Naval Architecture & Ocean Engineering

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Engineering Mathematics 2

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Fourier Series(2)
: Sturm-Liouville Problem

Sturm-Liouville Problem

▪**Review**

Linear Equations		**General solutions**
$y' + \alpha y = 0$		$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0$	$\alpha > 0$	$y = c_1 \cos \alpha x + c_2 \sin \alpha x$
$y'' - \alpha^2 y = 0$	$\alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, or \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x) \end{cases}$

← When x is an infinite or half finite interval (for $y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$)

← When x is a finite interval (for $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$)

Cauchy-Euler Equation		**General solutions** $x > 0$
$x^2 y'' + xy' - \alpha^2 y = 0$	$\alpha \geq 0$	$\begin{cases} y = c_1 x^{-\alpha} + c_2 x^{\alpha}, \alpha \neq 0 \\ y = c_1 + c_2 \ln x, \quad \alpha = 0 \end{cases}$

Linear Equations

Sturm-Liouville Problem

- **Review**

Parametric Bessel equation $\nu = 0$	**General solutions** $x > 0$
$x^2 y'' + y' + \alpha^2 x^2 y = 0$	$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$

Legendre's equation $n = 0, 1, 2, \ldots$	**Particular solutions are polynomials**
$(1 - x^2) y'' - 2xy' + n(n+1) y = 0$	$y = P_0(x) = 1,$ $y = P_1(x) = x,$ $y = P_2(x) = \frac{1}{2}(3x^2 - 1),$ $\vdots$

Sturm-Liouville Problem

- **Eigenvalues and Eigenfunctions**

Recall example 2 of section 3.9

$$y''+\lambda y=0, \quad y(0)=0, \; y(L)=0$$

When $\lambda > 0$ **(Case III)**

Write $\lambda=\alpha^2, \; \alpha>0$

Then roots of auxiliary equation is

$$m_1=i\alpha, \; m_2=-i\alpha$$

$$y=c_1\cos\alpha x+c_2\sin\alpha x$$

Eigenvalues

$$y(0)=0 \Rightarrow c_1=0$$

$$y(L)=0 \Rightarrow c_2=0 \; \textbf{or} \; \alpha L=n\pi, \; \lambda_n=\alpha_n{}^2=\left(\frac{n\pi}{L}\right)^2$$

Eigenfunctions

(nontrivial solution) $y_n=c_2\sin\frac{n\pi}{L}x$

Sturm-Liouville Problem

- **Eigenvalues and Eigenfunctions**

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(L) = 0$$

Eigenvalues

$$\lambda_n = \alpha_n{}^2 = \left(\frac{n\pi}{L}\right)^2$$

Eigenfunctions

$$y_n = c_2 \sin \frac{n\pi}{L} x$$

It is important to recognize the set of functions generated by this B.V.P

the orthogonal set of functions on the interval $(0, L)$ used as the basis

for the Fourier sine series

$$y'' + \lambda y = 0, \quad y'(0) = 0, \; y'(L) = 0 \longrightarrow$$ **the Fourier cosine series**

Sturm-Liouville Problem

☑ Example 1 Eigenvalues and Eigenfunctions

It is left as an exercise to show, by considering the three possible cases for the parameter λ (zero, negative, or positive; that is, $\lambda = 0,\ \lambda = -\alpha^2 < 0,\ \alpha > 0,\ and\ \lambda = \alpha^2 > 0,\ \alpha > 0$) that the eigenvalues and eigenfunctions for the boundary-value problem

$$y'' + \lambda y = 0,\ \ y'(0) = 0,\ \ y'(L) = 0$$

are, respectively, $\lambda_n = {\alpha_n}^2 = n^2\pi^2 / L^2,\ n = 0,1,2,\cdots,$ $y = c_1 \cos(n\pi x / L),\ c_1 \neq 0.$ $\lambda_0 = 0$ is an eigenvalue for this BVP and $y = 1$ is the corresponding eigenfunction. The latter comes from solving $y'' = 0$ subject to the same boundary conditions $y'(0) = 0,\ y'(L) = 0$. Note also that $y = 1$ can be incorporated into the family $y = \cos(n\pi x / L)$ by permitting $n = 0$. The set $\{\cos(n\pi x / L)\},$ $n = 0,1,2,3,\cdots,$ is orthogonal on the interval [0,L].

