Class Note for Structural Analysis 2

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Hae Sung Lee, Professor

School of Civil, Urban and Geosystem Engineering

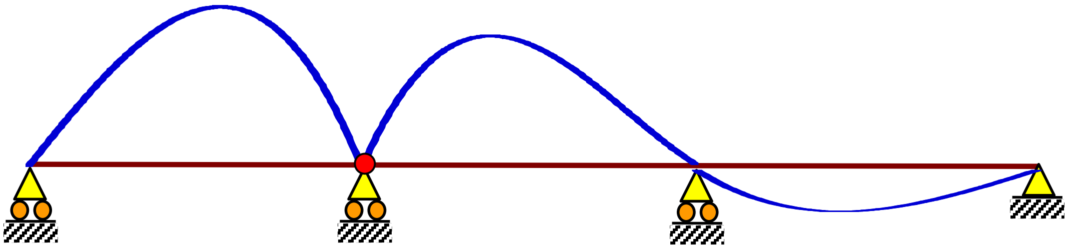
Seoul National University

Seoul, Korea

Contents

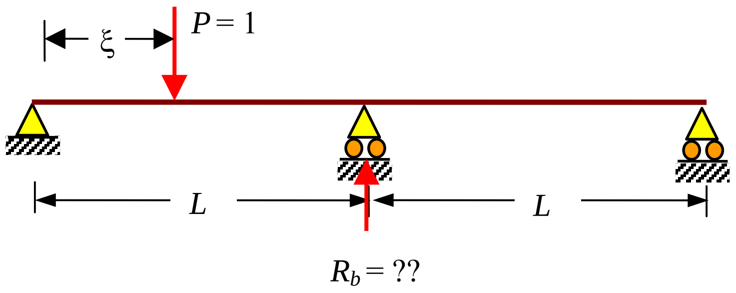
Chapter 1

Influence Lines for Indeterminate Beams

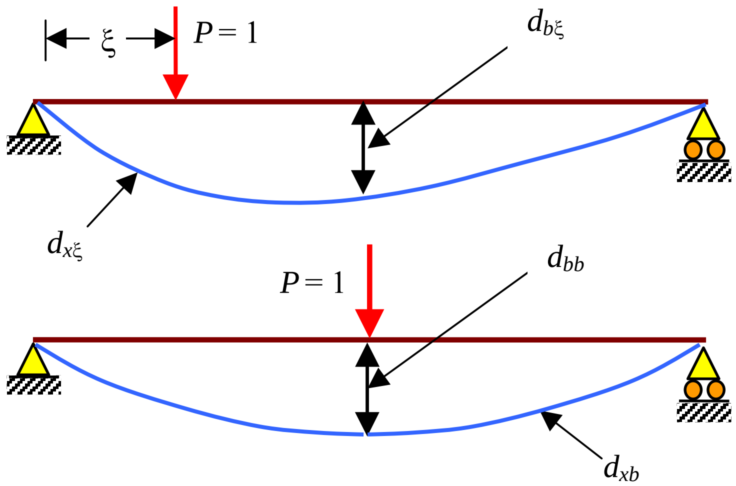


1.1 Influence Lines at Supports

1.1.1 Reaction Force



- **By the Flexibility Method**



- Compatibility Condition

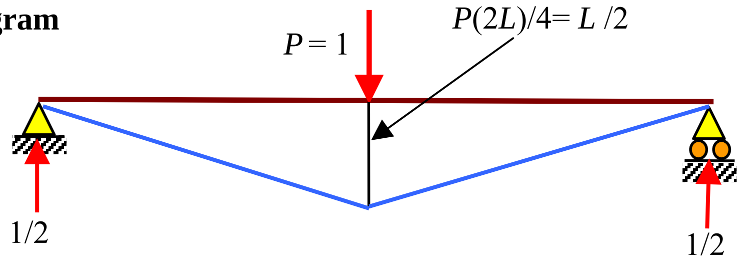
$$R_b \times d_{bb} + d_{b\xi} = 0 \rightarrow R_b = -\frac{d_{b\xi}}{d_{bb}}$$

- Betti-Maxwell's Reciprocal Theorem

$$\int_0^{2L} \delta(x-\xi) d_{xb} dx = \int_0^{2L} \delta(x-L) d_{x\xi} dx \rightarrow d_{\xi b} = d_{L\xi} = d_{b\xi}$$

- Influence Line : $R_b = -\dfrac{d_{b\xi}}{d_{bb}} = -\dfrac{d_{\xi b}}{d_{bb}}$

- **Moment Diagram**



$$d_{bb} = 2 \times \frac{2L}{3EI} M\overline{M} = 2 \times \frac{L}{3EI} \frac{L}{2} \frac{L}{2} = \frac{L^3}{6EI} = \frac{(2L)^3}{48EI}$$

- **Calculation of Deflection**

$$EIw'' = -M = -\frac{1}{2}x \rightarrow w = -\frac{1}{EI}\frac{x^3}{12} + ax + b$$

- Boundary conditions

$$w(0) = 0 \rightarrow b = 0$$

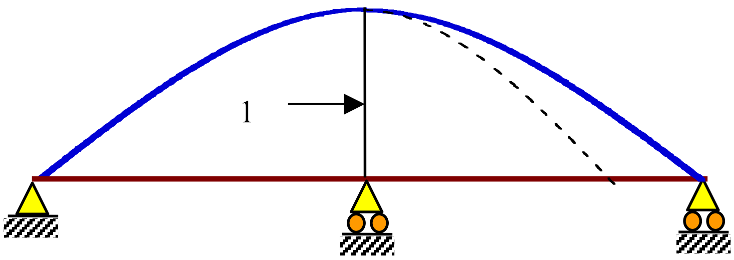
$$w'(L) = 0 \rightarrow -\frac{1}{EI}\frac{L^2}{4} + a = 0 \rightarrow a = \frac{L^2}{4EI}$$

- Deflection of the Beam

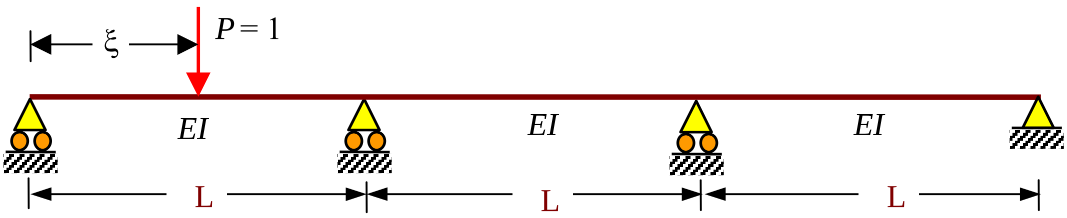
$$d_{xb} = w = \frac{1}{12EI}(-x^3 + 3L^2 x)$$

- **Influence Line**

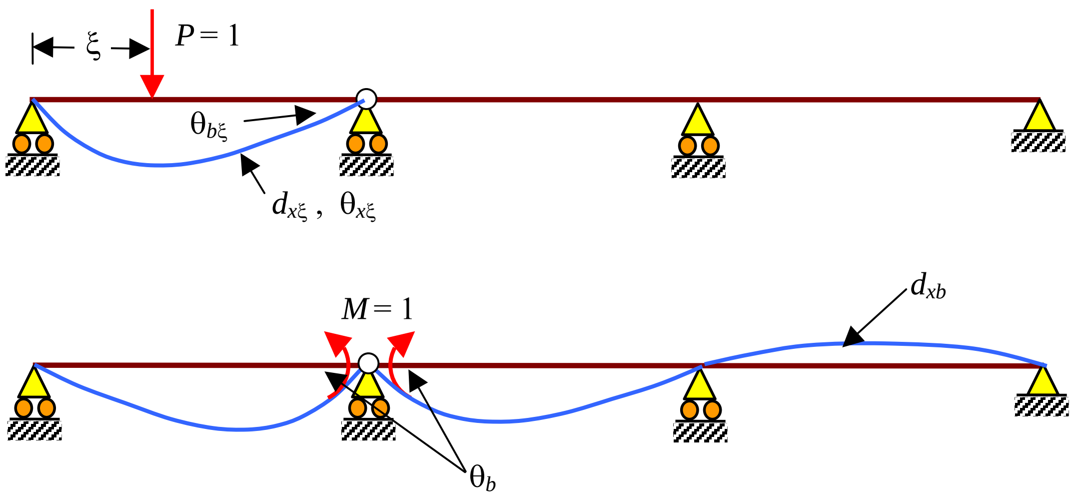
$$R_b = -\frac{d_{\xi b}}{d_{bb}} = -\frac{1}{12EI}(-x^3 + 3L^2 x) / \frac{L^3}{6EI} = \frac{1}{2}[(\frac{x}{L})^3 - 3(\frac{x}{L})]$$



1.1.2 Moment



- **By the force method**



- Compatibility Condition

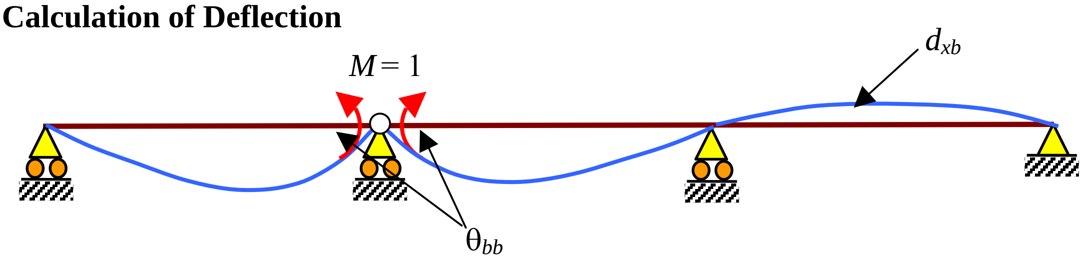
$$M_b \times \theta_{bb} + \theta_{b\xi} = 0 \rightarrow M_b = -\frac{\theta_{b\xi}}{\theta_{bb}}$$

- Betti-Maxwell's Reciprocal Theorem

$$\int_0^{3L} \delta(x-\xi) d_{xb} dx = \int_0^{3L} \delta(x-L)\theta_{x\xi} dx \rightarrow d_{\xi b} = \theta_{L\xi} = \theta_{b\xi}$$

- Influence Line : $M_b = -\frac{\theta_{b\xi}}{\theta_{bb}} = -\frac{d_{\xi b}}{\theta_{bb}}$

- **Calculation of Deflection**



i) Left span

$$EIw_L'' = -M = -\frac{x}{L} \rightarrow w_L = -\frac{x^3}{6LEI} + ax + b$$

- Boundary conditions

$$w_L(0) = 0 \rightarrow b = 0$$

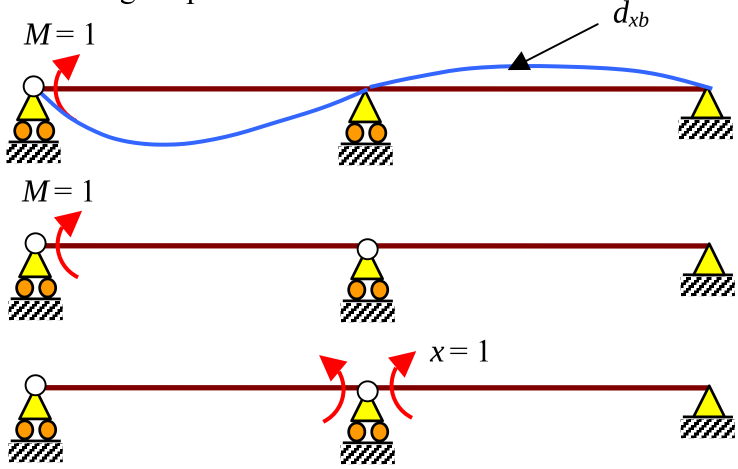
$$w_L(L) = 0 \rightarrow -\frac{L^2}{6EI} + aL = 0 \rightarrow a = \frac{L}{6EI}$$

- Deflection of the left span

$$d_{xb} = w_L = \frac{1}{6LEI}(-x^3 + L^2 x)$$

$$\theta_{bb}^L = -\frac{L}{3EI} \quad \text{(counterclockwise)}$$

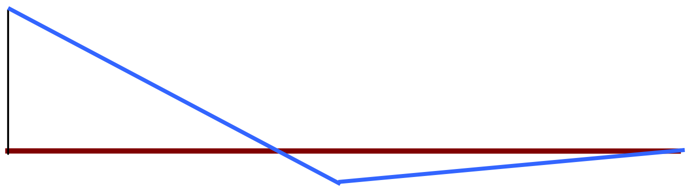
ii) Analysis of Center and Right Span



$$\theta_{cc} = \frac{2L}{3EI} \ , \ \ \theta_{cb} = \frac{L}{6EI}$$

- Compatibility condition: $\theta_{cb} + M_c\theta_{cc} = 0 \rightarrow M_c = -\frac{\theta_{cb}}{\theta_{cc}} = -\frac{1}{4}$

- Moment Diagram



iii) Deflection of Center span

$$EIw_c'' = -M = -(-\frac{5}{4L}x + 1) \rightarrow w_c = \frac{1}{EI}(\frac{5}{24L}x^3 - \frac{x^2}{2}) + ax + b$$

$$w_c(0) = 0 \rightarrow b = 0$$

$$w_c(L) = 0 \rightarrow \frac{1}{EI}(\frac{5L^2}{24} - \frac{L^2}{2}) + aL = 0 \rightarrow a = \frac{7L}{24EI}$$

$$\delta_{xb} = w_c = \frac{1}{24LEI}(5x^3 - 12x^2L + 7xL^2)$$

$$\theta_{bb}^R = w_c'(0) = \frac{7L}{24EI} \ \text{(Clockwise)}$$

$$\theta_{bb} = \theta_{bb}^L + \theta_{bb}^R = \frac{L}{3EI} + \frac{7L}{24EI} = \frac{5L}{8EI}$$

iv) Deflection of Right Span

$$EIw_R'' = -M = -(\frac{x}{4L} - \frac{1}{4}) \rightarrow w_R = \frac{1}{EI}(-\frac{x^3}{24L} + \frac{x^2}{8}) + ax + b$$

$$w_R(0) = 0 \rightarrow b = 0$$

$$w_R(L) = 0 \rightarrow \frac{1}{EI}(-\frac{L^2}{24} + \frac{L^2}{8}) + aL = 0 \rightarrow a = -\frac{L}{12EI}$$

$$\delta_{xb} = w_R = -\frac{1}{24LEI}(x^3 - 3x^2L + 2xL^2)$$

- **Final Influence Line**

i) left span :

$$M_b = -\frac{w_L}{\theta_{bb}} = -\frac{1}{6LEI}(-x^3 + L^2x) / \frac{5L}{8EI} = \frac{4}{15L^2}(x^3 - L^2x)$$

$$M_b' = \frac{4}{15L^2}(3x^2 - L^2) = 0 \to x = \sqrt{\frac{1}{3}}L = 0.577L$$

$$M_b(0.577L) = \frac{4L}{15}0.577(0.577-1)(0.577-1) = -0.103L$$

ii) Center Span :

$$M_b = -\frac{w_C}{\theta_{bb}} = -\frac{1}{24LEI}(5x^3 - 12x^2L + 7xL^2)/\frac{5L}{8EI}$$

$$= -\frac{1}{15L^2}(5x^3 - 12x^2L + 7xL^2) = -\frac{x}{15L^2}(5x - 7L)(x - L)$$

$$M_b' = -\frac{1}{15L^2}(15x^2 - 24xL + 7L^2) = 0 \to x = \frac{12-\sqrt{39}}{15}L = 0.384L$$

$$M_b(0.384L) = -\frac{0.384L}{15}(5\times 0.384 - 7)(0.384 - 1) = -0.080L$$

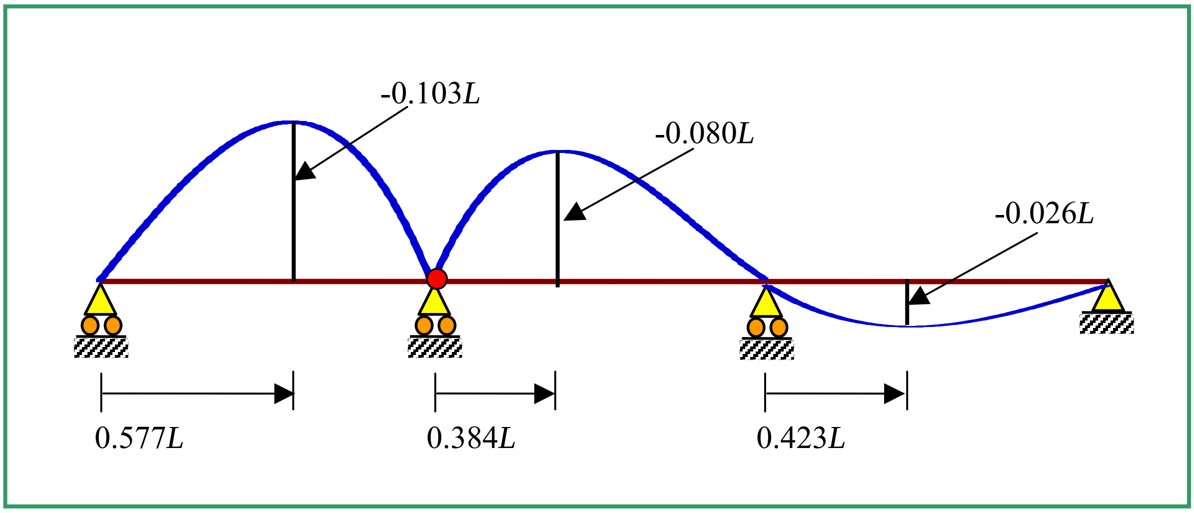
iii) Right Span:

$$M_b = -\frac{w_R}{\theta_{bb}} = \frac{1}{24LEI}(x^3 - 3x^2L + 2xL^2)/\frac{5L}{8EI}$$

$$= \frac{1}{15L^2}(x^3 - 3x^2L + 2xL^2) = \frac{1}{15L^2}x(x-2L)(x-L)$$

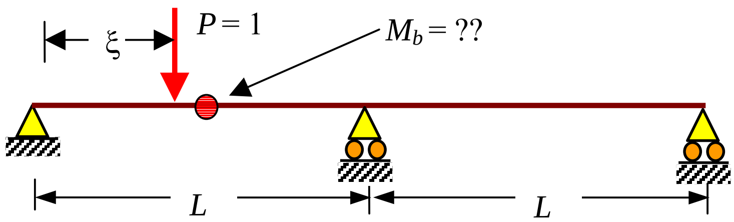
$$M_b' = \frac{1}{15L^2}(3x^2 - 6xL + 2L^2) = 0 \to x = \frac{3-\sqrt{3}}{3} = 0.423L$$

$$M_b = \frac{0.423L}{15}(0.423-2)(0.423-1) = 0.026L$$

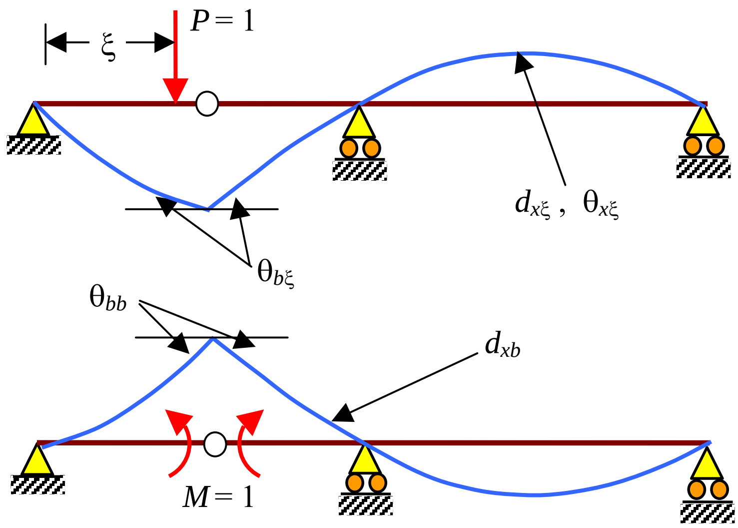


1.2 Inflence Lines in Members

1.2.1 Moment



- **By the Flexibility Method**



- Compatibility Condition: $M_b \times \theta_{bb} + \theta_{b\xi} = 0 \rightarrow M_b = -\dfrac{\theta_{b\xi}}{\theta_{bb}}$

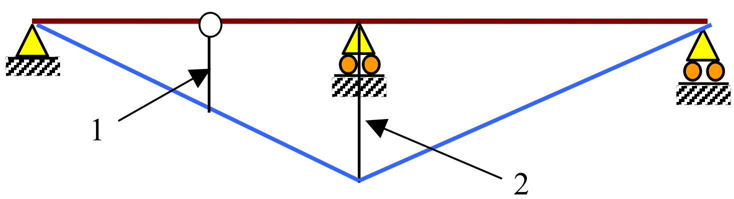
- Betti-Maxwell's Reciprocal Theorem

$$\int_0^{2L} \delta(x-\xi) d_{xb} dx = \int_0^{2L} \delta(x - L/2)\theta_{x\xi} dx \rightarrow d_{\xi b} = \theta_{\frac{L}{2}\xi} = \theta_{b\xi}$$

- Influence Line : $M_b = -\dfrac{\theta_{b\xi}}{\theta_{bb}} = -\dfrac{d_{\xi b}}{\theta_{bb}}$

- **Calculation of Deflection**

i) Moment Diagram



$$\theta_{bb} = \frac{8L}{3EI}$$

ii) Suspended span

$$EIw_S'' = -M = -\frac{2x}{L} \rightarrow w_S = -\frac{x^3}{3LEI} + ax + b$$

- Boundary conditions

$$w_S(0) = 0 \rightarrow b = 0$$

$$w_S(\frac{L}{2}) = w_O(0) \rightarrow -\frac{L^2}{24EI} + a\frac{L}{2} = ??$$

- Deflection of the suspended span

$$w_S = -\frac{x^3}{3LEI} + ax$$

iii) Overhanged span

$$EIw_O'' = -M = -(1+\frac{2x}{L}) \rightarrow w_O = -\frac{1}{EI}(\frac{x^3}{3L} + \frac{x^2}{2}) + cx + e$$

- Boundary conditions

$$w_O(0) = w_S(\frac{L}{2}) \rightarrow e = -\frac{L^2}{24EI} + a\frac{L}{2}$$

$$w_O(\frac{L}{2}) = 0 \rightarrow -\frac{L^2}{6EI} + c\frac{L}{2} + e = 0$$

iv) Right span

$$EIw_R'' = -M = -(2-\frac{2x}{L}) \rightarrow w_R = -\frac{1}{EI}(-\frac{x^3}{3L} + x^2) + fx + g$$

- Boundary conditions

$$w_R(0) = 0 \rightarrow g = 0$$

$$w_R(L) = 0 \rightarrow \frac{1}{EI}(-\frac{L^2}{3} + L^2) + fL = 0 \rightarrow f = \frac{2L}{3EI}$$

- Deflection

$$w_R = \frac{1}{3LEI}(x^3 - 3Lx^2 + 2L^2x)$$

v) Determination of a, c, e

$$\theta_O(\frac{L}{2}) = \theta_R(0) = \frac{2L}{3EI} \rightarrow -\frac{1}{EI}(\frac{L}{4} + \frac{L}{2}) + c = \frac{2L}{3EI} \rightarrow c = \frac{17L}{12EI}$$

$$-\frac{L^2}{6EI} + c\frac{L}{2} + e = 0 \rightarrow -\frac{L^2}{6EI} + \frac{17L^2}{24EI} + e = 0 \rightarrow e = -\frac{13L^2}{24EI}$$

$$-\frac{13L^2}{24EI} = -\frac{L^2}{24EI} + a\frac{L}{2} \rightarrow a = -\frac{L}{EI}$$

vi) Deflection of the left span

- Suspended span: $w_S = -\dfrac{x^3}{3LEI} + ax = -\dfrac{1}{3LEI}(x^3 + 3L^2 x)$

- Overhanged span: $w_O = -\dfrac{1}{24LEI}(8x^3 + 12Lx^2 - 34xL^2 + 13L^3)$

- **Final Influence Line**

i) Suspended span

$$M_b = -\frac{w_S}{\theta_{bb}} = \frac{1}{3LEI}(x^3 + 3L^2 x) / \frac{8L}{3EI} = \frac{1}{8L^2}(x^3 + 3L^2 x)$$

$$M_b(\frac{L}{2}) = \frac{1}{8L^2}((\frac{L}{2})^3 + 3\frac{L^3}{2}) = \frac{13}{64}L = 0.203L$$

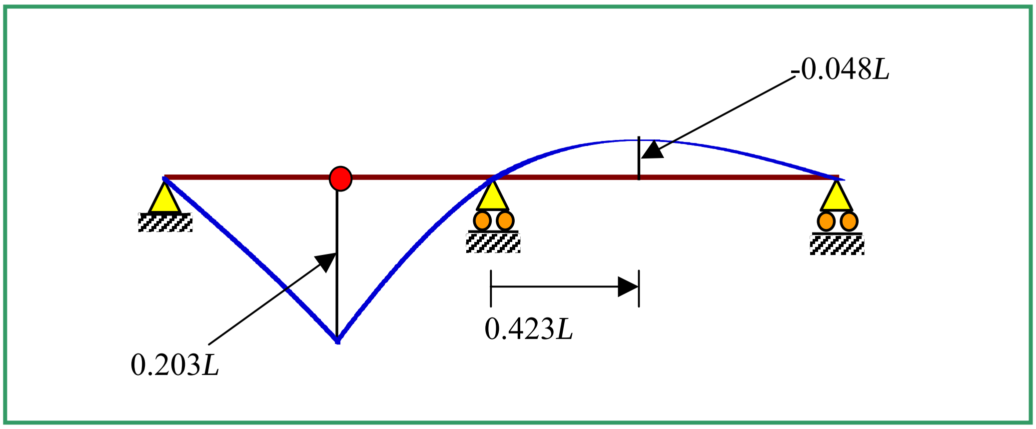
ii) Overhanged span

$$M_b = -\frac{w_O}{\theta_{bb}} = \frac{1}{64L^2}(8x^3 + 12Lx^2 - 34xL^2 + 13L^3)$$

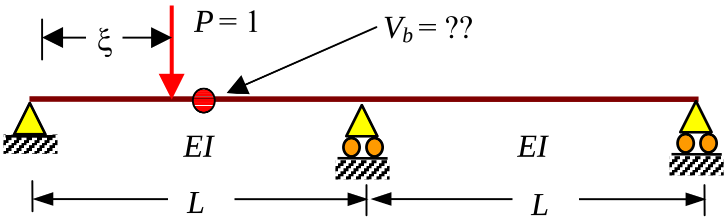
iii) Right span

$$M_b = -\frac{w_R}{\theta_{bb}} = -\frac{1}{8L^2}(x^3 - 3Lx^2 + 2L^2 x)$$

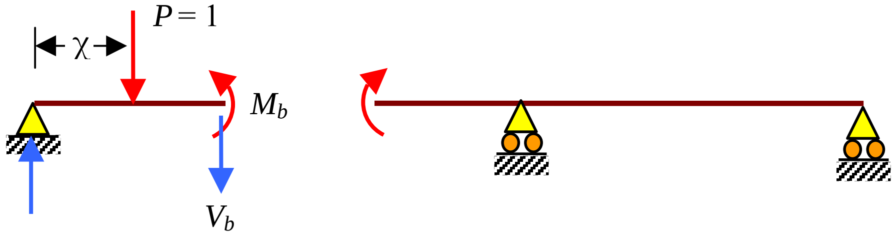
$$M_b' = -\frac{1}{8L^2}(3x^2 - 6Lx + 2L^2) = 0 \rightarrow x = 0.423L\,,\quad M_b(0.423L) = -0.048L$$



1.2.2 Influence Line of Shear Force using the Influence Line of Moment

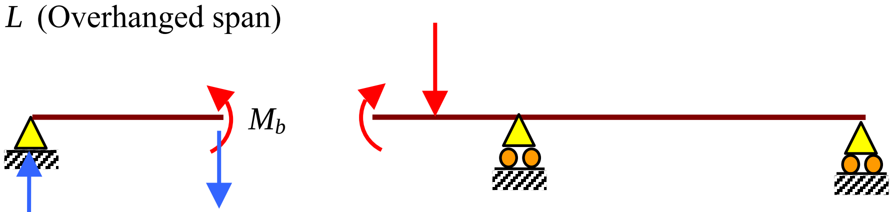


i) $\xi \le \frac{L}{2}$



$$V_b \times \frac{L}{2} + 1 \times x - M_b = 0 \rightarrow V_b = \frac{2M_b}{L} - \frac{2x}{L} = \frac{1}{4L^3}(x^3 + 3L^2 x) - \frac{2x}{L} = \frac{1}{4L^3}(x^3 - 5L^2 x)$$

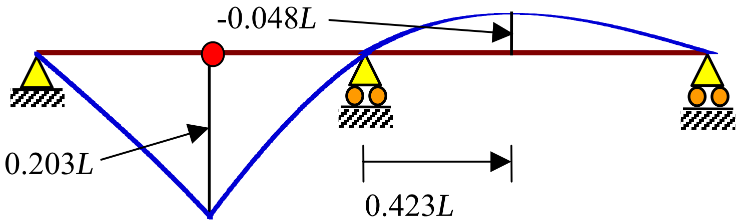
ii) $\frac{L}{2} \le \xi \le L$ (Overhanged span)



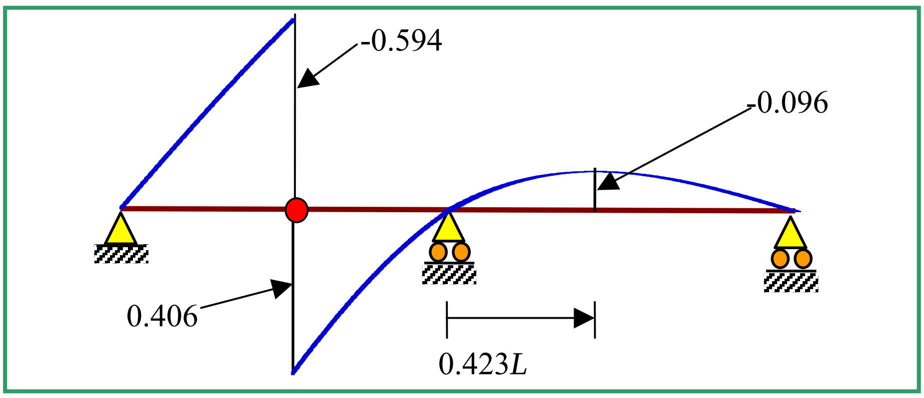
$$V_b \times \frac{L}{2} - M_b = 0 \rightarrow V_b = \frac{2M_b}{L} = \frac{1}{32L^3}(8x^3 + 12Lx^2 - 34xL^2 + 13L^3)$$

iii) $L \le \xi$ (Right span)

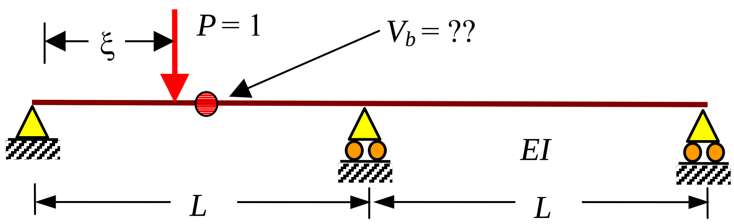
$$V_b \times \frac{L}{2} - M_b = 0 \rightarrow V_b = \frac{2M_b}{L} = -\frac{1}{4L^3}(x^3 - 3Lx^2 + 2L^2 x)$$



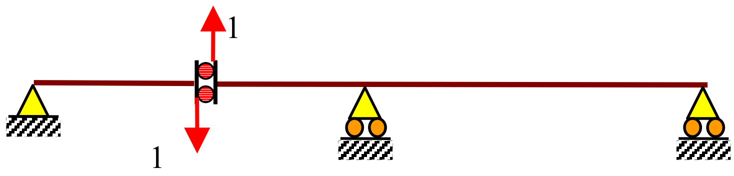
$$\times \frac{2}{L} - \frac{2x}{L}(x \le \frac{L}{2}) =$$

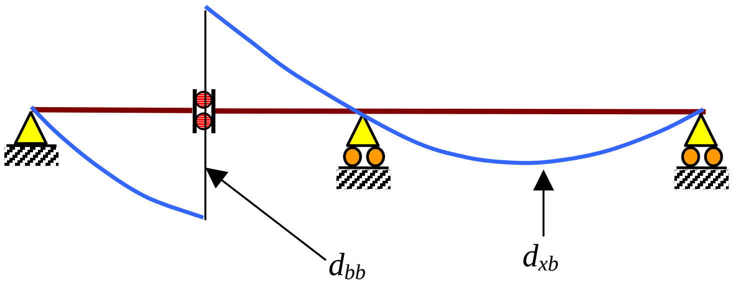


1.2.3 Influence lineof Shear Force by Müller –Breslau's Principle



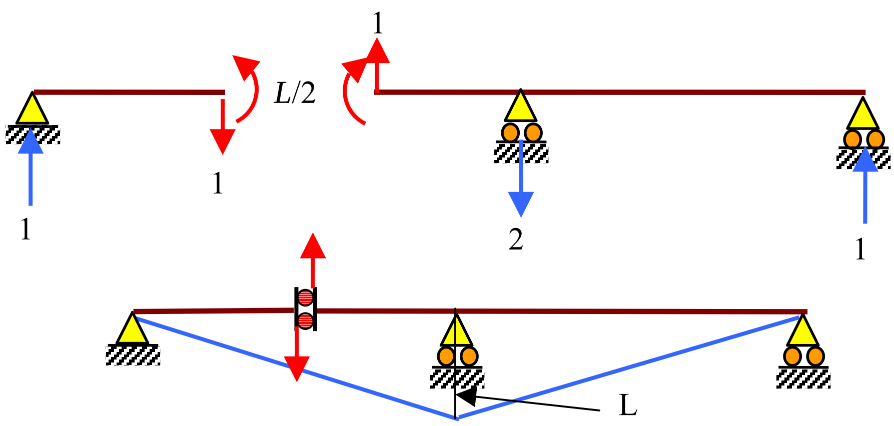
- **Remove Redundancy and Apply an Unit Load**





$$V_b = -\frac{d_{xb}}{d_{bb}}$$

- **Free Body Digram and Moment Diagram**



$$d_{bb} = \frac{4L^3}{6EI}$$

- **Deflection of the Beam**

i) Suspended span

$$EIw_S'' = -M = -x \rightarrow w_S = -\frac{x^3}{6EI} + ax + b$$

- Boundary conditions

$$w_S(0) = 0 \rightarrow b = 0$$
$$\theta_S(\frac{L}{2}) = \theta_O(0) \rightarrow -\frac{L^2}{8EI} + a = ??$$

- Deflection of the suspended span

$$w_S = -\frac{x^3}{6EI} + ax$$

ii) Overhanged span

$$EIw_O'' = -M = -(\frac{L}{2} + x) \rightarrow w_O = -\frac{1}{EI}(\frac{x^3}{6} + \frac{L}{4}x^2) + cx + e$$

- Boundary conditions

$$\theta_O(0) = \theta_S(\frac{L}{2}) \rightarrow c = -\frac{L^2}{8EI} + a$$
$$w_O(\frac{L}{2}) = 0 \rightarrow -\frac{L^2}{12EI} + c\frac{L}{2} + e = 0$$

iii) Right span

$$EIw_R'' = -M = -(L - x) \rightarrow w_R = -\frac{1}{EI}(-\frac{x^3}{6} + \frac{L}{2}x^2) + fx + g$$

- Boundary conditions

$$w_R(0) = 0 \rightarrow g = 0$$
$$w_R(L) = 0 \rightarrow -\frac{1}{EI}(-\frac{L^3}{6} + \frac{L^3}{2}) + fL = 0 \rightarrow f = \frac{L^2}{3EI}$$

- Deflection

$$w_R = \frac{1}{6EI}(x^3 - 3Lx^2 + 2L^2x)$$

iv) Determination of a, c, e

$$\theta_O(\frac{L}{2}) = \theta_R(0) = \frac{L^2}{3EI} \rightarrow -\frac{1}{EI}(\frac{L^2}{8} + \frac{L^2}{4}) + c = \frac{L^2}{3EI} \rightarrow c = \frac{17L^2}{24EI}$$
$$-\frac{L^3}{12EI} + c\frac{L}{2} + e = 0 \rightarrow -\frac{L^3}{12EI} + \frac{17L^3}{48EI} + e = 0 \rightarrow e = -\frac{13L^2}{48EI}$$
$$\frac{17L^2}{24EI} = -\frac{L^3}{8EI} + a \rightarrow a = \frac{5L^3}{6EI}$$

v) Deflection of the left span

- Suspended span : $w_S = -\dfrac{x^3}{6EI} + ax = -\dfrac{1}{6EI}(x^3 - 5L^2 x)$

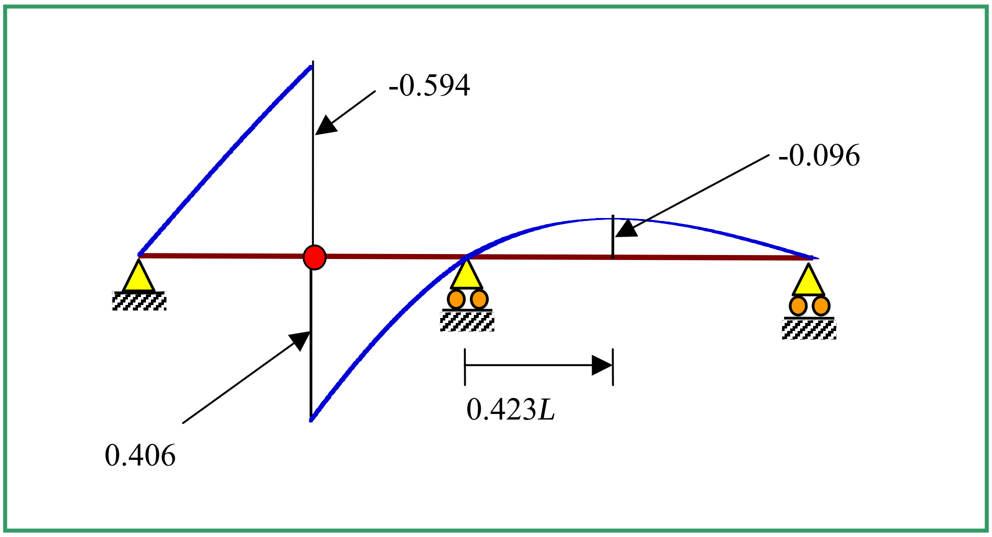
-Overhanged span : $w_O = -\dfrac{1}{48EI}(8x^3 + 12Lx^2 - 34xL^2 + 13L^3)$

- **Final Influence Line**

i) Suspended span : $V_b = -\dfrac{w_S}{d_{bb}} = \dfrac{1}{6EI}(x^3 - 5L^2 x) / \dfrac{4L^3}{6EI} = \dfrac{1}{4L^3}(x^3 - 5L^2 x)$

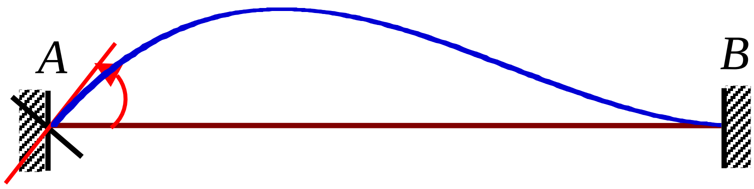
ii) Overhanged span : $V_b = -\dfrac{w_O}{d_{bb}} = \dfrac{1}{32L^3}(8x^3 + 12Lx^2 - 34xL^2 + 13L^3)$

