Naval Architecture & Ocean Engineering

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Engineering Mathematics 2

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Series Solutions (2) : Solutions about Singular Points

Solutions about Singular Points

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$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

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$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

$$y'' + P(x)y' + Q(x)y = 0 \cdots (2)$$

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Definition 5.2

Regular/Irregular Singular Points

A singular point x_0 **is said to be a regular singular point of the differential equation (1) if the functions** $p(x) = (x - x_0)P(x)$ **and** $q(x) = (x - x_0)^2 Q(x)$ **are both analytic at** x_0 **.**

A singular point that is not regular is said to be an irregular point of the equation

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analytic at x_0 means

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In case $P(x), Q(x)$ are not analytic at x_0 and $x = x_0$ is a regular singular point then we multiply (2) by $(x - x_0)^2$, then

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$$

Where $p(x), q(x)$ are analytic at $x = x_0$

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solution → **Method of Frobenius**

Where $p(x), q(x)$ **are analytic at** $x = x_0$

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Multiply

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Multiply $(x - x_0)^2$

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standard form   **multiply** x^2

$$x^3 y'' - 2xy' + 8y = 0 \Rightarrow y'' + \underbrace{\boxed{\left(-\frac{2}{x^2}\right)}}_{P(x)} y' + \underbrace{\boxed{\frac{8}{x^3}}}_{Q(x)} y = 0 \Rightarrow x^2 y'' + x\underbrace{\boxed{\left(\frac{-2}{x}\right)}}_{p(x)} y' + \underbrace{\boxed{\frac{8}{x}}}_{q(x)} y = 0$$

not analytic at $x = 0$ **not analytic at** $x = 0$

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standard form

$$xy'' - 2xy' + 8y = 0 \Rightarrow y'' + (-2)y' + \frac{8}{x}y = 0$$

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$$P(x) \qquad Q(x)$$

not analytic at $x = 0$

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standard form **multiply x^2**

$$xy'' - 2xy' + 8y = 0 \Rightarrow y'' + \boxed{(-2)}y' + \boxed{\frac{8}{x}}y = 0 \Rightarrow$$

$$\underset{P(x)}{} \qquad \underset{Q(x)}{}$$

$P(x)$ $Q(x)$

not analytic at $x = 0$

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standard form **multiply** x^2

$$xy'' - 2xy' + 8y = 0 \Rightarrow y'' + \boxed{(-2)}\,y' + \boxed{\frac{8}{x}}\,y = 0 \Rightarrow x^2y'' + x(-2x)y' + 8xy = 0$$

$$P(x) \qquad Q(x)$$

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$$(x^2 + 9)y'' - 3xy' + (1 - x)y = 0$$

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$$(x^2+9)y'' - 3xy' + (1-x)y = 0 \Rightarrow y'' + \left(\frac{-3x}{x^2+9}\right)y' + \left(\frac{1-x}{x^2+9}\right)y = 0$$

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not analytic at $x = \pm 3i$

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multiply $(x^2+9)^2$

not analytic at $x = \pm 3i$

Solutions about Singular Points

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

$$y'' + P(x)y' + Q(x)y = 0 \cdots (2)$$

analytic at x_0 means

continuous, differentiable and integrable at x_0

Definition 5.2

Regular/Irregular Singular Points

A singular point x_0 is said to be a regular singular point of the differential equation (1) if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0.

A singular point that is not regular is said to be an irregular point of the equation

standard form

$$(x^2+9)y'' - 3xy' + (1-x)y = 0 \quad \Rightarrow \quad y'' + \boxed{\left(\frac{-3x}{x^2+9}\right)} y' + \boxed{\left(\frac{1-x}{x^2+9}\right)} y = 0$$

$P(x)$ $\quad Q(x)$

multiply $(x^2+9)^2$

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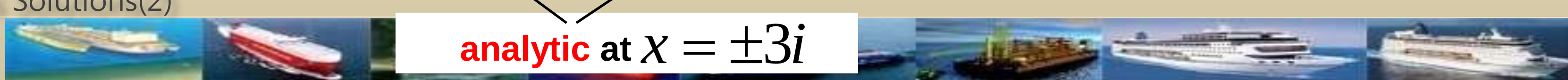