Sturm-Liouville Problem

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

p, q, r, r'

real-valued functions continuous on an interval $[a,b]$

$r(x) > 0,\ p(x) > 0$

for every x **in the interval** $[a,b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

Special case

$$p(x) = 1, q(x) = 0, r(x) = 1$$

$$A_1 = 1, B_1 = 0, A_2 = 1, B_2 = 0, a = 0, b = L$$

➡ $$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

$$A_1 = 0, B_1 = 1, A_2 = 0, B_2 = 1, a = 0, b = L$$

➡ $$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

Sturm-Liouville Problem :
Homogeneous Boundary Value problem

-> Trivial solution y=0

-> goal : find nontrivial solution y
(Eigenvalues, Eigenfunctions)

Sturm-Liouville Problem

▪Regular Sturm-Liouville Problem **B.V.P**

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Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$

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p, q, r, r'

real-valued functions continuous on an interval $[a, b]$

$r(x) > 0, \ p(x) > 0$

for every x **in the interval** $[a, b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

"Homogeneous"

⇨ Homogeneous D.E.+

Homogeneous B/C

Nonhomogeneous B/C

$$A_1 y(a) + B_1 y'(a) = C_2, C_2 : nonzero$$

12.5 Sturm-Liouville Problem

▪ **Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$

Subject to: $A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

p, q, r, r'

real-valued functions continuous on an interval $[a, b]$

$r(x) > 0, \ p(x) > 0$

for every x **in the interval** $[a, b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

"trivial solution is not our interest"

Homogeneous B.V.P always possesses the trivial solution $y = 0$

Sturm-Liouville Problem

Theorem 12.3

Properties of the Regular Sturm-Liouville Problem

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \lambda_n \to \infty$ such that $n \to \infty$ as

(b) For each eigenvalues there is only one eigenfunction (except for nonzero constant multiples)

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Solve : $$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $$\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$$

$$\frac{d}{dx}[r(x)y_m'] + [q(x) + \lambda_m p(x)]y_m = 0 \cdots (1)$$

$$\frac{d}{dx}[r(x)y_n'] + [q(x) + \lambda_n p(x)]y_n = 0 \cdots (2)$$

$$(1)\times y_n - (2)\times y_m : \; y_n \frac{d}{dx}[r(x)y_m'] - y_m \frac{d}{dx}[r(x)y_n'] + (\lambda_m - \lambda_n)p(x)y_n y_m = 0$$

$$(\lambda_n - \lambda_m)p(x)y_n y_m = y_n \frac{d}{dx}[r(x)y_m'] - y_m \frac{d}{dx}[r(x)y_n']$$

$$= y_n \frac{d}{dx}[r(x)y_m'] + [r(x)y_m']\frac{d}{dx}y_n - y_m \frac{d}{dx}[r(x)y_n'] - [r(x)y_n']\frac{d}{dx}y_m$$

$$r(x)y_n'y_m' \qquad\qquad -r(x)y_n'y_m'$$

$$= \frac{d}{dx}\left(y_n[r(x)y_m']\right) - \frac{d}{dx}\left(y_m[r(x)y_n']\right)$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $$\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$$