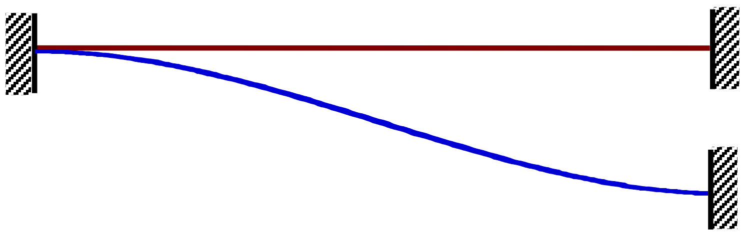
iii) Right span : $V_b = -\dfrac{w_R}{d_{bb}} = -\dfrac{1}{4L^3}(x^3 - 3Lx^2 + 2L^2 x)$



Chapter 2

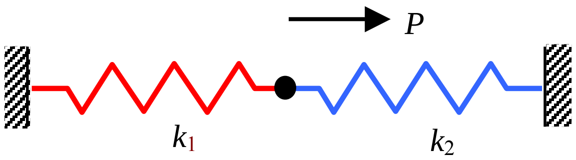
Slope Deflection Method



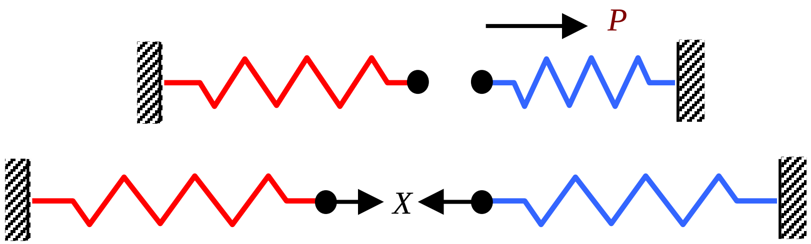


2.0 Comparison of Flexibility Method and Stiffness Method

Flexibility Method



- **Remove redundancy (Equilibrium)**

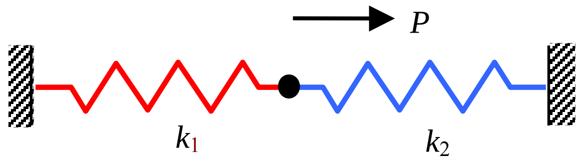


- **Compatibility**

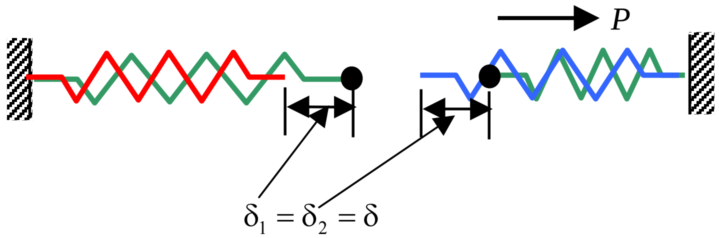
$$\delta_1 = \delta_2$$

$$\frac{X}{k_1} = \frac{P - X}{k_2} \rightarrow X = \frac{k_1}{k_1 + k_2} P$$

Stiffness Method



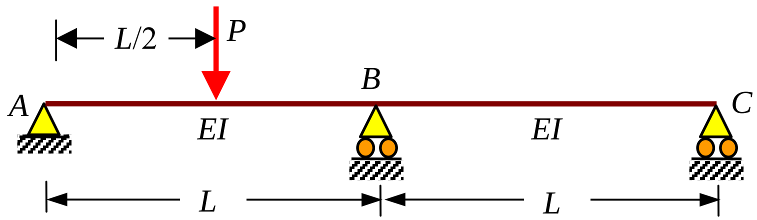
- **Compatibility**



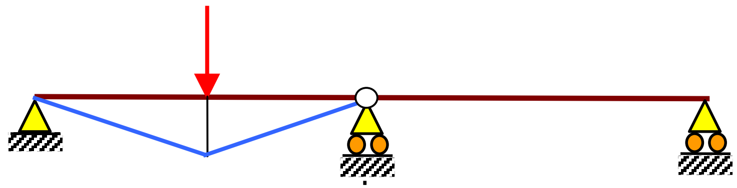
- **Equilibrium**

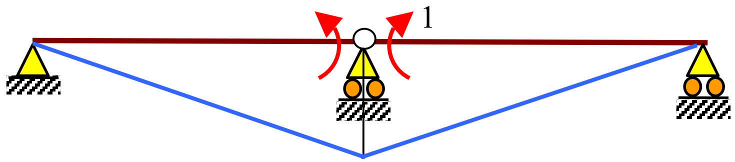
$$k_1\delta + k_2\delta = P \rightarrow \delta = \frac{P}{k_1 + k_2}$$

Flexibility Method



- **Remove redundancy (Equilibrium)**



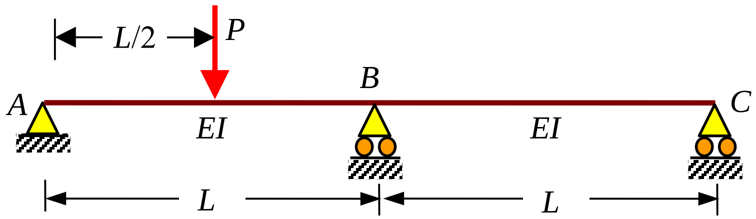


- **Compatibility**

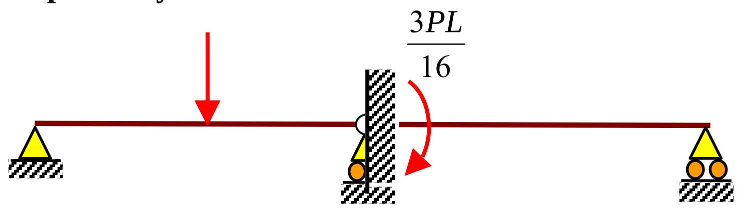
$$\delta_{B0} = \frac{L}{6EI}(1+\frac{1}{2})\frac{PL}{4}\times 1 = \frac{PL^2}{16EI}\ ,\quad \delta_{BB} = \frac{2L}{3EI}$$

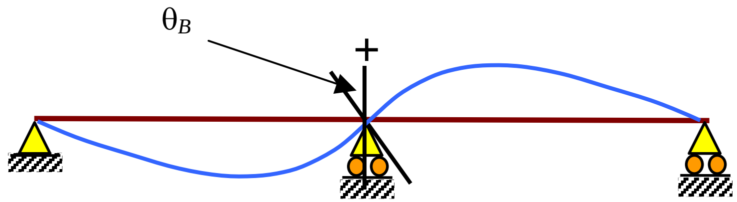
$$\delta_{B0} + \delta_{BB} M_B = 0 \rightarrow M_B = -\frac{\delta_{B0}}{\delta_{BB}} = -\frac{3PL}{32}$$

Stiffness Method



- **Compatibility**





$$\theta_{BA} = \theta_{BC} = \theta_B$$

- **Equilibrium**

$$M_{BA}^{f} = -\frac{3PL}{16},\ M_{BC}^{f} = 0,\ M_{BA}^{B} = M_{BC}^{B} = \frac{3EI}{L}\theta_B$$

$$\sum M_B = M_{BA}^{f} + M_{BC}^{f} + M_{BA}^{B} + M_{BC}^{B} = 0 \rightarrow$$

$$-\frac{3PL}{16} + \frac{6EI}{L}\theta_B = 0 \rightarrow \theta_B = \frac{PL^2}{32EI}$$

Flexibility Method

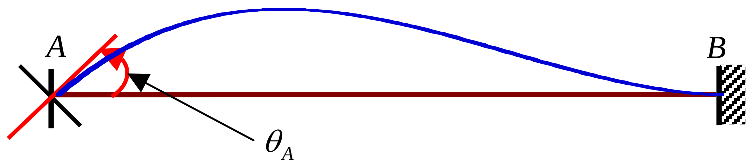
1. Release all redundancies.
2. Calculate displacements induced by external loads at the released redundancies.
3. Apply unit loads and calculate displacements at the released redundancies.
4. Construct the flexibility equation by superposing the displacement based on the compatibility conditions.
5. Solve the flexibility equation.
6. Calculate reactions and other quantities as needed.

Stiffness Method

1. Fix all Degrees of Freedom.
2. Calculate fixed end forces induced by external loads at the fixed DOF.
3. Apply unit displacements and calculate member end forces at the DOFs.
4. Construct the stiffness equation by superposing the member end forces based on the equilibrium equations.
5. Solve the stiffness equation.
6. Calculate reactions and other quantities as needed.

2.1 Analysis of Fundamental System

2.1.1 End Rotation



- **Flexibility Method**

i) $\theta_B = 0$

$$\left.\begin{aligned} \frac{L}{3EI}M_A + \frac{L}{6EI}M_B &= -\theta_A \\ \frac{L}{6EI}M_A + \frac{L}{3EI}M_B &= 0 \end{aligned}\right. \rightarrow M_A = -\frac{4EI}{L}\theta_A, \quad M_B = \frac{2EI}{L}\theta_A$$

ii) $\theta_A = 0$

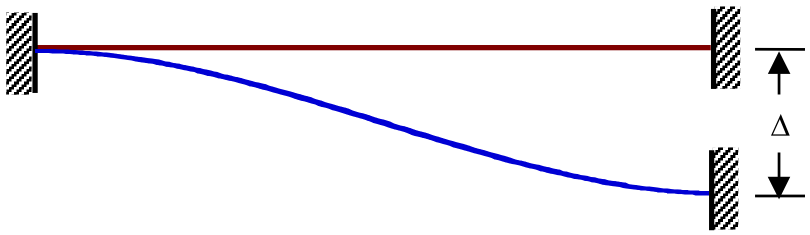
$$M_A = -\frac{2EI}{L}\theta_B, \quad M_B = \frac{4EI}{L}\theta_B$$

- **Sign Convention for M :Counterclockwise "+"**

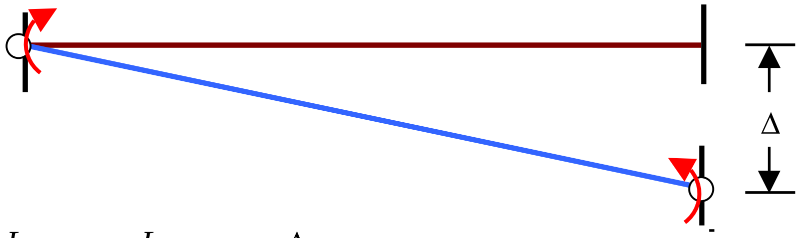
- $\theta_A \neq 0, \quad \theta_B \neq 0$

$$M_A = \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B$$
$$M_B = \frac{2EI}{L}\theta_A + \frac{4EI}{L}\theta_B$$

2.1.2 Relative motion of joints



- **Flexibility Method**



$$\frac{L}{3EI}M_A + \frac{L}{6EI}M_B = -\frac{\Delta}{L}$$
$$\frac{L}{6EI}M_A + \frac{L}{3EI}M_B = \frac{\Delta}{L} \rightarrow M_A = -\frac{6EI}{L}\frac{\Delta}{L},\quad M_B = \frac{6EI}{L}\frac{\Delta}{L}$$

or in the new sign convention : $M_A = \frac{6EI}{L}\frac{\Delta}{L},\quad M_B = \frac{6EI}{L}\frac{\Delta}{L}$

- **Final Slope-Deflection Equation**

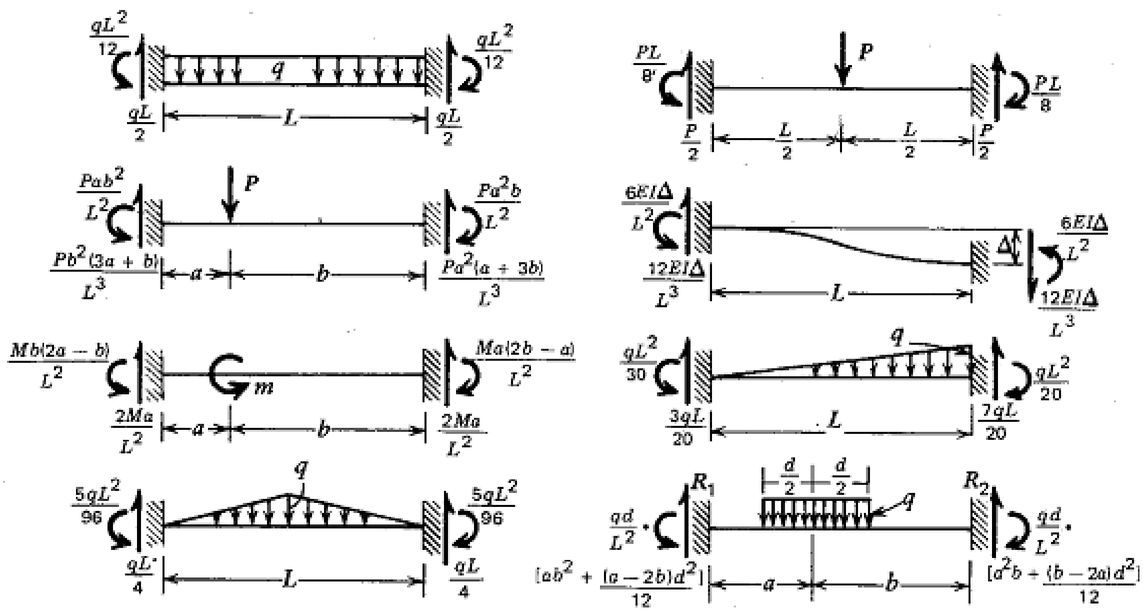
$$M_A = \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B + \frac{6EI}{L}\frac{\Delta}{L}$$
$$M_B = \frac{2EI}{L}\theta_A + \frac{4EI}{L}\theta_B + \frac{6EI}{L}\frac{\Delta}{L}$$

- **In Case an One End is Hinged**

$$M_A = \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B + \frac{6EI}{L}\frac{\Delta}{L} = 0 \rightarrow \frac{2EI}{L}\theta_A = -\frac{EI}{L}\theta_B - \frac{3EI}{L}\frac{\Delta}{L}$$
$$M_B = \frac{2EI}{L}\theta_A + \frac{4EI}{L}\theta_B + \frac{6EI}{L}\frac{\Delta}{L} = \frac{3EI}{L}\theta_B + \frac{3EI}{L}\frac{\Delta}{L}$$

2.1.3 Fixed End Force

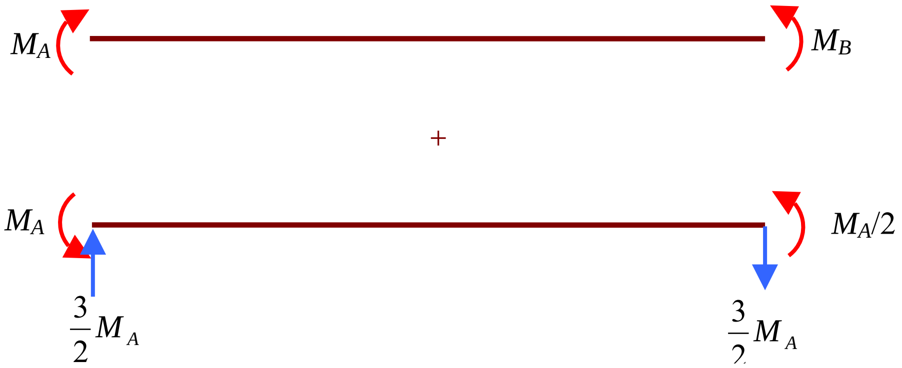
- **Both Ends Fixed**



Sign convention: All loads and reactive forces are positive in the directions shown.

$$R_1 = \frac{qd}{L^3}\left[(2a + L)b^2 + \left(\frac{a-b}{4}\right)d^2\right]$$
$$R_2 = \frac{qd}{L^3}\left[(2b + L)a^2 - \left(\frac{a-b}{4}\right)d^2\right]$$

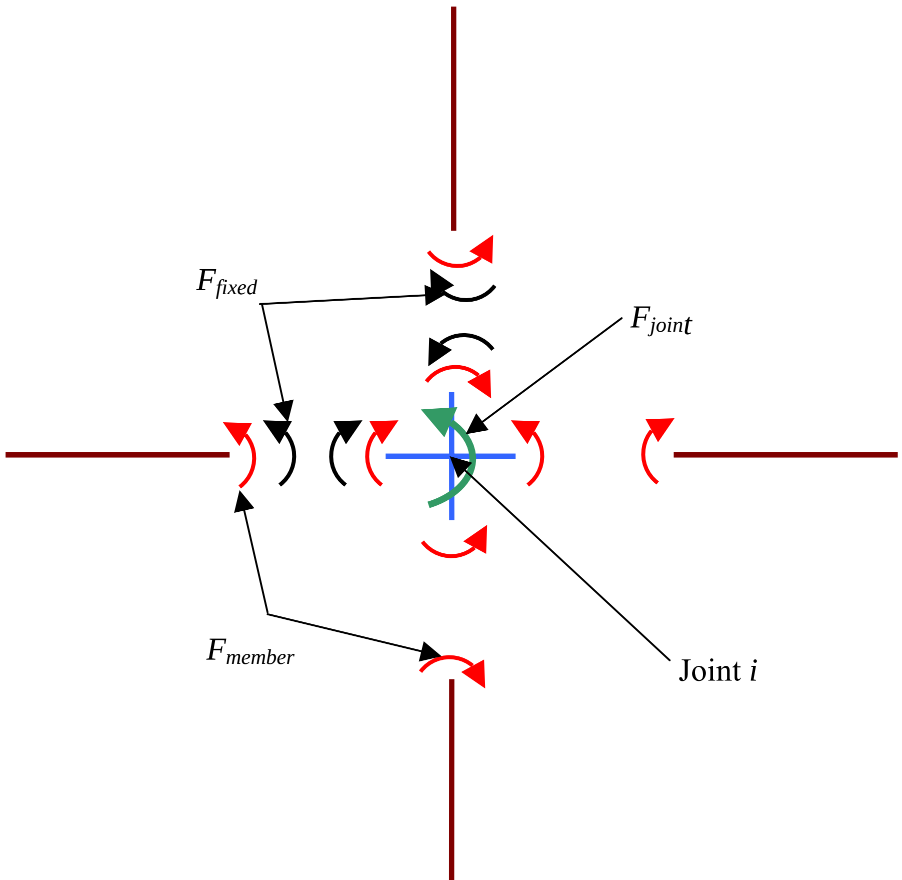
- **One End Hinged**



- **Ex.: Uniform load case with a hinged left end**

$$M_B^f = -\frac{qL^2}{12} - \frac{qL^2}{24} = -\frac{3qL^2}{24} = -\frac{qL^2}{8}\ , \quad M_A^f = 0$$

2.1.4 Joint Equilibrium



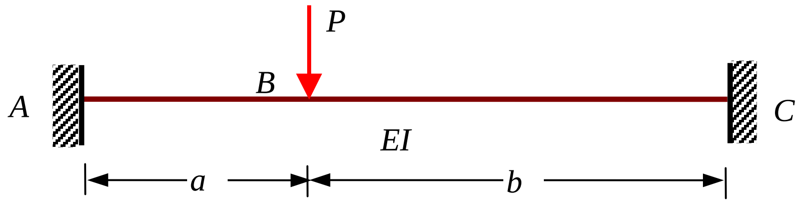
$$-\sum F_{fixed} - \sum F_{member} + \sum F_{joint} = 0$$

or

$$\boxed{\sum F_{fixed} + \sum F_{member} = \sum F_{joint}}$$

2.2 Analysis of Beams

2.2.1 A Fixed-fixed End Beam

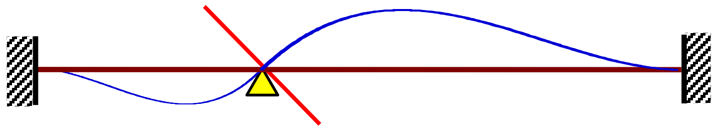


- **DOF :** θ_B **,** Δ_B

- **Analysis**

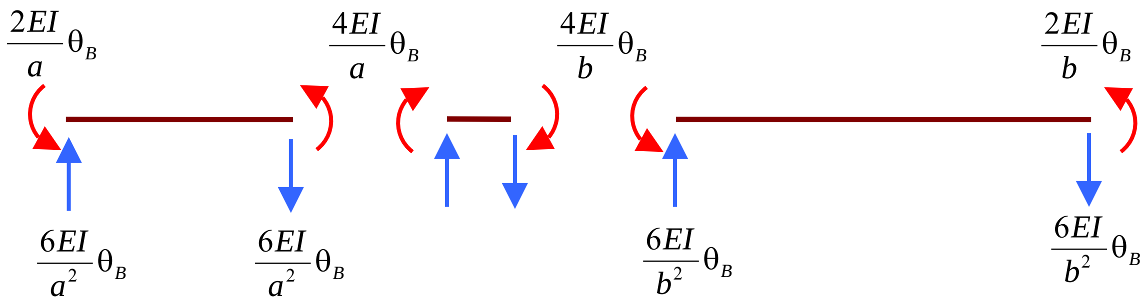
i) All fixed : No fixed end forces

ii) $\theta_B \neq 0$, $\Delta_B = 0$

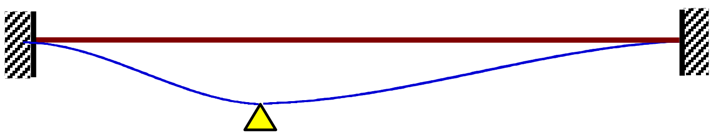


$$M_{AB}^1 = \frac{2EI}{a}\theta_B \ , \ M_{BA}^1 = \frac{4EI}{a}\theta_B \ , \ M_{BC}^1 = \frac{4EI}{b}\theta_B \ , \ M_{CB}^1 = \frac{2EI}{b}\theta_B$$

$$V_{BA}^1 = -V_{AB}^1 = \frac{6EI}{a^2}\theta_B \ , \ -V_{BC}^1 = V_{CB}^1 = \frac{6EI}{b^2}\theta_B$$

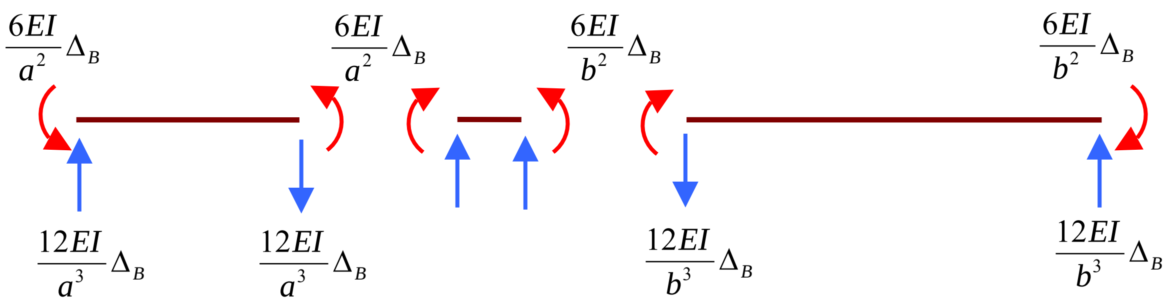


iii) $\theta_B = 0$, $\Delta_B \neq 0$



$$M_{AB}^2 = \frac{6EI}{a^2}\Delta_B \ , \ M_{BA}^2 = \frac{6EI}{a^2}\Delta_B \ , \ M_{BC}^2 = -\frac{6EI}{b^2}\Delta_B \quad M_{CB}^2 = -\frac{6EI}{b^2}\Delta_B$$

$$V_{BA}^2 = -V_{AB}^2 = \frac{12EI}{a^3}\Delta_B \ , \ V_{BC}^2 = -V_{CB}^2 = \frac{12EI}{b^2}\Delta_B$$



- **Construct the Stiffness Equation**

$$\sum M_B^i = 0 \rightarrow M_{BA}^1 + M_{BC}^1 + M_{BA}^2 + M_{BC}^2 = 0 \rightarrow \; 4EI(\frac{1}{a}+\frac{1}{b})\theta_B + 6EI(\frac{1}{a^2}-\frac{1}{b^2})\Delta_B = 0$$

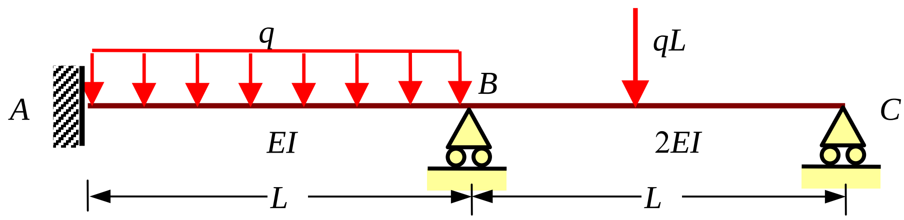
$$\sum V_B^i = P \rightarrow V_{BA}^1 + V_{BC}^1 + V_{BA}^2 + V_{BC}^2 = P \rightarrow \qquad 6EI(\frac{1}{a^2}-\frac{1}{b^2})\theta_B + 12EI(\frac{1}{a^3}+\frac{1}{b^3})\Delta_B = P$$

$$\theta_B = -\frac{(b-a)a^2b^2}{2EIl^3}P \;, \quad \Delta_B = \frac{a^3b^3}{3EIl^3}P$$

$$M_{AB} = M_{AB}^1 + M_{AB}^2 = \frac{2EI}{a}\theta_B + \frac{6EI}{a^2}\Delta_B = \frac{ab^2}{l^2}P \,,$$

$$M_{CB} = M_{CB}^1 + M_{CB}^2 = \frac{2EI}{b}\theta_B - \frac{6EI}{b^2}\Delta_B = -\frac{a^2b}{l^2}P$$

2.2.2 Analysis of a Two-span Continuous Beam (Approach I)



- **DOF :** θ_B , θ_C

- **Analysis**

i) Fix all DOFs and Calculate FEM.

$$M_{AB}^f = \frac{qL^2}{12} \,, \quad M_{BA}^f = -\frac{qL^2}{12} \,, \quad M_{BC}^f = \frac{qL^2}{8} \,, \quad M_{CB}^f = -\frac{qL^2}{8}$$

ii) $\theta_B \neq 0$, $\theta_C = 0$

$$M_{AB}^1 = \frac{2EI}{L}\theta_B \,, \quad M_{BA}^1 = \frac{4EI}{L}\theta_B \,, \quad M_{BC}^1 = \frac{8EI}{L}\theta_B \,, \quad M_{CB}^1 = \frac{4EI}{L}\theta_B$$

iii) $\theta_B = 0$, $\theta_C \neq 0$

$$M_{BC}^2 = \frac{4EI}{L}\theta_C \,, \quad M_{CB}^2 = \frac{8EI}{L}\theta_C$$

- **Construct the Stiffness Equation**

$$\sum M_B^i = 0 \rightarrow M_{BA}^f + M_{BC}^f + M_{BA}^1 + M_{BC}^1 + M_{BC}^2 = 0 \rightarrow \quad \frac{qL^2}{24} + 12\frac{EI}{L}\theta_B + 4\frac{EI}{L}\theta_C = 0$$

$$\sum M_C^i = 0 \rightarrow M_{CB}^f + M_{CB}^1 + M_{CB}^2 = 0 \rightarrow \qquad -\frac{qL^2}{8L} + 4\frac{EI}{L}\theta_B + 8\frac{EI}{L}\theta_C = 0$$

$$\theta_B = -\frac{qL^3}{96EI} \ , \ \theta_C = \frac{qL^3}{48EI}$$

- **Member End Forces**

$$M_{AB} = M_{AB}^f + M_{AB}^1 = \frac{qL^2}{12} + \frac{2EI}{L}\theta_B = \frac{3}{48}qL^2$$

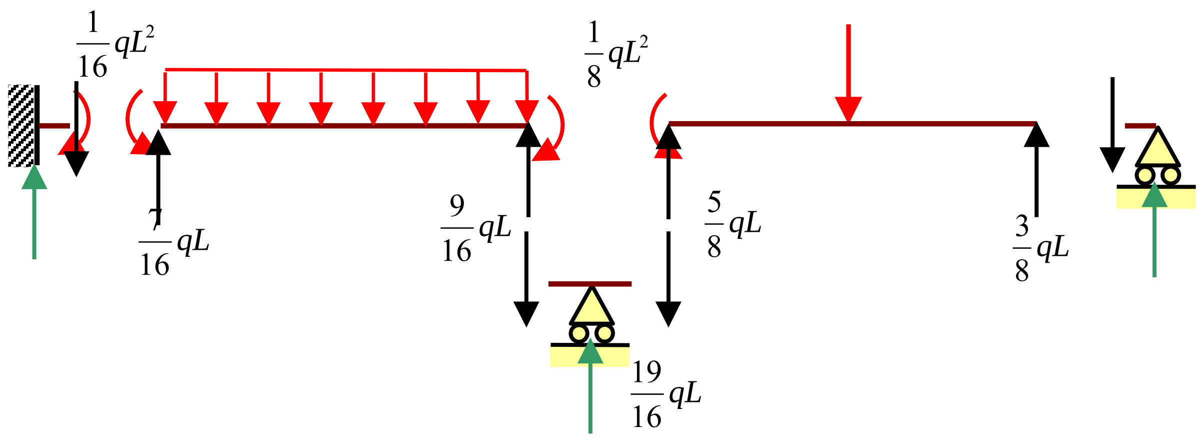
$$M_{BA} = M_{BA}^f + M_{BA}^1 = -\frac{qL^2}{12} + \frac{4EI}{L}\theta_B = -\frac{1}{8}qL^2$$

$$M_{BC} = M_{BC}^f + M_{BC}^1 + M_{BC}^2 = \frac{qL^2}{8} + \frac{8EI}{L}\theta_B + \frac{4EI}{L}\theta_C = \frac{1}{8}qL^2$$

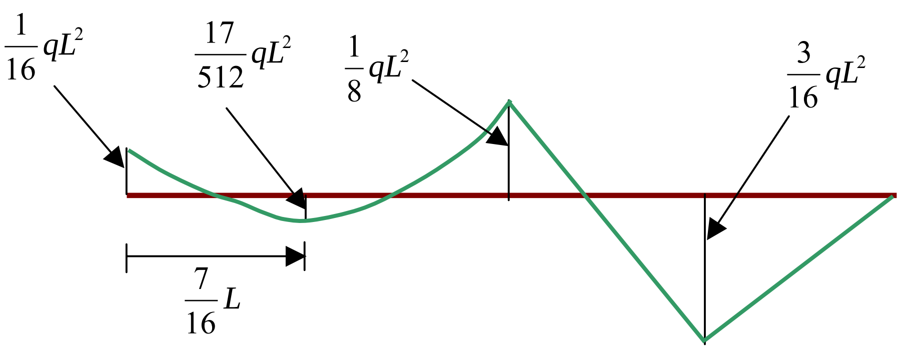
$$M_{CB} = M_{CB}^f + M_{CB}^1 + M_{CB}^2 = -\frac{qL^2}{8} + \frac{4EI}{L}\theta_B + \frac{8EI}{L}\theta_C = 0$$

- **Various Diagram**

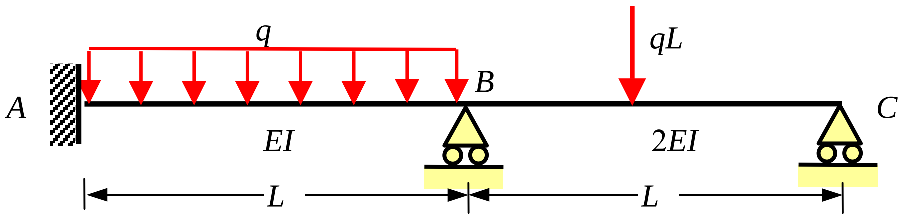
- Freebody Diagram



- Moment Diagram



2.2.3 Analysis of a Two-span Continuous Beam (Approach II)



- **DOF :** θ_B

- **Analysis**

i) Fix all DOFs and Calculate FEM.

$$M_{AB}^{f} = \frac{qL^2}{12}\ ,\quad M_{BA}^{f} = -\frac{qL^2}{12},\quad M_{BC}^{f} = \frac{qL^2}{8} + \frac{1}{2}\frac{qL^2}{8} = \frac{3qL^2}{16}$$

ii) $\theta_B \neq 0$

$$M_{AB}^{1} = \frac{2EI}{L}\theta_B\ ,\quad M_{BA}^{1} = \frac{4EI}{L}\theta_B\ ,\quad M_{BC}^{1} = \frac{6EI}{L}\theta_B$$

- **Construct Stiffness Equation**

$$\sum M_B = 0 \rightarrow M_{BA}^{f} + M_{BC}^{f} + M_{BA}^{1} + M_{BC}^{1} = 0$$

$$-\frac{qL^2}{12} + \frac{3qL^2}{16} + 4\frac{EI}{L}\theta_B + 6\frac{EI}{L}\theta_B = 0 \rightarrow \theta_B = -\frac{qL^2}{96EI}$$

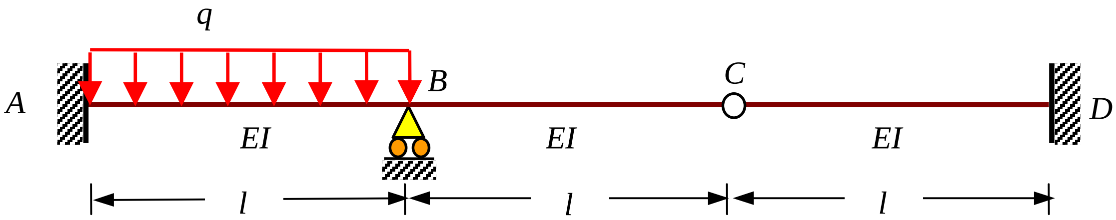
- **Member End Forces**

$$M_{AB} = M_{AB}^{f} + M_{AB}^{1} = \frac{qL^2}{12} + \frac{2EI}{L}\theta_B = \frac{3}{48}qL^2$$

$$M_{BA} = M_{BA}^{f} + M_{BA}^{1} = -\frac{qL^2}{12} + \frac{4EI}{L}\theta_B = -\frac{1}{8}qL^2$$

$$M_{BC} = M_{BC}^{f} + M_{BC}^{1} = \frac{3qL^2}{16} + \frac{6EI}{L}\theta_B = \frac{1}{8}qL^2$$

2.2.4 Analysis of a Beam with an Internal Hinge (**4 *DOFs System***)

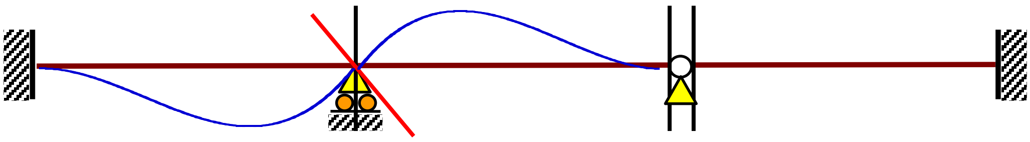


- **DOF :** θ_B **,** θ_C^L **,** θ_C^R **,** Δ_C
- **Analysis**

i) All fixed

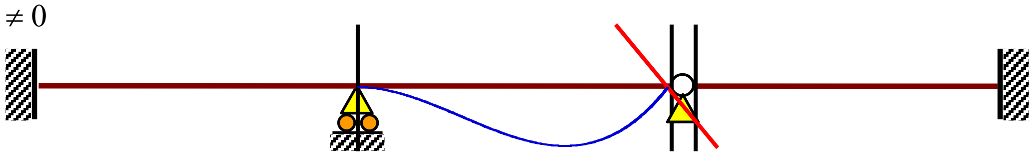
$$M_{AB}^{f} = \frac{ql^2}{12}\ ,\ \ M_{BA}^{f} = -\frac{ql^2}{12}$$

ii) $\theta_B \neq 0$



$$M_{AB}^{1} = \frac{2EI}{l}\theta_B\ ,\ \ M_{BA}^{1} = M_{BC}^{1} = \frac{4EI}{l}\theta_B\ ,\ \ M_{CB}^{1} = \frac{2EI}{l}\theta_B\ ,\ \ V_{CB}^{1} = \frac{6EI}{l^2}\theta_B$$

iii) $\theta_C^L \neq 0$

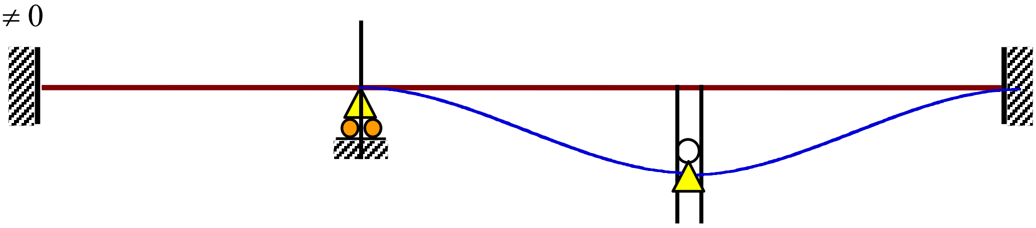


$$M_{BC}^{2} = \frac{2EI}{l}\theta_C^L\ ,\ \ M_{CB}^{2} = \frac{4EI}{l}\theta_C^L,\ \ V_{CB}^{2} = \frac{6EI}{l^2}\theta_C^L$$

iv) $\theta_C^R \neq 0$

$$M_{CD}^{3} = \frac{4EI}{l}\theta_C^R\ ,\ \ M_{DC}^{3} = \frac{2EI}{l}\theta_C^R,\ \ V_{CD}^{3} = -\frac{6EI}{l^2}\theta_C^L$$

v) $\Delta_C \neq 0$



$$M_{BC}^{4} = M_{CB}^{4} = \frac{6EI}{l^2}\Delta_C\ ,\ \ M_{CD}^{4} = M_{DC}^{4} = -\frac{6EI}{l^2}\Delta_C,\ \ V_{CD}^{4} = V_{CB}^{4} = \frac{12EI}{l^2}\Delta_C$$

- **Construct Stiffness Equation**

$$
\begin{aligned}
&\sum_i M_1^i = 0 \rightarrow -\frac{ql^2}{12} + 8\frac{EI}{l}\theta_B + 2\frac{EI}{l}\theta_C^L + \quad 0 \quad + 6\frac{EI}{l^2}\Delta_C = 0 \\
&\sum_i M_2^i = 0 \rightarrow \qquad 2\frac{EI}{l}\theta_B + 4\frac{EI}{l}\theta_C^L + \quad 0 \quad + 6\frac{EI}{l^2}\Delta_C = 0 \\
&\sum_i M_3^i = 0 \rightarrow \qquad\qquad 4\frac{EI}{l}\theta_C^R - 6\frac{EI}{l^2}\Delta_C = 0 \\
&\sum_i V_4^i = 0 \rightarrow \qquad 6\frac{EI}{l^2}\theta_B + 6\frac{EI}{l^2}\theta_C^L - 6\frac{EI}{l^2}\theta_C^R + 24\frac{EI}{l^3}\Delta_C = 0
\end{aligned}
$$

- **Elimination of θ_C^L and θ_C^R**

- 2nd and 3rd equation

$$2\frac{EI}{l}\theta_C^L = -(\frac{EI}{l}\theta_B + 3\frac{EI}{l^2}\Delta_C) \ , \ 2\frac{EI}{l}\theta_C^R = 3\frac{EI}{l^2}\Delta_C$$

- 1st equation

$$-\frac{ql^2}{12} + 8\frac{EI}{l}\theta_B + 2\frac{EI}{l}\theta_C^L + 6\frac{EI}{l^2}\Delta_C =$$

