Matrix Transformation

Human-Centered CAD Lab.

Transformation & Mapping

- Calculates world coordinates of points on a model when it (or its model coordinate system) is translated and rotated

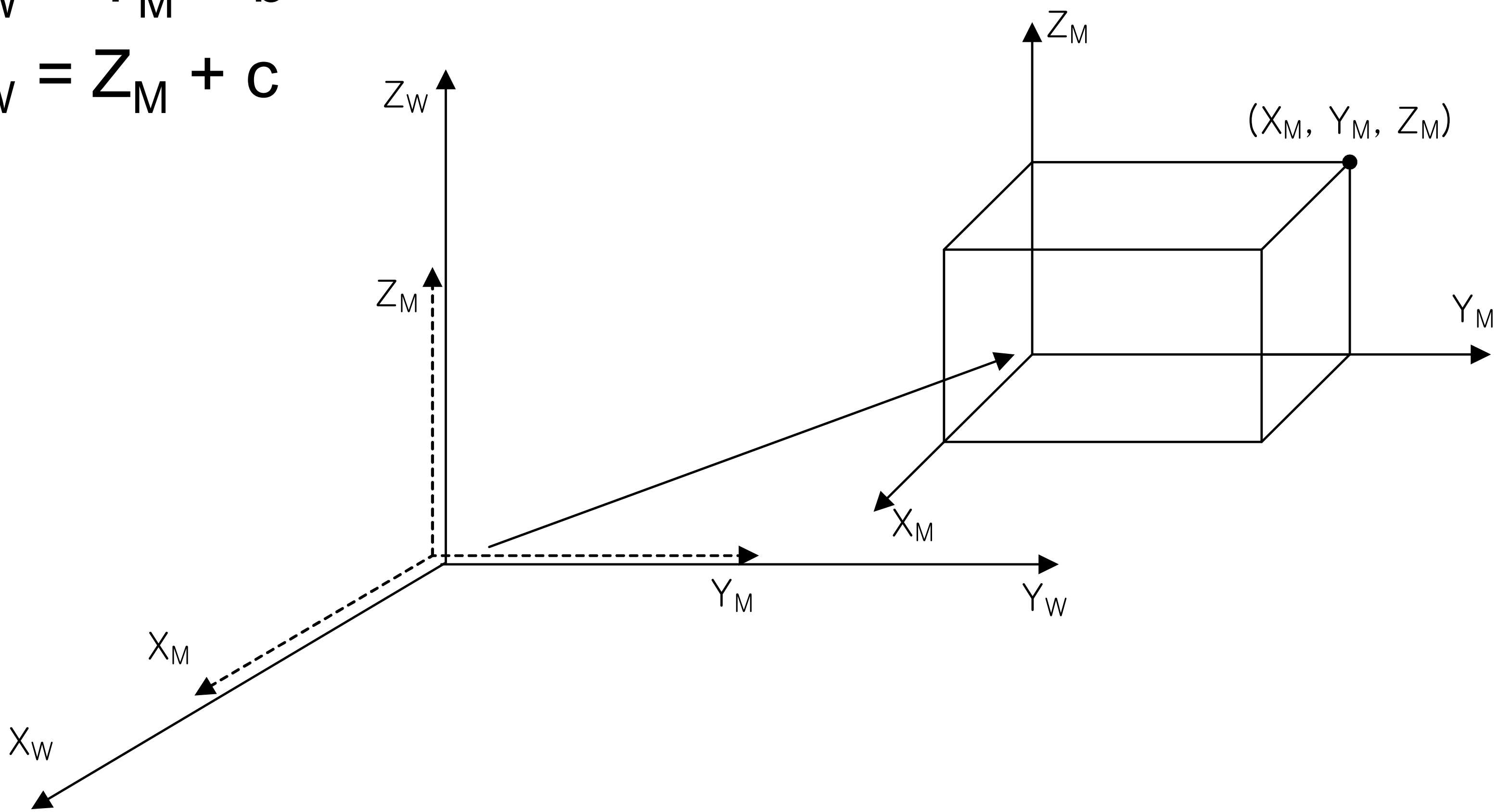
  ⇨ Transformation,
  MODELVIEW matrix mode in OpenGL

- Caculates viewing coordinates of points on a model from their world coordinates