Integrating $$(\lambda_n - \lambda_m)p(x)y_n y_m = \frac{d}{dx}\left(y_n[r(x)y'_m]\right) - \frac{d}{dx}\left(y_m[r(x)y'_n]\right)$$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = \int_a^b \left(\frac{d}{dx}\left(y_n[r(x)y'_m]\right) - \frac{d}{dx}\left(y_m[r(x)y'_n]\right)\right)dx$$

$$= y_n(b)[r(b)y'_m(b)] - y_n(a)[r(a)y'_m(a)]$$

$$-\left(y_m(b)[r(b)y'_n(b)] - y_m(a)[r(a)y'_n(a)]\right)$$

$$= r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)]$$

$$-r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$

Boundary Condition

$$A_1 y(a) + B_1 y'(a) = 0 \;\rightarrow\; A_1 y_m(a) + B_1 y'_m(a) = 0 \cdots (3)$$

$$A_1 y_n(a) + B_1 y'_n(a) = 0 \cdots (4)$$

$$A_2 y(b) + B_2 y'(b) = 0 \;\rightarrow\; A_2 y_m(b) + B_2 y'_m(b) = 0 \cdots (5)$$

$$A_2 y_n(b) + B_2 y'_m(b) = 0 \cdots (6)$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $$\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)]$$
$$- r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_1 y_m(a) + B_1 y_m'(a) = 0 \cdots (3)$$
$$A_1 y_n(a) + B_1 y_n'(a) = 0 \cdots (4)$$

$$\Rightarrow \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As A_1, B_1 **are not both zero**

$$\det \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} = y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_2 y_m(b) + B_2 y_m'(b) = 0 \cdots (5)$$
$$A_2 y_n(b) + B_2 y_n'(b) = 0 \cdots (6)$$

$$\Rightarrow \begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix}\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As A_2, B_2 **are not both zero**

$$\det\begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix} = y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

Sturm-Liouville Problem

(d) **The set of eigenfunctions** **corresponding to the set of eigenvalues** **is orthogonal** **with respect to the weight function** $p(x)$ **on interval** $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

zero

zero

From Boundary Condition:

$$y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$

$$y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

$$\therefore (\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = 0$$

$$\int_a^b p(x)y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Sturm-Liouville Problem

(d) **The set of eigenfunctions** corresponding to the set of eigenvalues **is orthogonal with respect to the weight function** $p(x)$ on interval $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

zero

zero

From Boundary Condition:

$$y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$

$$y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

$$\therefore (\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = 0$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Sturm-Liouville Problem

☑ Example 2
A Regular Sturm-Liouville Problem

Slove the boundary-value problem

$$y'' + \lambda y = 0,\ \ y(0) = 0,\ \ y(1) + y'(1) = 0.$$

$\lambda = 0$ *and* $\lambda = -\alpha^2 < 0$, *where* $\alpha > 0$,
the trivial solution $y = 0$

$\lambda = \alpha^2 > 0,\ \alpha > 0,$
the general solution of $y'' + \alpha^2 y = 0$
is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$

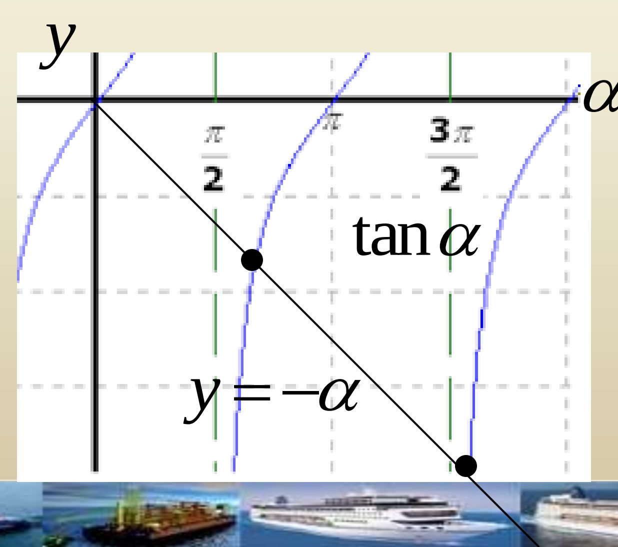
$y(0) = c_1 = 0 \ \therefore y = c_2 \sin \alpha x$

The second boundary condition $y(1) + y'(1) = 0$ **is satisfied if**

$$c_2 \sin \alpha + c_2 \alpha \cos \alpha = c_2(\sin \alpha + \alpha \cos \alpha) = 0$$