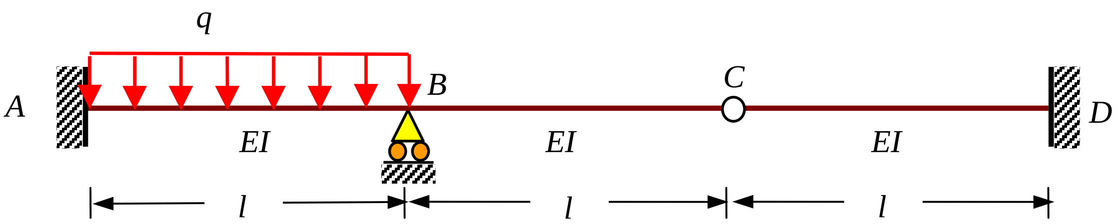
$$-\frac{ql^2}{12} + 8\frac{EI}{l}\theta_B - (\frac{EI}{l}\theta_B + 3\frac{EI}{l^2}\Delta_C) + 6\frac{EI}{l^2}\Delta_C = 0 \rightarrow \boxed{-\frac{ql^2}{12} + 7\frac{EI}{l}\theta_B + 3\frac{EI}{l^2}\Delta_C = 0}$$

- 4th equation

$$6\frac{EI}{l^2}\theta_B + 6\frac{EI}{l^2}\theta_C^L - 6\frac{EI}{l^2}\theta_C^R + 24\frac{EI}{l^3}\Delta_C =$$

$$6\frac{EI}{l^2}\theta_B - 3(\frac{EI}{l^2}\theta_B + 3\frac{EI}{l^3}\Delta_C) - 3(3\frac{EI}{l^3}\Delta_C) + 24\frac{EI}{l^3}\Delta_C = 0 \rightarrow \boxed{3\frac{EI}{l^2}\theta_B + 6\frac{EI}{l^3}\Delta_C = 0}$$

2.2.5 *Analysis of a Beam with an Internal Hinge (2 DOFs System)*



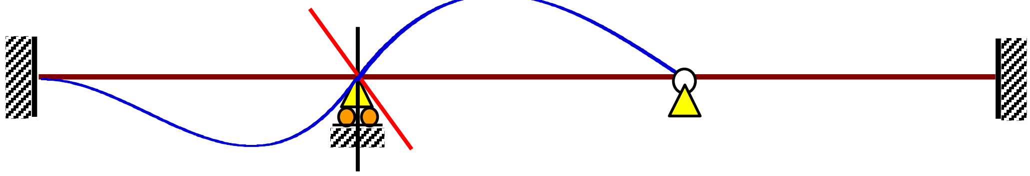
- **DOF : θ_B , Δ_C**

- **Analysis**

i) All fixed

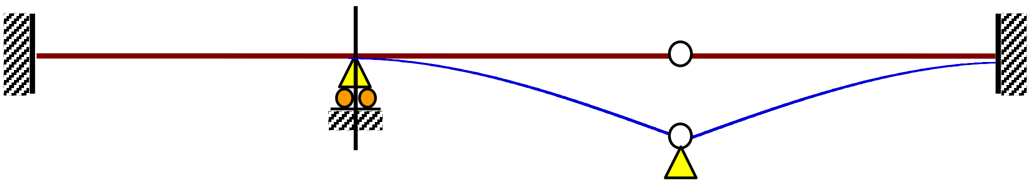
$$M_{AB}^f = \frac{ql^2}{12} \ , \ M_{BA}^f = -\frac{ql^2}{12}$$

ii) $\theta_B \neq 0$



$$M^1_{AB} = \frac{2EI}{l}\theta_B \;,\; M^1_{BA} = \frac{4EI}{l}\theta_B \;,\; M^1_{BC} = \frac{3EI}{l}\theta_B \;,\; V^1_{BC} = -V^1_{CB} = -\frac{3EI}{l^2}\theta_B$$

iii) $\Delta_C \neq 0$



$$M^2_{BC} = -M^2_{DC} = \frac{3EI}{l^2}\Delta_C \;,\; V^2_{BC} = -V^2_{CB} = -\frac{3EI}{l^2}\Delta_C \;,\; V^2_{CD} = -V^2_{DC} = \frac{3EI}{l^2}\Delta_C$$

- **Construct the Stiffness Equation**

$$\sum_i M^i_1 = 0 \rightarrow -\frac{ql^2}{12} + 7\frac{EI}{l}\theta_B + 3\frac{EI}{l^2}\Delta_C = 0$$

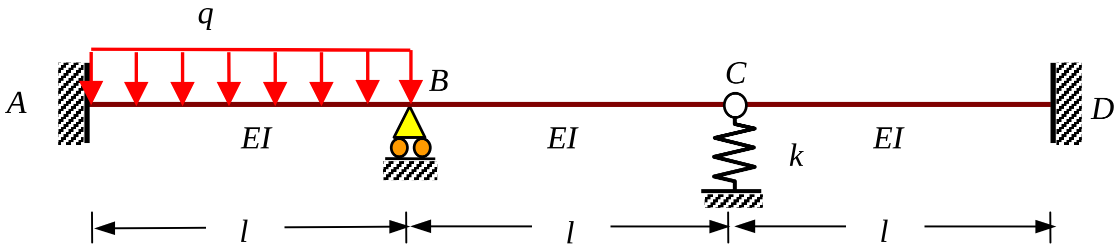
$$\sum_i V^i_2 = 0 \rightarrow \qquad 3\frac{EI}{l^2}\theta_B + 6\frac{EI}{l^3}\Delta_C = 0$$

$$\theta_B = \frac{ql^3}{66EI} \;,\; \Delta_C = -\frac{ql^4}{132EI}$$

$$\frac{2EI}{l}\theta^R_C = -\frac{3EI}{l}\left(-\frac{\Delta_C}{l}\right) \rightarrow \theta^R_C = \frac{3}{2}\frac{\Delta_C}{l} = -\frac{3}{264}\frac{ql^3}{EI}$$

$$\theta^L_C = -\frac{1}{2}\theta_B - \frac{3}{2}\frac{\Delta}{L} = -\frac{ql^3}{132EI} + \frac{3}{2}\frac{ql^3}{132EI} = \frac{ql^3}{264EI}$$

2.2.6 Beam with a Spring Support

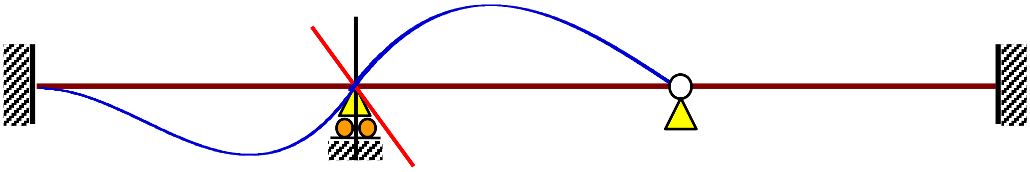


- **Analysis**

i) All fixed

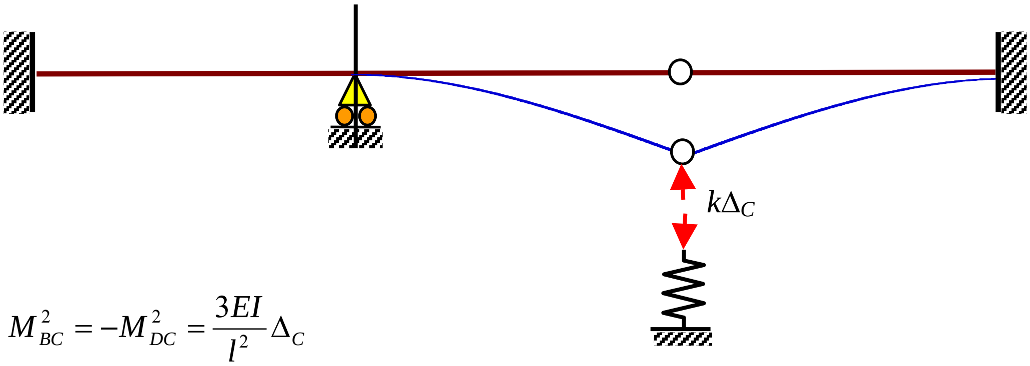
$$M^f_{AB} = \frac{ql^2}{12} \;,\; M^f_{BA} = -\frac{ql^2}{12}$$

ii) $\theta_B \neq 0$



$$M_{AB}^1 = \frac{2EI}{l}\theta_B \ , \ M_{BA}^1 = \frac{4EI}{l}\theta_B \ , \ M_{BC}^1 = \frac{3EI}{l}\theta_B \ , \ -V_{BC}^1 = V_{CB}^1 = \frac{3EI}{l^2}\theta_B$$

iii) $\Delta_C \neq 0$



$$M_{BC}^2 = -M_{DC}^2 = \frac{3EI}{l^2}\Delta_C$$

$$V_{BC}^2 = -V_{CB}^2 = -\frac{3EI}{l^2}\Delta_C \ , \ V_{CD}^2 = -V_{DC}^2 = \frac{3EI}{l^2}\Delta_C \ , \ V_S^2 = k\Delta_C$$

- **Construct the Stiffness Equation**

$$\sum_i M_1^i = 0 \rightarrow -\frac{ql^2}{12} + 7\frac{EI}{l}\theta_B \qquad + 3\frac{EI}{l^2}\Delta_C = 0$$

$$\sum_i V_2^i = 0 \rightarrow \qquad 3\frac{EI}{l^2}\theta_B + (6\frac{EI}{l^3} + k)\Delta_C = 0$$

$$\theta_B = \frac{1+\alpha}{1+14\alpha/11}\frac{ql^3}{66EI} \ , \ \Delta_C = -\frac{1}{(1+14\alpha/11)}\frac{ql^4}{132EI} \quad \text{where} \ \ k = \alpha\frac{6EI}{l^3}$$

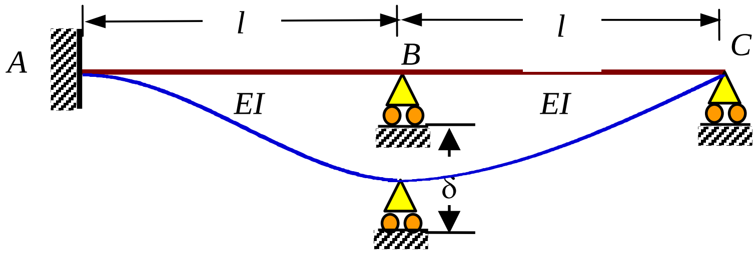
- $\alpha \rightarrow 0$

$$\theta_B = \frac{ql^3}{66EI} \ , \ \Delta_C = -\frac{ql^4}{132EI}$$

- $\alpha \rightarrow \infty$

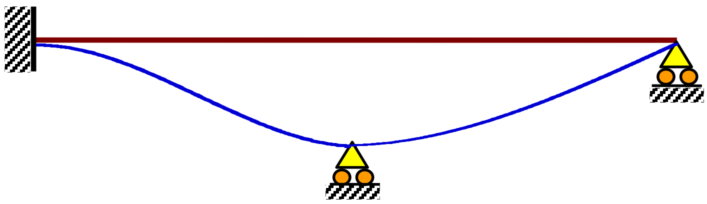
$$\theta_B = \frac{ql^3}{84EI} \ , \ \Delta_C = 0$$

2.2.7 Support Settlement



- **DOF :** θ_B

- **Analysis**

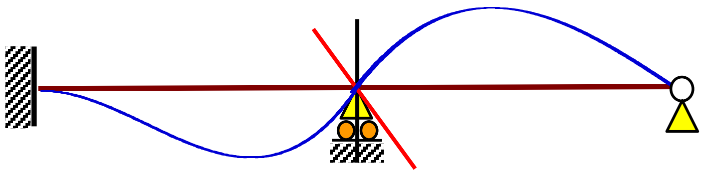


i) All fixed

$$M_{BA}^f = \frac{6EI}{l}\frac{\delta}{l} \ , \ M_{BC}^f = -\frac{3EI}{l}\frac{\delta}{l}$$

ii) $\theta_B \neq 0$

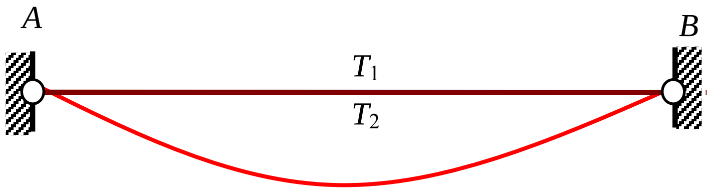
$$M_{BA}^1 = \frac{4EI}{l}\theta_B \ , \ M_{BC}^1 = \frac{3EI}{l}\theta_B$$



- **Construct the Equilibrium Equation**

$$\sum_i M_1^i = 0 \rightarrow 6\frac{EI}{l}\frac{\delta}{l} - 3\frac{EI}{l}\frac{\delta}{l} + 4\frac{EI}{l}\theta_B + 3\frac{EI}{l}\theta_B = 0 \rightarrow \theta_B = -\frac{3}{7}\frac{\delta}{l}$$

2.2.8 Temperature Change



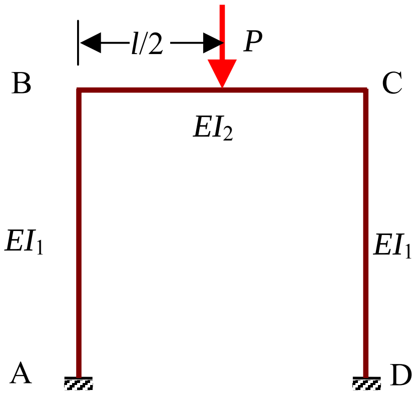
$$\theta_A = \frac{\alpha(T_2 - T_1)}{2h}l \ , \ \theta_B = -\frac{\alpha(T_2 - T_1)}{2h}l$$

- **Fixed End Moment**

$$M_A = \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B = \frac{\alpha(T_2 - T_1)}{h}EI$$
$$M_B = \frac{2EI}{L}\theta_A + \frac{4EI}{L}\theta_B = -\frac{\alpha(T_2 - T_1)}{h}EI$$

2.3 Analysis of Frames

2.3.1 A Portal Frame without Sidesway

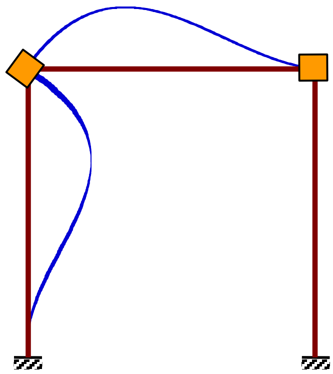


- **DOF :** θ_B **,** θ_C

- **Analysis**

i) All fixed

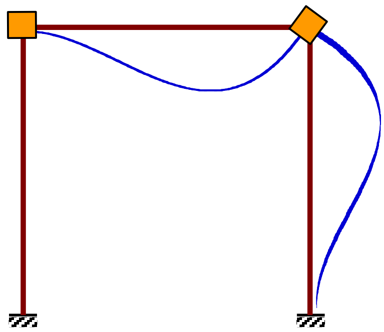
$$M_{BC}^{0} = \frac{Pl}{8} \;,\; M_{CB}^{0} = -\frac{Pl}{8}$$



ii) $\theta_B \neq 0$

$$M_{AB}^{1} = \frac{2EI_1}{l}\theta_B \;,\; M_{BA}^{1} = \frac{4EI_1}{l}\theta_B$$

$$M_{BC}^{1} = \frac{4EI_2}{l}\theta_B \;,\; M_{CB}^{1} = \frac{2EI_2}{l}\theta_B$$



iii) $\theta_C \neq 0$

$$M_{BC}^{2} = \frac{2EI_2}{l}\theta_C \;,\; M_{CB}^{2} = \frac{4EI_2}{l}\theta_C$$

$$M_{CD}^{2} = \frac{4EI_1}{l}\theta_C \;,\; M_{DC}^{2} = \frac{2EI_1}{l}\theta_C$$

- **Construct the Stiffness Equation**

$$\sum M_B^{i} = 0 \rightarrow \;\; \frac{Pl}{8} + (\frac{4EI_1}{l} + \frac{4EI_2}{l})\theta_B + \frac{2EI_2}{l}\theta_C = 0$$

$$\sum M_C^{i} = 0 \rightarrow -\frac{Pl}{8} + \frac{2EI_2}{l}\theta_B + (\frac{4EI_1}{l} + \frac{4EI_2}{l})\theta_C = 0$$

$$-\theta_B = \theta_C = \frac{1}{4EI_1 + 2EI_2}\frac{Pl^2}{8}$$

- **Member End Forces**

$$M_{AB} = \frac{2EI_1}{l}\theta_B = -\frac{2EI_1}{4EI_1 + 2EI_2}\frac{Pl}{8}$$

$$M_{BA} = \frac{4EI_1}{l}\theta_B = -\frac{4EI_1}{4EI_1 + 2EI_2}\frac{Pl}{8}$$

$$M_{BC} = \frac{Pl}{8} + \frac{4EI_2}{l}\theta_B + \frac{2EI_2}{l}\theta_C = \frac{4EI_1}{4EI_1 + 2EI_2}\frac{Pl}{8}$$

$$M_{CB} = -\frac{Pl}{8} + \frac{2EI_2}{l}\theta_B + \frac{4EI_2}{l}\theta_C = -\frac{4EI_1}{4EI_1 + 2EI_2}\frac{Pl}{8}$$

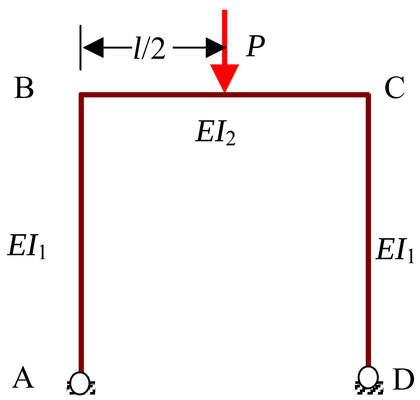
$$M_{CD} = \frac{4EI_1}{l}\theta_C = \frac{4EI_1}{4EI_1 + 2EI_2}\frac{Pl}{8}$$

$$M_{DC} = \frac{2EI_1}{l}\theta_C = \frac{2EI_1}{4EI_1 + 2EI_2}\frac{Pl}{8}$$

- **In case** $EI_1 = EI_2$

$$M_{AB} = -\frac{Pl}{24}\ ,\ M_{BA} = -\frac{Pl}{12}\ ,\ M_{BC} = \frac{Pl}{12}\ ,\ M_{CB} = -\frac{Pl}{12}\ ,\ M_{CD} = \frac{Pl}{12}\ ,\ M_{DC} = \frac{Pl}{24}$$

2.3.2 A Portal Frame without Sidesway – hinged suppoorts

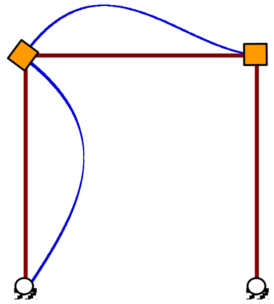


- **DOF :** θ_B **,** θ_C

- **Analysis**

i) All fixed

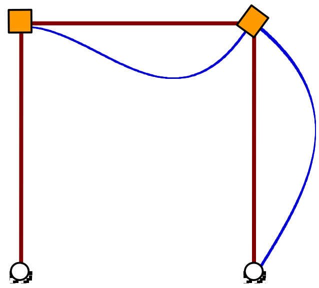
$$M_{BC}^0 = \frac{Pl}{8}\ ,\ M_{CB}^0 = -\frac{Pl}{8}$$



ii) $\theta_B \neq 0$

$$M_{BA}^1 = \frac{3EI_1}{l}\theta_B$$

$$M_{BC}^1 = \frac{4EI_2}{l}\theta_B \ , \quad M_{CB}^1 = \frac{2EI_2}{l}\theta_B$$



iii) $\theta_C \neq 0$

$$M_{BC}^2 = \frac{2EI_2}{l}\theta_C \ , \quad M_{CB}^2 = \frac{4EI_2}{l}\theta_C$$

$$M_{CD}^2 = \frac{3EI_1}{l}\theta_C$$

- **Construct the Stiffness Equation**

$$\sum M_B^i = 0 \rightarrow \quad \frac{Pl}{8} + (\frac{3EI_1}{l} + \frac{4EI_2}{l})\theta_B + \frac{2EI_2}{l}\theta_C = 0$$

$$\sum M_C^i = 0 \rightarrow -\frac{Pl}{8} + \frac{2EI_2}{l}\theta_B + (\frac{3EI_1}{l} + \frac{4EI_2}{l})\theta_C = 0$$

$$-\theta_B = \theta_C = \frac{1}{3EI_1 + 2EI_2}\frac{Pl^2}{8}$$

- **Member End Forces**

$$M_{AB} = 0$$

$$M_{BA} = \frac{3EI_1}{l}\theta_B = -\frac{3EI_1}{3EI_1 + 2EI_2}\frac{Pl}{8}$$

$$M_{BC} = \frac{Pl}{8} + \frac{4EI_2}{l}\theta_B + \frac{2EI_2}{l}\theta_C = \frac{3EI_1}{3EI_1 + 2EI_2}\frac{Pl}{8}$$

$$M_{CB} = -\frac{Pl}{8} + \frac{2EI_2}{l}\theta_B + \frac{4EI_2}{l}\theta_C = -\frac{3EI_1}{3EI_1 + 2EI_2}\frac{Pl}{8}$$

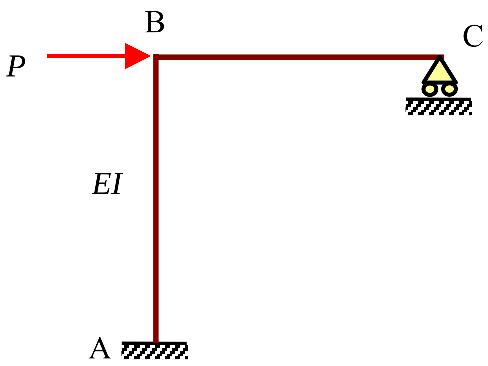
$$M_{CD} = \frac{3EI_1}{l}\theta_C = \frac{3EI_1}{3EI_1 + 2EI_2}\frac{Pl}{8}$$

$$M_{DC} = 0$$

- **In case of** $EI_1 = EI_2$

$$M_{AB} = 0 \ , \quad M_{BA} = -\frac{3}{40}Pl \ , \quad M_{BC} = \frac{3}{40}Pl \ , \quad M_{CB} = -\frac{3}{40}Pl \ , \quad M_{CD} = \frac{3}{40}Pl \ , \quad M_{DC} = 0$$

2.3.3 *A Frame with an horizontal force*



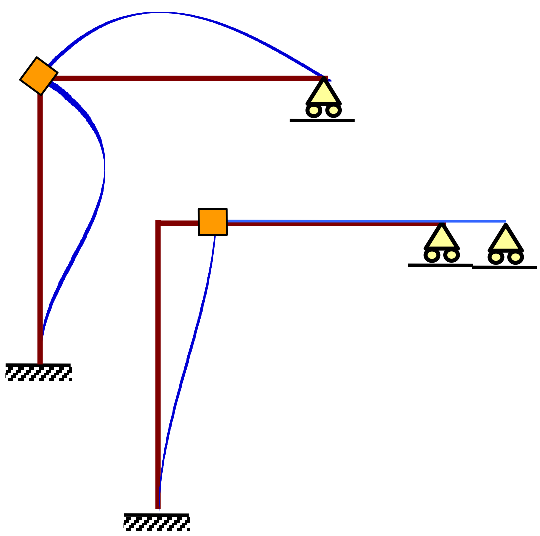
- **DOF :** θ_B **,** Δ
- **Analysis**

i) All fixed : None fixed end moment

ii) $\theta_B \neq 0$

$$M_{AB}^1 = \frac{2EI}{l}\theta_B \ , \ M_{BA}^1 = \frac{4EI}{l}\theta_B$$

$$M_{BC}^1 = \frac{3EI}{l}\theta_B \ , \ V_{BA}^1 = \frac{6EI}{l^2}\theta_B$$



iii) $\Delta \neq 0$

$$M_{AB}^2 = \frac{6EI}{l^2}\Delta \ , \ M_{BA}^2 = \frac{6EI}{l^2}\Delta$$

$$V_{BA}^2 = \frac{12EI}{l^3}\Delta$$

- **Construct the stiffness equation**

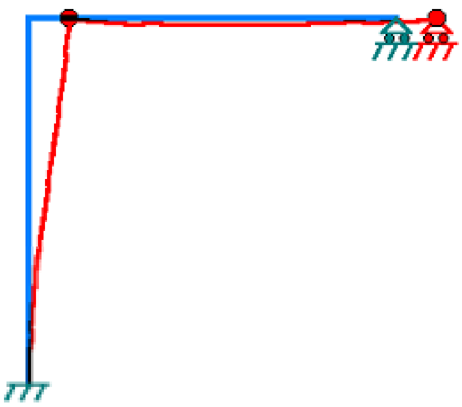
$$\sum M_B^i = 0 \rightarrow \quad (\frac{4EI}{l} + \frac{3EI}{l})\theta_B + \frac{6EI}{l^2}\Delta = 0$$

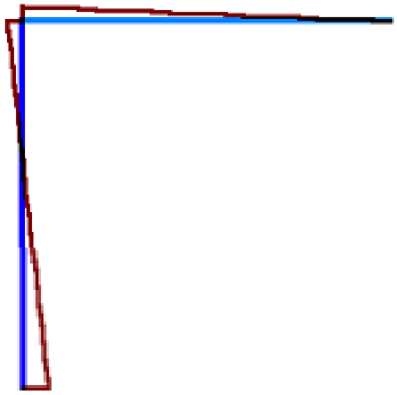
$$\sum V^i = P \rightarrow \quad \frac{6EI}{l^2}\theta_B + \frac{12EI}{l^3}\Delta = P$$

$$\theta_B = -\frac{Pl^2}{8EI}, \Delta = \frac{7Pl^3}{48EI}$$

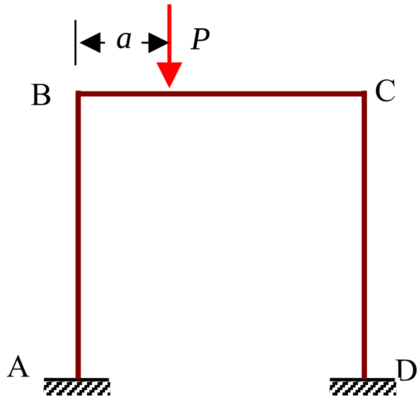
- **Member end forces**

$$M_{AB} = \frac{2EI}{l}\theta_B + \frac{6EI}{l^2}\Delta = \frac{5}{8}Pl \ , \ M_{BA} = \frac{4EI}{l}\theta_B + \frac{6EI}{l^2}\Delta = \frac{3}{8}Pl \ , \ M_{BC} = \frac{3EI}{l}\theta_B = -\frac{3}{8}Pl$$

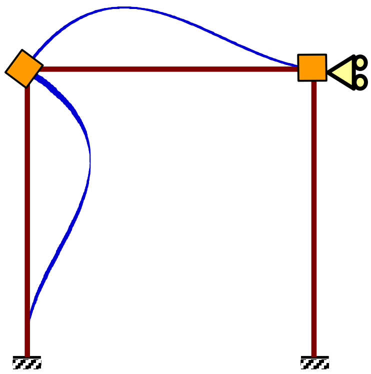




2.3.4 A Portal Frame with an Unsymmetrical Load



- **DOF :** θ_B , θ_C , Δ
- **Analysis**



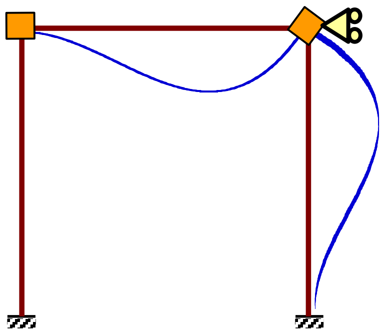
i) All fixed

$$M^0_{BC} = \frac{Pab^2}{l^2} \;,\quad M^0_{CB} = -\frac{Pa^2b}{l^2}$$

ii) $\theta_B \neq 0$

$$M^1_{AB} = \frac{2EI}{l}\theta_B \;,\quad M^1_{BA} = \frac{4EI}{l}\theta_B$$

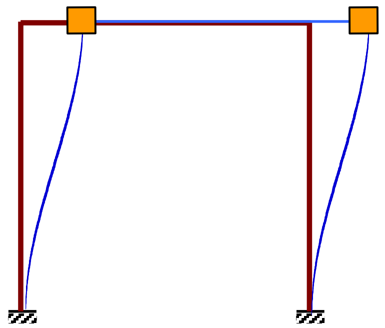
$$M^1_{BC} = \frac{4EI}{l}\theta_B \;,\quad M^1_{CB} = \frac{2EI}{l}\theta_B \;,\quad V^1_{BA} = \frac{6EI}{l^2}\theta_B$$



iii) $\theta_C \neq 0$

$$M^2_{BC} = \frac{2EI}{l}\theta_C \;,\quad M^2_{CB} = \frac{4EI}{l}\theta_C$$

$$M^2_{CD} = \frac{4EI}{l}\theta_C \;,\quad M^2_{DC} = \frac{2EI}{l}\theta_C \;,\quad V^2_{CD} = \frac{6EI}{l^2}\theta_C$$



iv) $\Delta \neq 0$

$$M_{AB}^{3} = M_{BA}^{3} = \frac{6EI}{l^2}\Delta\ , \quad M_{CD}^{3} = M_{DC}^{3} = \frac{6EI}{l^2}\Delta\ ,$$

$$V_{BA}^{3} = V_{CD}^{3} = \frac{12EI}{l^3}\Delta$$

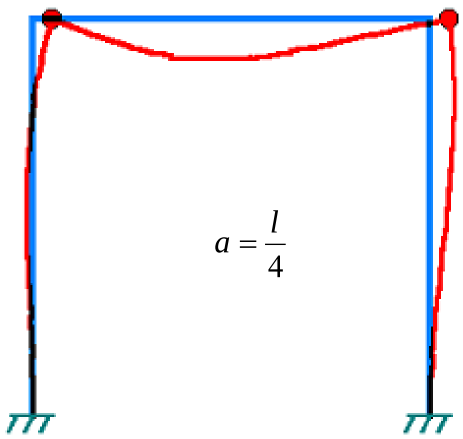
- **Construct the Stiffness Equation**

$$\sum M_B^i = 0 \rightarrow \quad \frac{Pab^2}{l^2} + \frac{8EI}{l}\theta_B + \frac{2EI}{l}\theta_c + \frac{6EI}{l^2}\Delta = 0$$

$$\sum M_C^i = 0 \rightarrow -\frac{Pa^2b}{l^2} + \frac{2EI}{l}\theta_B + \frac{8EI}{l}\theta_c + \frac{6EI}{l^2}\Delta = 0$$

$$\sum V^i = 0 \rightarrow \quad \frac{6EI}{l^2}\theta_B + \frac{6EI}{l^2}\theta_c + \frac{24EI}{l^3}\Delta = 0$$

$$\Delta = -\frac{l}{4}(\theta_B + \theta_C)$$



$$\frac{Pa^2b}{l^2} + \frac{13EI}{2l}\theta_B + \frac{EI}{2l}\theta_c = 0$$

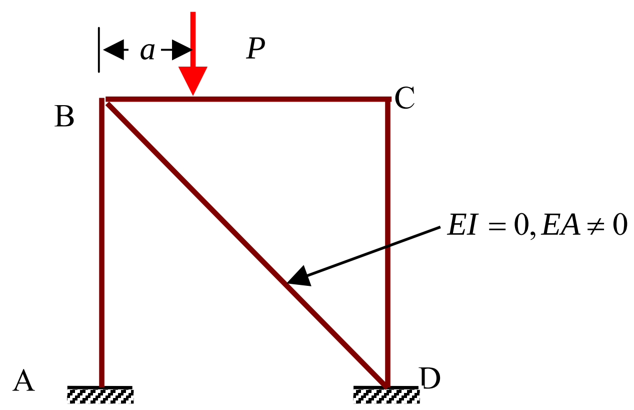
$$-\frac{Pab^2}{l^2} + \frac{EI}{2l}\theta_B + \frac{13EI}{2l}\theta_c = 0$$

$$\theta_B = -\frac{1}{84}\frac{Pab}{EI}\frac{(a+13b)}{l}$$

$$\theta_C = \frac{1}{84}\frac{Pab}{EI}\frac{(13a+b)}{l}$$

$$\Delta = \frac{1}{28}\frac{Pab}{EI}(b-a)$$

2.3.5 A Portal Frame with a Bracing (Vertical Load)



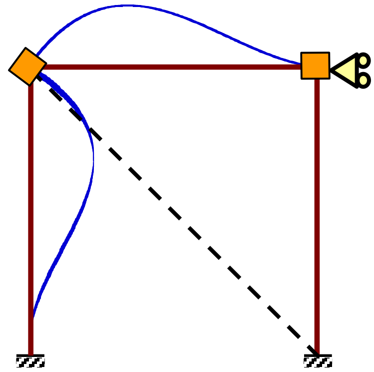
- **DOF :** θ_B , θ_C , Δ

- **Analysis**

i) All fixed

$$M_{BC}^{0} = \frac{Pab^2}{l^2}\ , \quad M_{CB}^{0} = -\frac{Pa^2b}{l^2}$$

ii) $\theta_B \neq 0$

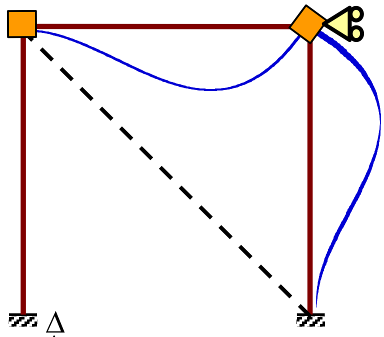


$$M_{AB}^1 = \frac{2EI}{l}\theta_B \ , \ M_{BA}^1 = \frac{4EI}{l}\theta_B$$

$$M_{BC}^1 = \frac{4EI}{l}\theta_B \ , \ M_{CB}^1 = \frac{2EI}{l}\theta_B$$

$$V_{BA}^1 = \frac{6EI}{l^2}\theta_B$$

iii) $\theta_C \neq 0$

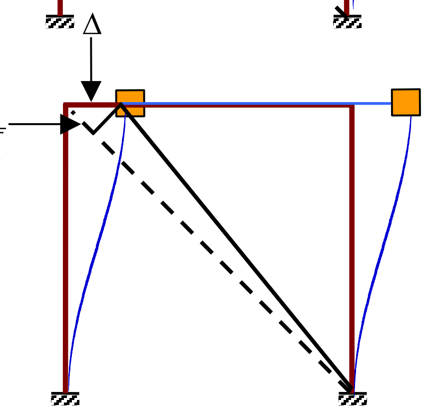


$$M_{BC}^2 = \frac{2EI}{l}\theta_C \ , \ M_{CB}^2 = \frac{4EI}{l}\theta_C$$

$$M_{CD}^2 = \frac{4EI}{l}\theta_C \ , \ M_{DC}^2 = \frac{2EI}{l}\theta_C$$

$$V_{CD}^2 = \frac{6EI}{l^2}\theta_C$$

iv) $\Delta \neq 0$



$$M_{AB}^3 = M_{BA}^3 = \frac{6EI}{l^2}\Delta ,$$

$$M_{CD}^3 = M_{DC}^3 = \frac{6EI}{l^2}\Delta ,$$

$$V_{BA}^3 = V_{CD}^3 = \frac{12EI}{l^3}\Delta$$

$$A_{BD} = \frac{EA}{\sqrt{2}l}\frac{\Delta}{\sqrt{2}} \ (C) \ \rightarrow V_{BD} = \frac{EA}{\sqrt{2}l}\frac{\Delta}{\sqrt{2}}\frac{1}{\sqrt{2}} = V_{BD} = \frac{EA}{\sqrt{2}l}\frac{\Delta}{2}$$

- **Construct the Stiffness Equation**

$$\sum M_B^i = 0 \rightarrow \ \frac{Pab^2}{l^2} + \frac{8EI}{l}\theta_B + \frac{2EI}{l}\theta_c \ + \ \frac{6EI}{l^2}\Delta = 0$$

$$\sum M_C^i = 0 \rightarrow -\frac{Pa^2b}{l^2} + \frac{2EI}{l}\theta_B + \frac{8EI}{l}\theta_c \ + \ \frac{6EI}{l^2}\Delta = 0$$

$$\sum V^i = 0 \rightarrow \ \frac{6EI}{l^2}\theta_B + \frac{6EI}{l^2}\theta_c + \frac{24EI}{l^3}(1+\alpha)\Delta = 0$$

$$\Delta = -\frac{1}{1+\alpha}\frac{l}{4}(\theta_B + \theta_C) \quad , \quad \alpha = \frac{EAl^2}{48\sqrt{2}EI}$$

- **Solution for** $b = 3a$

$$\theta_B = -\frac{40+52\alpha}{256(7+10\alpha)}\frac{Pl^2}{EI}, \ \theta_C = -\frac{16+28\alpha}{256(7+10\alpha)}\frac{Pl^2}{EI} \ , \ \Delta = \frac{3}{128(7+10\alpha)}\frac{Pl^3}{EI}$$

For a $w \times h$ rectangular section and $l = 20h$, $\alpha = 50\sqrt{2}$.

$$\theta_B = -0.0203\frac{Pl^2}{EI}, \ \theta_C = 0.0109\frac{Pl^2}{EI} \quad \Delta = 0.3282 \times 10^{-4}\frac{Pl^3}{EI}$$

- **Performance**

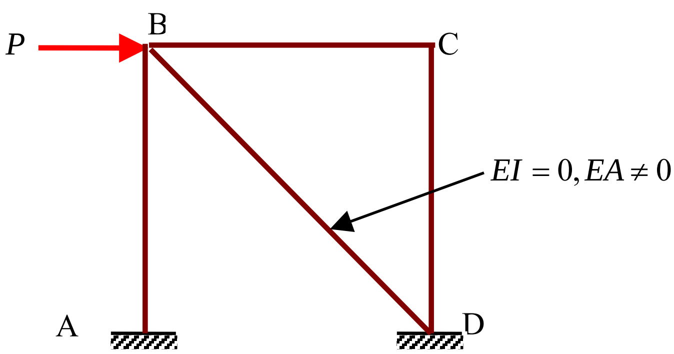
Response	with Bracing ($\alpha = 50\sqrt{2}$)	w/o bracing ($\alpha = 0$)	Ratio(%)
$\theta_B(\times Pl^2/EI)$	-0.0203	-0.0223	91.03
$\theta_C(\times Pl^2/EI)$	0.0109	0.0089	122.47
$\Delta(\times Pl^3/EI)$	0.3282×10^{-4}	0.0033	0.99
M_{AB} (Pl)	-0.0404	-0.0248	162.90
M_{BA} (Pl)	-0.0810	-0.0694	116.71
M_{CD} (Pl)	0.0438	0.0554	79.06
M_{DC} (Pl)	0.0220	0.0376	58.51
M_P (Pl)	0.1158	0.1216	95.23
$A_{BD}(P)$	0.0788	-	-
$P_{\max}(P_{all})$*	0.0720	0.0685	105.1
$P_{\max}$/vol.	0.0163	0.0228	71.5

*) $P_{all} = \sigma_{all} wh, M_{all} = P_{all} h/6$

$$\text{Unbalanced shear force in the columns} = \frac{6EI}{l^2}(\theta_B + \theta_C) = 0.0564P$$

The bracing carries 99 % of the unbalanced shear force between the two columns.