  ⇨ mapping
  GL_PROJECTION matrix mode in OpenGL

Translation

- $X_W = X_M + a$
- $Y_W = Y_M + b$
- $Z_W = Z_M + c$



Translation -continued

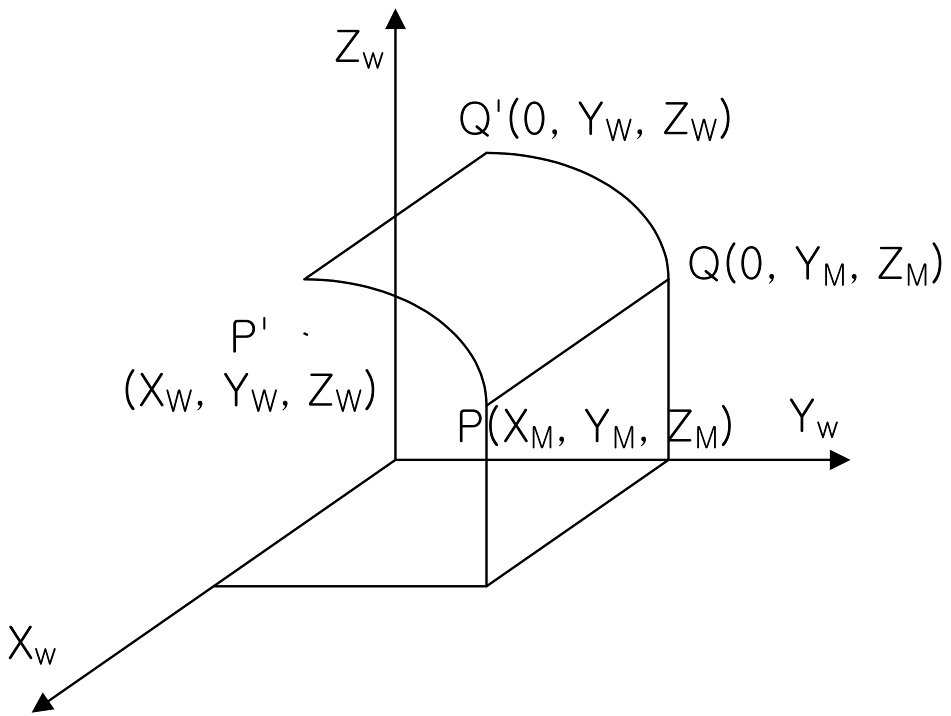
- For 2D, set Z_W, Z_M = 0 or use 3×3 transformation matrix

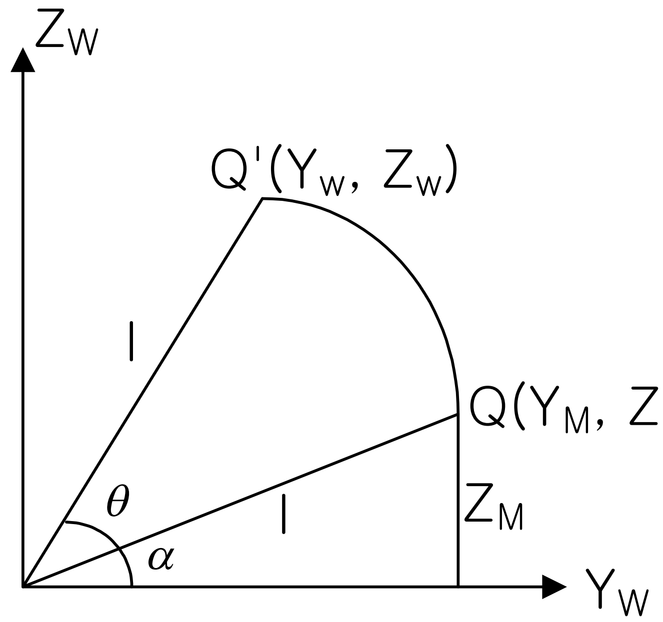
$$\underbrace{\begin{bmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathrm{Trans}(a,b,c)} \cdot \begin{bmatrix} X_M \\ Y_M \\ Z_M \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Homogeneous Transformation Matrix

Rotation


Zw
Q'(0, YW, ZW)
Q(0, YM, ZM)
P'
(XW, YW, ZW)
P(XM, YM, ZM)
Yw
Xw


ZW
Q'(Yw, Zw)
l
Q(YM, ZM)
θ
l
ZM
α
YW

Rotation -continued

- Y_W $= l\cos(\theta + \alpha)$
  $= l(\cos\theta \cdot \cos\alpha - \sin\theta \cdot \sin\alpha)$
  $= l\cos\alpha\cos\theta - l\sin\alpha\sin\theta$
  $= Y_M\cos\theta - Z_M\sin\theta$

