad \text{at} \begin{cases} y = y_L = 0 \\ y = y_R = 1 \end{cases}$$

meaning:

$$\Longrightarrow \quad u^0(x, y) = \qquad \qquad (8)$$

[2] $O\left(\frac{1}{\epsilon}\right)$ term:

$$\int_{\Omega} \frac{1}{L_y^c}\left[\int_Y E\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right)\frac{\partial v}{\partial y}\, dy\right] dx = 0 \qquad (9)$$

if ______________ is chosen,

i.e.:

(w) $\boxed{}$ (10)

⊕ periodicity (i.e, $E(y_L) = E(y_R)$) etc

- For simplicity, assume

$$E(x) = E(x, y) \equiv E(y)$$

- $\frac{\partial u^0}{\partial x} = \frac{du^0(x)}{dx}$,

$$\Rightarrow \int_Y E(y)\left[\frac{du^0(x)}{dx} + \frac{\partial u^1(x,y)}{\partial y}\right]\frac{dv}{dy}\, dy = 0 \qquad (10)'$$

Meaning: u^1 must be in $F(x)\, G(y)$ form!

Thus

Let $\boxed{}$ (11)

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Then (10)$'$ becomes

$$\frac{du^0(x)}{dx}\int_Y E(y)\left[1-\frac{d\chi(y)}{dy}\right]\frac{dv}{dy}\,dy=0$$

i.e. : $\boxed{}$ (12)

microscopic equilibrium

For given E, χ is determined except a rigid-body displacement (may set $\chi(y_L)=\chi(y_R)=0$)

Q : what does Eq. (12) imply?

→ Check its strong form

Integrate (12) by part (or Divergence)

$$\Big[\qquad \Big] v \, dy = 0 \qquad (*)$$

For (*) to hold ; ($v(y)$ = arbitrary, $v(Y_R) = v(y_L)$))

$$\begin{cases} \left(E \frac{dx}{dy} - E\right) v \Big|_{Y_R} - \left(E \frac{dx}{dy} - E\right) v \Big|_{Y_L} = 0 & \text{(a)} \\ \frac{d}{dy}\left(E \frac{dx}{dy}\right) = \frac{dE}{dy} \quad \text{in } Y & \text{(b)} \end{cases}$$

Since $v|_{Y_R} = v|_{Y_L}$

(a) $\rightarrow$ $\qquad = \qquad$ (a)' and

=

$$(a)' \rightarrow \quad \left(E \frac{dx}{dy}\right)_{y_R} = \left(E \frac{dx}{dy}\right)_{y_L} \qquad (a)''$$

Thus (12) is equivalent to solving

$$\boxed{\begin{array}{ll} \frac{d}{dy}\left(E \frac{dx}{dy}\right) = \frac{dE}{dy} \quad \text{in } Y & (12a)' \\ \oplus \; x|_{y_L} = x|_{y_R}, \; \frac{dx}{dy}\Big|_{y_L} = \frac{dx}{dy}\Big|_{y_R} & (12b)' \\ \text{(Thus } x \text{ has a rigid-body motion;} & \\ \quad \text{may set } x(y_R) = x(y_L) = 0 \text{)} & \end{array}}$$

Integrating (12a)';

Thus

$$x(y) = y + a\int_{y_L}^{y} \frac{dy}{E(y)} + b \qquad (c)$$

* use the periodicity condition

$$x(y_L) = x(y_R) \qquad — (*)$$

$$(**)\begin{cases} x(y=y_L) = y_L + b \\ x(y=y_R) = y_R + a\int_{y_L}^{y_R} \frac{dy}{E(y)} + b \end{cases}$$

$$\int_{Y} \frac{dy}{E(y)}$$

$(**) \to (*)$

$$\boxed{a = \frac{-L_y^c}{\int_Y \frac{dy}{E(y)}} = -\frac{1}{\left[\frac{1}{L_y^c}\int_Y \frac{dy}{E(y)}\right]}} \quad (d)$$

SUMMARY

※ Remark: if $u^0(x)$ is known, $u'(x,y)$ is determined (but not yet known!!)