2.3.6 A Portal Frame with a Bracing (Horizontal Load)



- **DOF :** θ_B , θ_C , Δ

- **Analysis**

i) All fixed: No fixed end forces

ii)-iv) the same as the previous case

- **Construct the Stiffness Equation**

$$\sum M_B^i = 0 \rightarrow \quad \frac{8EI}{l}\theta_B + \frac{2EI}{l}\theta_c + \frac{6EI}{l^2}\Delta = 0$$

$$\sum M_C^i = 0 \rightarrow \quad \frac{2EI}{l}\theta_B + \frac{8EI}{l}\theta_c + \frac{6EI}{l^2}\Delta = 0$$

$$\sum V^i = 0 \rightarrow \quad \frac{6EI}{l^2}\theta_B + \frac{6EI}{l^2}\theta_c + \frac{24EI}{l^3}(1+\alpha)\Delta = P$$

$$\theta_B = \theta_C \ , \quad \Delta = -\frac{5}{3}\theta_B l = -\frac{5}{3}\theta_C l$$

- **Solution**

$$\theta_B = \theta_C = -\frac{1}{(28+40\alpha)}\frac{Pl^2}{EI}, \quad \Delta = \frac{5}{3(28+40\alpha)}\frac{Pl^3}{EI}$$

For $\alpha = 50\sqrt{2}$,

$$\theta_B = \theta_C = -0.3501\times 10^{-3}\frac{Pl^2}{EI} \ , \quad \Delta = 0.5835\times 10^{-3}\frac{Pl^3}{EI}$$

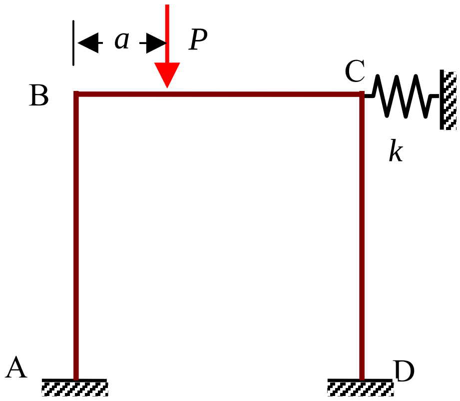
- **Performance**

Response	with Bracing ($\alpha = 50\sqrt{2}$)	w/o bracing ($\alpha = 0$)	Ratio(%)
$\theta_B (\times Pl^2/EI)$	-0.3501×10^{-3}	-0.3571×10^{-1}	0.98
$\theta_C (\times Pl^2/EI)$	-0.3501×10^{-3}	-0.3571×10^{-1}	0.98
$\Delta (\times Pl^3/EI)$	0.5835×10^{-3}	0.5952×10^{-1}	0.98
M_{AB} (Pl)	0.2801×10^{-2}	0.2857	0.98
M_{BA} (Pl)	0.2101×10^{-2}	0.2143	0.98
M_{CD} (Pl)	0.2101×10^{-2}	0.2143	0.98
M_{DC} (Pl)	0.2801×10^{-2}	0.2857	0.98
A_{BD} (P)	1.4004	-	-
$P_{max}(P_{all})$*	0.7141	0.0292	2448
P_{max}/vol.	0.1617	0.0097	1670

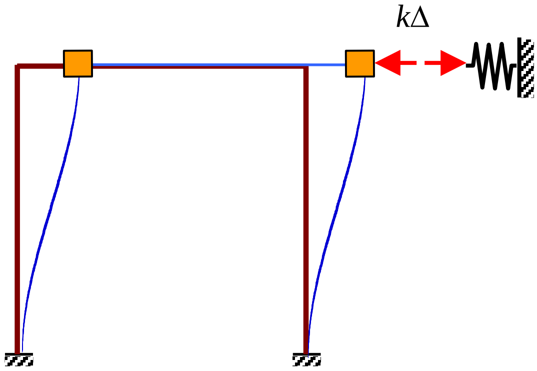
*) Governed by A_{BD} for the structure with bracing, and by M_{DC} for the structure without bracing. $P_{all} = \sigma_{all} wh$, $M_{all} = P_{all} h/6$

The bracing carries about 99% of the external horizontal load.

2.3.7 *A Portal Frame with a Spring*



- **DOF :** θ_B , θ_C , Δ
- **Analysis**



iv) $\Delta \neq 0$

$$M_{AB}^3 = M_{BA}^3 = \frac{6EI}{l^2}\Delta \ , \quad M_{CD}^3 = M_{DC}^3 = \frac{6EI}{l^2}\Delta \ ,$$

$$V_{BA}^3 = V_{CD}^3 = \frac{24EI}{l^3}\Delta \ , \quad V_S^3 = k\Delta$$

- **Construct the Stiffness Equation**

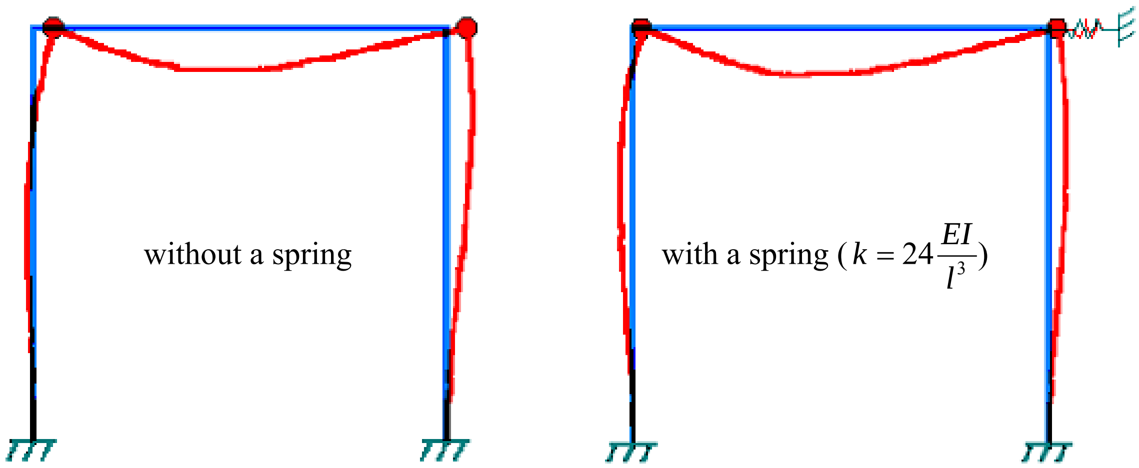
$$\sum M_B^i = 0 \rightarrow \quad \frac{Pab^2}{l^2} + \frac{8EI}{l}\theta_B + \frac{2EI}{l}\theta_c + \frac{6EI}{l^2}\Delta = 0$$

$$\sum M_C^i = 0 \rightarrow -\frac{Pa^2b}{l^2} + \frac{2EI}{l}\theta_B + \frac{8EI}{l}\theta_c + \frac{6EI}{l^2}\Delta = 0$$

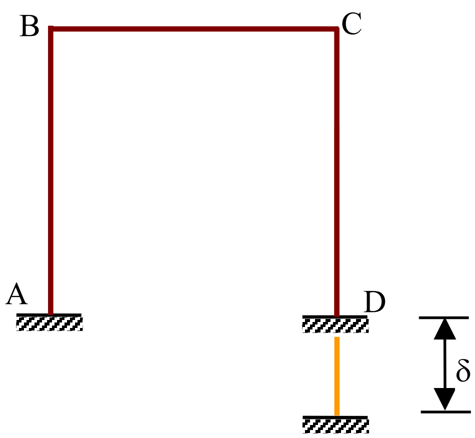
$$\sum V^i = 0 \rightarrow \quad \frac{6EI}{l^2}\theta_B + \frac{6EI}{l^2}\theta_c + \frac{24EI}{l^3}\Delta = -k\Delta$$

$$\frac{6EI}{l^2}\theta_B + \frac{6EI}{l^2}\theta_c + (\frac{24EI}{l^3} + k)\Delta = 0$$

- **Deformed Shapes** ($\frac{\Delta_S}{\Delta} = 0.41$)



2.3.8 A Portal Frame Subject to Support Settlement



- **DOF :** θ_B , θ_C , Δ
- **Analysis**

i) All fixed

$$M^0_{BC} = M^0_{CB} = \frac{6EI}{l^2}\delta$$

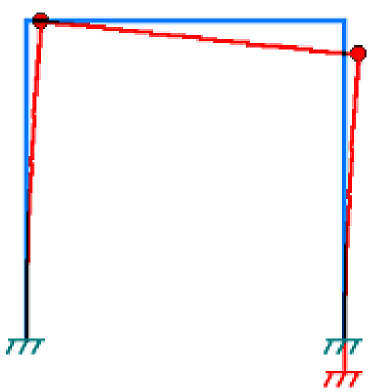
ii)-iv) the same as the previous problem

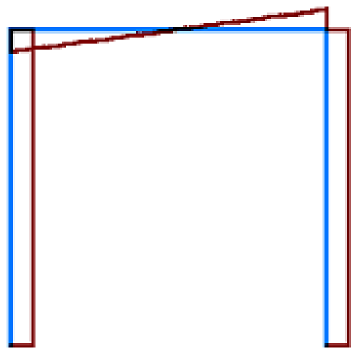
- **Construct the Stiffness Equation**

$$\sum M^i_B = 0 \rightarrow \quad \frac{6EI}{l^2}\delta + \frac{8EI}{l}\theta_B + \frac{2EI}{l}\theta_c \; + \frac{6EI}{l^2}\Delta = 0$$

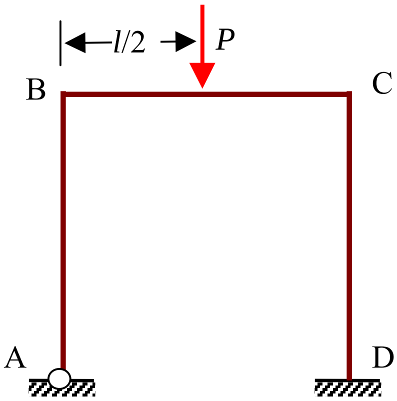
$$\sum M^i_C = 0 \rightarrow \quad \frac{6EI}{l^2}\delta + \frac{2EI}{l}\theta_B + \frac{8EI}{l}\theta_c \; + \frac{6EI}{l^2}\Delta = 0$$

$$\sum V^i = 0 \rightarrow \quad \frac{6EI}{l^2}\theta_B + \frac{6EI}{l^2}\theta_c + \frac{24EI}{l^3}\Delta = 0$$





2.3.9 A Portal Frame with Unsymmetrical Supports

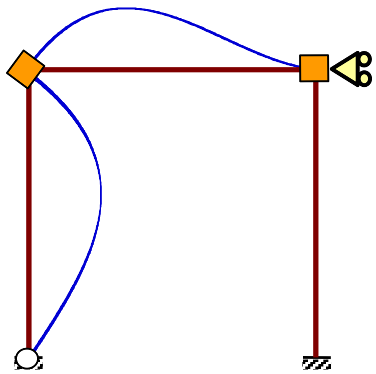


- **DOF :** θ_B , θ_C , Δ
- **Analysis**

i) All fixed

$$M_{BC}^0 = \frac{Pl}{8} \; , \quad M_{CB}^0 = -\frac{Pl}{8}$$

ii) $\theta_B \neq 0$

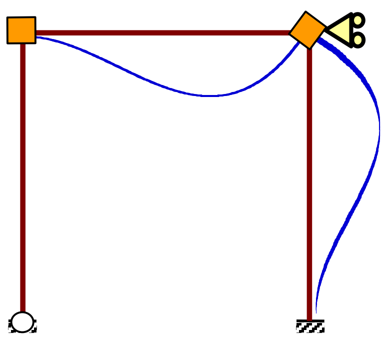


$$M_{BA}^1 = \frac{3EI}{l}\theta_B$$

$$M_{BC}^1 = \frac{4EI}{l}\theta_B \; , \quad M_{CB}^1 = \frac{2EI}{l}\theta_B$$

$$V_{BA}^1 = \frac{3EI}{l^2}\theta_B$$

iii) $\theta_C \neq 0$

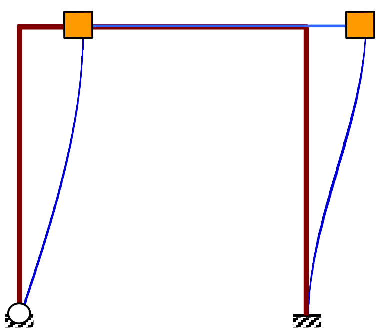


$$M_{BC}^2 = \frac{2EI}{l}\theta_C \; , \quad M_{CB}^2 = \frac{4EI}{l}\theta_C$$

$$M_{CD}^2 = \frac{4EI}{l}\theta_C \; , \quad M_{DC}^2 = \frac{2EI}{l}\theta_C$$

$$V_{CD}^2 = \frac{6EI}{l^2}\theta_C$$

iv) $\Delta \neq 0$



$$M_{BA}^3 = \frac{3EI}{l^2}\Delta \; ,$$

$$M_{CD}^3 = M_{DC}^3 = \frac{6EI}{l^2}\Delta \; ,$$

$$V_{BA}^3 = \frac{3EI}{l^3}\Delta \, , \quad V_{CD}^3 = \frac{12EI}{l^3}\Delta$$

- **Construct the Stiffness Equation**

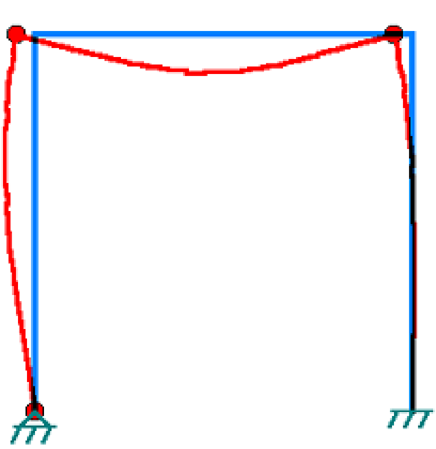
$$\sum M_B^i = 0 \rightarrow \quad \frac{Pl}{8} + \frac{7EI}{l}\theta_B + \frac{2EI}{l}\theta_c + \frac{3EI}{l^2}\Delta = 0$$

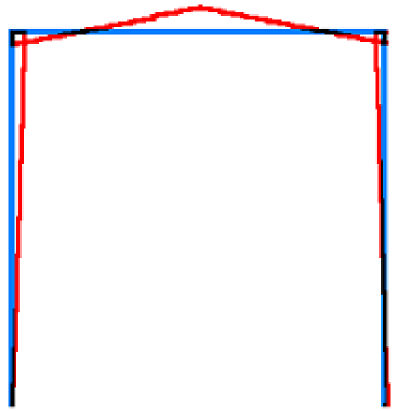
$$\sum M_C^i = 0 \rightarrow \quad -\frac{Pl}{8} + \frac{2EI}{l}\theta_B + \frac{8EI}{l}\theta_c + \frac{6EI}{l^2}\Delta = 0$$

$$\sum V^i = 0 \rightarrow \quad \frac{3EI}{l^2}\theta_B + \frac{6EI}{l^2}\theta_c + \frac{15EI}{l^3}\Delta = 0$$

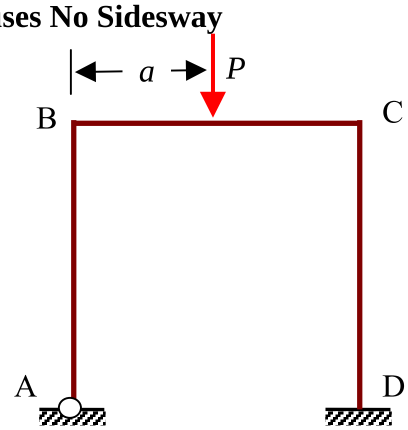
$$\Delta = -\frac{l}{5}(\theta_B + 2\theta_C)$$

$$\theta_B = -\frac{1}{44}\frac{Pl^2}{EI}\ , \quad \theta_C = \frac{1}{44}\frac{9}{8}\frac{Pl^2}{EI}\ , \quad \Delta = -\frac{1}{176}\frac{Pl^3}{EI}$$





- **Load Location that Causes No Sidesway**



$$\Delta = -\frac{l}{5}(\theta_B + 2\theta_C) = 0 \rightarrow \theta_B = -2\theta_C$$

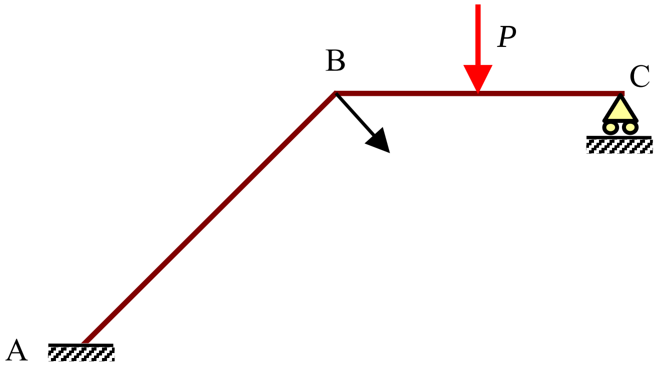
- Stiffness equation

$$\sum M_B^i = 0 \rightarrow \quad \frac{Pab^2}{l^2} + \frac{7EI}{l}\theta_B + \frac{2EI}{l}\theta_c = 0\ , \quad \sum M_C^i = 0 \rightarrow -\frac{Pa^2b}{l^2} + \frac{2EI}{l}\theta_B + \frac{8EI}{l}\theta_c = 0$$

$$\frac{Pab^2}{l^2} - \frac{12EI}{l}\theta_C = 0\ , \quad -\frac{Pa^2b}{l^2} + \frac{4EI}{l}\theta_c = 0$$

$$\frac{Pab^2}{l^2} = 3\frac{Pa^2b}{l^2} \rightarrow b = 3a$$

2.3.10 A Frame with a Skewed Member



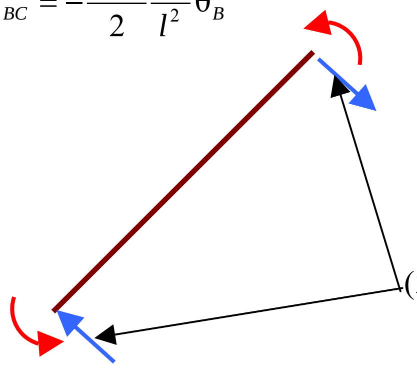
- **DOF :** θ_B, Δ
- **Analysis**

i) All fixed : $M_{BC}^{0} = \frac{Pl}{8} + \frac{Pl}{16} = \frac{3}{16}Pl$, $V_{BC}^{0} = -\frac{\sqrt{2}}{2}\frac{11}{16}P$

ii) $\theta_B \neq 0$

$$M_{AB}^{1} = \frac{2EI}{\sqrt{2}l}\theta_B = \sqrt{2}\frac{EI}{l}\theta_B \ , \ M_{BA}^{1} = 2\sqrt{2}\frac{EI}{l}\theta_B \ , \ M_{BC}^{1} = \frac{3EI}{l}\theta_B \ , \ V_{BA}^{1} = 3\frac{EI}{l^2}\theta_B$$

$$V_{BC}^{1} = -\frac{3\sqrt{2}}{2}\frac{EI}{l^2}\theta_B$$



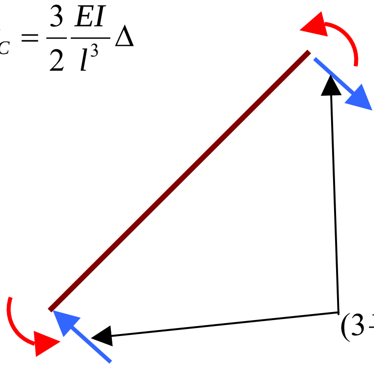
$$3\frac{EI}{l}\theta_B\frac{1}{l}\frac{\sqrt{2}}{2} = 3\frac{\sqrt{2}}{2}\frac{EI}{l^2}\theta_B$$

$$(2\sqrt{2}\frac{EI}{l}\theta_B + \sqrt{2}\frac{EI}{l}\theta_B)\frac{1}{\sqrt{2}l} = 3\frac{EI}{l^2}\theta_B$$

iii) $\Delta \neq 0$

$$M_{BA}^{2} = M_{AB}^{2} = \frac{6EI}{\sqrt{2}l}\frac{\Delta}{\sqrt{2}l} = \frac{3EI}{l^2}\Delta \quad M_{BC}^{2} = -\frac{3EI}{l}\frac{\Delta}{\sqrt{2}l} = -\frac{3\sqrt{2}}{2}\frac{EI}{l^2}\Delta, \ V_{BA}^{2} = 3\sqrt{2}\frac{EI}{l^3}\Delta \ ,$$

$$V_{BC}^{2} = \frac{3}{2}\frac{EI}{l^3}\Delta$$



$$3\frac{\sqrt{2}}{2}\frac{EI}{l^2}\Delta\frac{1}{l}\frac{1}{\sqrt{2}} = \frac{3}{2}\frac{EI}{l^3}\Delta$$

$$(3\frac{EI}{l^2}\Delta + 3\frac{EI}{l^2}\Delta)\frac{1}{\sqrt{2}l} = 3\sqrt{2}\frac{EI}{l^3}\theta_B$$

- **Construct the stiffness equation**

$$\sum M_B^i = 0 \rightarrow (2\sqrt{2}+3)\frac{EI}{l}\theta_B + (3-\frac{3\sqrt{2}}{2})\frac{EI}{l^2}\Delta = -\frac{3}{16}Pl$$

$$\sum V^i = 0 \rightarrow (3-\frac{3}{2}\sqrt{2})\frac{EI}{l^2}\theta_B + (3\sqrt{2}+\frac{3}{2})\frac{EI}{l^3}\Delta = \frac{\sqrt{2}}{2}\frac{11}{16}P$$

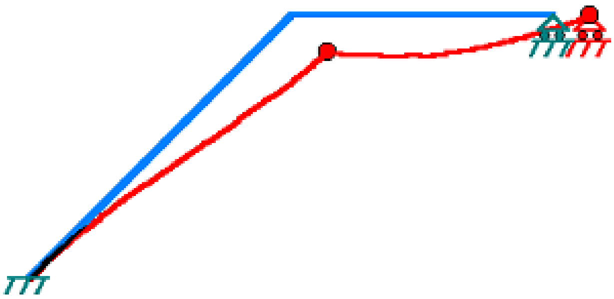
$$5.8284\frac{EI}{l}\theta_B + 0.8787\frac{EI}{l^2}\Delta = -0.1875Pl$$

$$0.8787\frac{EI}{l^2}\theta_B + 5.7426\frac{EI}{l^3}\Delta = 0.4861P$$

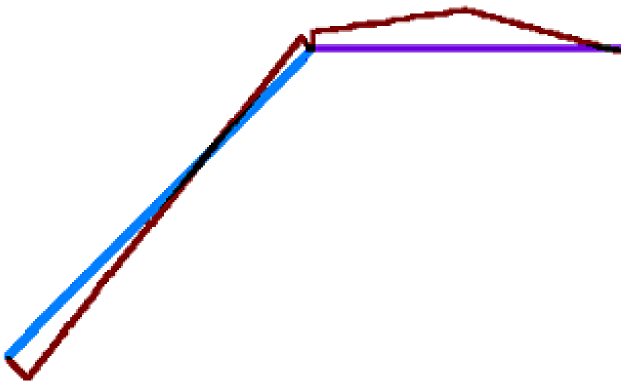
$$\theta_B = -0.0460\frac{Pl^2}{EI}, \quad \Delta = 0.0917\frac{Pl^3}{EI}$$

- **Results**

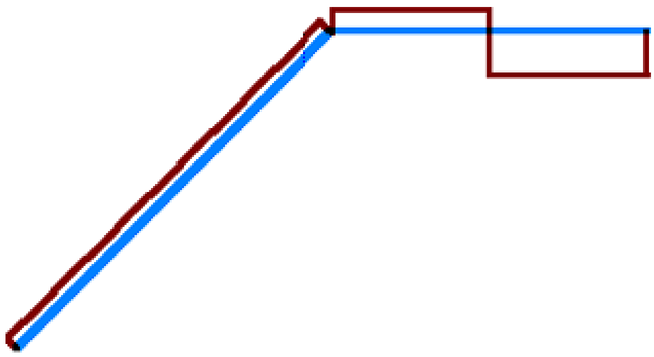
- Deformed shape



- Moment diagram

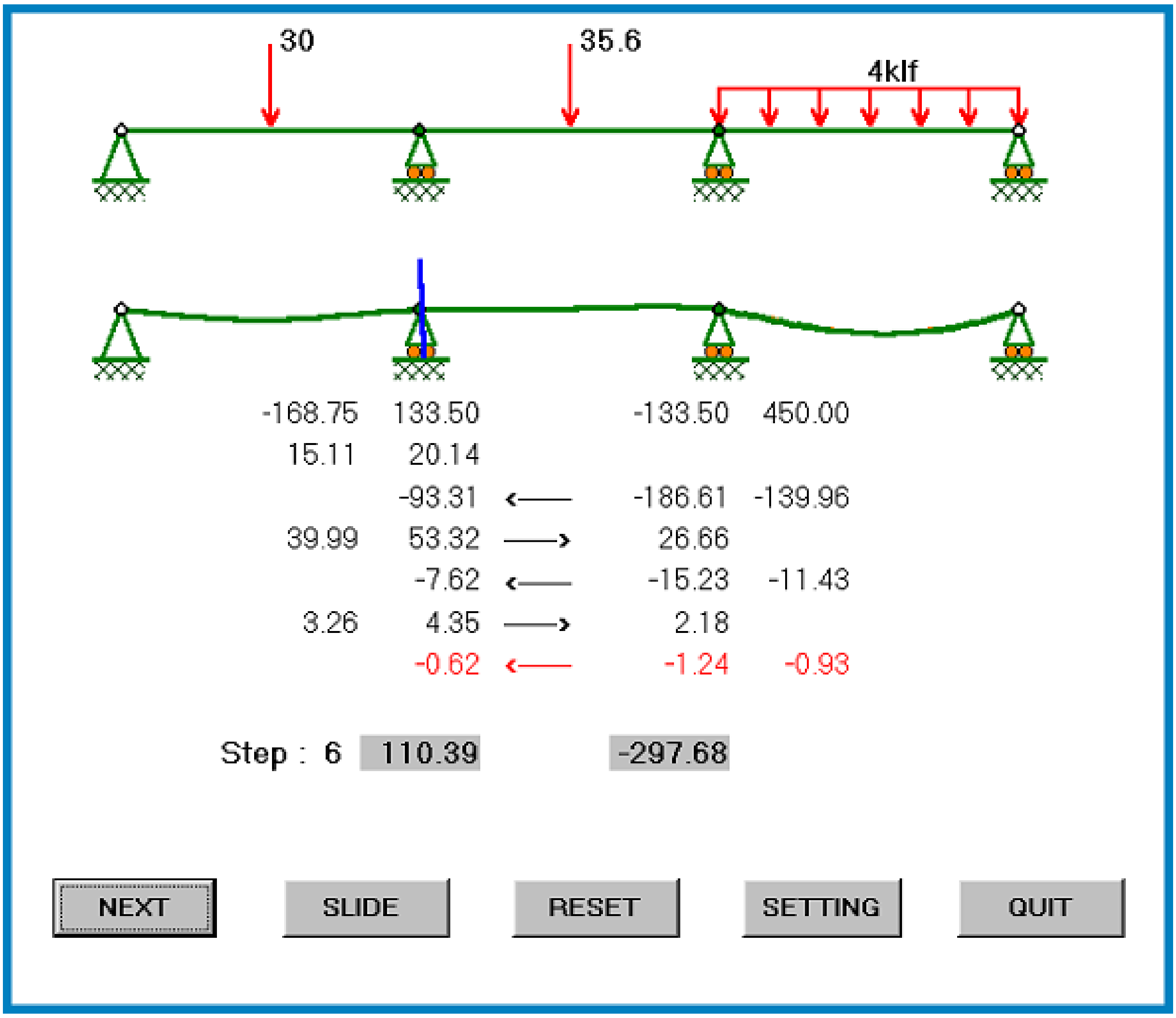


- Shear force diagram



Chapter 3

Iterative Solution Method & Moment Distribution Method



3.1 Solution Method for Linear Algebraic Equations

3.1.1 Direct Method – Gauss Elimination

$$\left.\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1i}X_i + \cdots + a_{1n}X_n = b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2i}X + \cdots + a_{2n}X_n = b_2 \\ \vdots \\ a_{i1}X_1 + a_{i2}X_2 + \cdots + a_{ii}X_i + \cdots + a_{in}X_n = b_i \\ \vdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{ni}X_i + \cdots + a_{nn}X_n = b_n \end{array}\right\} \rightarrow \sum_{j=1}^{n} a_{ij}X_j = b_i \quad \text{for } i = 1 \cdots n$$

or in a matrix form

$$[\mathbf{A}](\mathbf{X}) = (\mathbf{b})$$

By multiplying $\dfrac{a_{i1}}{a_{11}}$ to the first equation and subtracting the resulting equation from the i-th equation for $2 \le i \le n$, the first unknown X_1 is eliminated from the second equation as follows.

$$\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1i}X_i + \cdots + a_{1n}X_n = b_1 \\ \qquad a_{22}^{(2)}X_2 + \cdots + a_{2i}^{(2)}X + \cdots + a_{2n}^{(2)}X_n = b_2^{(2)} \\ \qquad \vdots \\ \qquad a_{i2}^{(2)}X_2 + \cdots + a_{ii}^{(2)}X_i + \cdots + a_{in}^{(2)}X_n = b_i^{(2)} \\ \qquad \vdots \\ \qquad a_{n2}^{(2)}X_2 + \cdots + a_{n2}^{(2)}X_i + \cdots + a_{nn}^{(2)}X_n = b_n^{(2)} \end{array}$$

where $a_{ij}^{(2)} = a_{ij} - \dfrac{a_{i1}a_{1j}}{a_{11}}$. Again, the second unknown X_2 is eliminated from the third equation by multiplying $\dfrac{a_{i2}^{(2)}}{a_{22}^{(2)}}$ to the second equation and subtracting the resulting equation from the i-th equation for $3 \le i \le n$. The aforementioned procedures are repeated until the last unknown remains in the last equation.

$$\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1i}X_i + \cdots + a_{1n}X_n = b_1 \\ \qquad a_{22}^{(2)}X_2 + \cdots + a_{2i}^{(2)}X + \cdots + a_{2n}^{(2)}X_n = b_2^{(2)} \\ \qquad\qquad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ \qquad\qquad a_{ii}^{(i)}X_i + \cdots + a_{in}^{(i)}X_n = b_i^{(i)} \\ \qquad\qquad\qquad\qquad \vdots \qquad\quad \vdots \\ \qquad\qquad\qquad\qquad a_{nn}^{(n)}X_n = b_n^{(n)} \end{array}$$

where $a_{ij}^{(k)} = a_{ij}^{(k-1)} - \dfrac{a_{i,k-1}^{(k-1)} a_{k-1,j}^{(k-1)}}{a_{k-1,k-1}^{k-1}}$ $k \le i, j \le n$, and $a_{ij}^{1} = a_{ij}$. Once the system matrix is tri-angularized, the solution of the given system is easily obtained by the back-substitution.

$$a_{nn}^{(n)} X_n = b_n^{(n)} \rightarrow \qquad X_n = \frac{b_n^{(n)}}{a_{nn}^{(n)}}$$

$$a_{n-1,n-1}^{(n-1)} X_{n-1} + a_{n-1,n}^{(n-1)} X_n = b_{n-1}^{(n-1)} \rightarrow \quad X_{n-1} = \frac{b_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)} X_n}{a_{n-1,n-1}^{n-2}}$$

$$a_{ii}^{(i)} X_i + \cdots + a_{i,n}^{(i)} X_n = b_i^{(i)} \rightarrow \quad X_i = \frac{b_i^{(i)} - \sum_{k=n}^{i+1} a_{ik}^{(i)} X_k}{a_{ii}^{i-1}} \quad \text{for } 1 \le i \le n-1$$

3.1.2 Iterative Method – Gauss-Jordan Method

A system of linear algebraic equations may be solved by iterative method. For this purpose, the given system is rearranged as follows.

$$X_1 = \frac{b_1 - (a_{12} X_2 + \cdots + a_{1n} X_n)}{a_{11}}$$
$$X_2 = \frac{b_2 - (a_{21} X_1 + a_{23} X_3 + \cdots + a_{2n} X_n)}{a_{22}}$$
$$X_i = \frac{b_i - (a_{i1} X_1 + \cdots + a_{i,i-1} X_{i-1} + a_{i,i+1} X_{i-1} + \cdots + a_{in} X_n)}{a_{ii}}$$
$$X_n = \frac{b_n - (a_{n1} X_1 + \cdots + a_{n,n-1} X_{n-1})}{a_{nn}}$$

Suppose we substitute an approximate solution $(\mathbf{X})_{k-1}$ into the right-hand side of the above equation, a new approximate solution $(\mathbf{X})_k$, which is not the same as $(\mathbf{X})_{k-1}$, is obtained. This procedure is repeated until the solution converges.

$$(X_i)_k = \frac{1}{a_{ii}} (b_i - \sum_{\substack{j=1 \\ i \ne j}}^{n} a_{ij} (X_j)_{k-1})$$

where the subscript *k* denotes the iterational count.

3.1.3 Iterative Method – Gauss-Siedal Method

When we calculate a new X_i value in the k-th iteration of Gauss-Jordan iteration, the values of $X_1, \cdots, X_{i-1}$ are already updated, and we can utilize the updated values to accelerate convergence rate, which leads to the Gauss-Siedal Method.

$$(X_1)_k = \frac{b_1 - (a_{12}(X_2)_{k-1} + \cdots + a_{1n}(X_n)_{k-1})}{a_{11}}$$

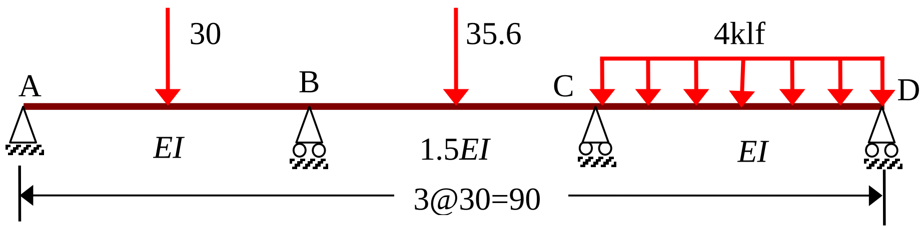
$$(X_2)_k = \frac{b_2 - (a_{21}(X_1)_k + a_{23}X_3)_{k-1} + \cdots + (a_{2n}X_n)_{k-1})}{a_{22}}$$

$$(X_i)_k = \frac{b_i - (a_{i1}(X_1)_k + \cdots + a_{i,i-1}(X_{i-1})_k + a_{i,i+1}(X_{i+1})_{k-1} + \cdots + a_{in}(X_n)_{k-1})}{a_{ii}}$$

$$(X_n)_k = \frac{b_n - (a_{n1}(X_1)_k + \cdots + a_{n,n-1}(X_{n-1})_k)}{a_{nn}}$$

$$(X_i)_k = \frac{1}{a_{ii}}(b_i - \sum_{\substack{j=1 \\ i>1}}^{i-1} a_{ij}(X_j)_k - \sum_{\substack{j=i+1 \\ i<n}}^{n} a_{ij}(X_j)_{k-1})$$

3.1.4 Example



- **Stiffness Equation**

$$-168.7 + 133.3 + (\frac{3EI}{l} + \frac{6EI}{l})\theta_B + \frac{3EI}{l}\theta_C = 0$$

$$-133.3 + 450.0 + \frac{3EI}{l}\theta_B + (\frac{6EI}{l} + \frac{3EI}{l})\theta_C = 0$$

For the simplicity of derivation, $\frac{EI}{l}\theta_B \to \theta_B$, $\frac{EI}{l}\theta_C \to \theta_C$. The stiffness equation becomes

$$-35.4 + 9\theta_B + 3\theta_C = 0$$

$$316.7 + 3\theta_B + 9\theta_C = 0$$

- **Gauss-Jordan Iteration**

$$(\theta_B)_k = \frac{1}{9}(35.4 - 3(\theta_C)_{k-1})$$

$$(\theta_C)_k = \frac{1}{9}(316.7 - 3(\theta_B)_{k-1})$$

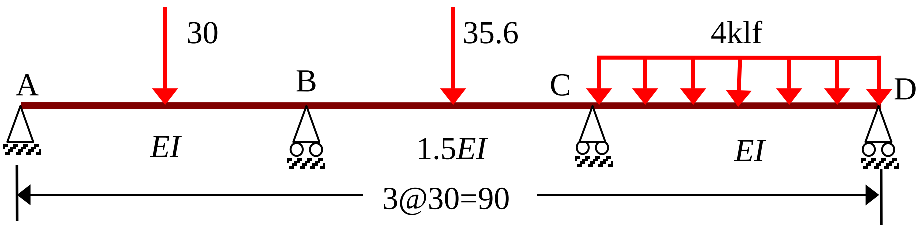
- **Gauss-Siedal Iteration**

$$(\theta_B)_k = \frac{1}{9}(35.4 - 3(\theta_C)_{k-1})$$

$$(\theta_C)_k = \frac{1}{9}(316.7 - 3(\theta_B)_k)$$

Gauss-Seidal	Gauss-Jordan
GAUSS-SIEDAL ITERATION	GAUSS-Jordan ITERATION
======================	======================
***** Iteration 1*****	***** Iteration 1*****
X(1) = 0.3933334E+01	X(1) = 0.3933334E+01
X(2) = -0.3650000E+02	X(2) = -0.3518889E+02
ERROR = 0.1000000E+01	ERROR = 0.1000000E+01
***** Iteration 2*****	***** Iteration 2*****
X(1) = 0.1610000E+02	X(1) = 0.1566296E+02
X(2) = -0.4055556E+02	X(2) = -0.3650000E+02
ERROR = 0.2939146E+00	ERROR = 0.2971564E+00
***** Iteration 3*****	***** Iteration 3*****
X(1) = 0.1745185E+02	X(1) = 0.1610000E+02
X(2) = -0.4100617E+02	X(2) = -0.4040988E+02
ERROR = 0.3197497E-01	ERROR = 0.9044394E-01
***** Iteration 4*****	***** Iteration 4*****
X(1) = 0.1760206E+02	X(1) = 0.1740329E+02
X(2) = -0.4105624E+02	X(2) = -0.4055556E+02
ERROR = 0.3544420E-02	ERROR = 0.2971564E-01
***** Iteration 5*****	***** Iteration 5*****
X(1) = 0.1761875E+02	X(1) = 0.1745185E+02
X(2) = -0.4106181E+02	X(2) = -0.4098999E+02
ERROR = 0.3937214E-03	ERROR = 0.9812154E-02
****** MOMENT ******	****** MOMENT ******
MBA =-115.84375	MBA =-116.34444
MBC = 115.82707	MBC = 115.04115
MCB =-326.81460	MCB =-326.88437
MCD = 326.81458	MCD = 327.03004

3.2 Moment Distribution Method



- **At the Joint B**

\- Moment distribution

$$M_B = \left(\frac{3EI_{AB}}{L_{AB}} + \frac{4EI_{BC}}{L_{BC}}\right)\theta_B = M^1_{BA} + M^1_{BC} \rightarrow \theta_B = \frac{M_B}{\frac{3EI_{AB}}{L_{AB}} + \frac{4EI_{BC}}{L_{BC}}}$$

$$M^1_{BA} = \frac{3EI_{AB}}{L_{AB}}\theta_B = \frac{\frac{3EI_{AB}}{L_{AB}}}{\frac{3EI_{AB}}{L_{AB}} + \frac{4EI_{BC}}{L_{BC}}} M_B = D_{BA} M_B$$

$$M^1_{BC} = \frac{4EI_{BC}}{L_{BC}}\theta_B = \frac{\frac{4EI_{BC}}{L_{BC}}}{\frac{3EI_{AB}}{L_{AB}} + \frac{4EI_{BC}}{L_{BC}}} M_B = D_{BC} M_B$$

\- Moment carry over to joint C: $M^1_{CB} = \dfrac{2EI_{BC}}{L_{BC}}\theta_B = \dfrac{1}{2} D_{BC} M_B$