Solutions about Singular Points

☑ Example 1

It should be clear that x=2 and x=-2 are singular points of

$$(x^2-4)^2 y'' + 3(x-2)y' + 5y = 0$$

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Solutions about Singular Points

☑ **Method of Frobenius**

Solutions about Singular Points

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Solutions about Singular Points

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$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

Theorem 5.2

Frobenius's Theorem

If $x = x_0$ is an <u>regular singular point</u> of the differential equation (1), then there exists at least one solution of the form $y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$

where the number r is a constant to be determined . The series will converge at least on some interval $0 < x - x_0 < R$

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Solutions about Singular Points

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find the unknown exponent r **(indicial equation)**

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determine the unknown coefficient c_n **by a recursion relation**

Solutions about Singular Points

☑ Example 2

Two Series Solutions

$$3xy'' + y' - y = 0$$

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Which implies $r(3r-2)c_0 = 0$

$$(k+r+1)(3k+3r+1)c_{k+1} - c_k = 0,$$
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Since nothing is gained by taking $c_0 = 0$

Solutions about Singular Points

☑ Example 2

Two Series Solutions

$$3xy'' + y' - y = 0$$

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$$c_1 = \frac{c_0}{5\cdot 1}$$

$$c_2 = \frac{c_1}{8\cdot 2} = \frac{c_0}{2!5\cdot 8}$$

$$c_3 = \frac{c_2}{11\cdot 3} = \frac{c_0}{3!5\cdot 8\cdot 11}$$

$$c_4 = \frac{c_3}{14\cdot 4} = \frac{c_0}{4!5\cdot 8\cdot 11\cdot 14}$$

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Solutions about Singular Points

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$$c_2 = \frac{c_1}{2\cdot 4} = \frac{c_0}{2!1\cdot 4}$$

$$c_3 = \frac{c_2}{3\cdot 7} = \frac{c_0}{3!1\cdot 4\cdot 7}$$

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$$y_1(x) = x^{2/3}[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,5 \cdot 8 \cdot 11 \cdots (3n+2)} x^n]$$

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Solutions about Singular Points

☑ Indicial Equation

Solutions about Singular Points

☑ **Indicial Equation**

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

Solutions about Singular Points

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Theorem 5.2 **Frobenius's Theorem**

If $x = x_0$ is an <u>regular singular point</u> of the differential equation (1), then there exists at least one solution of the form $y = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ **where the number r is a constant to be determined . The series will converge at least on some interval** $0 < x - x_0 < R$

Solutions about Singular Points

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Solutions about Singular Points

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Solutions about Singular Points

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Theorem 5.2

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If $x = x_0$ **is an <u>regular singular point</u> of the differential equation (1), then there exists at least one solution of the form** $y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$

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$$p(x) = xP(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$q(x) = x^2 Q(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

Solutions about Singular Points

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Solutions about Singular Points

☑ Example 3

Solve

$$2xy'' + (1+x)y' + y = 0$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ **gives**

Solutions about Singular Points

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$$2xy'' + (1+x)y' + y$$

$$= 2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r}$$

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&+ \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} \\
&+ \sum_{n=0}^{\infty} c_n x^{n+r}
\end{aligned}$$

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&+ \sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} \\
&+ \sum_{n=0}^{\infty}(n+r+1)c_n x^{n}] \\
&= x^r[r(2r-1)c_0 x^{-1} \\
&+ \sum_{k=0}^{\infty}[(k+r+1)(2k+2r+1)c_{k+1} \\
&+ (k+r+1)c_k]x^k]
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Solutions about Singular Points

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$$y_1(x)$$

$$= x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+1/2}, \ x \geq 0$$

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$$y_2(x)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)} x^n, \ |x| < \infty$$