Choosing $c_2 \neq 0$**, we see that the last equation is equivalent to** $\tan \alpha = -\alpha$

The eigenvalues of problem are then $\lambda_n = \alpha_n^2$**, where** α_n**,** $n = 1, 2, 3, \cdots$ **, are the consecutive positive roots** $\alpha_1, \alpha_2, \alpha_3, \cdots$ **of** $\tan \alpha = -\alpha$



Sturm-Liouville Problem

☑ Example 2 A Regular Sturm-Liouville Problem

Slove the boundary-value problem

$$y'' + \lambda y = 0,\ \ y(0) = 0,\ \ y(1) + y'(1) = 0.$$

$\lambda = 0$ *and* $\lambda = -\alpha^2 < 0$, *where* $\alpha > 0$,

the trivial solution $y = 0$

$\lambda = \alpha^2 > 0,\ \alpha > 0$,

the general solution of $y'' + \alpha^2 y = 0$

is $y = c_1 \cos\alpha x + c_2 \sin\alpha x$

$$y(0) = c_1 = 0 \ \therefore y = c_2 \sin\alpha x$$

The second boundary condition $y(1) + y'(1) = 0$ **is satisfied if**

$$c_2 \sin\alpha + c_2 \alpha\cos\alpha = c_2(\sin\alpha + \alpha\cos\alpha) = 0$$

Choosing $c_2 \neq 0$**, we see that the last equation is equivalent to** $\tan\alpha = -\alpha$

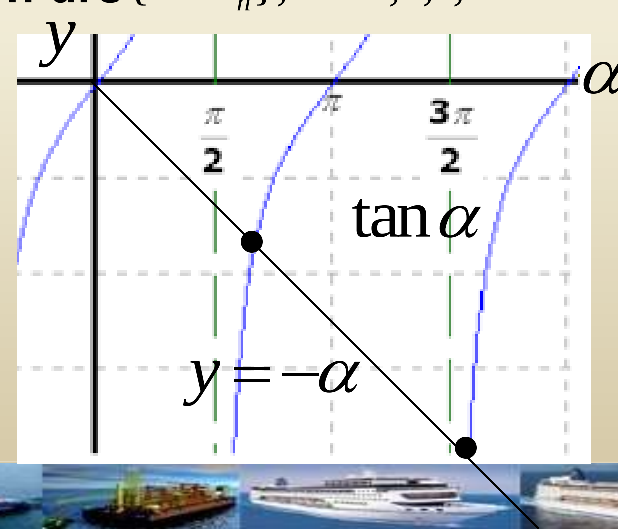
$\alpha_1 = 2.0288,\ \alpha_2 = 4.9132,\ \alpha_3 = 7.9787,\ \alpha_4 = 11.0855,$

and the corresponding solutions are

$y_1 = \sin 2.0288x,\ \ y_2 = \sin 4.9132x,\ \ y_3 = \sin 7.9787x,$

$y_4 = \sin 11.0855x$

In general, the eigenfunctions of the problem are $\{\sin\alpha_n\},\ n = 1,2,3,\cdots$



Sturm-Liouville Problem

☑ Example 2
A Regular Sturm-Liouville Problem

Slove the boundary-value problem

- **Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\dfrac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$

Subject to: $A_1 y(a)+B_1 y'(a)=0$

$A_2 y(b)+B_2 y'(b)=0$

$$y''+\lambda y=0,\ \ y(0)=0,\ \ y(1)+y'(1)=0.$$

$$r(x)=1, q(x)=0, p(x)=1$$

$$A_1=1, B_1=0, A_2=1, B_2=1$$

In general, the eigenfunctions of the problem are $\{\sin\alpha_n\},\ n=1,2,3,\cdots$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx=0,\ \ \lambda_n \neq \lambda_m$$

Regular Sturm-Liouville Problem

$\{\sin\alpha_n\},\ n=1,2,3,\cdots$ **is an orthogonal set with respect to the weight function** $p(x)=1$ **on the interval [0,1].**