- **At the Joint C**

\- Moment distribution

$$M_C = \left(\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}\right)\theta_C = M^2_{CB} + M^2_{CD} \rightarrow \theta_C = \frac{M_C}{\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}}$$

$$M^2_{CB} = \frac{4EI_{BC}}{L_{BC}}\theta_C = \frac{\frac{4EI_{BC}}{L_{BC}}}{\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}} M_C = D_{CB} M_C$$

$$M^2_{CD} = \frac{3EI_{CD}}{L_{CD}}\theta_C = \frac{\frac{3EI_{CD}}{L_{CD}}}{\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}} M_C = D_{CD} M_C$$

\- Moment carry over to joint B: $M^2_{BC} = \dfrac{2EI_{BC}}{L_{BC}}\theta_C = \dfrac{1}{2} D_{CB} M_C$

- **Stiffness Equation in terms of Moment at Joints**

$$\left.\begin{aligned} -35.4 + M_B + \frac{1}{2} D_{CB} M_C &= 0 \\ 316.7 + \frac{1}{2} D_{BC} M_B + M_C &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} M_B &= 35.4 - \frac{1}{2} D_{CB} M_C \\ M_C &= -316.7 - \frac{1}{2} D_{BC} M_B \end{aligned}$$

- Gauss-Siedal Approach

$$(M_B)_k = 35.4 - \frac{1}{2} D_{CB} (M_C)_{k-1}$$

$$(M_C)_k = -316.7 - \frac{1}{2} D_{BC} (M_B)_k$$

- Gauss-Jordan Approach

$$(M_B)_k = 35.4 - \frac{1}{2} D_{CB} (M_C)_{k-1}$$

$$(M_C)_k = -316.7 - \frac{1}{2} D_{BC} (M_B)_{k-1}$$

- **For the given structure**

$$D_{BA} = D_{CD} = \frac{1}{3} \quad , \quad D_{BC} = D_{CB} = \frac{2}{3}$$

- **Incremental form for the Gauss-Siedal Method**

- For $k = 1$

$(M_B)_0 = (M_C)_0 = 0$ because we assume all degrees of freedom are fixed for step 0.

$$(M_B)_1 = 35.4 - \frac{1}{2} D_{CB} (M_c)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4$$

$$(M_C)_1 = -316.7 - \frac{1}{2} D_{BC} (M_B)_1 = -316.7 - \frac{1}{2}\frac{2}{3} 35.4 \rightarrow (\Delta M_C)_1 = -328.5$$

- For $k > 1$

$$\begin{aligned} (M_B)_k &= 35.4 - \frac{1}{2} D_{CB} (M_c)_{k-1} = 35.4 - \frac{1}{2} D_{CB} (M_c)_{k-2} - \frac{1}{2} D_{CB} (\Delta M_c)_{k-1} \\ &= (M_B)_{k-1} - \frac{1}{2} D_{CB} (\Delta M_c)_{k-1} \rightarrow (\Delta M_B)_k = -\frac{1}{2} D_{CB} (\Delta M_c)_{k-1} \end{aligned}$$

$$\begin{aligned} (M_C)_k &= -316.7 - \frac{1}{2} D_{BC} (M_B)_k = -316.7 - \frac{1}{2} D_{BC} (M_B)_{k-1} - \frac{1}{2} D_{BC} (\Delta M_B)_k \\ &= (M_C)_{k-1} - \frac{1}{2} D_{BC} (\Delta M_B)_k \rightarrow (\Delta M_C)_k = -\frac{1}{2} D_{BC} (\Delta M_B)_k \end{aligned}$$

- Iteration 1

$$(M_B)_0 = 0\,,\ \ (M_C)_0 = 0$$

$$(M_B)_1 = 35.4 - \frac{1}{2} D_{CB} (M_C)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4$$

$$(M_{BA})_1 = M_{BA}^f + D_{BA} (\Delta M_B)_1 = -168.7 + 0.33 \times 35.4 = -168.7 + 11.8 \rightarrow (\Delta M_{BA})_1 = 11.8$$

$$(M_{BC})_1 = M_{BC}^f + D_{BC} (\Delta M_B)_1 = 133.3 + 0.67 \times 35.4 = 133.3 + 23.6 \rightarrow (\Delta M_{BC})_1 = 23.6$$

$$(M_C)_1 = -316.7 - \frac{1}{2} D_{BC} (\Delta M_B)_1 = -316.7 - \frac{1}{2}\frac{2}{3} 35.4 = -328.5 \rightarrow (\Delta M_c)_1 = -328.5$$

$$(M_{CB})_1 = M_{CB}^f + \frac{1}{2} D_{BC} (\Delta M_B)_1 + D_{CB} (\Delta M_C)_1 = -133.3 + 11.8 - 219.0 \rightarrow$$

$$(\Delta M_{CB})_1 = 11.8 - 219.0$$

$$(M_{CD})_1 = M_{CD}^f + D_{CD} (\Delta M_C)_1 = 450 - 0.33 \times 328.5 = 450 - 109.5 \rightarrow (\Delta M_{CD})_1 = -109.5$$

- Iteration 2

$$(\Delta M_B)_2 = -\frac{1}{2} D_{CB} (\Delta M_c)_1 = 109.5 \rightarrow \begin{cases} (\Delta M_{BA})_2 = D_{BA} (\Delta M_B)_2 = 36.5 \\ (\Delta M_{BC})_2 = \frac{1}{2} D_{CB} (\Delta M_c)_1 + D_{BC} (\Delta M_B)_2 \\ \qquad = -109.5 + 73.0 \end{cases}$$

$$(\Delta M_C)_2 = -\frac{1}{2} D_{BC} (\Delta M_B)_2 = -36.5 \rightarrow \begin{cases} (\Delta M_{CB})_2 = \frac{1}{2} D_{BC} (\Delta M_B)_2 + D_{CB} (\Delta M_C)_2 \\ \qquad = 36.5 - 24.3 \\ (\Delta M_{CD})_2 = D_{CD} (\Delta M_C)_2 = -12.2 \end{cases}$$

- Iteration 3

$$(\Delta M_B)_3 = -\frac{1}{2} D_{CB} (\Delta M_c)_2 = 12.2 \rightarrow \begin{cases} (\Delta M_{BA})_3 = D_{BA} (\Delta M_B)_3 = 4.1 \\ (\Delta M_{BC})_3 = \frac{1}{2} D_{CB} (\Delta M_c)_2 + D_{BC} (\Delta M_B)_3 \\ \qquad = -12.2 + 8.2 \end{cases}$$

$$(\Delta M_C)_3 = -\frac{1}{2} D_{BC} (\Delta M_B)_3 = -4.1 \rightarrow \begin{cases} (\Delta M_{CB})_3 = \frac{1}{2} D_{BC} (\Delta M_B)_3 + D_{CB} (\Delta M_C)_3 = 4.1 - 2.7 \\ (\Delta M_{CD})_3 = D_{CD} (\Delta M_C)_3 = -1.4 \end{cases}$$

- Final Moments

$$M_{BA} = M_{BA}^f + \sum_k (\Delta M_{BA})_k = -168.7 + 11.8 + 36.5 + 4.1 = -116.3$$

$$M_{BC} = M_{BC}^f + \sum_k (\Delta M_{BC})_k = 133.3 + 23.6 + (-109.5 + 73.0) + (-12.2 + 8.2) = 116.4$$

$$M_{CB} = M_{CB}^f + \sum_k (\Delta M_{CB})_k = -133.3 + (11.8 - 219.0) + (36.5 - 24.3) + (4.1 - 2.7) = -326.9$$

$$M_{CD} = M_{CD}^f + \sum_k (\Delta M_{CD})_k = 450.0 - 109.5 - 12.2 - 1.4 = 326.9$$

0.33	**0.66**	**0.67**	**0.33**
-168.7	**133.3**	**-133.3**	**450.0**
11.8	23.6	11.8	
	-109.5	-219.0	-109.5
36.5	73.0	36.5	
	-12.2	-24.3	-12.2
4.1	8.2	4.1	
	-1.4	-2.7	-1.4
0.5	0.9	**-326.9**	**326.9**
-115.8	**115.9**		

- **Incremental form for the Gauss-Jordan Method**

- For $k=1$

$(M_B)_0 = (M_C)_0 = 0$ because we assume all degrees of freedom are fixed for step 0.

$$(M_B)_1 = 35.4 - \frac{1}{2} D_{CB} (M_c)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4$$

$$(M_C)_1 = -316.7 - \frac{1}{2} D_{BC} (M_B)_0 = -316.7 \rightarrow (\Delta M_C)_1 = -316.7$$

- For $k>1$

$$(M_B)_k = 35.4 - \frac{1}{2} D_{CB} (M_c)_{k-1} = 35.4 - \frac{1}{2} D_{CB} (M_c)_{k-2} - \frac{1}{2} D_{CB} (\Delta M_c)_{k-1}$$

$$= (M_B)_{k-1} - \frac{1}{2} D_{CB} (\Delta M_c)_{k-1} \rightarrow (\Delta M_B)_k = -\frac{1}{2} D_{CB} (\Delta M_c)_{k-1}$$

$$(M_C)_k = -316.7 - \frac{1}{2} D_{BC} (M_B)_{k-1} = -316.7 - \frac{1}{2} D_{BC} (M_B)_{k-2} - \frac{1}{2} D_{BC} (\Delta M_B)_{k-1}$$

$$= (M_C)_{k-1} - \frac{1}{2} D_{BC} (\Delta M_B)_{k-1} \rightarrow (\Delta M_C)_k = -\frac{1}{2} D_{BC} (\Delta M_B)_{k-1}$$

- Iteration 1

$(M_B)_0 = 0$, $(M_C)_0 = 0$

$$(M_B)_1 = 35.4 - \frac{1}{2} D_{CB} (M_C)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4$$

$$(M_{BA})_1 = M_{BA}^f + D_{BA} (M_B)_1 = -168.7 + 0.33 \times 35.4 = -168.7 + 11.8 \rightarrow (\Delta M_{BA})_1 = 11.8$$
$$(M_{BC})_1 = M_{BC}^f + D_{BC} (M_B)_1 = 133.3 + 0.67 \times 35.4 = 133.3 + 23.6 \rightarrow (\Delta M_{BC})_1 = 23.6$$

$$(M_C)_1 = -316.7 - \frac{1}{2}D_{BC}(M_B)_0 = -316.7 = -316.7 \rightarrow (\Delta M_c)_1 = -316.7$$

$$(M_{CB})_1 = M_{CB}^f + D_{CB}(M_C)_1 = -133.3 - 0.67 \times 316.7 = -133.3 - 212.2 \rightarrow$$

$$(\Delta M_{CB})_1 = -212.2$$

$$(M_{CD})_1 = M_{CD}^f + D_{CD}(M_C)_1 = 450 - 0.33 \times 316.7 = 450 - 104.5 \rightarrow (\Delta M_{CD})_1 = -104.5$$

- Iteration 2

$$(\Delta M_B)_2 = -\frac{1}{2}D_{CB}(\Delta M_c)_1 = 106.1 \rightarrow \begin{cases} (\Delta M_{BA})_2 = D_{BA}(\Delta M_B)_2 = 35.0 \\ (\Delta M_{BC})_2 = \frac{1}{2}D_{CB}(\Delta M_c)_1 + D_{BC}(\Delta M_B)_2 \\ \qquad = -106.1 + 71.1 \end{cases}$$

$$(\Delta M_C)_2 = -\frac{1}{2}D_{BC}(\Delta M_B)_1 = -11.8 \rightarrow \begin{cases} (\Delta M_{CB})_2 = \frac{1}{2}D_{BC}(\Delta M_B)_1 + D_{CB}(\Delta M_C)_2 \\ \qquad = 11.8 - 7.9 \\ (\Delta M_{CD})_2 = D_{CD}(\Delta M_C)_2 = -3.9 \end{cases}$$

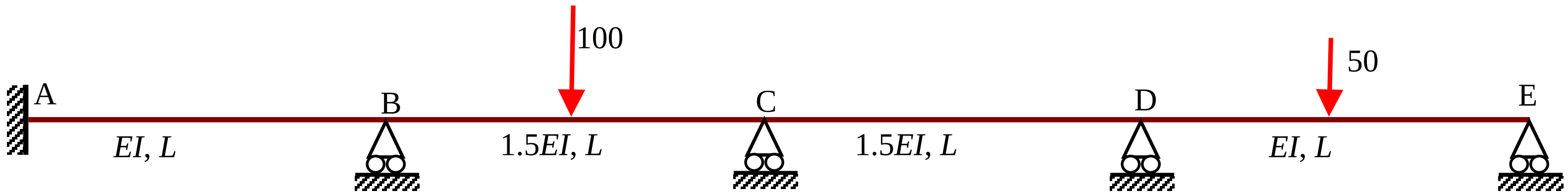
- Iteration 3

$$(\Delta M_B)_3 = -\frac{1}{2}D_{CB}(\Delta M_C)_2 = 4.0 \rightarrow \begin{cases} (\Delta M_{BA})_3 = D_{BA}(\Delta M_B)_3 = 1.3 \\ (\Delta M_{BC})_3 = \frac{1}{2}D_{CB}(\Delta M_C)_2 + D_{BC}(\Delta M_B)_3 \\ \qquad = -4.0 + 2.7 \end{cases}$$

$$(\Delta M_C)_3 = -\frac{1}{2}D_{BC}(\Delta M_B)_2 = -35.6 \rightarrow \begin{cases} (\Delta M_{CB})_3 = \frac{1}{2}D_{BC}(\Delta M_B)_3 + D_{CB}(\Delta M_C)_3 \\ \qquad = 35.6 - 23.9 \\ (\Delta M_{CD})_3 = D_{CD}(\Delta M_C)_3 = -11.7 \end{cases}$$

0.33	**0.66**	**0.67**	**0.33**
-168.7	**133.3**	**-133.3**	**450.0**
11.8	23.6	-212.2	-104.5
	-106.1	11.8	
35.0	71.1	-7. 9	-3.9
	-4.0	35.6	
1.3	2. 7	-23.9	-11.7
	-12.0	1.4	
4.0	8.0	-0.9	-0.5
	-0.5	4.0	
0.2	0.3	-2.7	-1.3
	-1.4	0.2	
0.5	0.9	0.1	0.1
-115.9	**115.9**	**-328.0**	**328.2**

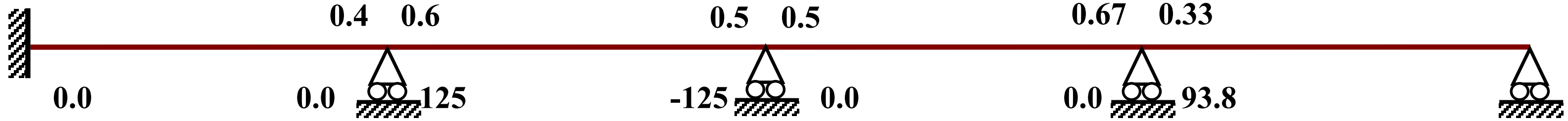
3.3 Example - MDM for a 4-span Continuous Beam



$$D_{BA} = 4\frac{EI}{L}\Big/(4\frac{EI}{L}+4\frac{1.5EI}{L}) = 0.4\,,\quad D_{BC} = 4\frac{1.5EI}{L}\Big/(4\frac{EI}{L}+4\frac{1.5EI}{L}) = 0.6\,,\quad D_{CB} = 4\frac{1.5EI}{L}\Big/(4\frac{1.5EI}{L}+4\frac{1.5EI}{L}) = 0.5\,,$$

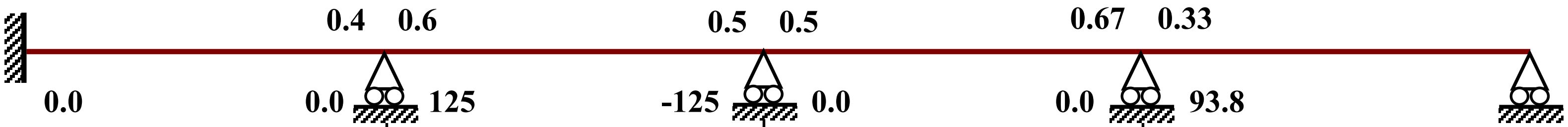
$$D_{CD} = 4\frac{1.5EI}{L}\Big/(4\frac{1.5EI}{L}+4\frac{1.5EI}{L}) = 0.5\,,\quad D_{DC} = 4\frac{1.5EI}{L}\Big/(4\frac{1.5EI}{L}+3\frac{EI}{L}) = 0.67\,,\quad D_{DE} = 3\frac{EI}{L}\Big/(4\frac{1.5EI}{L}+3\frac{EI}{L}) = 0.33$$

- **Gauss-Siedal Approach**



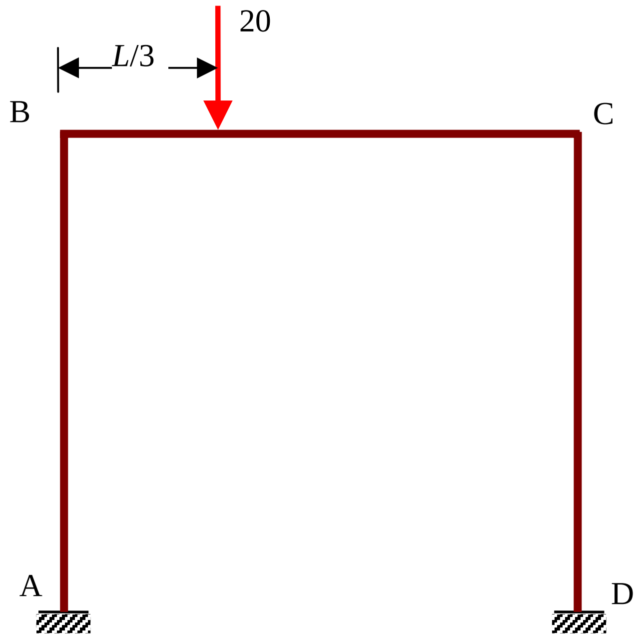
	0.4	**0.6**	**0.5**	**0.5**	**0.67**	**0.33**
0.0	**0.0**	**125**	**-125**	**0.0**	**0.0**	**93.8**
-25.0	-50.0	-75	-37.5			
		40.6	81.3	81.3	40.6	
				-45.0	-90.0	-44.3
		11.3	22.5	22.5	11.3	
-10.4	-20.8	-31.1	-15.6			
		3.9	7.8	7.8	3.9	
				-5.1	-10.2	-5.0
		1.3	2.6	2.6	1.3	
-1.1	-2.1	-3.1			-0.9	-0.4
36.5	**-72.9**	**72.9**	**-63.9**	**64.1**	**-44.0**	**44.1**

- **Gauss-Jordan Approach**



0.0	**0.0**	**125**	**-125**	**0.0**	**0.0**	**93.8**
	-50	-75	62.5	62.5	-62.8	-31.0
-25.0		31.3	-37.5	-31.4	31.3	
	-12.5	-18.8	34.5	34.5	-21.0	-10.3
-6.3		17.3	-9.4	-10.5	17.3	
	-6.9	-10.4	10.0	10.0	-11.6	-5.7
-3.5		5.0	-5.2	-5.8	5.0	
	-2.0	-3.0	5.5	5.5	-3.4	-1.6
-1.0		2.8	-1.5	-1.7	2.8	
	-1.1	-1.7	1.6	1.6	-1.9	-0.9
-0.6		0.8	-0.9	-1.0	0.8	
	-0.3	-0.5	1.0	1.0	-0.5	-0.3
-36.4	**-72.8**	**72.8**	**-64.4**	**64.7**	**-44.0**	**44.0**

3.4 Direct Solution Scheme by Partitioning



- **Slope deflection (Stiffness) Equation**

$$\begin{bmatrix} \frac{8EI}{L} & \frac{2EI}{L} & \frac{6EI}{L} \\ \frac{2EI}{L} & \frac{8EI}{L} & \frac{6EI}{L} \\ \frac{6EI}{L} & \frac{6EI}{L} & \frac{24EI}{L} \end{bmatrix} \begin{pmatrix} \theta_B \\ \theta_C \\ \Delta \end{pmatrix} + \begin{pmatrix} 88.8 \\ -44.4 \\ 0.0 \end{pmatrix} = 0 \rightarrow \begin{bmatrix} [\mathbf{K}_{\theta\theta}] & (\mathbf{K}_{\theta\Delta}) \\ (\mathbf{K}_{\Delta\theta}) & \mathbf{K}_{\Delta\Delta} \end{bmatrix} \begin{pmatrix} (\Theta) \\ \Delta \end{pmatrix} + \begin{pmatrix} (\mathbf{P}) \\ 0 \end{pmatrix} = 0$$

$$[\mathbf{K}_{\theta\theta}](\Theta) = -(\mathbf{P}) - (\mathbf{K}_{\theta\Delta})\Delta \rightarrow (\Theta) = -[\mathbf{K}_{\theta\theta}]^{-1}(\mathbf{P}) - [\mathbf{K}_{\theta\theta}]^{-1}(\mathbf{K}_{\theta\Delta})\Delta = (\Theta)^P + (\Theta)^\Delta$$

$$(\mathbf{K}_{\Delta\theta})(\Theta)^P + (\mathbf{K}_{\Delta\theta})(\Theta)^\Delta + \mathbf{K}_{\Delta\Delta}\Delta = 0$$

- **Direct Solution Procedure by Partitioning**
  - Assume $\Delta = 0$ and calculate $(\Theta)^P$.
  - Assume an arbitrary $\overline{\Delta} = \dfrac{\Delta}{\alpha}$ and calculate $(\overline{\Theta})^\Delta$.
  - By linearity, $(\overline{\Theta})^\Delta = \dfrac{(\Theta)^\Delta}{\alpha}$
  - Calculate α by the second equation by $\overline{\Delta}$ and $(\overline{\Theta})^\Delta$.

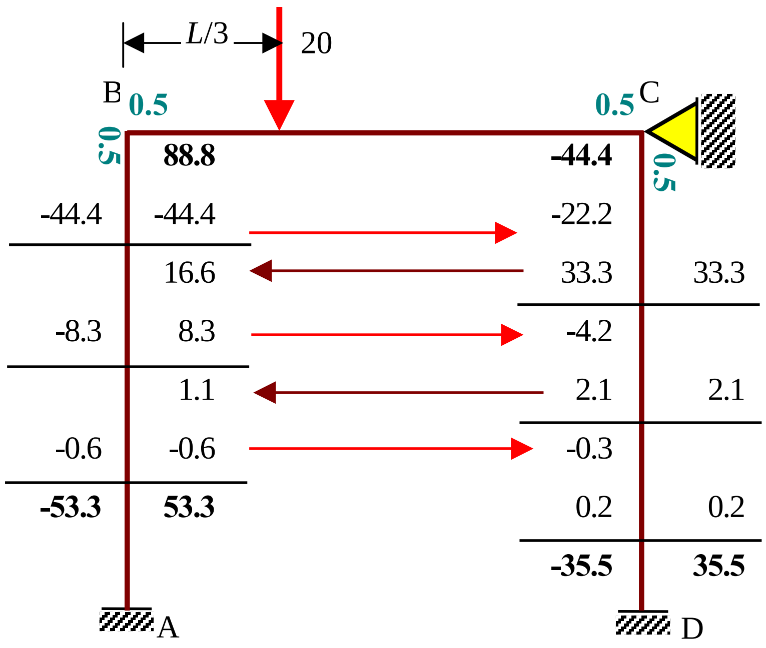
$$\alpha = -\frac{(\mathbf{K}_{\Delta\theta})(\Theta)^P}{\mathbf{K}_{\Delta\Delta}\overline{\Delta} + (\mathbf{K}_{\Delta\theta})(\overline{\Theta})^\Delta}$$

  - Obtain (Θ) by $(\Theta) = (\Theta)^P + \alpha(\overline{\Theta})^\Delta$.

3.5 Moment Distribution Method for Frames

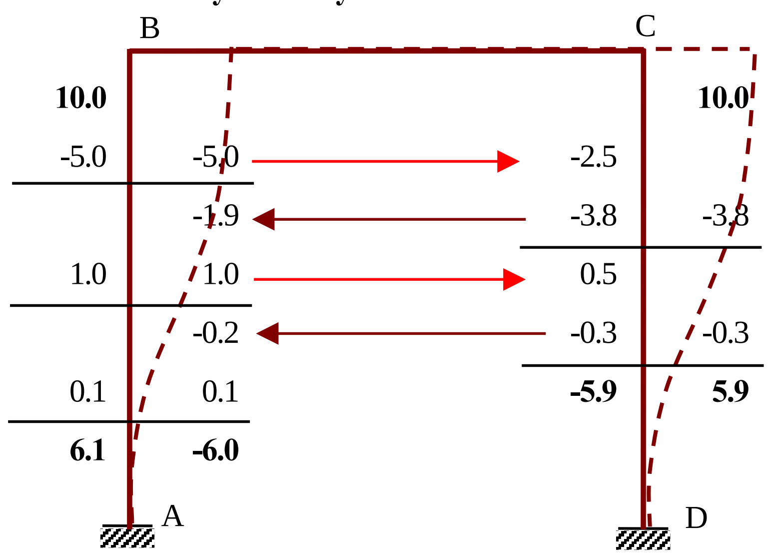
- **Solution Procedure**
  - Assume there is no sideway and do the MDM.
  - Perform the MDM again for an assumed sidesway.
  - Adjust the Moment obtained by the second MDM to satisfy the second equation.
  - Add the adjusted moment to the moment by the first MDM.

- **First MDM – with no Sidesway**



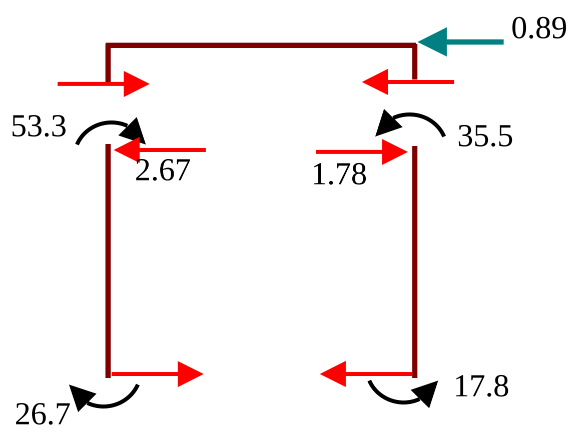
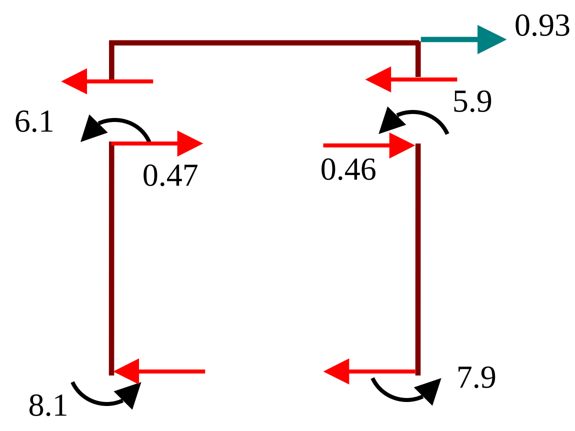
M_{AB} = -53.3/2 = -26.7, M_{DC} = -35.5/2 = 17.8, V_{AB} = 2.67, V_{DC} = -1.78, V^P = 0.89

- **Second MDM – with an Arbitrary Sidesway**



M_{AB} = 10 – 3.9/2.0 = 8.1, M_{DC} = 10 – 4.1/2.0 = 7.9, V_{AB} =-0.47, V_{DC} = -0.46, V^{Δ} = -0.93

- **Shear Equilibrium Condition (the 2nd equation)**

No Sidesway (1st MDM)	**Sidesway Only (2nd MDM)**
	

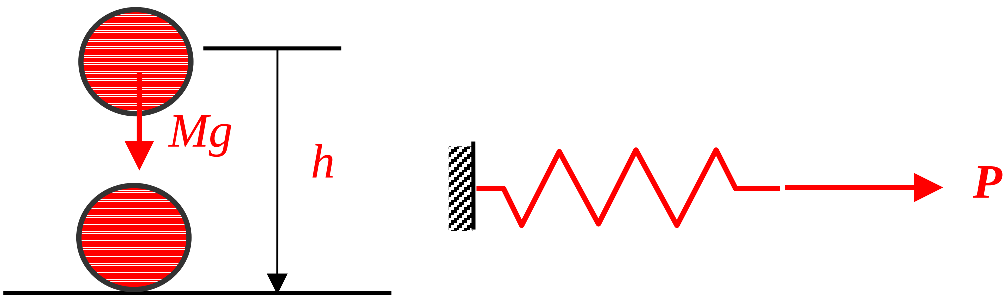
$$\alpha = -0.89/(-0.93) = 0.97$$

Total Moment = 1st Moment + α 2nd Moment

Chapter 4

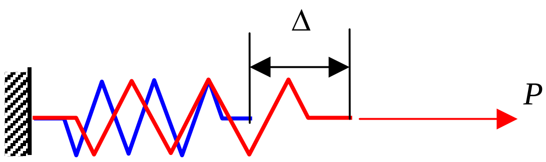
Energy Principles

Principle of Minimum Potential Energy and Principle of Virtual Work



Read Chapter 11 (pp.420~ 428) of Elementary Structural Analysis 4th Edition by C .H. Norris *et al* very carefully. In this note an overbarred variable denotes a virtual quantity. The virtual displacement field should satisfy the displacement boundary conditions of supports if specified. For beam problems, displacement boundary conditions include boundary conditions for rotational angle. Variables with superscript *e* denote the exact solution that satisfies the equilibrium equation(s).

4.1 Spring-Force Systems



- **Total Potential energy**

The energy required to return a mechanical system to a reference status

$$\Pi_{\text{int}} = -\int_{\Delta}^{0} k(\Delta - u)du = \int_{0}^{\Delta} k(\Delta - u)du = \frac{1}{2}k\Delta^2 \text{ , } \Pi_{ext} = -P\Delta$$

$$\Pi_{total} = \Pi_{\text{int}} + \Pi_{ext} = \frac{1}{2}k\Delta^2 - P\Delta$$

- **Equilibrium Equation**

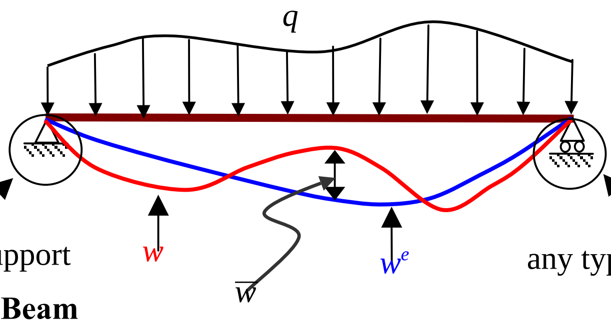
$k\Delta^e = P$

- **Principle of Minimum Potential Energy for an arbitrary displacement** $\Delta = \Delta^e + \overline{\Delta}$.

$$\begin{aligned}
\Pi_{total} &= \frac{1}{2}k(\Delta^e + \overline{\Delta})^2 - P(\Delta^e + \overline{\Delta}) \\
&= \frac{1}{2}k(\Delta^e)^2 + k\Delta^e\overline{\Delta} + \frac{1}{2}k(\overline{\Delta})^2 - P(\Delta^e + \overline{\Delta}) \\
&= \frac{1}{2}k(\Delta^e)^2 - P\Delta^e + \frac{1}{2}k(\overline{\Delta})^2 + \boxed{\overline{\Delta}(k\Delta^e - P)} \\
&= \frac{1}{2}k(\Delta^e)^2 - P\Delta^e + \frac{1}{2}k(\overline{\Delta})^2 \\
&= \Pi^e_{total} + \frac{1}{2}k(\overline{\Delta})^2 \geq \Pi^e_{total}
\end{aligned}$$

In the above equation, the equality sign holds if and only if $\overline{\Delta} = 0$. Therefore the total potential energy of the spring-force system becomes minimum when displacement of spring satisfies the equilibrium equation.

4.2 Beam Problems



- **Potential Energy of a Beam**

$$\Pi_{\text{int}} = \frac{1}{2}\int_0^l \frac{M^2}{EI}dx = \frac{1}{2}\int_0^l EI(\frac{d^2w}{dx^2})^2 dx\,, \Pi_{ext} = -\int_0^l qwdx$$

$$\Pi_{total} = \Pi_{\text{int}} + \Pi_{ext} = \frac{1}{2}\int_0^l EI(\frac{d^2w}{dx^2})^2 dx - \int_0^l qwdx$$

- **Equilibrium Equation**

$$EI\frac{d^2M^e}{dx^2} = -q \quad \text{or} \quad EI\frac{d^4w^e}{dx^4} = q$$

- **Principle of Minimum Potential Energy for a virtual displacement** $w = w^e + \overline{w}$**.**

$$\Pi^h = \frac{1}{2}\int_0^l (\frac{d^2(w^e+\overline{w})}{dx^2} EI \frac{d^2(w^e+\overline{w})}{dx^2})dx - \int_0^l (w^e+\overline{w})qdx$$

$$= \frac{1}{2}\int_0^l (\frac{d^2w^e}{dx^2} EI \frac{d^2w^e}{dx^2})dx - \int_0^l w^e qdx +$$

$$\frac{1}{2}\int_0^l (\frac{d^2\overline{w}}{dx^2} EI \frac{d^2\overline{w}}{dx^2})dx + \int_0^l (\frac{d^2\overline{w}}{dx^2} EI \frac{d^2w^e}{dx^2})dx - \int_0^l \overline{w}qdx$$

$$= \Pi^e + \frac{1}{2}\int_0^l (\frac{d^2\overline{w}}{dx^2} EI \frac{d^2\overline{w}}{dx^2})dx + \int_0^l (-EI\frac{d^2\overline{w}}{dx^2})\frac{1}{EI}(-EI\frac{d^2w^e}{dx^2})dx - \int_0^l \overline{w}qdx$$

$$= \Pi^e + \frac{1}{2}\int_0^l (\frac{d^2\overline{w}}{dx^2} EI \frac{d^2\overline{w}}{dx^2})dx + \boxed{\int_0^l \frac{\overline{M}M^e}{EI}dx - \int_0^l \overline{w}qdx}$$

$$= \Pi^e + \frac{1}{2}\int_0^l (\frac{d^2\overline{w}}{dx^2} EI \frac{d^2\overline{w}}{dx^2})dx \geq \Pi^e \quad \text{for all virtual } \overline{w}$$

Since the equation in the box represents the total virtual work in a beam, the total potential energy of a beam becomes minimum for all virtual displacement fields when the principle of virtual work holds. In the above equation, the equality sign holds if and only if $\overline{w} = 0$.