Solutions about Singular Points

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$y_1(x)$

$$= x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+1/2}, \; x \geq 0$$

$y_2(x)$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)} x^n, \; |x| < \infty$$

$$\therefore y = C_1 y_1(x) + C_2 y_2(x) \quad ,(0,\infty)$$

Solutions about Singular Points

☑ Example 4

Only One Series Solution

Solve

$$xy'' + y = 0$$

☑ Example 4

Only One Series Solution

Solve

$$xy'' + y = 0$$

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1}$$

$$= x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

Solutions about Singular Points

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Only One Series Solution

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Case 1 : If r_1 and r_2 are distinct and do not differ by an integer

Solutions about Singular Points

☑ Example 4
Only One Series Solution

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Solutions about Singular Points

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Case 2 : If r_1-r_2=N is a positive integer

Solutions about Singular Points

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$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0$$

Solutions about Singular Points

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Solutions about Singular Points

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Solutions about Singular Points

☑ Example 4
Only One Series Solution

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Case 3 : If r1=r2

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0$$

Solutions about Singular Points

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Solutions about Singular Points

☑ **Method of Frobenius**

Solutions about Singular Points

☑ Method of Frobenius

$$x^2 y'' + x[xP(x)]y' + [x^2 Q(x)]y = 0 \longleftarrow y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

$\downarrow$

find the unknown exponent r (indicial equation)

Solutions about Singular Points

☑ Method of Frobenius

$$x^2y'' + x[xP(x)]y' + [x^2Q(x)]y = 0 \longleftarrow \quad y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

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find the unknown exponent r **(indicial equation)**

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Solutions about Singular Points

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Solutions about Singular Points

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indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}, y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$

Solutions about Singular Points

☑ Method of Frobenius

$$x^2 y'' + x[xP(x)]y' + [x^2 Q(x)]y = 0 \longleftarrow y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

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Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}, y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$
Case II	**if** $r_1 - r_2 = N$ N : **positive integer** $r_1 > r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}, c_0 \neq 0,$ Frobenius fail to give a second series solution $y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, b_0 \neq 0, C \text{ could be zero}$

Solutions about Singular Points

☑ Method of Frobenius

$$x^2y'' + x[xP(x)]y' + [x^2Q(x)]y = 0 \longleftarrow y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

find the unknown exponent r (indicial equation)

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}, \; y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
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Solutions about Singular Points

☑ Method of Frobenius

indicial equation

$$r(r-1)+a_0 r+b_0=0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x)=\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
Case II	**if** $r_1-r_2=N$ N **: positive integer** $r_1>r_2$	$y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0\neq 0,$ Frobenius fail to give a second series solution $y_2(x)=Cy_1(x)\ln x+\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2},\ b_0\neq 0, C \text{ could be zero}$
Case III	**if** $r_1=r_2$	$y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0\neq 0 \longrightarrow y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ Frobenius fail to give a second series solution $y_2(x)=y_1(x)\ln x+\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0$$

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
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Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0 \longrightarrow$$

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}, y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
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Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0 \longrightarrow r(2r-1) = 0 \to r_1 = \frac{1}{2},\ r_2 = 0$$

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
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Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0 \longrightarrow r(2r-1)=0 \rightarrow r_1 = \frac{1}{2},\ r_2 = 0$$

Two Series Solution

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
Case II	**if** $r_1 - r_2 = N$ N : **positive integer** $r_1 > r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0 \neq 0,$ Frobenius fail to give a second series solution $y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2},\ b_0 \neq 0, C \text{ could be zero}$
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Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0 \longrightarrow r(2r-1) = 0 \rightarrow r_1 = \frac{1}{2},\ r_2 = 0$$

Two Series Solution

$$xy'' + y = 0$$

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
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Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0 \longrightarrow r(2r-1) = 0 \rightarrow r_1 = \frac{1}{2},\ r_2 = 0$$

Two Series Solution

$$xy'' + y = 0 \longrightarrow$$

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
Case II	**if** $r_1 - r_2 = N$ N : **positive integer** $r_1 > r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0 \neq 0,$ Frobenius fail to give a second series solution $y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2},\ b_0 \neq 0, C \text{ could be zero}$
Case III	**if** $r_1 = r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0 \neq 0 \longrightarrow y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$ Frobenius fail to give a second series solution $y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0 \longrightarrow r(2r-1)=0 \rightarrow r_1 = \frac{1}{2},\ r_2 = 0$$