For future use, we compute dx/dy explicitly:

$$\frac{dx}{dy} = 1 + \frac{a}{E(y)} \qquad (g)$$

[3] $O(1)$ term:

$$\int_\Omega \frac{1}{L_y^c}\left[\int_Y E\left\{\left(\frac{\partial u^0}{\partial x}+\frac{\partial u^1}{\partial y}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial u^1}{\partial x}+\frac{\partial u^2}{\partial y}\right)\frac{\partial v}{\partial y}\right\} dy\right] dx$$

du^0/dx

$$= \int_\Omega \left[\frac{1}{L_y^c}\int_Y \gamma\, v\, dy\right] dx + \hat{t}\, v(L) \tag{13}$$

Choose

Then (13) becomes

$$\boxed{} \tag{14}$$

Substitute $u'(x,y)$ in page 12 into (14)

$$\int_{\Omega}$$

$$= \int_{\Omega} \qquad + $$

Let

- $E^H \triangleq$

$\underset{\text{page 12, (9)}}{=}$

$=$ (15)

- $b(x) \triangleq \frac{1}{L_Y^c} \int_Y \gamma(x,y)\,dy$ (16)

Then

$$(W)\quad \boxed{\int_\Omega \qquad = \int_\Omega \qquad + \qquad} \quad (17)$$

Macroscopic equation for $u^0(x)$ (with E^H, $b(x)$)

Integrating (17) by part to obtain strong form

$$\int_\Omega \left[\frac{d}{dx}\left(E^H \frac{du^0}{dx}\right) + b(x)\right] v(x)\, dx$$

$$-\left(E^H(x)\frac{du^0}{dx} - \hat{t}\right) v(L) = 0$$

$$(\because v(0) = 0)$$

)

Thus

$$(S)\ \Big\{$$

How about $\sigma^0(x,y)$?

$$① \quad \sigma(X) = \sigma^0(x,y) + \varepsilon\,\sigma^1(x,y) + \cdots$$

$$② \quad \sigma^0(x,y) = E\,\varepsilon^0(x,y) \quad \text{etc}$$

To find $\varepsilon^0(x,y)$, consider

$$③ \quad \epsilon(X) = \epsilon(x,y) \overset{\Delta}{=} \epsilon^0(x,y) + \epsilon\,\epsilon^1(x,y) + \cdots$$

$$④ \quad \frac{\partial u}{\partial X} = \left(\frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y}\right)\left(u^0 + \epsilon u^1 + \cdots\right)$$

$$= \frac{1}{\epsilon}\cancel{\frac{\partial u^0}{\partial y}} + \underbrace{\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right)} + \epsilon(\cdots)$$

because $u^0 = u^0(x)$,

From ③ = ④

$$\epsilon^0(x,y) = (\qquad\qquad)$$

$$= (\qquad\qquad\qquad)$$

use $\left(\frac{dX}{dy} = 1 - \frac{a}{E}\right)$

$$= \left(1 - \frac{\partial X}{\quad}\right)\frac{du^0}{dx} = \frac{a}{E(y)}\frac{du^0}{dx}$$

$$=$$

Thus

$$\sigma^{o}(x,y) = E\,\varepsilon^{o}(x,y)$$

$$= \underbrace{E^{H}\frac{du^{o}(x)}{dx}}_{\text{ftn of } x \text{ only.}} \equiv \sigma^{o}(x)$$

Because $\sigma^{h}(x)$ is usually defined as

$$\sigma^{h}(x) =$$

$\sigma^{h}(x)$ can be shown to be

$$\sigma^{h}(x) =$$

in general case.

In the present case,

$$\sigma^{h}(x) = \underbrace{\sigma^{o}(x,y)}_{\sigma^{o}(x)}$$