- **Principle of Virtual Work**

If a beam is in equilibrium, the principle of the virtual work holds for the beam,.

$$\int_0^l \overline{w}(EI\frac{d^4w^e}{dx^4} - q)dx = 0 \text{ for all virtual displacement } \overline{w}$$

$$\int_0^l \frac{d^2\overline{w}}{dx^2} EI \frac{d^2w^e}{dx^2} dx - \int_0^l \overline{w}qdx + \frac{d\overline{w}}{dx} EI \frac{d^2w^e}{dx^2}\Big|_0^l - \overline{w}EI\frac{d^3w^e}{dx^3}\Big|_0^l = 0$$

$$\int_0^l \frac{d^2\overline{w}}{dx^2} EI \frac{d^2w^e}{dx^2} dx - \int_0^l \overline{w}qdx = \int_0^l \frac{\overline{M}M^e}{EI}dx - \int_0^l \overline{w}qdx = 0$$

In case that there is no support settlement, the boundary terms in above equation vanishes identically since either virtual displacement including virtual rotational angle or corresponding forces (moment and shear) vanish at supports. The principle of virtual work yields the displacement of an arbitrary point $\tilde{x}$ in a beam by applying an unit load at $\tilde{x}$ and by using the reciprocal theorem.

$$\int_0^l \overline{w}qdx = \int_0^l w\overline{q}dx = \int_0^l w\delta(x-\tilde{x})dx = \boxed{w(\tilde{x}) = \int_0^l \frac{\overline{M}M^e}{EI}dx}$$

- **Approximation using the principle of minimum potential energy**

- Approximation of displacement field

$$w = \sum_{i=1}^n a_i g_i$$

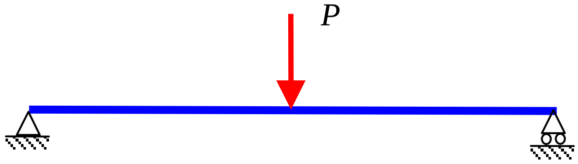
- Total potential energy by the assumed displacement field

$$\Pi^h = \frac{1}{2}\int_0^l (\frac{d^2w}{dx^2} EI \frac{d^2w}{dx^2})dx - \int_0^l wqdx = \frac{1}{2}\int_0^l (\sum_{i=1}^n a_i g_i'')EI(\sum_{j=1}^n a_j g_j'')dx - \int_0^l \sum_{i=1}^n a_i g_i qdx$$

- The first-order Necessary Condition

$$\begin{aligned}\frac{\partial \Pi^h}{\partial a_k} &= \frac{\partial}{\partial a_k}(\frac{1}{2}\int_0^l (\sum_{i=1}^n a_i g_i'')EI(\sum_{i=1}^n a_i g_i'')dx - \int_0^l \sum_{i=1}^n a_i g_i qdx) \\ &= \frac{1}{2}[\int_0^l g_k'' EI(\sum_{i=1}^n a_j g_j'')dx + \int_0^l (\sum_{i=1}^n a_i g_i'')EIg_k''dx] - \int_0^l g_k qdx) \quad \text{or} \quad \mathbf{Ka} = \mathbf{f} \\ &= \sum_{i=1}^n \int_0^l g_k'' EIg_i''dxa_i - \int_0^l g_k qdx = \sum_{i=1}^n K_{ki}a_i - f_k = 0\end{aligned}$$

- **Example**



i) with one unknown

$$w = ax(x-l) = a(x^2 - lx) \rightarrow w'' = 2a$$

$$\Pi_{total} = \frac{1}{2}\int_0^l EI(w'')^2 dx - \int_0^l P\delta(x-\frac{l}{2})wdx = \frac{1}{2}\int_0^l EI(2a)^2 dx + aP\frac{l^2}{4} = \frac{1}{2}EI4a^2l + aP\frac{l^2}{4}$$

$$\frac{\partial \Pi_{total}}{\partial a} = 0 \rightarrow 4aEIl + P\frac{l^2}{4} = 0 \rightarrow a = -\frac{Pl}{16EI} \rightarrow w = -\frac{Pl}{16EI}(x^2 - xl)$$

$$w(\frac{l}{2}) = \frac{Pl^3}{64EI} \;,\quad w^e(\frac{l}{2}) = \frac{Pl^3}{48EI} = 0.0208\frac{Pl^3}{EI} \;,\quad \text{Error} = \frac{w^e(\frac{l}{2}) - w(\frac{l}{2})}{w^e(\frac{l}{2})} = 0.25$$

ii) with two unknowns

$$w = ax(x-l) + bx(x^2 - l^2) \rightarrow w'' = 2a + 6bx$$

$$\Pi_{total} = \frac{1}{2}\int_0^l EI(w'')^2 dx - \int_0^l P\delta(x-\frac{l}{2})wdx = \frac{1}{2}\int_0^l EI(2a+6bx)^2 dx - P(-a\frac{l^2}{4} - b\frac{3l^3}{8})$$

$$= \frac{1}{2}EI(4a^2l + 24ab\frac{l^2}{2} + 36b^2\frac{l^3}{3}) + P(a\frac{l^2}{4} + b\frac{3l^3}{8})$$

$$\frac{\partial \Pi_{total}}{\partial a} = 0 \rightarrow EI(4la + 6l^2b) = -P\frac{l^2}{4}$$

$$\frac{\partial \Pi_{total}}{\partial b} = 0 \rightarrow EI(6l^2a + 12l^3b) = -P\frac{3l^3}{8}$$

$$\rightarrow a = -\frac{1}{16}\frac{Pl}{EI} \;,\quad b = 0 \quad ???$$

iii) with three unknowns

$$w = ax(x-l) + bx(x^2 - l^2) + cx(x^3 - l^3) \rightarrow w'' = 2a + 6bx + 12cx^2$$

$$\Pi_{total} = \frac{1}{2}\int_0^l EI(w'')^2 dx - \int_0^l P\delta(x-\frac{l}{2})wdx$$

$$= \frac{1}{2}\int_0^l EI(2a+6bx+12cx^2)^2 dx - P(-a\frac{l^2}{4} - b\frac{3l^3}{8} - c\frac{7l^4}{16})$$

$$= \frac{1}{2}EI(4a^2l + 36b^2\frac{l^3}{3} + 144c^2\frac{l^5}{5} + 24ab\frac{l^2}{2} + 48ac\frac{l^3}{3} + 144bc\frac{l^4}{4}) +$$

$$P(a\frac{l^2}{4} + b\frac{3l^3}{8} + c\frac{7l^4}{16})$$

$$\begin{aligned}
\frac{\partial \Pi_{total}}{\partial a} &= 0 \rightarrow EI(4la + 6l^2 b + 8l^3 c) = -P\frac{l^2}{4} \\
\frac{\partial \Pi_{total}}{\partial b} &= 0 \rightarrow EI(6l^2 a + 12l^3 b + 18l^4 c) = -P\frac{3l^3}{8} \\
\frac{\partial \Pi_{total}}{\partial c} &= 0 \rightarrow EI(8l^3 a + 18l^4 b + \frac{144}{5}l^5 c) = -P\frac{7l^4}{16}
\end{aligned}
\rightarrow \begin{cases} a = \frac{1}{64}\frac{Pl}{EI} \\ b = -\frac{5}{32}\frac{P}{EI} \\ \mathrm{c} = \frac{5}{64}\frac{P}{EIl} \end{cases}$$

$$w = \frac{1}{64}\frac{Pl}{EI}x(x-l) - \frac{5}{32}\frac{P}{EI}x(x^2 - l^2) + \frac{5}{64}\frac{P}{EIl}x(x^3 - l^3)$$

$$w(\frac{l}{2}) = \frac{Pl^3}{EI}(-\frac{1}{64}\frac{1}{4} + \frac{5}{32}\frac{3}{8} - \frac{5}{64}\frac{7}{16}) = \frac{21}{1024}\frac{Pl^3}{EI} = 0.0205\frac{Pl^3}{EI} \text{ , Error} = 0.0144$$

iv) with one sin function

$$w = a\sin\frac{\pi}{l}x \Longrightarrow w'' = a(\frac{\pi}{l})^2 \sin\frac{\pi}{l}x$$

$$\begin{aligned}
\Pi_{total} &= \frac{1}{2}\int_0^l EI(w'')^2 dx - \int_0^l P\delta(x - \frac{l}{2})wdx = \frac{1}{2}\int_0^l EI(a(\frac{\pi}{l})^2 \sin\frac{\pi}{l}x)^2 dx - aP \\
&= \frac{1}{2}EIa^2(\frac{\pi}{l})^4 \int_0^l \sin^2\frac{\pi}{l}xdx + aP = \frac{1}{2}EIa^2(\frac{\pi}{l})^4 \frac{l}{2} + aP
\end{aligned}$$

$$\frac{\partial \Pi_{total}}{\partial a} = 0 \rightarrow EIa(\frac{\pi}{l})^4 \frac{l}{2} - P = 0 \rightarrow a = \frac{2}{\pi^4}\frac{Pl^3}{EI} \rightarrow w = \frac{2}{\pi^4}\frac{Pl^3}{EI}\sin\frac{\pi}{l}x$$

$$w(\frac{l}{2}) = \frac{1}{48.7045}\frac{Pl^3}{EI} \quad , \quad w^e(\frac{l}{2}) = \frac{Pl^3}{48EI} \quad , \quad \text{Error} = 0.0145$$

v) with two sin function

$$w = a\sin\frac{\pi}{l}x + b\sin\frac{3\pi}{l}x \Longrightarrow w'' = a(\frac{\pi}{l})^2 \sin\frac{\pi}{l}x + b(\frac{3\pi}{l})^2 \sin\frac{3\pi}{l}x$$

$$\begin{aligned}
\Pi_{total} &= \frac{1}{2}\int_0^l EI(w'')^2 dx - \int_0^l P\delta(x - \frac{l}{2})wdx \\
&= \frac{1}{2}\int_0^l EI(a(\frac{\pi}{l})^2 \sin\frac{\pi}{l}x + b(\frac{3\pi}{l})^2 \sin\frac{3\pi}{l}x)^2 dx - aP + bP \\
&= \frac{1}{2}EIa^2(\frac{\pi}{l})^4 \int_0^l \sin^2\frac{\pi}{l}xdx + EIab(\frac{\pi}{l})^2(\frac{3\pi}{l})^2 \int_0^l \sin\frac{\pi}{l}\sin\frac{3\pi}{l}xdx + \\
&\quad \frac{1}{2}EIb^2(\frac{3\pi}{l})^4 \int_0^l \sin^2\frac{3\pi}{l}xdx - aP + bP \\
&= \frac{1}{2}EIa^2(\frac{\pi}{l})^4 \frac{l}{2} + \frac{1}{2}EIb^2(\frac{3\pi}{l})^4 \frac{l}{2} - aP + bP
\end{aligned}$$

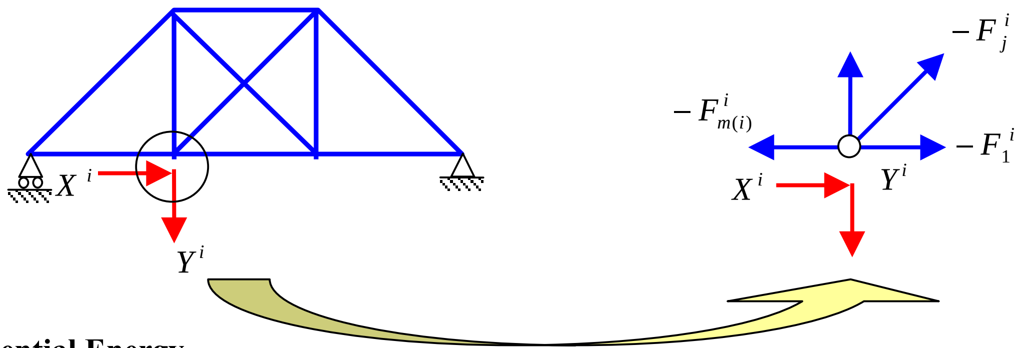
$$\frac{\partial \Pi_{total}}{\partial a} = 0 \rightarrow EIa\left(\frac{\pi}{l}\right)^4 \frac{l}{2} - P = 0 \rightarrow a = \frac{2}{\pi^4}\frac{Pl^3}{EI}$$

$$\frac{\partial \Pi_{total}}{\partial b} = 0 \rightarrow EIb\left(\frac{3\pi}{l}\right)^4 \frac{l}{2} + P = 0 \rightarrow b = -\frac{2}{(3\pi)^4}\frac{Pl^3}{EI}$$

$$w = \frac{2}{\pi^4}\frac{Pl^3}{EI}\left(\sin\frac{\pi}{l}x - \frac{1}{81}\sin\frac{3\pi}{l}x\right)$$

$$w\left(\frac{l}{2}\right) = 0.0205(1+0.0123)\frac{Pl^3}{EI} = 0.0208\frac{Pl^3}{EI}\ ,\quad w^e\left(\frac{l}{2}\right) = 0.0208\frac{Pl^3}{EI}\ ,\ \text{Error} \cong 0$$

4.3 Truss problems



- **Potential Energy**

$$\Pi_{\text{int}} = \frac{1}{2}\sum_{i=1}^{nmb}\frac{(F_i)^2 l_i}{EA^i}\ ,\quad \Pi_{ext} = -\sum_{i=1}^{njn}(X_i u_i + Y_i v_i)$$

$$\Pi_{total} = \Pi_{\text{int}} + \Pi_{ext} = \frac{1}{2}\sum_{i=1}^{nmb}\frac{(F_i)^2 l_i}{EA^i} - \sum_{i=1}^{njn}(X_i u_i + Y_i v_i)$$

where *nmb* and *njn* denotes the total number of members and the total numbers of joints in a truss.

- **Equilibrium Equations**

$$-\sum_{j=1}^{m(i)} H_j^i + X^i = 0\ ,\quad -\sum_{j=1}^{m(i)} V_j^i + Y^i = 0 \qquad \text{for} \quad i = 1,\cdots,njn$$

where $m(i)$, H_j^i and V_j^i are the number of member connected to joint i, the horizontal component and the vertical component of the bar force of j-th member connected to joint i, respectively.

- **Principle of Minimum Potential Energy**

$$\Pi_{total} = \frac{1}{2}\sum_{i=1}^{nmb}\frac{(F_i^e + \overline{F}_i)^2 l_i}{EA_i} - \sum_{i=1}^{njn}(X_i(u_i + \overline{u}_i) + Y_i(v_i + \overline{v}_i))$$

$$= \frac{1}{2}\sum_{i=1}^{nmb}\frac{(F_i^e)^2 l_i}{EA_i} - \sum_{i=1}^{njn}(X_i u_i + Y_i v_i) + \frac{1}{2}\sum_{i=1}^{nmb}\frac{(\overline{F}_i)^2 l_i}{EA_i} + \boxed{\sum_{i=1}^{nmb}\frac{F_i^e \overline{F}_i l_i}{EA_i} - \sum_{i=1}^{njn}(X_i\overline{u}_i + Y_i\overline{v}_i)}$$

$$= \frac{1}{2}\sum_{i=1}^{nmb}\frac{(F_i^e)^2 l_i}{EA_i} - \sum_{i=1}^{njn}(X_i u_i + Y_i v_i) + \sum_{i=1}^{nmb}\frac{(\overline{F}_i)^2 l_i}{EA_i}$$

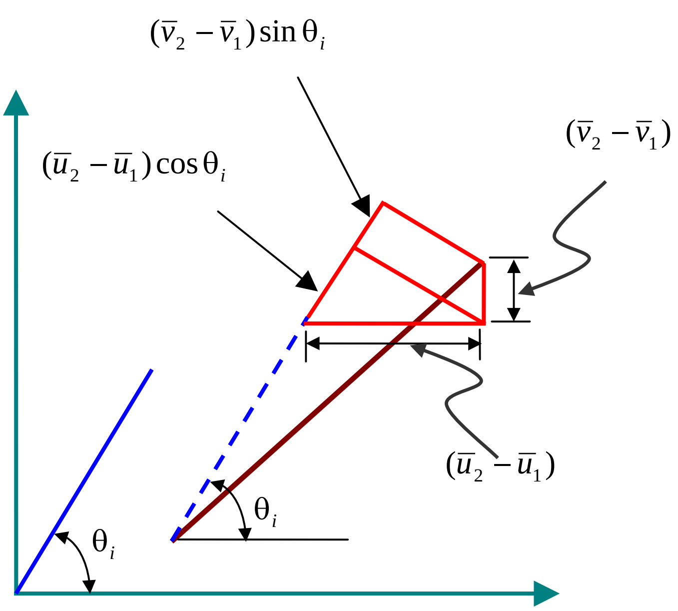
$$= \Pi^e + \frac{1}{2}\sum_{i=1}^{nmb}\frac{(\overline{F}_i)^2 l_i}{EA_i} \geq \Pi^e \qquad \text{for all virtual displacement fields}$$

where F_i^e and $\overline{F}_i$ are the bar force of *i*-th member induced by the real displacement of joints and virtual displacement induced by the virtual displacement of joints. Since the equation in the box represents the total virtual work in a truss, the total potential energy of a truss becomes minimum for all virtual displacement fields when the principle of virtual work holds. In the above equation, the equality sign holds if and only if the virtual displacements at all joints are zero.

- **Virtual Work Expression**

If a truss is in equilibrium, the principle of the virtual work holds for the truss,.

$$\sum_{i=1}^{njn}((-\sum_{j=1}^{m(i)} H_j^i + X^i)\overline{u}^i + (-\sum_{j=1}^{m(i)} V_j^i + Y^i)\overline{v}^i) = 0$$



$$\sum_{i=1}^{njn}((-\sum_{j=1}^{m(i)}F_j^i\cos\theta_j + X_i)\bar{u}_i + (-\sum_{j=1}^{m(i)}F_j^i\sin\theta_j + Y_i)\bar{v}_i) = 0$$

$$\sum_{i=1}^{njn}(\bar{u}^i\sum_{j=1}^{m(i)}F_j^i\cos\theta_j + \delta v^i\sum_{j=1}^{m(i)}F_j^i\sin\theta_j) = \sum_{i=1}^{njn}(X_i\bar{u}_i + Y_i\bar{v}_i)$$

$$\sum_{i=1}^{nmb}(F_i^e\cos\theta_i(\bar{u}_i^2 - \bar{u}_i^1) + F_i^e\sin\theta_i(\bar{v}_i^2 - \bar{v}_i^1)) = \sum_{i=1}^{njn}(X_i\bar{u}_i + Y_i\bar{v}_i)$$

$$\sum_{i=1}^{nmb}F_i^e(\cos\theta_i(\bar{u}_i^2 - \bar{u}_i^1) + \sin\theta_i(\bar{v}_i^2 - \bar{v}_i^1)) = \sum_{i=1}^{nmb}F_i^e\Delta\bar{l}_i = \sum_{i=1}^{nmb}F_i^e\frac{\bar{F}_i\ l_i}{(EA_i)} = \sum_{i=1}^{nmb}\frac{F_i^e\bar{F}_i\ l_i}{(EA_i)}$$

$$\sum_{i=1}^{nmb}\frac{F_i^e\bar{F}_i\ l_i}{(EA_i)} = \sum_{i=1}^{n}(X_i\ \bar{u}_i + Y_i\ \bar{v}_i)$$

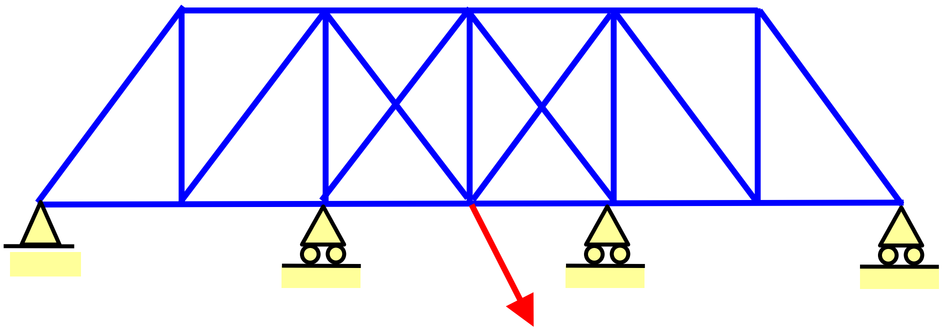
The principle of virtual work yields the displacement of a joint *k* in a truss by applying an unit load at a joint k in an arbitrary direction and by using the reciprocal theorem.

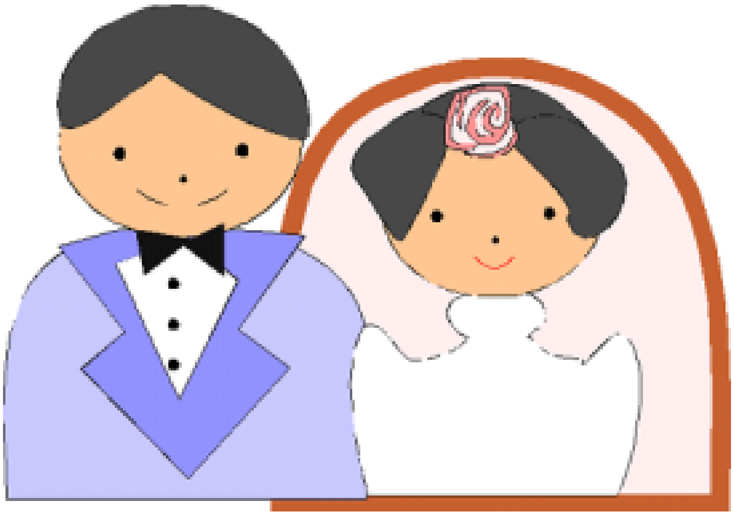
$$\bar{X}_k u_k + \bar{Y}_k\ v_k = \|\mathbf{X}\|\|\mathbf{u}_k\|\cos\alpha = \|\mathbf{u}_k\|\cos\alpha = \sum_{i=1}^{nmb}\frac{F_i^e\bar{F}_i\ l_i}{(EA_i)}$$

Since α represnts the angle between the applied unit load and the displacement vector, $\|\mathbf{u}_k\|\cos\alpha$ are the displacement of the joint *k* in the direction of the applied unit load.

Chapter 5

Matrix Structural Analysis





Mr. Force & Ms. Displacement

Matchmaker: **Stiffness Matrix**

5.1 Truss Problems

5.1.1 Member Stiffness Matrix

- **Force – Displacement relation at Member ends**

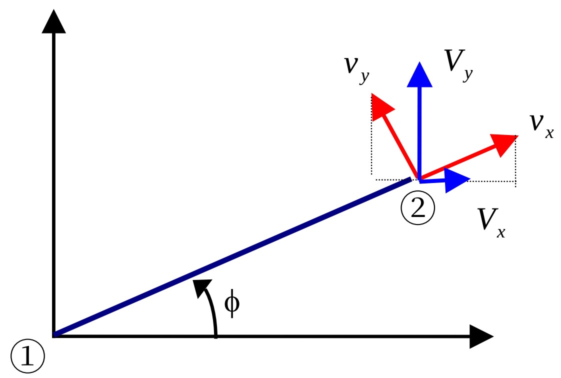
$$
\begin{aligned}
f_x^L &= -\frac{EA}{L}(\delta_x^R - \delta_x^L) \\
f_x^R &= \frac{EA}{L}(\delta_x^R - \delta_x^L) \\
f_y^L &= f_y^R = 0
\end{aligned}
$$

- **Member Stiffness Matrix in Local Coordinate System**

$$
\begin{pmatrix} f_x^L \\ f_y^L \\ f_x^R \\ f_y^R \end{pmatrix}^e = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \delta_x^R \\ \delta_y^R \end{pmatrix}^e
$$

$$
\underline{\underline{(\mathbf{f})^e = [\mathbf{k}]^e(\delta)^e}}
$$

- Transformation Matrix



$$
\left.\begin{aligned}
V_x &= \cos\phi v_x - \sin\phi v_y \\
V_y &= \sin\phi v_x + \cos\phi v_y
\end{aligned}\right\} \rightarrow \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}
$$

$$
(\mathbf{V}) = [\gamma]^T(\mathbf{v}) \rightarrow (\mathbf{v}) = [\gamma](\mathbf{V})
$$

- **Member End Force**

$$\begin{pmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{pmatrix}^e = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} f_x^L \\ f_y^L \\ f_x^R \\ f_y^R \end{pmatrix}^e \rightarrow \underline{\underline{(\mathbf{F})^e = [\Gamma]^T(\mathbf{f})^e}}$$

- **Member End Displacement**

$$\begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \delta_x^R \\ \delta_y^R \end{pmatrix}^e = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} \Delta_x^1 \\ \Delta_y^1 \\ \Delta_x^2 \\ \Delta_y^2 \end{pmatrix}^e \rightarrow \underline{\underline{(\delta)^e = [\Gamma](\Delta)^e}}$$

- **Member Stiffness Matrix in Global Coordinate**

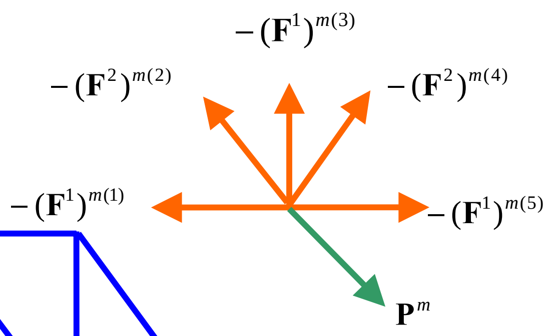
$$(\mathbf{F})^e = [\Gamma]^T(\mathbf{f})^e = [\Gamma]^T[\mathbf{k}]^e(\delta)^e = [\Gamma]^T[\mathbf{k}]^e[\Gamma](\Delta)^e$$

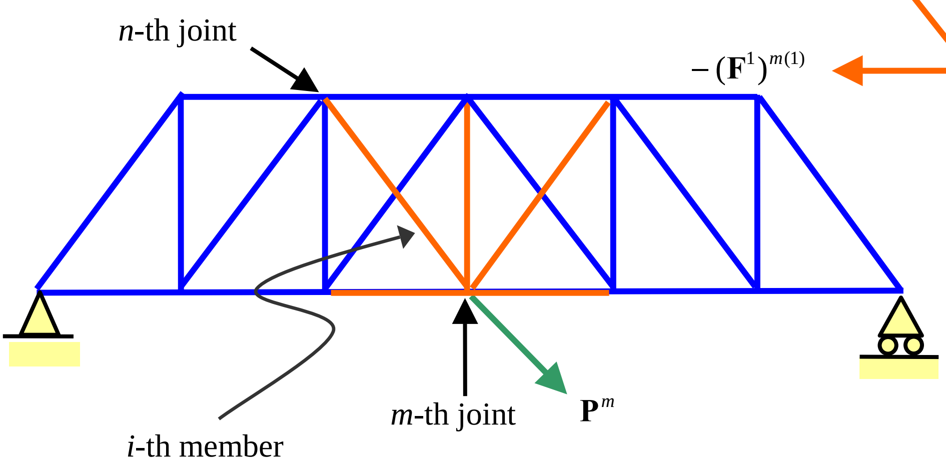
$$\underline{\underline{(\mathbf{F})^e = [\mathbf{K}]^e(\Delta)^e}}$$

$$[\mathbf{K}]^e = \frac{EA}{L}\left[\begin{array}{cc|cc} \cos^2\phi & \sin\theta\cos\phi & -\cos^2\phi & -\sin\phi\cos\phi \\ \sin\theta\cos\phi & \sin^2\phi & -\sin\phi\cos\phi & -\sin^2\phi \\ \hline -\cos^2\phi & -\sin\theta\cos\phi & \cos^2\phi & \sin\phi\cos\phi \\ -\sin\phi\cos\phi & -\sin^2\phi & \sin\phi\cos\phi & \sin^2\phi \end{array}\right] = \begin{bmatrix} [\mathbf{K}_{11}]^e & [\mathbf{K}_{12}]^e \\ [\mathbf{K}_{21}]^e & [\mathbf{K}_{22}]^e \end{bmatrix}$$

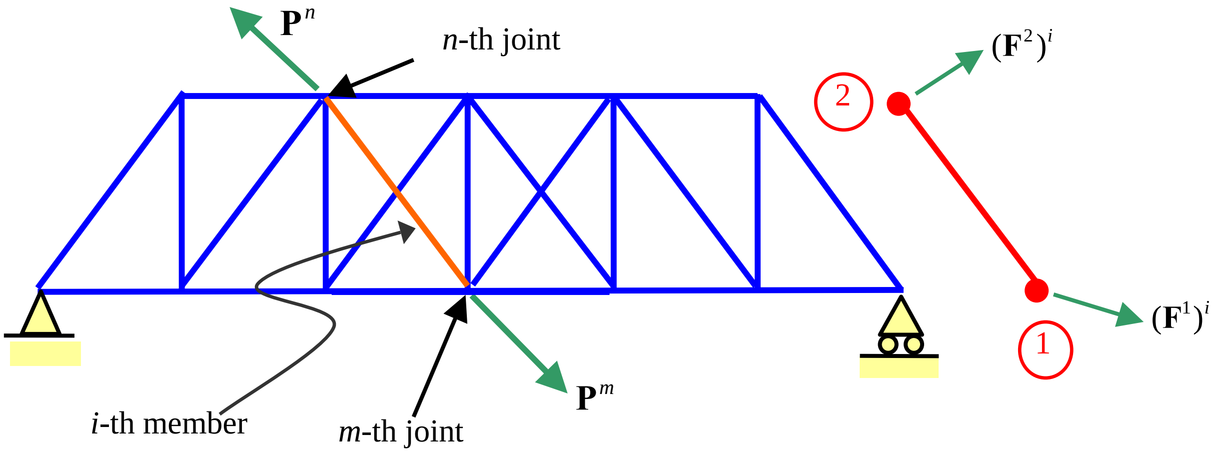
5.1.2 Global Stiffness Equation

- **Nodal Equilibrium**





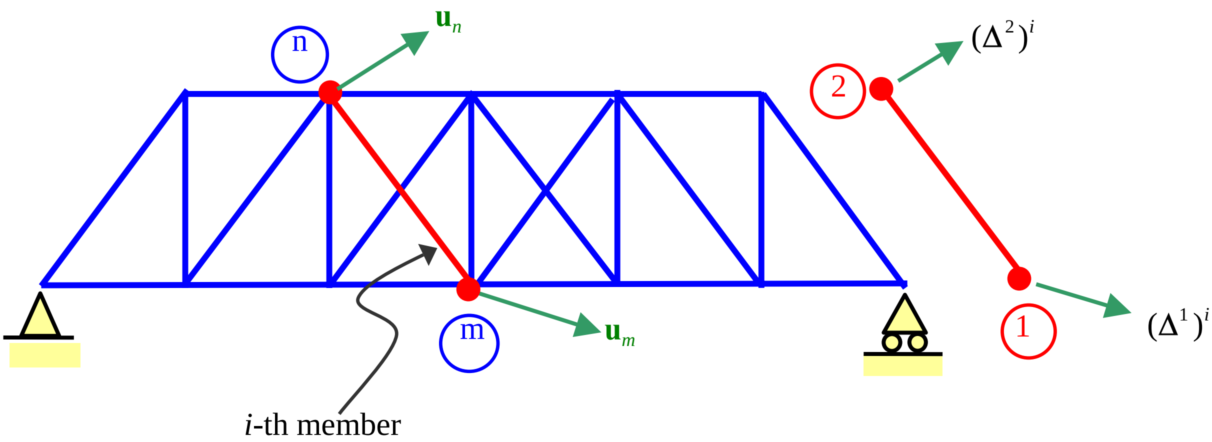
$$(\mathbf{P})^m = (\mathbf{F}^1)^{m(1)} + (\mathbf{F}^2)^{m(2)} + (\mathbf{F}^1)^{m(3)} + (\mathbf{F}^2)^{m(4)} + (\mathbf{F}^1)^{m(5)} = \sum_{k=1}^{nm(m)} (\mathbf{F}^{1\,\mathrm{or}\,2})^{m(k)}$$



$$(\mathbf{P}) = \begin{pmatrix} (\mathbf{P})^1 \\ \vdots \\ (\mathbf{P})^m \\ \vdots \\ (\mathbf{P})^q \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{nm(1)} (\mathbf{F}^{1\,\text{or}\,2})^{1(k)} \\ \vdots \\ \sum_{k=1}^{nm(m)} (\mathbf{F}^{1\,\text{or}\,2})^{m(k)} \\ \vdots \\ \sum_{k=1}^{nm(q)} (\mathbf{F}^{1\,\text{or}\,2})^{q(k)} \end{pmatrix} = \sum_{i=1}^{p} \begin{pmatrix} 0 \\ \vdots \\ (\mathbf{F}^1)^i \\ \vdots \\ (\mathbf{F}^2)^i \\ \vdots \\ 0 \end{pmatrix} = \sum_{i=1}^{p} \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \mathbf{I} & 0 \\ \vdots & \vdots \\ 0 & \mathbf{I} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ m\text{-th row} \\ \\ n\text{-th row} \\ \\ \\ \end{array} \begin{pmatrix} (\mathbf{F}^1)^i \\ (\mathbf{F}^2)^i \end{pmatrix} = \sum_{i=1}^{p} [\mathbf{E}]^i (\mathbf{F})^i = [\mathbf{E}](\mathbf{F})$$

$$[\mathbf{E}] = \left[[\mathbf{E}]^1 \ \cdots \ [\mathbf{E}]^i \ \cdots \ [\mathbf{E}]^p \right], \quad (\mathbf{F}) = \begin{pmatrix} (\mathbf{F})^1 \\ \vdots \\ (\mathbf{F})^i \\ \vdots \\ (\mathbf{F})^p \end{pmatrix}$$

- **Compatability Condition**



$$(\Delta^1)^i = \mathbf{u}_m\ ,\ (\Delta^2)^i = \mathbf{u}_n \rightarrow (\Delta)^i = \begin{pmatrix} (\Delta^1)^i \\ (\Delta^2)^i \end{pmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{pmatrix} \mathbf{u}_m \\ \mathbf{u}_n \end{pmatrix}$$

$$(\Delta)^i = \begin{pmatrix} (\Delta^1)^i \\ (\Delta^2)^i \end{pmatrix} = \begin{array}{c} \qquad\qquad m\text{-th column} \\ \begin{bmatrix} 0 & \cdots & \mathbf{I} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & \mathbf{I} & \cdots & 0 \end{bmatrix} \\ \qquad\qquad\qquad n\text{-th column} \end{array} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \\ \vdots \\ \mathbf{u}_n \\ \vdots \\ \mathbf{u}_q \end{pmatrix} = [\mathbf{C}]^i(\mathbf{u})$$

$$(\Delta) = \begin{pmatrix} (\Delta)^1 \\ \vdots \\ (\Delta)^p \end{pmatrix} = \begin{bmatrix} [\mathbf{C}]^1 \\ \vdots \\ [\mathbf{C}]^p \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_q \end{pmatrix} = \underbrace{[\mathbf{C}]}_{\text{Compatibility Matrix}}(\mathbf{u})$$

- **Contragradient**

$$(\mathbf{P})^T \cdot (\mathbf{u}) = (\mathbf{F})^T \cdot (\Delta) \rightarrow (\mathbf{P})^T \cdot (\mathbf{u}) = (\mathbf{F})^T [\mathbf{C}](\mathbf{u}) \rightarrow$$

$$((\mathbf{P})^T - (\mathbf{F})^T[\mathbf{C}])(\mathbf{u}) = 0 \ \text{ for all possible } \ (\mathbf{u}) \rightarrow (\mathbf{P})^T = (\mathbf{F})^T[\mathbf{C}]$$

$$\underline{\underline{(\mathbf{P}) = [\mathbf{C}]^T(\mathbf{F}) = [\mathbf{E}](\mathbf{F}) \rightarrow [\mathbf{C}]^T = [\mathbf{E}]}}$$

- **Unassembled Member Stiffness Equation**

$$\begin{pmatrix} (\mathbf{F})^1 \\ \vdots \\ (\mathbf{F})^i \\ \vdots \\ (\mathbf{F})^p \end{pmatrix} = \begin{bmatrix} [\mathbf{K}]^1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \\ 0 & \cdots & [\mathbf{K}]^i & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & [\mathbf{K}]^p \end{bmatrix} \begin{pmatrix} (\Delta)^1 \\ \vdots \\ (\Delta)^i \\ \vdots \\ (\Delta)^p \end{pmatrix} \rightarrow \underline{\underline{(\mathbf{F}) = [\overline{\mathbf{K}}](\Delta)}}$$

- **Global Stiffness Equation**

$$(\mathbf{P}) = [\mathbf{C}]^T(\mathbf{F}) = [\mathbf{C}]^T[\overline{\mathbf{K}}](\Delta) = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}](\mathbf{u})$$

$$\underline{\underline{(\mathbf{P}) = [\mathbf{K}](\mathbf{u})}} \quad \text{where} \quad \underline{\underline{[\mathbf{K}] = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}]}}$$

- **Direct Stiffness Method**

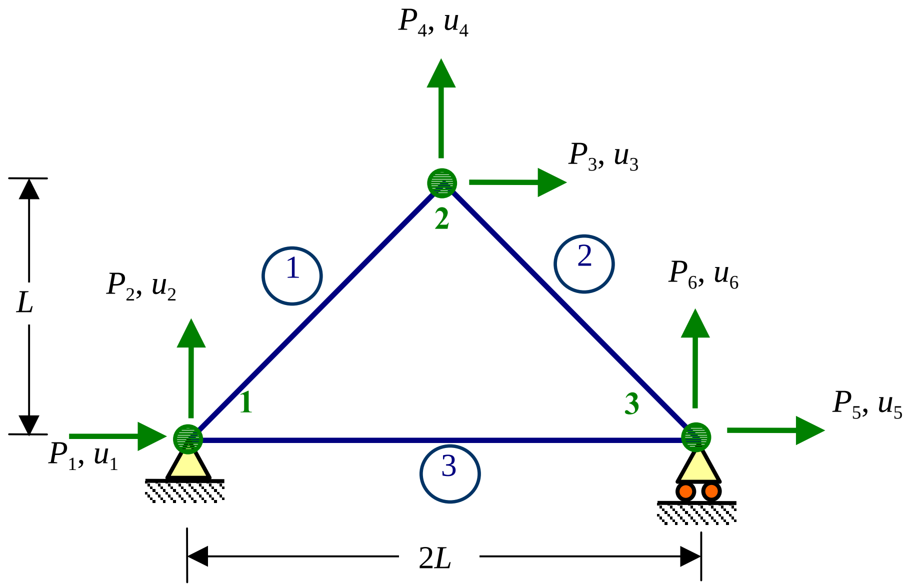
$$[\mathbf{K}] = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}] = \left[[\mathbf{C}]^{1^T} \cdots [\mathbf{C}]^{i^T} \cdots [\mathbf{C}]^{p^T}\right] \begin{bmatrix} [\mathbf{K}]^1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \\ 0 & \cdots & [\mathbf{K}]^i & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & [\mathbf{K}]^p \end{bmatrix} \begin{bmatrix} [\mathbf{C}]^1 \\ \vdots \\ [\mathbf{C}]^i \\ \vdots \\ [\mathbf{C}]^p \end{bmatrix}$$

$$= [\mathbf{C}]^{1^T}[\mathbf{K}]^1[\mathbf{C}]^1 + \cdots + [\mathbf{C}]^{i^T}[\mathbf{K}]^i[\mathbf{C}]^i + \cdots + [\mathbf{C}]^{p^T}[\mathbf{K}]^p[\mathbf{C}]^p$$

$$[\mathbf{C}]^{i^T}[\mathbf{K}]^i[\mathbf{C}]^i = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \mathbf{I} & 0 \\ \vdots & \vdots \\ \vdots & \mathbf{I} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\mathbf{K}_{11}]^i & [\mathbf{K}_{12}]^i \\ [\mathbf{K}_{21}]^i & [\mathbf{K}_{22}]^i \end{bmatrix} \begin{bmatrix} 0 & \cdots & \mathbf{I} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & \mathbf{I} & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \mathbf{I} & 0 \\ \vdots & \vdots \\ \vdots & \mathbf{I} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & [\mathbf{K}_{11}]^i & \cdots & [\mathbf{K}_{12}]^i & \cdots & 0 \\ 0 & \cdots & [\mathbf{K}_{21}]^i & \cdots & [\mathbf{K}_{22}]^i & \cdots & 0 \end{bmatrix} = \begin{array}{r} \\ m\text{-th row} \\ \\ n\text{-th row} \\ \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & [\mathbf{K}_{11}]^i & 0 & [\mathbf{K}_{12}]^i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & [\mathbf{K}_{21}]^i & 0 & [\mathbf{K}_{22}]^i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad m\text{-th column} \quad n\text{-th column} \end{array}$$

5.1.3 Example



- **Member Stiffness Matrix**

$$[\mathbf{K}]^e = \frac{EA}{L}\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi & -\cos^2\phi & -\sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi & -\sin\phi\cos\phi & -\sin^2\phi \\ -\cos^2\phi & -\sin\phi\cos\phi & \cos^2\phi & \sin\phi\cos\phi \\ -\sin\phi\cos\phi & -\sin^2\phi & \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} = \begin{bmatrix} [\mathbf{K}_{11}]^e & [\mathbf{K}_{12}]^e \\ [\mathbf{K}_{21}]^e & [\mathbf{K}_{22}]^e \end{bmatrix}$$

- Member 1: $\phi = 45^\circ$

$$[\mathbf{K}]^1 = \frac{EA}{\sqrt{2}L}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

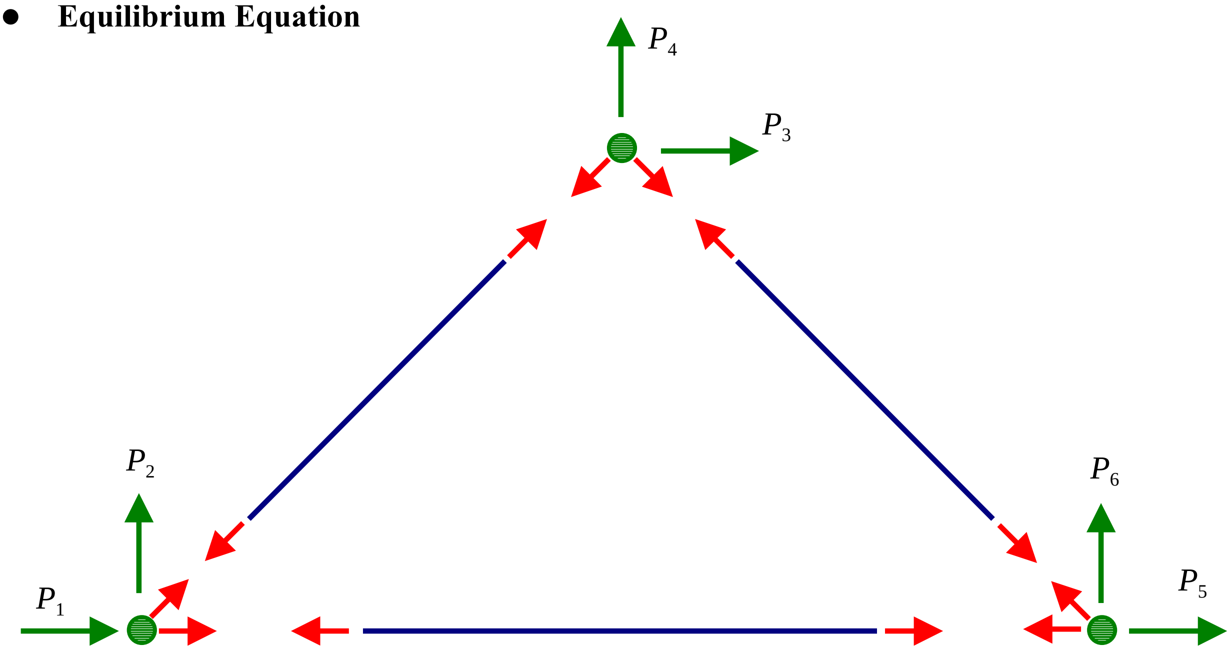
- Member 2: $\phi = -45^\circ$

$$[\mathbf{K}]^2 = \frac{EA}{\sqrt{2}L}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- Member 3: ϕ = 0°

$$[\mathbf{K}]^3 = \frac{EA}{2L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Equilibrium Equation**