Two Series Solution

$$xy'' + y = 0 \longrightarrow r(r-1)=0 \rightarrow r_1 = 1,\ r_2 = 0$$

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
Case II	**if** $r_1 - r_2 = N$ N : **positive integer** $r_1 > r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0 \neq 0,$ Frobenius fail to give a second series solution $y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2},\ b_0 \neq 0, C \text{ could be zero}$
Case III	**if** $r_1 = r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0 \neq 0 \longrightarrow y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ Frobenius fail to give a second series solution $y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

Solutions about Singular Points

☑ Method of Frobenius

$$2xy'' + (1+x)y' + y = 0 \longrightarrow r(2r-1) = 0 \rightarrow r_1 = \frac{1}{2},\ r_2 = 0$$

Two Series Solution

$$xy'' + y = 0 \longrightarrow r(r-1) = 0 \rightarrow r_1 = 1,\ r_2 = 0$$

Only One Series Solution

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
Case II	**if** $r_1 - r_2 = N$ N : **positive integer** $r_1 > r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0 \neq 0,$ Frobenius fail to give a second series solution $y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2},\ b_0 \neq 0, C \text{ could be zero}$
Case III	**if** $r_1 = r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0 \neq 0 \longrightarrow y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$ Frobenius fail to give a second series solution $y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

Solutions about Singular Points

☑ **Method of Frobenius**

Solutions about Singular Points

☑ Method of Frobenius

$$r(r-1)+a_0 r+b_0=0 \longrightarrow r_1, r_2$$

Solutions about Singular Points

☑ Method of Frobenius

$$r(r-1)+a_0r+b_0=0 \longrightarrow r_1, r_2$$

Solutions about Singular Points

☑ Method of Frobenius

$$r(r-1)+a_0 r+b_0=0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	**Two linearly independent series solution** $y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x)=\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

Solutions about Singular Points

☑ Method of Frobenius

$$r(r-1)+a_0 r+b_0=0 \longrightarrow r_1, r_2$$

Case I	**distinct and do not differ by an integer**	**Two linearly independent series solution** $y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ y_2(x)=\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
Case II	**if** $r_1-r_2=N$ N : **positive integer** $r_1>r_2$	

Solutions about Singular Points

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Solutions about Singular Points

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Case III	**if** $r_1=r_2$	**Two linearly independent series solution** Frobenius fail to give a second series solution $y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1},\ c_0\neq 0 \longrightarrow y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ $y_2(x)=y_1(x)\ln x+\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

Classification of Singular Points

☑ Example 5

Example 4 Revisited- Using a CAS

Find the general solution of

$$xy'' + y = 0$$

Classification of Singular Points

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Find the general solution of

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From the known solution given in Example 4,

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

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Find the general solution of

$$xy'' + y = 0$$

From the known solution given in Example 4,

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int 0\,dx}}{[y_1(x)]^2}dx$$

$$= y_1(x)\int \frac{dx}{[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots]^2}$$

$$= y_1(x)\int \frac{dx}{[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots]}$$

$$= y_1(x)\int [\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots]dx$$

$$= y_1(x)[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots]$$

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or

Classification of Singular Points

☑ Example 5

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Find the general solution of

$$xy'' + y = 0$$

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$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int 0\,dx}}{[y_1(x)]^2}dx$$

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$$= y_1(x)\int \frac{dx}{[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots]}$$

$$= y_1(x)\int [\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots]dx$$

$$= y_1(x)[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots]$$

or $y_2(x) = y_1(x)\ln x$

$$+ y_1(x)[-\frac{1}{x} + \frac{7}{12}x + \frac{9}{144}x^2 + \cdots]$$

example

example

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.*

$$8x^2y'' + 10xy' + (x-1)y = 0$$

1) x=0 이 ordinary point 인지 regular singular point 인지 다음의 정의를 참고하여 판별하시오.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

standard form $$y'' + P(x)y' + Q(x)y = 0 \cdots (2)$$

Definition 5.2

Regular/Irregular Singular Points

A singular point x_0 **is said to be a regular singular point of the differential equation (1) if the functions** $p(x) = (x - x_0)P(x)$ **and** $q(x) = (x - x_0)^2 Q(x)$ **are both analytic at** x_0**.**