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- ***In some circumstances, we can prove the orthogonality of the solutions of*** $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$ ***without the necessity of specifying a boundary condition at x=a and at x=b***

→ Singular Sturm-Liouville Problem

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a)+B_1 y'(a)=0$$
$$A_2 y(b)+B_2 y'(b)=0$$

$$(\lambda_n-\lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b)-y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a)-y_m(a)y_n'(a)]$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx=0,\ \lambda_n\neq\lambda_m$$

should be zero for Orthogonal relation

- **Singular Sturm-Liouville Problem**

If $r(a)=0$ **then** $x=a$ **may be a singular and the equation**

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$$

may become unbounded as $x\to a$ $,(a,b]$

however

zero

$$r(b)[y_m'(b)y_n(b)-y_m(b)y_n'(b)]-r(a)[y_m'(a)y_n(a)-y_m(a)y_n'(a)]$$

dropped from the problem
: no boundary condition at $x=a$

Orthogonal relation hold on $[a,b]$

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- **Singular Sturm-Liouville Problem**

If $r(b) = 0$ **then** $x = b$ **may be a singular and the equation**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

may become unbounded as $x \to b$ $,[a,b)$

however

zero

$$r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

dropped from the problem
: no boundary condition at $x = b$

➡ **Orthogonal relation hold on** $[a,b]$

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \ \lambda_n \neq \lambda_m$$

▪**Singular Sturm-Liouville Problem**

example*) Legendre's equation is a Sturm-Liouville equation

$$\left[(1-x^2)y'\right]' + \lambda y = 0 \Leftrightarrow \underbrace{(1-x^2)}_{r(x)} y'' - 2xy' + \lambda y = 0 \qquad \lambda = n(n+1)$$

Since $r(\pm 1) = 0$ **need no boundary conditions, but have a singular Sturm-Liouville problem on the interval** $-1 \leq x \leq 1$ **. We know that , the Legendre polynomials** $P_n(x)$ **are solutions of the problem for** $n = 0,1,2,\ldots(\lambda = 0, 1\cdot 2, 2\cdot 3, \ldots)$

Hence these are the eigenfunctions. They are orthogonal on the interval

$$\int_{-1}^{1} p_m(x) p_n(x) dx = 0, \ (m \neq n)$$

*Kreyszig E. ,Advanced Engineering Mathematics, 9th edition, Willey, 2006, p207 example 5

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$

should be zero for Orthogonal relation

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

- **Periodic Sturm-Liouville Problem**

If $r(a) = r(b)$ **then**

$$r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$
$$= r(a)\left[(y'_m(b)y_n(b) - y'_m(a)y_n(a)) + (y_m(a)y'_n(a) - y_m(b)y'_n(b))\right]$$

$\therefore$ **Orthogonal relation hold on** $[a,b]$ **with** $y(a) = y(b),\ y'(a) = y'(b)$

Sturm-Liouville Problem

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx$$

$$= r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

▪**Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

▪**By assuming the solution (y) are bounded on the closed interval [a,b] , then**

Orthogonal relation hold on.. [a,b]

$$\int_a^b p(x) y_n y_m dx = 0, \; \lambda_n \neq \lambda_m \quad [a,b]$$

- **Regular** $r(x) \neq 0$ **with boundary Condition:** $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$
- **Singular** $r(a) = 0$ **without B/C at x=a** $A_2 y(b) + B_2 y'(b) = 0$

  $r(b) = 0$ **without B/C at x=b** $A_1 y(a) + B_1 y'(a) = 0$
- **Periodic** $r(a) = r(b)$ **with** $y(a) = y(b), \; y'(a) = y'(b)$

Sturm-Liouville Problem

- **Self-Adjoint Form**

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$$

If the coefficient are continuous and $a(x)\neq 0$ for all x in some interval, then any second-order differential equation

$$a(x)y''+b(x)y'+(c(x)+\lambda d(x))y=0$$

can be recast into the so-called '**self-adjoint form**'.