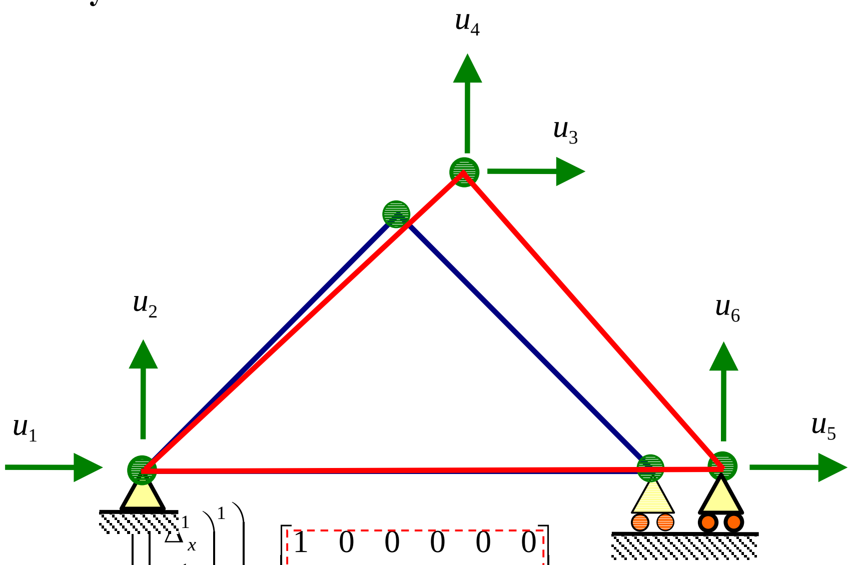


$$\begin{aligned} P_1 &= (F_x^1)^1 + (F_x^2)^3 \\ P_2 &= (F_y^1)^1 + (F_y^2)^3 \\ P_3 &= (F_x^2)^1 + (F_x^1)^2 \\ P_4 &= (F_y^2)^1 + (F_y^1)^2 \\ P_5 &= (F_x^2)^2 + (F_x^1)^3 \\ P_6 &= (F_y^2)^2 + (F_y^1)^3 \end{aligned}$$

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{pmatrix} = \left[\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}_{[\mathbf{E}]^1} \; \underbrace{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{[\mathbf{E}]^2} \; \underbrace{\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}}_{[\mathbf{E}]^3} \right] \begin{pmatrix} \begin{pmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{pmatrix}^1 \\ \begin{pmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{pmatrix}^2 \\ \begin{pmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{pmatrix}^3 \end{pmatrix}$$

$$\underline{\underline{(\mathbf{P}) = [[\mathbf{E}^1]\ [\mathbf{E}^2]\ [\mathbf{E}^2]](\mathbf{F}) = [\mathbf{E}](\mathbf{F})}}$$

- **Compatibility Condition**



$$
\begin{pmatrix}
\begin{pmatrix}\Delta_x^1\\ \Delta_y^1\\ \Delta_x^2\\ \Delta_y^2\end{pmatrix}^1\\
\begin{pmatrix}\Delta_x^1\\ \Delta_y^1\\ \Delta_x^2\\ \Delta_y^2\end{pmatrix}^2\\
\begin{pmatrix}\Delta_x^1\\ \Delta_y^1\\ \Delta_x^2\\ \Delta_y^2\end{pmatrix}^3
\end{pmatrix}
=
\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\ \hline
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\ \hline
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{l}
\leftarrow [\mathbf{C}]^1\\ \\ \\ \\
\leftarrow [\mathbf{C}]^2\\ \\ \\ \\
\leftarrow [\mathbf{C}]^3\\ \\ \\ \\
\end{array}
\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{pmatrix}
=
\begin{bmatrix}[\mathbf{C}]^1\\ [\mathbf{C}]^2\\ [\mathbf{C}]^3\end{bmatrix}
\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{pmatrix}
$$

$$\underline{\underline{(\Delta) = [\mathbf{C}](\mathbf{u})}}$$

$$[\mathbf{E}^1] = [\mathbf{C}^1]^T,\ \ [\mathbf{E}^2] = [\mathbf{C}^2]^T,\ \ [\mathbf{E}^3] = [\mathbf{C}^3]^T \ \rightarrow\ [\mathbf{E}] = [\mathbf{C}]^T$$

- **Global Stiffness Matrix**

$$[\mathbf{K}] = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}] = [\mathbf{C}]^{1^T}[\mathbf{K}]^1[\mathbf{C}]^1 + [\mathbf{C}]^{2^T}[\mathbf{K}]^2[\mathbf{C}]^2 + \cdots + [\mathbf{C}]^{3^T}[\mathbf{K}]^3[\mathbf{C}]^3$$

$$[\mathbf{C}^1]^T[\mathbf{K}^1][\mathbf{C}^1] = \frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} = \frac{EA}{\sqrt{2}L}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{C}^2]^T[\mathbf{K}^2][\mathbf{C}^2] = \frac{EA}{\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{EA}{\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[\mathbf{C}^3]^T[\mathbf{K}^3][\mathbf{C}^3] = \frac{EA}{2L}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
= \frac{EA}{2L}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \frac{EA}{2L}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
[\mathbf{K}] = \frac{EA}{\sqrt{2}L}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EA}{\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}
$$

$$
+ \frac{EA}{2L}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= \frac{EA}{L}\begin{bmatrix} \frac{1}{2\sqrt{2}}+\frac{1}{2} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}+\frac{1}{2} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}
$$

- **Stiffness Equation**

$$
\begin{array}{r}
\text{Unknown} \rightarrow \\ \text{Unknown} \rightarrow \\ \text{Known} \rightarrow \\ \text{Known} \rightarrow \\ \text{Known} \rightarrow \\ \text{Unknown} \rightarrow
\end{array}
\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{pmatrix} = \frac{EA}{L}
\begin{bmatrix}
\frac{1+\sqrt{2}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & 0 \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
-\frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}
\end{bmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}
\begin{array}{l}
\leftarrow \text{Known} \\ \leftarrow \text{Known} \\ \leftarrow \text{Unknown} \\ \leftarrow \text{Unknown} \\ \leftarrow \text{Unknown} \\ \leftarrow \text{Known}
\end{array}
$$

- **Application of Support Conditions (Boundary Conditions)**

$$
\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{pmatrix} = \frac{EA}{L}
\begin{bmatrix}
\frac{1+\sqrt{2}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & 0 \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
-\frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}
\end{bmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}
$$

- **Final Stiffness Equation**

$$
\begin{pmatrix} P_3 \\ P_4 \\ P_5 \end{pmatrix} = \frac{EA}{L}
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
-\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{2\sqrt{2}}
\end{bmatrix}
\begin{pmatrix} u_3 \\ u_4 \\ u_5 \end{pmatrix}
\rightarrow
\begin{pmatrix} u_3 \\ u_4 \\ u_5 \end{pmatrix} = \frac{L}{EA}
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
-\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{2\sqrt{2}}
\end{bmatrix}^{-1}
\begin{pmatrix} P_3 \\ P_4 \\ P_5 \end{pmatrix}
$$

5.2 Beam Problems

5.2.1 Member Stiffness Matrix

- **Force-Displacement Relation at Member Ends**

$$m^L = \frac{4EI_e}{L_e}\theta^L + \frac{2EI_e}{L_e}\theta^R + \frac{6EI_e}{L_e^2}\delta_y^L - \frac{6EI_e}{L_e^2}\delta_y^R$$

$$m^R = \frac{2EI_e}{L_e}\theta^L + \frac{4EI_e}{L_e}\theta^R + \frac{6EI_e}{L_e^2}\delta_y^L - \frac{6EI_e}{L_e^2}\delta_y^R$$

$$f_y^L = \frac{M_e^L + M_e^R}{L_e} = \frac{6EI_e}{L_e^2}\theta^L + \frac{6EI_e}{L_e^2}\theta^R + \frac{12EI_e}{L_e^3}\delta_y^L - \frac{12EI_e}{L_e^3}\delta_y^R$$

$$f_y^R = -\frac{M_e^L + M_e^R}{L_e} = -\frac{6EI_e}{L_e^2}\theta^L - \frac{6EI_e}{L_e^2}\theta^R - \frac{12EI_e}{L_e^3}\delta_y^L + \frac{12EI_e}{L_e^3}\delta_y^R$$

- **Transformation Matrix is not required**

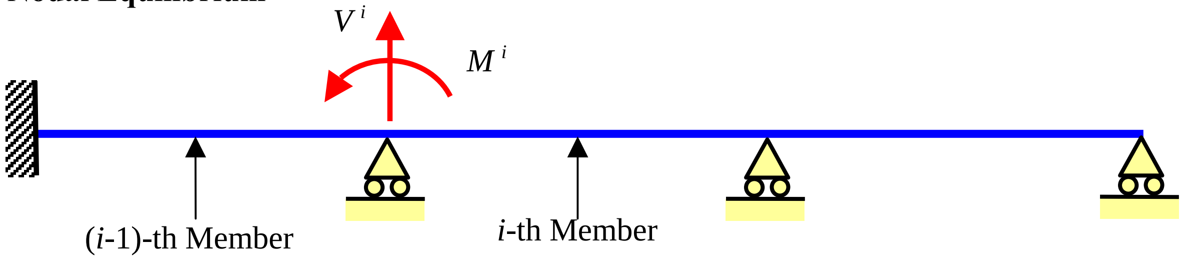
$$f \to F\,, \quad m \to M\,, \quad \delta \to \Delta \;, \theta \to \Theta$$

- **Member Stiffness Matrix**

$$\begin{pmatrix} F_y^1 \\ M^1 \\ F_y^2 \\ M^2 \end{pmatrix}^e = \frac{EI_e}{L_e}\begin{bmatrix} \frac{12}{L_e^2} & \frac{6}{L_e} & -\frac{12}{L_e^2} & \frac{6}{L_e} \\ \frac{6}{L_e} & 4 & -\frac{6}{L_e} & 2 \\ -\frac{12}{L_e^2} & -\frac{6}{L_e} & \frac{12}{L_e^2} & -\frac{6}{L_e} \\ \frac{6}{L_e} & 2 & -\frac{6}{L_e} & 4 \end{bmatrix}\begin{pmatrix} \Delta_y^1 \\ \Theta^1 \\ \Delta_y^2 \\ \Theta_y^2 \end{pmatrix}^e \quad \text{or} \quad \underline{\underline{(\mathbf{F})^e = [\mathbf{K}]^e(\Delta)^e}}$$

5.1.2 Global Stiffness Matrix

- **Nodal Equilibrium**

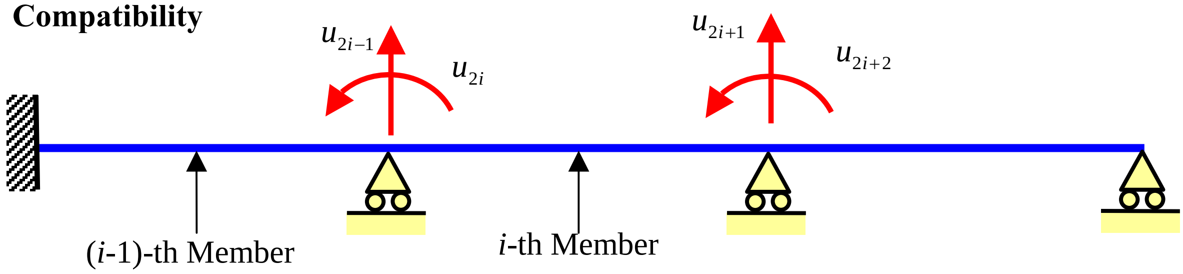


$$\left.\begin{aligned} V^i &= (F_y^2)_{i-1} + (F_y^1)_i \\ M^i &= (M^2)_{i-1} + (M^1)_i \end{aligned}\right\} \rightarrow \mathbf{P}^i = (\mathbf{F})_{i-1}^R + (\mathbf{F})_i^L$$

$$\begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \vdots \\ \mathbf{P}^{i-1} \\ \mathbf{P}^i \\ \mathbf{P}^{i+1} \\ \vdots \\ \mathbf{P}^p \\ \mathbf{P}^{p+1} \end{pmatrix} = \begin{pmatrix} (\mathbf{F})_1^L \\ (\mathbf{F})_1^R + (\mathbf{F})_2^L \\ \vdots \\ (\mathbf{F})_{i-2}^R + (\mathbf{F})_{i-1}^L \\ (\mathbf{F})_{i-1}^R + (\mathbf{F})_i^L \\ (\mathbf{F})_i^R + (\mathbf{F})_{i+1}^L \\ \vdots \\ (\mathbf{F})_{p-1}^R + (\mathbf{F})_p^L \\ (\mathbf{F})_p^R \end{pmatrix} = \begin{bmatrix} [\mathbf{E}]_1 & \cdots & [\mathbf{E}]_i & \cdots & [\mathbf{E}]_p \end{bmatrix} \begin{pmatrix} (\mathbf{F})_1 \\ \vdots \\ (\mathbf{F})_i \\ \vdots \\ (\mathbf{F})_p \end{pmatrix} = [\mathbf{E}](\mathbf{F})$$

$$\begin{array}{r} \\ \\ i\text{-th row} \\ (i+1)\text{-th row} \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ (\mathbf{F})_i^L \\ (\mathbf{F})_i^R \\ \vdots \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \mathbf{I} & 0 \\ 0 & \mathbf{I} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{pmatrix} (\mathbf{F})_i^L \\ (\mathbf{F})_i^R \end{pmatrix} = [\mathbf{E}]_i (\mathbf{F})_i$$

- **Compatibility**



$$\begin{aligned} (\Delta_y^1)_i &= u_{2i-1} \\ (\Theta^1)_i &= u_{2i} \\ (\Delta_y^2)_i &= u_{2i+1} \\ (\Theta^2)_i &= u_{2i+2} \end{aligned} \rightarrow \quad (\Delta)_i^L = \mathbf{u}^i \,,\ (\Delta)_i^R = \mathbf{u}^{i+1} \rightarrow (\Delta)^i = \begin{pmatrix} (\Delta)_i^L \\ (\Delta)_i^R \end{pmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{u}^i \\ \mathbf{u}^{i+1} \end{pmatrix}$$

$$\begin{array}{c} \begin{array}{cccccc} & & i\text{-th column} & & & \end{array} \\ (\Delta)_i = \begin{pmatrix}(\Delta)_i^L\\(\Delta)_i^R\end{pmatrix} = \begin{bmatrix}0&\cdots&\mathbf{I}&0&\cdots&0\\0&\cdots&0&\mathbf{I}&\cdots&0\end{bmatrix}\begin{pmatrix}\mathbf{u}^1\\\vdots\\\mathbf{u}^i\\\mathbf{u}^{i+1}\\\vdots\\\mathbf{u}^{p+1}\end{pmatrix} = [\mathbf{C}]_i(\mathbf{u}) \\ \begin{array}{cccccc} & & & (i+1)\text{-th column} & & \end{array} \end{array}$$

$$(\Delta) = \begin{pmatrix}(\Delta)_1\\\vdots\\(\Delta)_p\end{pmatrix} = \begin{bmatrix}[\mathbf{C}]_1\\\vdots\\[\mathbf{C}]_p\end{bmatrix}\begin{pmatrix}\mathbf{u}^1\\\vdots\\\mathbf{u}^{p+1}\end{pmatrix} = [\mathbf{C}](\mathbf{u})$$

- **Unassembled Member Stiffness Equation**

$$\begin{pmatrix}(\mathbf{F})_1\\\vdots\\(\mathbf{F})_i\\\vdots\\(\mathbf{F})_p\end{pmatrix} = \begin{bmatrix}[\mathbf{K}]_1&\cdots&0&\cdots&0\\\vdots&\cdots&\vdots&\cdots&\\0&\cdots&[\mathbf{K}]_i&\cdots&0\\\vdots&\cdots&\vdots&\cdots&\vdots\\0&\cdots&0&\cdots&[\mathbf{K}]_p\end{bmatrix}\begin{pmatrix}(\Delta)_1\\\vdots\\(\Delta)_i\\\vdots\\(\Delta)_p\end{pmatrix} \rightarrow \underline{\underline{(\mathbf{F}) = [\overline{\mathbf{K}}](\Delta)}}$$

- **Global Stiffness Equation**

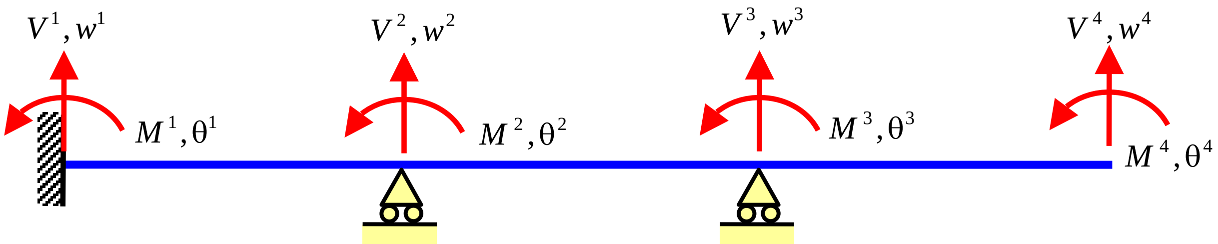
$$(\mathbf{P}) = [\mathbf{C}]^T(\mathbf{F}) = [\mathbf{C}]^T[\overline{\mathbf{K}}](\Delta) = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}](\mathbf{u})$$

$$\underline{\underline{(\mathbf{P}) = [\mathbf{K}](\mathbf{u})}} \quad \text{where} \quad \underline{\underline{[\mathbf{K}] = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}]}}$$

- **Direct Stiffness Method**

$$[\mathbf{C}]_i^T[\mathbf{K}]_i[\mathbf{C}]_i = \begin{array}{c}\begin{bmatrix}0&0&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&\cdots&[\mathbf{K}_{11}]_i&[\mathbf{K}_{12}]_i&\cdots&0\\0&\cdots&[\mathbf{K}_{21}]_i&[\mathbf{K}_{22}]_i&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&0&0\end{bmatrix}\begin{array}{l}\\\\i\text{-th row}\\i+1\text{-th row}\\\\\\\end{array}\\\begin{array}{cccccc}&&&i\text{-th column}&i+1\text{-th column}&\end{array}\end{array}$$

5.2.3 Example



- **Equilibrium Equation**

$$
\begin{aligned}
V^1 &= V_1^L &, M^1 &= M_1^L \\
V^2 &= V_1^R + V_2^L &, M^2 &= M_1^R + M_2^L \\
V^3 &= V_2^R + V_3^L &, M^3 &= M_2^R + M_3^L \\
V^4 &= V_4^R &, M^4 &= M_4^R
\end{aligned}
$$

$$
(\mathbf{P}) = \begin{pmatrix} V^1 \\ M^1 \\ V^2 \\ M^2 \\ V^3 \\ M^3 \\ V^4 \\ M^4 \end{pmatrix} = \left[\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}_{[\mathbf{E}]_1} \;\; \underbrace{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}_{[\mathbf{E}]_2} \;\; \underbrace{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{[\mathbf{E}]_3} \right] \begin{pmatrix} V_1^L \\ M_1^L \\ V_1^R \\ M_1^R \\ V_2^L \\ M_2^L \\ V_2^R \\ M_2^R \\ V_3^L \\ M_3^L \\ V_3^R \\ M_3^R \end{pmatrix} = [\mathbf{E}](\mathbf{F})
$$

- **Compatability Condition**

$$
\begin{aligned}
& w_1^L = w^1, \theta_1^L = \theta^1 \;,\; w_1^R = w^2, \theta_1^R = \theta^2 \\
& w_2^L = w^2, \theta_2^L = \theta^2 \;,\; w_2^R = w^3, \theta_2^R = \theta^3 \\
& w_3^L = w^3, \theta_3^L = \theta^3 \;,\; w_3^R = w^4, \theta_3^R = \theta^4
\end{aligned}
$$

$$(\boldsymbol{\Delta}) = \begin{pmatrix} w_1^L \\ \theta_1^L \\ w_1^R \\ \theta_1^R \\ w_2^L \\ \theta_2^L \\ w_2^R \\ \theta_2^R \\ w_3^L \\ \theta_3^L \\ w_3^R \\ \theta_3^R \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} w^1 \\ \theta^1 \\ w^2 \\ \theta^2 \\ w^3 \\ \theta^3 \\ w^4 \\ \theta^4 \end{pmatrix} = \begin{bmatrix} [\mathbf{C}]_1 \\ [\mathbf{C}]_2 \\ [\mathbf{C}]_3 \end{bmatrix} \begin{pmatrix} w^1 \\ \theta^1 \\ w^2 \\ \theta^2 \\ w^3 \\ \theta^3 \\ w^4 \\ \theta^4 \end{pmatrix} = [\mathbf{C}](\mathbf{u})$$

$[\mathbf{C}]_1$ $[\mathbf{C}]_3$ $[\mathbf{C}]_2$

- **Unassembled Member Stiffness Matrix**

$$(\mathbf{F}) = \begin{pmatrix} (\mathbf{F})_1 \\ (\mathbf{F})_2 \\ (\mathbf{F})_3 \end{pmatrix} = \begin{bmatrix} [\mathbf{K}]_1 & 0 & 0 \\ 0 & [\mathbf{K}]_2 & 0 \\ 0 & 0 & [\mathbf{K}]_3 \end{bmatrix} \begin{pmatrix} (\boldsymbol{\Delta})_1 \\ (\boldsymbol{\Delta})_2 \\ (\boldsymbol{\Delta})_3 \end{pmatrix} = [\overline{\mathbf{K}}](\boldsymbol{\Delta})$$

- **Global Stiffness Equation**

$$(\mathbf{P}) = [\mathbf{E}](\mathbf{F}) = [\mathbf{E}][\overline{\mathbf{K}}](\mathbf{u}) = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}](\mathbf{u}) = [\mathbf{K}](\mathbf{u})$$

$$(\mathbf{P}) = \left[[\mathbf{C}]_1^T \;,\; [\mathbf{C}]_2^T \;,[\mathbf{C}]_3^T\right] \begin{bmatrix} [\mathbf{K}]_1 & 0 & 0 \\ 0 & [\mathbf{K}]_2 & 0 \\ 0 & 0 & [\mathbf{K}]_3 \end{bmatrix} \begin{bmatrix} [\mathbf{C}]_1 \\ [\mathbf{C}]_2 \\ [\mathbf{C}]_3 \end{bmatrix} (\mathbf{u})$$

$$= [\mathbf{K}](\mathbf{u}) = (\,[\mathbf{C}]_1^T[\mathbf{K}]_1[\mathbf{C}]_1 + [\mathbf{C}]_2^T[\mathbf{K}]_2[\mathbf{C}]_2 + [\mathbf{C}]_3^T[\mathbf{K}]_3[\mathbf{C}]_3)(\mathbf{u})$$

- Member 1

$$\frac{EI_1}{L_1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\frac{12}{L^2}&\frac{6}{L_1}&-\frac{12}{L_1^2}&\frac{6}{L_1}\\\frac{6}{L_1}&4&-\frac{6}{L_1}&2\\-\frac{12}{L_1^2}&-\frac{6}{L_1}&\frac{12}{L_1^2}&-\frac{6}{L_1}\\\frac{6}{L_1}&2&-\frac{6}{L_1}&4\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\end{bmatrix}=$$

$$\frac{EI_1}{L_1}\begin{bmatrix}\frac{12}{L_1^2}&\frac{6}{L_1}&-\frac{12}{L_1^2}&\frac{6}{L_1}&0&0&0&0\\\frac{6}{L_1}&4&-\frac{6}{L_1}&2&0&0&0&0\\-\frac{12}{L_1^2}&-\frac{6}{L_1}&\frac{12}{L_1^2}&\frac{6}{L_1}&0&0&0&0\\\frac{6}{L_1}&2&-\frac{6}{L_1}&4&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

- Member 2

$$\frac{EI_2}{L_2}\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\frac{12}{L_2^2}&\frac{6}{L_2}&-\frac{12}{L_2^2}&\frac{6}{L_2}\\\frac{6}{L_2}&4&-\frac{6}{L_2}&2\\-\frac{12}{L_2^2}&-\frac{6}{L_2}&\frac{12}{L_2^2}&-\frac{6}{L_2}\\\frac{6}{L_2}&2&-\frac{6}{L_2}&4\end{bmatrix}\begin{bmatrix}0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}=$$

$$\frac{EI_2}{L_2}\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&\frac{12}{L_2^2}&\frac{6}{L_2}&-\frac{12}{L_2^2}&\frac{6}{L_2}&0&0\\0&0&\frac{6}{L_2}&4&-\frac{6}{L_2}&2&0&0\\0&0&-\frac{12}{L_2^2}&-\frac{6}{L_2}&\frac{12}{L_2^2}&-\frac{6}{L_2}&0&0\\0&0&\frac{6}{L_2}&2&-\frac{6}{L_2}&4&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

- Member 3

$$\frac{EI_3}{L_3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{12}{L_3^2} & \frac{6}{L_3} & -\frac{12}{L_3^2} & \frac{6}{L_3} \\ \frac{6}{L_3} & 4 & -\frac{6}{L_3} & 2 \\ -\frac{12}{L_3^2} & -\frac{6}{L_3} & \frac{12}{L_3^2} & -\frac{6}{L_3} \\ \frac{6}{L_3} & 2 & -\frac{6}{L_3} & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\frac{EI_3}{L_3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{12}{L_3^2} & \frac{6}{L_3} & -\frac{12}{L_3^2} & \frac{6}{L_3} \\ 0 & 0 & 0 & 0 & \frac{6}{L_3} & 4 & -\frac{6}{L_3} & 2 \\ 0 & 0 & 0 & 0 & -\frac{12}{L_3^2} & -\frac{6}{L_3} & \frac{12}{L_3^2} & -\frac{6}{L_3} \\ 0 & 0 & 0 & 0 & \frac{6}{L_3} & 2 & -\frac{6}{L_3} & 4 \end{bmatrix}$$

- **Global Stiffness Matrix**

$$E\begin{bmatrix} \frac{12I_1}{L_1^3} & \frac{6I_1}{L_1^2} & -\frac{12I_1}{L_1^3} & \frac{6I_1}{L_1^2} & 0 & 0 & 0 & 0 \\ \frac{6I_1}{L_1^2} & \frac{4I_1}{L_1} & -\frac{6I_1}{L_1^2} & \frac{2I_1}{L_1} & 0 & 0 & 0 & 0 \\ -\frac{12I_1}{L_1^3} & -\frac{6I_1}{L_1^2} & \frac{12I_1}{L_1^3}+\frac{12I_2}{L_2^3} & -\frac{6I_1}{L_1^2}+\frac{6I_2}{L_2^2} & -\frac{12I_2}{L_2^3} & \frac{6I_2}{L_2^2} & 0 & 0 \\ \frac{6I_1}{L_1^2} & \frac{2I_1}{L_1} & -\frac{6I_1}{L_1^2}+\frac{6I_2}{L_2^2} & \frac{4I_1}{L_1}+\frac{4I_2}{L_2} & -\frac{6I_2}{L_1^2} & \frac{2I_2}{L_2} & 0 & 0 \\ 0 & 0 & -\frac{12I_2}{L_2^3} & -\frac{6I_2}{L_2^2} & \frac{12I_2}{L_2^3}+\frac{12I_3}{L_3^3} & -\frac{6I_2}{L_2^2}+\frac{6I_3}{L_3^2} & -\frac{12I_3}{L_3^3} & \frac{6I_3}{L_3^2} \\ 0 & 0 & \frac{6I_2}{L_2^2} & \frac{2I_2}{L_2} & -\frac{6I_2}{L_2^2}+\frac{6I_3}{L_3^2} & \frac{4I_2}{L_2}+\frac{4I_3}{L_3} & -\frac{6I_3}{L_3^2} & \frac{2I_3}{L_3} \\ 0 & 0 & 0 & 0 & -\frac{12I_3}{L_3^3} & -\frac{6I_3}{L_3^2} & \frac{12I_3}{L_3^3} & -\frac{6I_3}{L_3^2} \\ 0 & 0 & 0 & 0 & \frac{6I_3}{L_3^2} & \frac{2I_3}{L_3} & -\frac{6I_3}{L_3^2} & \frac{4I_3}{L_3} \end{bmatrix}$$

- **Application of support Conditions**

$$\begin{pmatrix} V^1 \\ M^1 \\ V^2 \\ M^2 \\ V^3 \\ M^3 \\ V^4 \\ M^4 \end{pmatrix} = E \begin{bmatrix} \frac{12I_1}{L_1^3} & \frac{6I_1}{L_1^2} & -\frac{12I_1}{L_1^3} & \frac{6I_1}{L_1^2} & 0 & 0 & 0 & 0 \\ \frac{6I_1}{L_1^2} & \frac{4I_1}{L_1} & -\frac{6I_1}{L_1^2} & \frac{2I_1}{L_1} & 0 & 0 & 0 & 0 \\ -\frac{12I_1}{L_1^3} & -\frac{6I_1}{L_1^2} & \frac{12I_1}{L_1^3}+\frac{12I_2}{L_2^3} & -\frac{6I_1}{L_1^2}+\frac{6I_2}{L_2^2} & -\frac{12I_2}{L_2^3} & \frac{6I_2}{L_2^2} & 0 & 0 \\ \frac{6I_1}{L_1^2} & \frac{2I_1}{L_1} & -\frac{6I_1}{L_1^2}+\frac{6I_2}{L_2^2} & \frac{4I_1}{L_2}+\frac{4I_2}{L_2} & -\frac{6I_2}{L_1^2} & \frac{2I_2}{L_2} & 0 & 0 \\ 0 & 0 & -\frac{12I_2}{L_2^3} & -\frac{6I_2}{L_2^2} & \frac{12I_2}{L_2^3}+\frac{12I_3}{L_3^3} & -\frac{6I_2}{L_2^2}+\frac{6I_3}{L_3^2} & -\frac{12I_3}{L_3^3} & \frac{6I_3}{L_3^2} \\ 0 & 0 & \frac{6I_2}{L_2^2} & \frac{2I_2}{L_2} & -\frac{6I_2}{L_2^2}+\frac{6I_3}{L_3^2} & \frac{4I_2}{L_2}+\frac{4I_3}{L_3} & -\frac{6I_3}{L_3^2} & \frac{2I_3}{L_3} \\ 0 & 0 & 0 & 0 & -\frac{12I_3}{L_3^3} & -\frac{6I_3}{L_3^2} & \frac{12I_3}{L_3^3} & -\frac{6I_3}{L_3^2} \\ 0 & 0 & 0 & 0 & \frac{6I_3}{L_3^2} & \frac{2I_3}{L_3} & -\frac{6I_3}{L_3^2} & \frac{4I_3}{L_3} \end{bmatrix} \begin{pmatrix} w^1 \\ \theta^1 \\ w^2 \\ \theta^2 \\ w^3 \\ \theta^3 \\ w^4 \\ \theta^4 \end{pmatrix}$$

- **Final Stiffness Equation**

$$\begin{pmatrix} M^2 \\ M^3 \\ V^4 \\ M^4 \end{pmatrix} = E \begin{bmatrix} \frac{4I_1}{L_2}+\frac{4I_2}{L_2} & \frac{2I_2}{L_2} & 0 & 0 \\ \frac{2I_2}{L_2} & \frac{4I_2}{L_2}+\frac{4I_3}{L_3} & -\frac{6I_3}{L_3^2} & \frac{2I_3}{L_3} \\ 0 & -\frac{6I_3}{L_3^2} & \frac{12I_3}{L_3^3} & -\frac{6I_3}{L_3^2} \\ 0 & \frac{2I_3}{L_3} & -\frac{6I_3}{L_3^2} & \frac{4I_3}{L_3} \end{bmatrix} \begin{pmatrix} \theta^2 \\ \theta^3 \\ w^4 \\ \theta^4 \end{pmatrix}$$

5.3 Frame Problems

5.3.1 Member Stiffness Matrix

- **Force-Displacement Relation at Member Ends**

  - Beam action

$$m^L = \frac{4EI_e}{L_e}\theta_e^L + \frac{2EI_e}{L_e}\theta_e^R + \frac{6EI_e}{L_e^2}\delta_y^L - \frac{6EI_e}{L_e^2}\delta_y^R$$

$$m^R = \frac{2EI_e}{L_e}\theta_e^L + \frac{4EI_e}{L_e}\theta_e^R + \frac{6EI_e}{L_e^2}\delta_y^L - \frac{6EI_e}{L_e^2}\delta_y^R$$

$$f_y^L = \frac{M_e^L + M_e^R}{L_e} = \frac{6EI_e}{L_e^2}\theta_e^L + \frac{6EI_e}{L_e^2}\theta_e^R + \frac{12EI_e}{L_e^3}\delta_y^L - \frac{12EI_e}{L_e^3}\delta_y^R$$

$$f_y^R = -\frac{M_e^L + M_e^R}{L_e} = -\frac{6EI_e}{L_e^2}\theta_e^L - \frac{6EI_e}{L_e^2}\theta_e^R - \frac{12EI_e}{L_e^3}\delta_y^L + \frac{12EI_e}{L_e^3}\delta_y^R$$

  - Truss action

$$f_x^L = -\frac{EA}{L}(\delta_x^R - \delta_x^L)$$

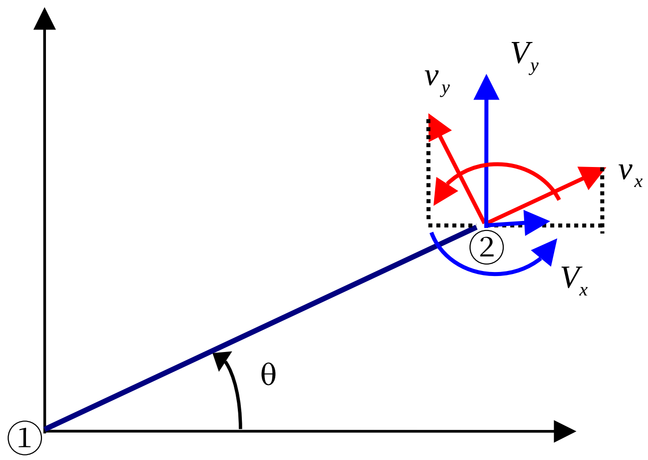
$$f_x^R = \frac{EA}{L}(\delta_x^R - \delta_x^L)$$

$$V_e^L = V_e^R = 0$$

- **Member Stiffness Matrix**

$$\begin{pmatrix} f_x^L \\ f_y^L \\ m^L \\ f_x^R \\ f_y^R \\ m^R \end{pmatrix} = \frac{E}{L_e}\begin{bmatrix} A_e & 0 & 0 & -A_e & 0 & 0 \\ 0 & \frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} & 0 & -\frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 4I_e & 0 & -\frac{6I_e}{L_e} & 2I_e \\ -A_e & 0 & 0 & A_e & 0 & 0 \\ 0 & -\frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} & 0 & \frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 2I_e & 0 & -\frac{6I_e}{L_e} & 4I_e \end{bmatrix}\begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \theta_e^L \\ \delta_x^R \\ \delta_y^R \\ \theta_e^R \end{pmatrix} \quad \text{or} \quad \underline{\underline{(\mathbf{f})^e = [\mathbf{k}]^e(\delta)^e}}$$

- **Transformation Matrix**



①②

$$\left.\begin{aligned} V_x &= \cos\theta v_x - \sin\theta v_y \\ V_y &= \sin\theta v_x + \cos\theta v_y \\ M &= m \end{aligned}\right\} \rightarrow \begin{pmatrix} V_x \\ V_y \\ M \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ v_y \\ m \end{pmatrix}$$

$$(\mathbf{V}) = [\gamma]^T(\mathbf{v}) \rightarrow (\mathbf{v}) = [\gamma](\mathbf{V})$$

- **Member End Force**

$$\begin{pmatrix} F_x^1 \\ F_y^1 \\ M^1 \\ F_x^2 \\ F_y^2 \\ M^2 \end{pmatrix}^e = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} f_x^L \\ f_y^L \\ m^L \\ f_x^R \\ f_y^R \\ m^R \end{pmatrix}^e \rightarrow \underline{\underline{(\mathbf{F})^e = [\Gamma]^T(\mathbf{f})^e}}$$

- **Member End Displacement**

$$\begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \theta_e^L \\ \delta_x^R \\ \delta_y^R \\ \theta_e^R \end{pmatrix}^e = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \Delta_x^1 \\ \Delta_y^1 \\ \theta^1 \\ \Delta_x^2 \\ \Delta_y^2 \\ \theta^1 \end{pmatrix}^e \rightarrow \underline{\underline{(\delta)^e = [\Gamma](\Delta)^e}}$$

- **Member Stiffness Matrix in Global Coordinate**

$$(\mathbf{F})^e = [\Gamma]^T(\mathbf{f})^e = [\Gamma]^T[\mathbf{k}]^e(\delta)^e = [\Gamma]^T[\mathbf{k}]^e[\Gamma](\Delta)^e$$

$$\underline{\underline{(\mathbf{F})^e = [\mathbf{K}]^e(\Delta)^e}} \text{ where } [\mathbf{K}]^e = \begin{bmatrix} [\mathbf{K}_{11}]^e & [\mathbf{K}_{12}]^e \\ [\mathbf{K}_{21}]^e & [\mathbf{K}_{22}]^e \end{bmatrix}$$

- **Nodal Equilibrium & Compatibility**

The same as the truss problems.

- **Global Stiffness Matrix**

$$(\mathbf{P}) = [\mathbf{C}]^T(\mathbf{F}) = [\mathbf{C}]^T[\overline{\mathbf{K}}](\Delta) = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}](\mathbf{u})$$

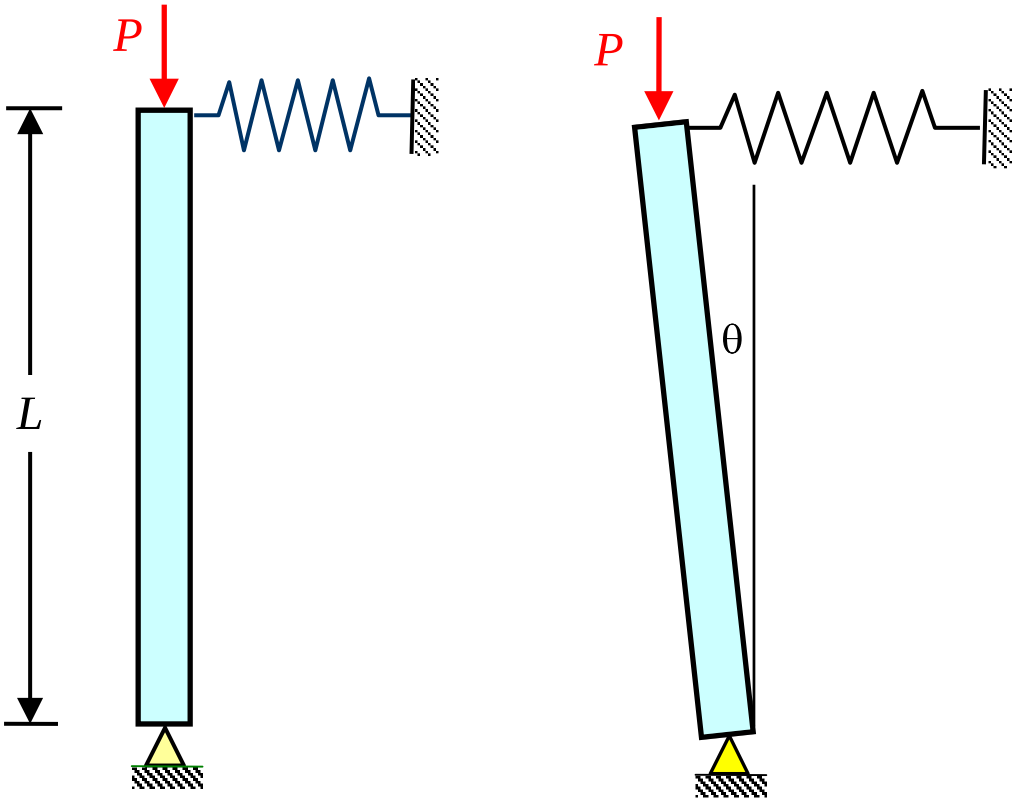
$$\underline{\underline{(\mathbf{P}) = [\mathbf{K}](\mathbf{u})}} \quad \text{where} \quad \underline{\underline{[\mathbf{K}] = [\mathbf{C}]^T[\overline{\mathbf{K}}][\mathbf{C}]}}$$

- **Direct Stiffness Method**

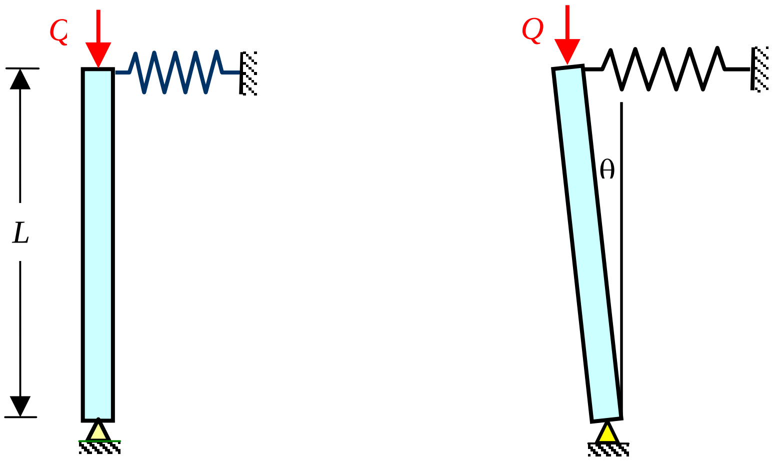
The same as the truss problems.