A singular point that is not regular is said to be an irregular point of the equation

*Bronson R., Costa G., Schaum's Outlines, Differential Equations Third Edition, Mc Graw Hill, 2006, p278, 28.6문제 응용

Problem

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$$8x^2y'' + 10xy' + (x-1)y = 0$$

2) 해의 형태를 가정하고 그렇게 가정한 이유를 1)의 결과와 연관지어 설명하시오.

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

4) equation에서 linearly independent 한 solution을 몇 개 찾을 수 있는지 3)의 indicial equation의 형태와 연관하여 설명하시오.

5) x=0 부근에서 General Solution을 구하시오. 단 series를 나타낼 때는 처음 세 번째 항까지 표시하시오.

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Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2 y'' + 10xy' + (x-1)y = 0$$

1) x=0 이 ordinary point 인지 regular singular point 인지 판별하시오.

Standard form을 바꾸면 $\quad y'' + \dfrac{5}{4x} y' + \dfrac{(x-1)}{8x^2} y = 0$

$P(x) = \dfrac{5}{4x}$ $, Q(x) = \dfrac{(x-1)}{8x^2}$ 는 x=0에서 singular point 이다.

$(x-0)^2$ 을 곱하면 $\quad x^2 y'' + x(\dfrac{10}{8})y' + \dfrac{(x-1)}{8} y = 0$

$p(x) = \dfrac{5}{4}$ $, q(x) = \dfrac{(x-1)}{8}$ 는 x=0에서 analytic 하다.

∴x=0은 regular singular point이다

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

2) 해의 형태를 가정하고 그렇게 가정한 이유를 1)의 결과와 연관지어 설명하시오.

x=0에서 regular singular point이기 때문에 Frobenius method를 적용하면 적어도 하나의 solution을 구할 수 있다. 따라서 해를 다음과 같이 가정한다.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

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식에 대입하면

$$8x^2y''+10xy'+(x-1)y$$

$$=8x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}+10x\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}+(x-1)\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$=\sum_{n=0}^{\infty}8(n+r)(n+r-1)c_nx^{n+r}+\sum_{n=0}^{\infty}10(n+r)c_nx^{n+r}+\sum_{n=0}^{\infty}c_nx^{n+r+1}-\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$=x^r\left\{\sum_{n=0}^{\infty}8(n+r)(n+r-1)c_nx^n+\sum_{n=0}^{\infty}10(n+r)c_nx^n+\sum_{n=0}^{\infty}c_nx^{n+1}-\sum_{n=0}^{\infty}c_nx^n\right\}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

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Continue ...

$$= x^r\left\{\sum_{n=0}^{\infty} 8(n+r)(n+r-1)c_n x^n + \sum_{n=0}^{\infty} 10(n+r)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n\right\}$$

$$= x^r\left\{\sum_{n=0}^{\infty} 8(n+r)(n+r-1)c_n x^n + \sum_{n=0}^{\infty} 10(n+r)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n\right\}$$

$$= x^r\left\{\left[8(r)(r-1)c_0x^0 + 10(r)c_0x^0 - c_0x^0\right] + \right.$$

$$\left.\left[\sum_{n=1}^{\infty} 8(n+r)(n+r-1)c_n x^n + \sum_{n=1}^{\infty} 10(n+r)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=1}^{\infty} c_n x^n\right]\right\}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2 y'' + 10xy' + (x-1)y = 0$$