Recall, ch. 2.3 integrating factor

$$a_1(x)y'+a_0(x)y=0 \longrightarrow \frac{d}{dx}[\mu y]=0$$

$$\mu=e^{\int p(x)dx},\ p(x)=\frac{a_0(x)}{a_1(x)}$$

Sturm-Liouville Problem

▪**Self-Adjoint Form**

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

$$y'' + \frac{b(x)}{a(x)}y' + \left(\frac{c(x)}{a(x)} + \lambda\frac{d(x)}{a(x)}\right)y = 0$$

divided by $a(x)$

multiply $e^{\int \frac{b(x)}{a(x)}dx}$

$$e^{\int \frac{b(x)}{a(x)}dx}y'' + \frac{b(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}y' + \left(e^{\int \frac{b(x)}{a(x)}dx}\frac{c(x)}{a(x)} + \lambda e^{\int \frac{b(x)}{a(x)}dx}\frac{d(x)}{a(x)}\right)y = 0$$

$$\frac{d}{dt}\Bigg[\underbrace{e^{\int \frac{b(x)}{a(x)}dx}}_{r(x)}y'\Bigg] + \left(\underbrace{\frac{c(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}}_{q(x)} + \lambda\underbrace{\frac{d(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}}_{p(x)}\right)y = 0$$

Sturm-Liouville Problem

- **Self-Adjoint Form**

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

$$r(x) = e^{\int \frac{b(x)}{a(x)}dx}$$

$$q(x) = \frac{c(x)}{a(x)} e^{\int \frac{b(x)}{a(x)}dx}$$

$$p(x) = \frac{d(x)}{a(x)} e^{\int \frac{b(x)}{a(x)}dx}$$

Example) $3y'' + 6y' + \lambda y = 0$

$$y'' + \frac{6}{3}y' + \frac{\lambda}{3}y = 0$$

$$e^{\int \frac{b(x)}{a(x)}dx} = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y'' + 2e^{2x}y' + \frac{\lambda}{3}e^{2x}y = 0$$

$$\frac{d}{dx}\left[e^{2x}y'\right] + \lambda \frac{e^{2x}}{3} y = 0$$

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \qquad [a,b]$$

$$\int_a^b p(x) y_n y_m dx = 0, \ \lambda_n \neq \lambda_m$$

▪**Self-Adjoint Form**

Ex.)Parametric Bessel Series*

$$x^2 y'' + xy' + (\alpha^2 x^2 - n^2)y = 0, \quad n = 0,1,2,\ldots$$

General solution $y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$

$J_n(x)$ converges on $[0,\infty)$ when $n \geq 0$

$Y_n(x)$ converges on $(0,\infty)$

---

divided by x^2

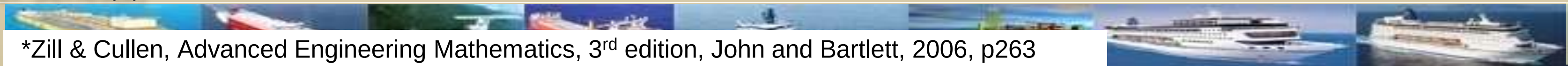
$$y'' + \frac{1}{x} y' + (\alpha^2 - \frac{n^2}{x^2})y = 0 \longrightarrow xy'' + y' + (x\alpha^2 - \frac{n^2}{x})y = 0$$

multiply $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$x > 0$

$$\frac{d}{dx}[xy'] + (-\frac{n^2}{x} + \alpha^2 x)y = 0$$

$r(x)$ $q(x)$ λ $p(x)$



*Zill & Cullen, Advanced Engineering Mathematics, 3rd edition, John and Bartlett, 2006, p263