Chapter 6

Buckling of Structures



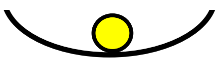
6.0 Stability of Structures



- **Stable state**

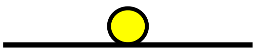
$$K\theta L \times L > Q\theta L \rightarrow KL > Q$$

The structure would return its original equilibrium position for a small perturbation in θ.



- **Critical state**

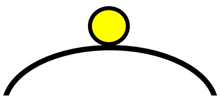
$$K\theta L \times L = Q\theta L \rightarrow KL = Q$$



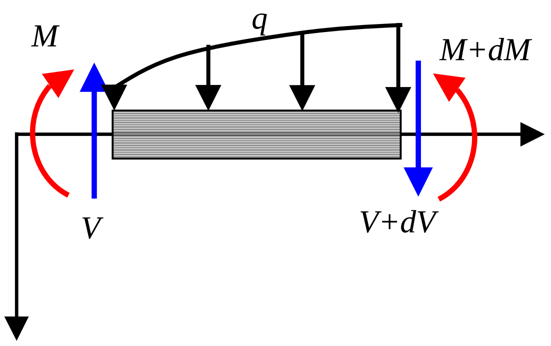
- **Unstable state**

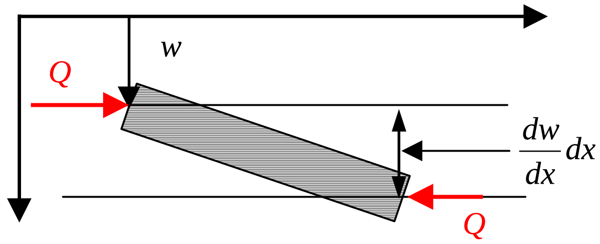
$$K\theta L \times L < Q\theta L \rightarrow KL < Q$$

The structure would not return its original equilibrium position for a small perturbation in θ.



6.1 Governing Equation for a Beam with Axial Force





- **Equilibrium for vertical force**

$$(V+dV)-V+qdx=0 \rightarrow \frac{dV}{dx}=-q$$

- **Equilibrium for moment**

$$(M+dM)-M-Vdx+qdx\frac{dx}{2}-P\frac{dw}{dx}=0 \rightarrow \frac{dM}{dx}-Q\frac{dw}{dx}=V$$

- **Elimination of shear force**

$$\frac{d^2M}{dx^2}-Q\frac{d^2w}{dx^2}=-q$$

- **Strain-displacement relation**

$$\varepsilon=-\frac{d^2w}{dx^2}y+\frac{du}{dx}$$

- **Stress-strain relation (Hooke law)**

$$\sigma=E\varepsilon=-E\frac{d^2w}{dx^2}y+E\frac{du}{dx}$$

- **Definition of Moment**

$$M=\int_A \sigma y dA=\int_A E\varepsilon y dA=-\int_A (E\frac{d^2w}{dx^2}y^2-E\frac{du}{dx}y)dA=-EI\frac{d^2w}{dx^2}$$

- **Beam Equation with Axial Force**

$$EI\frac{d^4w}{dx^4}+Q\frac{d^2w}{dx^2}=q$$

6.2 Homogeneous Solutions

- **Characteristic Equation for** $P > 0$

$$w = e^{\lambda x}$$
$$e^{\lambda x}(\lambda^4 + \beta^2\lambda^2) = 0 \rightarrow \lambda = \pm\beta i,\ 0 \quad \text{where} \quad \beta^2 = \frac{Q}{EI}$$

- **Homogeneous solution**

$$w = Ae^{\beta i x} + Be^{-\beta i x} + Cx + D$$

- **Exponential Function with Complex Variable**

$$e^{ix} = 1 + ix + \frac{i^2}{2!}x^2 + \frac{i^3}{3!}x^3 + \frac{i^4}{4!}x^4 + \frac{i^5}{5!}x^5 + \frac{i^6}{6!}x^6 + \cdots$$
$$e^{-ix} = 1 - ix + \frac{(-i)^2}{2!}x^2 + \frac{(-i)^3}{3!}x^3 + \frac{(-i)^4}{4!}x^4 + \frac{(-i)^5}{5!}x^5 + \frac{(-i)^6}{6!}x^6 + \cdots$$
$$e^{ix} + e^{-ix} = 2(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots) = 2\cos x$$
$$e^{ix} - e^{-ix} = 2(ix - \frac{i}{3!}x^3 + \frac{i}{5!}x^5 - \frac{i}{7!}x^7 + \cdots) = 2i\sin x$$
$$e^{ix} = \cos x + i\sin x \ , \quad e^{-ix} = \cos x - i\sin x$$

- **Homogeneous solution**

$$\begin{aligned} w &= A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x) + Cx + D \\ &= (A+B)\cos\beta x + i(A-B)\sin\beta x + Cx + D \\ &= A\cos\beta x + B\sin\beta x + Cx + D \end{aligned}$$

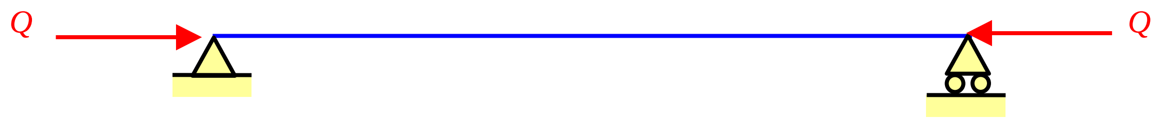
- **Characteristic Equation for** $P < 0$

$$w = e^{\lambda x}$$
$$e^{\lambda x}(\lambda^4 - \beta^2\lambda^2) = 0 \rightarrow \lambda = \pm\beta,\ 0 \quad \text{where} \quad \beta^2 = \left|\frac{Q}{EI}\right|$$

- **Homogeneous solution for** $P < 0$

$$\begin{aligned} w &= Ae^{\beta x} + Be^{-\beta x} + Cx + D \\ &= (A+B)\frac{e^{\beta x} + e^{-\beta x}}{2} + (A-B)\frac{e^{\beta x} - e^{-\beta x}}{2} + Cx + D \\ &= A\cosh\beta x + B\sinh\beta x + Cx + D \end{aligned}$$

- **Simple Beam**



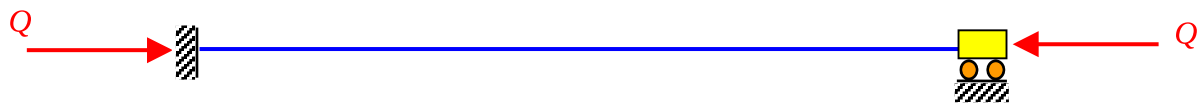
– Boundary Condition

$$w(0) = A + D = 0\ ,\quad w''(0) = -A\beta^2 = 0 \to A = 0$$
$$w(L) = B\sin\beta L + CL = 0\ ,\quad w''(L) = -B\beta^2 \sin\beta L = 0 \to B = C = 0$$

– Characteristic Equation

$$A = B = C = D = 0 \to w = 0 \ \text{(trivial solution) or}$$
$$\beta L = n\pi \to Q = \frac{n^2\pi^2 EI}{L^2}\ ,\quad n = 1,2,3\cdots$$
$$w = B\sin\beta x = B\sin\frac{n\pi}{L}x$$

- **Fixed-Fixed Beam**



– Boundary Condition

$$w(0) = A + D = 0$$
$$w'(0) = \beta B + C = 0$$
$$w(L) = A\cos\beta L + B\sin\beta L + CL + D = 0$$
$$w'(L) = -A\beta\sin\beta L + B\beta\cos\beta L + C = 0$$

– Characteristic Equation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{bmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to Det(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{bmatrix}) = 0$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{vmatrix} = \begin{vmatrix} \beta & 1 & 0 \\ \sin\beta L & L & 1 \\ \beta\cos\beta L & 1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & \beta & 1 \\ \cos\beta L & \sin\beta L & L \\ -\beta\sin\beta L & \beta\cos\beta L & 1 \end{vmatrix}$$

$$-\beta-(-\beta\cos\beta L)-(-\beta(\cos\beta L+\beta L\sin\beta L)+\beta)=\beta(2\cos\beta L-2+\beta L\sin\beta L)=0$$

$$2\cos\beta L-2+\beta L\sin\beta L=2(\cos\beta L-1)+\beta L\sin\beta L=$$

$$-4\sin^2\frac{\beta L}{2}+2\beta L\sin\frac{\beta L}{2}\cos\frac{\beta L}{2}=$$

$$\sin\frac{\beta L}{2}\left(\frac{\beta L}{2}\cos\frac{\beta L}{2}-\sin\frac{\beta L}{2}\right)=0\rightarrow\sin\frac{\beta L}{2}=0\ \text{ or }\ \frac{\beta L}{2}\cos\frac{\beta L}{2}-\sin\frac{\beta L}{2}=0$$

– Eigenvalues

Symmetric modes

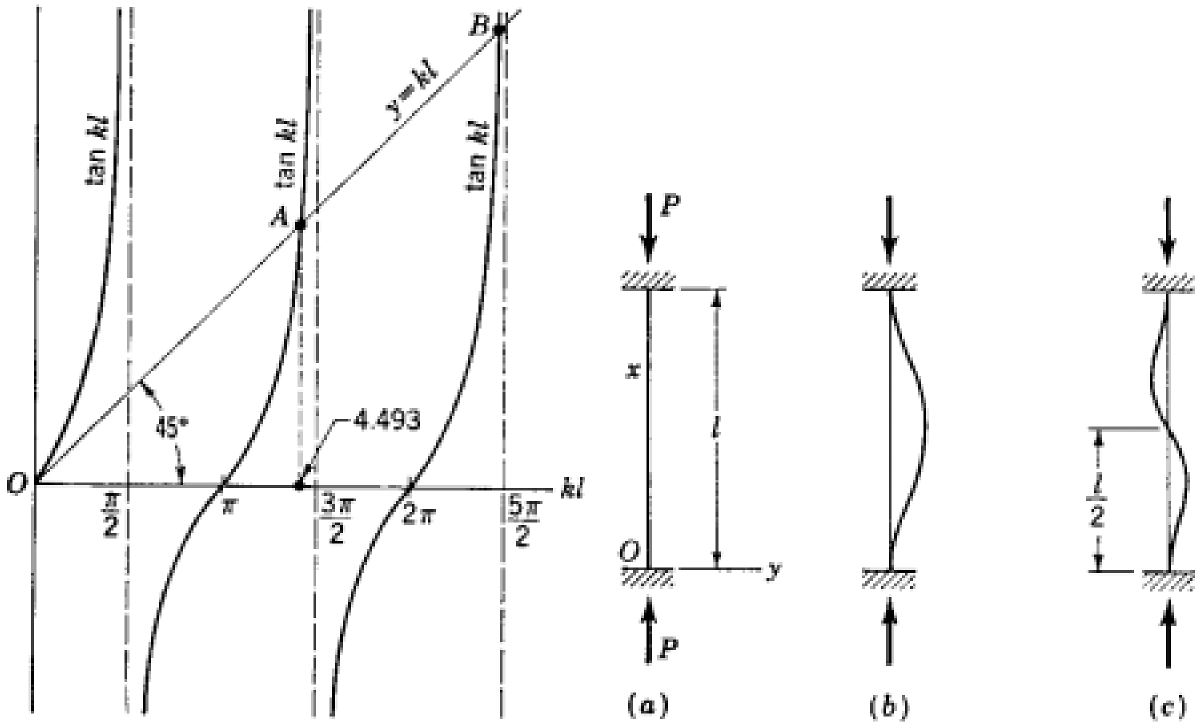
$$\sin\frac{\beta L}{2}=0\ \rightarrow\frac{\beta L}{2}=n\pi\rightarrow Q=\frac{4n^2\pi^2EI}{L^2},\quad n=1,2,3\cdots$$

$$w(0)=A+D=0,\ w'(0)=w'(L)=\beta B+C=0,\ w(L)=A+CL+D=0$$

$$A+D=0\rightarrow A=-D\rightarrow w=A(\cos\frac{2n\pi}{L}x-1)\ \text{ for }\ A\neq 0$$

Anti-symmetric modes

$$\frac{\beta L}{2}\cos\frac{\beta L}{2}-\sin\frac{\beta L}{2}=0\rightarrow\frac{\beta L}{2}=\tan\frac{\beta L}{2}\rightarrow Q=\frac{8.18\pi^2EI}{L^2}$$



- **Cantilever Beam**

– Boundary Condition

$$w(0) = A + D = 0$$

$$w'(0) = \beta B + C = 0$$

$$M(L) = -EIw''(L) = -EI(-A\beta^2 \cos\beta L - B\beta^2 \sin\beta L) = 0$$

$$V(L) = -EI\frac{d^3w}{dx^3} - P\frac{dw}{dx} = 0$$

$$Q = \frac{n^2\pi^2 EI}{4L^2}, \quad n = 1,2,3\cdots$$

6.3.Homogeneous and Particular solution

$$w = w_h + w_p = A\cos\beta x + B\sin\beta x + Cx + D + w_p$$

$$EI\frac{d^4(w_h + w_p)}{dx^4} + Q\frac{d^2(w_h + w_p)}{dx^2} = EI\frac{d^4 w_h}{dx^4} + Q\frac{d^2 w_h}{dx^2} + EI\frac{d^4 w_p}{dx^4} + Q\frac{d^2 w_p}{dx^2}$$

$$= EI\frac{d^4 w_p}{dx^4} + Q\frac{d^2 w_p}{dx^2} = q$$

- Four Boundary Conditions for Simple Beams

$$w(0) = A + D + w_p(0) = 0\ , \quad M(0) = -EIw''(0) = -EI(-A\beta^2 + w_p''(0)) = 0$$

$$w(L) = A\cos\beta L + B\sin\beta L + CL + D + w_p(L) = 0$$

$$M(L) = -EIw''(L) = -EI(-A\beta^2\cos\beta L - B\beta^2\sin\beta L + w_p''(L)) = 0$$

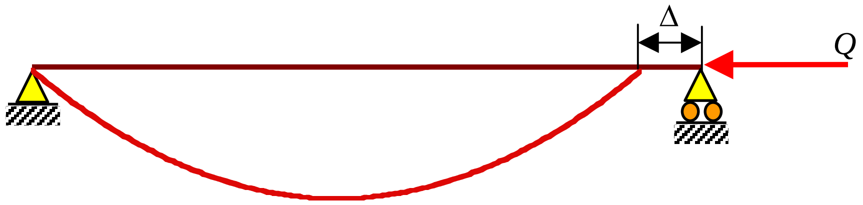
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -\beta^2 & 0 & 0 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta^2\cos\beta L & -\beta^2\sin\beta L & 0 & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \begin{pmatrix} w_p(0) \\ w_p''(0) \\ w_p(L) \\ w_p''(L) \end{pmatrix} = 0 \rightarrow \mathbf{KX} + \mathbf{F} = 0$$

- The homogenous solution is for the boundary conditions, while the particular solution is for the equilibrium.

6.4.Energy Method

- **Total Potential Energy**

$$\Pi = \frac{1}{2}\int_0^l \frac{d^2w}{dx^2} EI \frac{d^2w}{dx^2}dx - \frac{1}{2}Q\int_0^l \frac{dw}{dx}\frac{dw}{dx}dx - \int_0^l wqdx$$



$$L = \int_0^{L-\Delta} ds = \int_0^{L-\Delta} \sqrt{1+(w')^2}dx \approx \int_0^{L-\Delta} (1+\frac{1}{2}(w')^2)dx$$

$$L = L - \Delta + \int_0^{L-\Delta} \frac{1}{2}(w')^2 dx \rightarrow \Delta = \int_0^{L-\Delta} \frac{1}{2}(w')^2 dx \approx \frac{1}{2}\int_0^L (w')^2 dx \ \text{ for } \ \Delta << L$$

- **Principle of the Minimum Potential Energy**

$$\begin{aligned}
\Pi^h &= \frac{1}{2}\int_0^l \frac{d^2(w^e+\bar{w})}{dx^2} EI \frac{d^2(w^e+\bar{w})}{dx^2}dx - \frac{1}{2}\int_0^l \frac{d(w^e+\bar{w})}{dx} Q \frac{d(w^e+\bar{w})}{dx}dx - \int_0^l (w^e+\bar{w})qdx \\
&= \frac{1}{2}\int_0^l (\frac{d^2w^e}{dx^2}EI\frac{d^2w^e}{dx^2} - \frac{dw^e}{dx}Q\frac{dw^e}{dx})dx - \int_0^l w^e qdx + \int_0^l (\frac{d^2w^e}{dx^2}EI\frac{d^2\bar{w}}{dx^2} - \frac{dw^e}{dx}Q\frac{d\bar{w}}{dx})dx \\
&\quad - \int_0^l \bar{w}qdx + \frac{1}{2}\int_0^l (\frac{d^2\bar{w}}{dx^2}EI\frac{d^2\bar{w}}{dx^2} - \frac{d\bar{w}}{dx}Q\frac{d\bar{w}}{dx})dx \\
&= \Pi^e + \int_0^l \bar{w}(EI\frac{d^4w^e}{dx^4} + Q\frac{d^2w^e}{dx^2} - q)dx + \frac{1}{2}\int_0^l (\frac{d^2\bar{w}}{dx^2}EI\frac{d^2\bar{w}}{dx^2} - \frac{d\bar{w}}{dx}Q\frac{d\bar{w}}{dx})dx \\
&= \Pi^e + \frac{1}{2}\int_0^l (\frac{d^2\bar{w}}{dx^2}EI\frac{d^2\bar{w}}{dx^2} - \frac{d\bar{w}}{dx}Q\frac{d\bar{w}}{dx})dx
\end{aligned}$$

- The principle of minimum potential energy holds if and only if

$$\int_0^l (\frac{d^2\bar{w}}{dx^2}EI\frac{d^2\bar{w}}{dx^2} - \frac{d\bar{w}}{dx}Q\frac{d\bar{w}}{dx})dx > 0 \text{ for all possible } \bar{w}$$

- The principle of the minimum potential energy is not valid for the following cases.

$$\int_0^l (\frac{d^2\bar{w}}{dx^2}EI\frac{d^2\bar{w}}{dx^2} - \frac{d\bar{w}}{dx}Q\frac{d\bar{w}}{dx})dx \le 0 \text{ for some } \bar{w}$$

- **The critical status of a structure is defined as**

$$\int_0^l \left(\frac{d^2\overline{w}}{dx^2} EI \frac{d^2\overline{w}}{dx^2} - \frac{d\overline{w}}{dx} Q \frac{d\overline{w}}{dx}\right) dx = 0$$

- **Approximation**

– Approximation of displacement: $\overline{w} = \sum_{i=1}^{n} a_i g_i$

– Critical Status

$$\int_0^l \left(\sum_{i=1}^n a_i g_i''\right) EI \left(\sum_{j=1}^n a_j g_j''\right) dx - P \int_0^l \left(\sum_{i=1}^n a_i g_i'\right)\left(\sum_{j=1}^n a_j g_j'\right) dx =$$

$$\sum_{i=1}^n \sum_{j=1}^n a_i \int_0^l g_i'' EI g_i'' dx a_j - Q \sum_{i=1}^n \sum_{j=1}^n a_i \int_0^l g_i' g_j' dx a_j = \sum_{i=1}^n \sum_{j=1}^n a_i K_{ij} a_j - Q \sum_{i=1}^n \sum_{j=1}^n a_i K_{ij}^G a_j =$$

$$(a)^T (\mathbf{K} - \mathbf{K}^G)(a)^T = 0 \rightarrow Det(\mathbf{K} - \mathbf{K}^G) = 0$$

- **Example - Simple Beam**

– with a parabola: $w = ax(x - l) \rightarrow g_1' = 2x - l, \; g_1'' = 2$

$$\int_0^l g_i'' EI g_i'' dx = \int_0^l 2EI\, 2dx = 4EIl$$

$$\int_0^l g_i' EI g_i' dx = \int_0^l (2x - l)^2 dx = \int_0^l (4x^2 - 4xl + l^2) dx = \left(\frac{4}{3} - 2 + 1\right) l^3 = \frac{1}{3} l^3$$

$$Det\left(4EIl - Q\frac{1}{3} l^3\right) = 0 \rightarrow Q_{cr} = \frac{12EI}{l^2} \quad \left(\text{exact}: \frac{\pi^2 EI}{l^2} = 9.86 \frac{EI}{l^2}, \; \text{error} = 22\%\right)$$

– with one sine curve: $w = a \sin\frac{\pi x}{l} \rightarrow g_1' = \frac{\pi}{l} \cos\frac{\pi x}{l}, \; g_1'' = \left(\frac{\pi}{l}\right)^2 \sin\frac{\pi x}{l}$

$$\int_0^l g_i'' EI g_i'' dx = EI\left(\frac{\pi}{l}\right)^4 \int_0^l \sin^2\frac{\pi x}{l} dx = EI\left(\frac{\pi}{l}\right)^4 \frac{l}{2}$$

$$\int_0^l g_i' EI g_i' dx = \left(\frac{\pi}{l}\right)^2 \int_0^l \cos^2\frac{\pi x}{l} dx = \left(\frac{\pi}{l}\right)^2 \frac{l}{2}$$

$$Det\left(EI\left(\frac{\pi}{l}\right)^4 \frac{l}{2} - Q\left(\frac{\pi}{l}\right)^2 \frac{l}{2}\right) = 0 \rightarrow Q = \frac{\pi^2 EI}{l^2} \text{(exact)}$$

- **Example – Cantilever Beam**

– with one unknown:

$$w = ax^2 \rightarrow g_1' = 2x\,,\ g_1'' = 2$$

$$\int_0^l g_i'' EI g_i'' dx = \int_0^l 2EI\,2dx = 4EIl\ ,\ \int_0^l g_i' EI g_i' dx = \int_0^l 4x^2 dx = \frac{4}{3}l^3$$

$$Det(4EIl - Q\frac{4}{3}l^3) = 0 \rightarrow Q_{cr} = \frac{3EI}{l^2} \quad (\text{exact}: \frac{\pi^2 EI}{4l^2} = 2.46\frac{EI}{l^2},\ \text{error} = 22\%)$$

– with two unknowns:

$$w = ax^2 + bx^3 \rightarrow g_1' = 2x\,, g_2' = 3x^2\,,\ g_1'' = 2\,,\ g_2'' = 6x$$

$$K_{11}^G = \int_0^l 4x^2 dx = \frac{4}{3}l^3\,,\ K_{12}^G = K_{21}^G = \int_0^l 6x^3 dx = \frac{6}{4}l^4\,,\ K_{22}^G = \int_0^l 9x^4 dx = \frac{9}{5}l^5$$

$$K_{11} = \int_0^l 4dx = 4l\,,\ K_{12} = K_{21} = \int_0^l 12x dx = 6l^2\,,\ K_{22} = \int_0^l 36x^2 dx = 12l^3$$

$$Det(EI\begin{bmatrix} 4l & 6l^2 \\ 6l^2 & 12l^3 \end{bmatrix} - Q\frac{1}{30}\begin{bmatrix} 40l^3 & 45l^4 \\ 45l^4 & 54l^5 \end{bmatrix}) = 0 \rightarrow \begin{vmatrix} 4l - 40l^3\alpha & 6l^2 - 45l^4\alpha \\ 6l^2 - 45l^4\alpha & 12l^3 - 54l^5\alpha \end{vmatrix} = 0\,, \alpha = \frac{1}{30}\frac{Q}{EI}$$

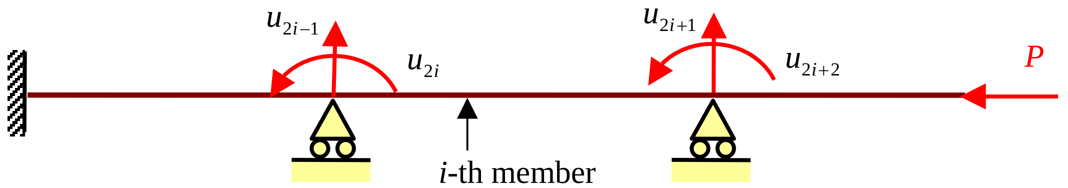
$$(4l - 40l^3\alpha)(12l^3 - 54l^5\alpha) - (6l^2 - 45l^4\alpha)^2 = 0 \rightarrow 4l^4 - 52l^6\alpha + 45l^8\alpha^2 = 0$$

$$\alpha = \frac{26l^6 \pm \sqrt{26^2 l^{12} - 180l^{12}}}{45l^8} = \frac{26 \pm 22.27}{45l^2} = \frac{0.0829}{l^2}\ \text{or}\ \frac{1.0727}{l^2} \rightarrow Q_{cr} = 2.487\frac{EI}{l^2}\ \text{or}\ 32.181\frac{EI}{l^2}$$

$$Q_{exact} = 2.49\frac{EI}{l^2}\,(\text{error} = 1.2\%)\ \text{or}\ \ Q_{exact} = 22.19\frac{EI}{l^2}\,(\text{error} = 45\%)$$

6.5 Approximation with the Homogeneous Beam Solutions

- **Homogeneous Solution of Beam**



$$w^i = N_1\delta_y^L + N_2\theta_z^L + N_3\delta_y^R + N_4\theta_z^R = (N_1, N_2, N_3, N_4)\begin{pmatrix}\delta_y^L \\ \theta_z^L \\ \delta_y^R \\ \theta_z^R\end{pmatrix} = (N)(\Delta_i) = (\mathbf{N})(\Delta_i)$$

$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\ ,\quad N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2}\ ,\quad N_3 = \frac{3x^2}{L} - \frac{2x^3}{L^2}\ ,\quad N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

6.5.1 Beam Analysis

- **Total Potential Energy**

$$\Pi = \sum_{i=1}^{p}\frac{1}{2}\int_0^l \frac{d^2w_i}{dx^2} EI \frac{d^2w_i}{dx^2}dx - \sum_{i=1}^{p}\frac{1}{2}Q\int_0^l \frac{dw_i}{dx}\frac{dw_i}{dx}dx - \sum_{i=1}^{p}\int_0^l w_i q_i dx$$

$$= \sum_{i=1}^{p}\frac{1}{2}\int_0^l (\frac{d^2w_i}{dx^2})^T EI \frac{d^2w_i}{dx^2}dx - \sum_{i=1}^{p}\frac{1}{2}Q\int_0^l (\frac{dw_i}{dx})^T \frac{dw^i}{dx}dx - \sum_{i=1}^{p}\int_0^l (w_i)^T q_i dx$$

$$= \sum_{i=1}^{p}\frac{1}{2}(\Delta_i)^T\int_0^l (\frac{d^2\mathbf{N}}{dx^2})^T EI \frac{d^2\mathbf{N}}{dx^2}dx(\Delta_i) - \sum_{i=1}^{p}\frac{1}{2}Q(\Delta_i)^T\int_0^l (\frac{d\mathbf{N}}{dx})^T \frac{d\mathbf{N}}{dx}dx(\Delta_i) - \sum_{i=1}^{p}(\Delta_i)^T\int_0^l (\mathbf{N})^T q_i dx$$

$$= \frac{1}{2}\sum_{i=1}^{p}(\Delta_i)^T([\mathbf{K}_i^0] - [\mathbf{K}_i^G)(\Delta_i) - \sum_{i=1}^{p}(\Delta_i)^T(\mathbf{f}_i)$$

$$= \frac{1}{2}(\mathbf{u})^T\sum_{i=1}^{p}[\mathbf{C}_i]^T([\mathbf{K}_i^0] - [\mathbf{K}_i^G])[\mathbf{C}_i](\mathbf{u}) - \sum_{i=1}^{p}(\mathbf{u})^T[\mathbf{C}_i]^T(\mathbf{f}_i)$$

$$= \frac{1}{2}(\mathbf{u})^T\sum_{i=1}^{p}[\mathbf{C}_i]^T[\mathbf{K}_i][\mathbf{C}_i](\mathbf{u}) - (\mathbf{u})^T\sum_{i=1}^{p}[\mathbf{C}_i]^T(\mathbf{f}_i)$$

$$= \frac{1}{2}(\mathbf{u})^T[\mathbf{K}](\mathbf{u}) - (\mathbf{u})^T(\mathbf{P})$$

$$\mathbf{K}_i^0 = \frac{EI_i}{L_i}\begin{bmatrix} \frac{12}{L_i^2} & \frac{6}{L_i} & -\frac{12}{L_i^2} & \frac{6}{L_i} \\ \frac{6}{L_i} & 4 & -\frac{6}{L_i} & 2 \\ -\frac{12}{L_i^2} & -\frac{6}{L_i} & \frac{12}{L_i^2} & -\frac{6}{L_i} \\ \frac{6}{L_i} & 2 & -\frac{6}{L_i} & 4 \end{bmatrix},\quad \mathbf{K}_i^G = Q\begin{bmatrix} \frac{6}{5L_i} & \frac{1}{10} & -\frac{6}{5L_i} & \frac{1}{10} \\ \frac{1}{10} & \frac{2L_i}{15} & \frac{1}{10} & -\frac{L_i}{30} \\ -\frac{6}{5L_i} & \frac{1}{10} & \frac{6}{5L_i} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{L_i}{30} & -\frac{1}{10} & \frac{2L_i}{15} \end{bmatrix}$$

- **Principle of Minimum Potential Energy for** $Q < Q_{cr}$

$$\Pi = \frac{1}{2}(\mathbf{u})^T[\mathbf{K}](\mathbf{u}) - (\mathbf{u})^T(\mathbf{P}) = \frac{1}{2}\sum_{i=1}^{n} u_i \sum_{j=1}^{n} K_{ij}u_j - \sum_{i=1}^{n} u_i P_i$$

$$\begin{aligned}
\frac{\partial \Pi}{\partial u_k} &= \frac{1}{2}\sum_{i=1}^{n}\frac{\partial u_i}{\partial u_k}\sum_{j=1}^{n} K_{ij}u_j + \frac{1}{2}\sum_{i=1}^{n} u_i \sum_{j=1}^{n} K_{ij}\frac{\partial u_j}{\partial u_k} - \sum_{i=1}^{n} u_i P_i \\
&= \frac{1}{2}\sum_{j=1}^{n} K_{kj}u_j + \frac{1}{2}\sum_{i=1}^{n} u_i K_{ik} - P_k = \frac{1}{2}\sum_{j=1}^{n} K_{kj}u_j + \frac{1}{2}\sum_{i=1}^{n} K_{ki}u_i - P_k \\
&= \frac{1}{2}\sum_{j=1}^{n} K_{kj}u_j + \frac{1}{2}\sum_{j=1}^{n} K_{kj}u_j - P_k = \sum_{j=1}^{n} K_{kj}u_j - P_k = 0 \ \text{ for } \ k = 1, \cdots, n \rightarrow [\mathbf{K}](\mathbf{u}) = (\mathbf{P})
\end{aligned}$$

- **Calculation of the Critical Load**

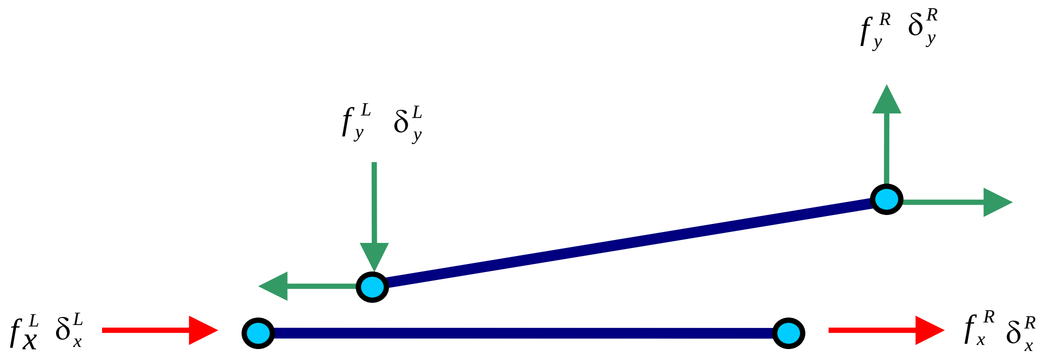
$$Det([\mathbf{K}]) = Det([\mathbf{K}^0] - [\mathbf{K}^G]) = 0$$

- **Frame Members**

$$\begin{pmatrix} f_x^L \\ f_y^L \\ m^L \\ f_x^R \\ f_y^R \\ m^R \end{pmatrix} = \left(\frac{E}{L_e}\begin{bmatrix} A_e & 0 & 0 & A_e & 0 & 0 \\ 0 & \frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} & 0 & -\frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 4I_e & 0 & -\frac{6I_e}{L_e} & 2I_e \\ A_e & 0 & 0 & A_e & 0 & 0 \\ 0 & -\frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} & 0 & \frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 2I_e & 0 & -\frac{6I_e}{L_e} & 4I_e \end{bmatrix} - P\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5L_e} & \frac{1}{10} & 0 & -\frac{6}{5L_e} & \frac{1}{10} \\ 0 & \frac{1}{10} & \frac{2L_e}{15} & 0 & \frac{1}{10} & -\frac{L_e}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5L_e} & \frac{1}{10} & 0 & \frac{6}{5L_e} & -\frac{1}{10} \\ 0 & \frac{1}{10} & -\frac{L_e}{30} & 0 & -\frac{1}{10} & \frac{2L_e}{15} \end{bmatrix}\right)\begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \theta_e^L \\ \delta_x^R \\ \delta_y^R \\ \theta_e^R \end{pmatrix}$$

$$P = EA\frac{\delta_x^L - \delta_x^R}{L_e}$$

6.6 Nonlinear Analysis of Truss



- **Force – Displacement relation at Member ends**

$$f_x^L = -\frac{EA}{L}(\delta_x^R - \delta_x^L)$$

$$f_x^R = \frac{EA}{L}(\delta_x^R - \delta_x^L)$$

$$f_y^R = -f_y^L = \frac{\delta_y^R - \delta_y^L}{l} f_x^R$$

- **Member Stiffness Matrix**

$$\begin{pmatrix} f_x^L \\ f_y^L \\ f_x^R \\ f_y^R \end{pmatrix}^e = \frac{EA}{l}\left[\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\delta_x^R - \delta_x^L}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right]\begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \delta_x^R \\ \delta_y^R \end{pmatrix}^e$$

$$(\mathbf{f})^e = ([\mathbf{k}]_0^e + f_x^R[\mathbf{k}]_g^e)(\delta)^e = ([\mathbf{k}]_0^e + p^e[\mathbf{k}]_g^e)(\delta)^e$$

- **Equilibrium Analysis**

$$(\mathbf{f})_i^e = ([\mathbf{k}]_0^e + p_i^e[\mathbf{k}]_g^e)(\delta)_i^e$$

$$(\mathbf{F})^e = [\Gamma]^T(\mathbf{f})^e = [\Gamma]^T[\mathbf{k}]^e(\delta)^e = [\Gamma]^T([\mathbf{k}]_0^e + p_i^e[\mathbf{k}]_g^e)[\Gamma](\Delta)^e = [\mathbf{K}]^e(\Delta)^e$$

- **Successive substitution**

$$(\mathbf{F})_{i-1}^e \approx [\Gamma]^T([\mathbf{k}]_0^e + p_{i-1}^e[\mathbf{k}]_g^e)[\Gamma](\Delta)^e = [\mathbf{K}]_{i-1}^e(\Delta)^e$$

$$(\mathbf{P}) = ([\mathbf{K}]_0 + [\mathbf{K}(p_{i-1}^e)]_G)(\mathbf{u})_i$$

- **Newton-Raphson Method**

$$
\begin{aligned}
(\mathbf{f})_i^e &= ([\mathbf{k}]_0^e + p_i^e[\mathbf{k}]_g^e)(\delta)_i^e \\
&= ([\mathbf{k}]_0^e + (p_{i-1}^e + \Delta p_i^e)[\mathbf{k}]_g^e)(\delta_{i-1}^e + \Delta\delta_i^e) \\
&= ([\mathbf{k}]_0^e + p_{i-1}^e[\mathbf{k}]_g^e)\delta_{i-1}^e + ([\mathbf{k}]_0^e + p_{i-1}^e[\mathbf{k}]_g^e)\Delta\delta_i^e + \Delta p_i^e[\mathbf{k}]_g^e(\delta_{i-1}^e + \Delta\delta_i^e) \\
&\approx ([\mathbf{k}]_0^e + p_{i-1}^e[\mathbf{k}]_g^e)\delta_{i-1}^e + ([\mathbf{k}]_0^e + p_{i-1}^e[\mathbf{k}]_g^e)\Delta\delta_i^e + \Delta p_i^e[\mathbf{k}]_g^e\delta_{i-1}^e \\
&= (\mathbf{f})_{i-1}^e + ([\mathbf{k}]_0^e + p_{i-1}^e[\mathbf{k}]_g^e + [\mathbf{k}]_\sigma^e)\Delta\delta_i^e
\end{aligned}
$$

$$([\mathbf{k}]_0^e + p_{i-1}^e[\mathbf{k}]_g^e + [\mathbf{k}]_\sigma^e)\Delta\delta_i^e = (\mathbf{f})_i^e - (\mathbf{f})_{i-1}^e = (\Delta\mathbf{f})_i^e$$

$$([\mathbf{K}]_0 + [\mathbf{K}(p_{i-1}^e)]_G + [\mathbf{K}]_\sigma)\Delta\mathbf{u} = (\Delta\mathbf{P})_i$$

$$
\begin{aligned}
\Delta p_i^e[\mathbf{k}]_g^e\delta_{i-1}^e &= \frac{EA}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \delta_x^R \\ \delta_y^R \end{pmatrix}_{i-1}^e \frac{\Delta\delta_x^R - \Delta\delta_x^L}{l} \\
&= \frac{EA}{l^2}\begin{pmatrix} 0 \\ \delta_y^L - \delta_y^R \\ 0 \\ \delta_y^R - \delta_y^L \end{pmatrix}_{i-1}^e \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \Delta\delta_x^L \\ \Delta\delta_y^L \\ \Delta\delta_x^R \\ \Delta\delta_y^R \end{pmatrix}_i^e \\
&= \frac{EA}{l^2}(\delta_y^R - \delta_y^L)_{i-1}^e \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} \Delta\delta_x^L \\ \Delta\delta_y^L \\ \Delta\delta_x^R \\ \Delta\delta_y^R \end{pmatrix}_i^e \\
&= [\mathbf{k}]_\sigma^e \Delta\delta_i^e
\end{aligned}
$$