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

Continue …

$$= x^r \left\{ \left[8r(r-1) + 10r - 1 \right] c_0 x^0 + \right.$$

$$\left. \left[\sum_{k=1}^{\infty} 8(k+r)(k+r-1) c_k x^k + \sum_{k=1}^{\infty} 10(k+r) c_k x^k + \sum_{k=1}^{\infty} c_{k-1} x^k - \sum_{k=1}^{\infty} c_k x^k \right] \right\}$$

$$= x^r \left\{ \left[8r^2 + 2r - 1 \right] c_0 x^0 + \right.$$

$$\left. \left[x^k \sum_{k=1}^{\infty} \left(8(k+r)(k+r-1) + 10(k+r) - 1 \right) c_k + c_{k-1} \right] \right\}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

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Continue …

$$= x^r\left\{\left[(4r-1)(2r+1)\right]c_0x^0 + \left[x^k\sum_{k=1}^{\infty}\left(8(k+r)(k+r-1)+10(k+r)-1\right)c_k + c_{k-1}\right]\right\}$$

$$when,\ \ r = \frac{1}{4},$$

$$2k(4k+3)c_k + c_{k-1} = 0$$

$$\therefore c_k = -\frac{1}{2k(4k+3)}c_{k-1}\ \ ,(k = 1,2,3,..)$$

$$when,\ \ r = -\frac{1}{2},$$

$$2k(4k-3)c_k + c_{k-1} = 0$$

$$\therefore c_k = -\frac{1}{2k(4k-3)}c_{k-1}\ \ ,(k = 1,2,3,..)$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

4) equation에서 linearly independent 한 solution을 몇 개 찾을 수 있는지 3)의 indicial equation의 형태와 연관하여 설명하시오.

indicial equation $(4r-1)(2r-1)$ **에서** $r = \dfrac{1}{4}, r = -\dfrac{1}{2}$ **은**

서로 다르고 차이가 정수가 아니므로 linearly independent 한 두 개의

Solution이 존재한다.

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

5) x=0 부근에서 General Solution을 구하시오. 단 series를 나타낼 때는 처음 세 번째 항까지 표시하시오.

$$r = \frac{1}{4},$$

$$c_k = -\frac{1}{2k(4k+3)}c_{k-1} \ ,(k = 1,2,3,..)$$

$$c_1 = -\frac{1}{2\cdot 7}c_0 = -\frac{1}{14}c_0$$

$$c_2 = -\frac{1}{4\cdot 11}c_1 = \frac{1}{4\cdot 11}\cdot\frac{1}{2\cdot 7}c_0 = \frac{1}{616}c_0$$

$$y = x^r\sum_{n=0}^{\infty}c_nx^n$$

$$y_1 = x^{1/4}\left(c_0x^0 + c_1x^1 + c_2x^2 + \cdots\right)$$

$$= c_0x^{1/4}\left(x^0 - \frac{1}{14}x^1 + \frac{1}{616}x^2 + \cdots\right)$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

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$$r = -\frac{1}{2},$$

$$c_k = -\frac{1}{2k(4k-3)}c_{k-1} \quad ,(k = 1,2,3,..)$$

$$c_1 = -\frac{1}{2\cdot 1}c_0 = -\frac{1}{2}c_0$$

$$c_2 = -\frac{1}{4\cdot 5}c_1 = \frac{1}{4\cdot 5}\cdot\frac{1}{2\cdot 1}c_0 = \frac{1}{40}c_0$$

$$y = x^r\sum_{n=0}^{\infty}c_n x^n$$

$$y_2 = x^{-1/2}\left(c_0x^0 + c_1x^1 + c_2x^2 + \cdots\right)$$

$$= c_0x^{-1/2}\left(x^0 - \frac{1}{2}x^1 + \frac{1}{40}x^2 + \cdots\right)$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2 y'' + 10xy' + (x-1)y = 0$$

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$$\therefore y = C_0 y_1 + C_1 y_2$$

$$y_1 = x^{1/4}\left(x^0 - \frac{1}{14}x^1 + \frac{1}{616}x^2 + \cdots\right)$$

$$y_2 = x^{-1/2}\left(x^0 - \frac{1}{2}x^1 + \frac{1}{40}x^2 + \cdots\right)$$