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \qquad [a,b]$$

$$\int_a^b p(x) y_n y_m dx = 0, \ \lambda_n \neq \lambda_m$$

- Singular $r(a) = 0$: Orthogonal relation holds on without B/C at x=a

▪**Self-Adjoint Form**

Ex.)Parametric Bessel Series*

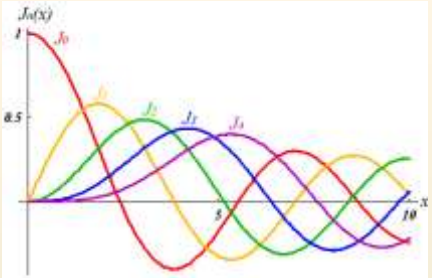
$$x^2 y'' + xy' + (\alpha^2 x^2 - n^2) y = 0, \quad n = 0, 1, 2, \dots$$

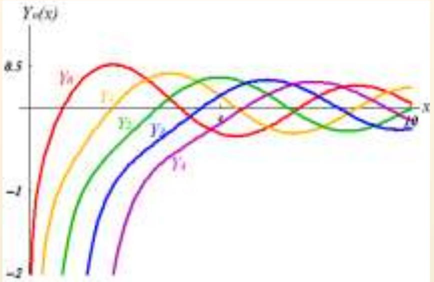
General solution $y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$ $\quad J_n(x)$ converges on $[0, \infty)$ when $n \geq 0$

$Y_n(x)$ converges on $(0, \infty)$

$$\frac{d}{dx}[xy'] + \left(-\frac{n^2}{x} + \alpha^2 x\right) y = 0$$

$r(x)$ $\quad q(x)$ $\quad \lambda \ p(x)$





$r(0) = 0$

only $J_n(\alpha x)$ is bounded at $x = 0$

of the two solutions $J_n(\alpha x), Y_n(\alpha x)$

$(Y_n \to -\infty \ as \ x \to 0)$

Recall, singular Sturm-Liouville Problem, the set $\{J_n(\alpha_i x)\}, i = 1, 2, 3\dots,$ is orthogonal with respect to the weight function $p(x) = x$ on an interval $[0, b]$

The orthogonality relation is

$$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x) dx = 0, \ \lambda_i \neq \lambda_j \ (\lambda = \alpha^2)$$

*Zill & Cullen, Advanced Engineering Mathematics, 3rd edition, John and Bartlett, 2006, p263

Sturm-Liouville Problem

▪**Self-Adjoint Form**

Ex.)Parametric Bessel Series*

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

$$\frac{d}{dx}[xy'] + \left(-\frac{n^2}{x} + \alpha^2 x\right)y = 0$$

$r(x)$ $\quad q(x)$ $\quad \lambda \; p(x)$

$\{J_n(\alpha_i x)\}, i = 1, 2, 3...,$ **: orthogonal set**

orthogonal relation

$$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x) dx = 0, \quad \lambda_i \neq \lambda_j \; (\lambda = \alpha^2)$$

Provided the α_i, and hence the eigenvalues $\lambda_i = \alpha_i^2, \; i = 1, 2, 3, \cdots$ are defined by means of a boundary condition at $x = b$ of the type given in $A_2 y(b) + B_2 y'(b) = 0$:

$$A_2 J_n(\alpha b) + B_2 \alpha J_n{}'(\alpha b) = 0$$

→ **Roots** $x_i (= \alpha_i b)$

→ **Eigenvalues** $\lambda_i = (\alpha_i)^2 = \left(\frac{x_i}{b}\right)^2$

Sturm-Liouville Problem

- **Self-Adjoint Form**

Ex.) Legendre's Equation*

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a, b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \;\; \lambda_n \neq \lambda_m$$

$$\left[(1-x^2)y'\right]' + \lambda y = 0 \;\; \Leftrightarrow \;\; (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$q(x) = 0, \;\; p(x) = 1$$

$$\lambda = n(n+1)$$

Legendre's D.E.
→ polynomial solutions $P_n(x)$ $\quad n = 0, 1, 2, \cdots$

As $P_n(x)$ is the only solutions of the equation that are bounded on the closed interval [-1,1], and $r(-1) = r(1) = 0$ (no boundary condition required) that the set $P_n(x)$ is orthogonal with respect to the weight function on [-1,1], The orthogonality relation is $\int_{-1}^{1} 1 \cdot P_m(x) P_n(x) dx = 0, \;\; m \neq n$