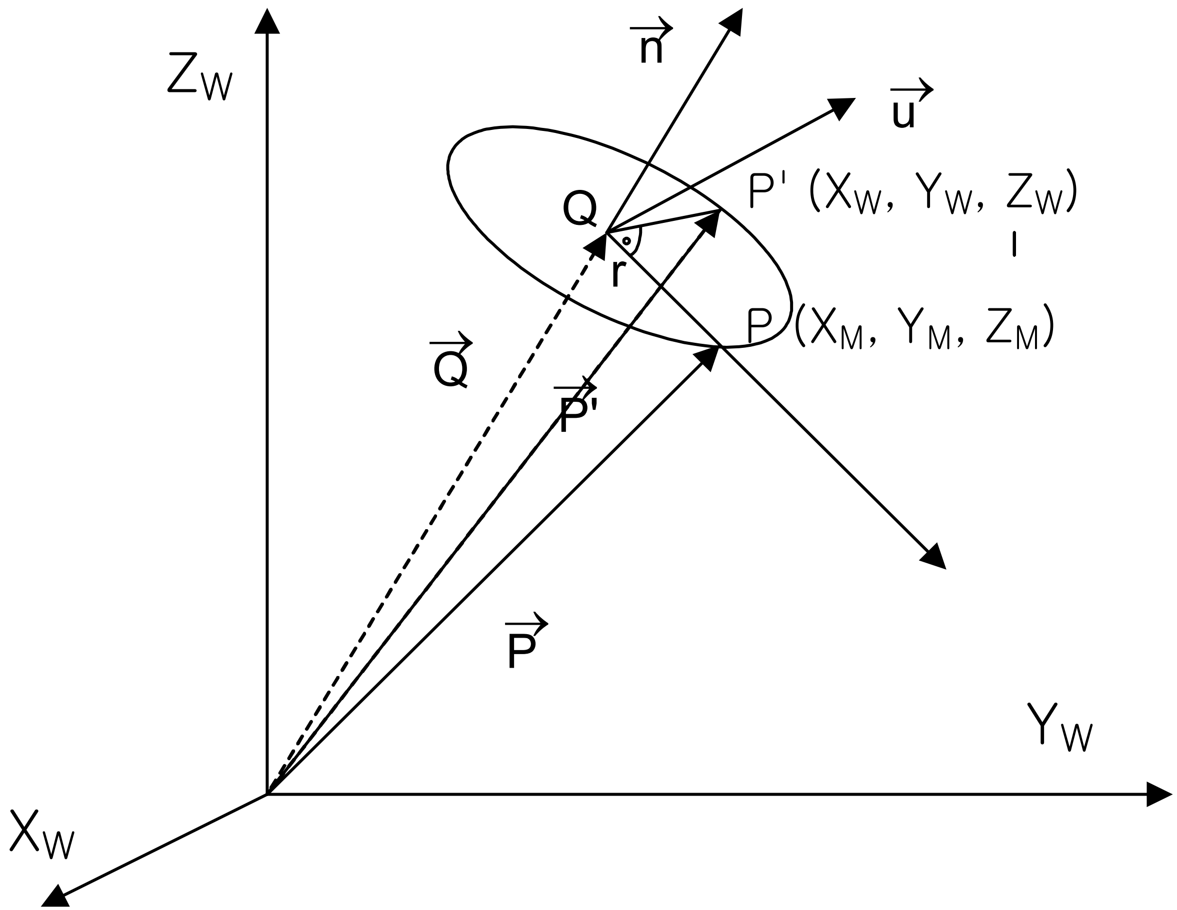
- Z_W $= l\sin(\theta+\alpha)$
  $= l(\sin\theta \cdot \cos\alpha + \cos\theta \cdot \sin\alpha)$
  $= l\cos\alpha\sin\theta - l\sin\alpha\cos\theta$
  $= Y_M\sin\theta - Z_M\cos\theta$

Rotation -continued

$$\begin{bmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot(X,\theta)} \cdot \begin{bmatrix} X_M \\ Y_M \\ Z_M \\ 1 \end{bmatrix}$$

$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Rot(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about arbitrary axis



(a) $\mathbf{P'} = \mathbf{Q} + r\cos\theta\ \mathbf{v} + \mathrm{r}\sin\theta\ \mathbf{u}$

(b) $r\mathbf{v} = \mathbf{P} - \mathbf{Q}$

(c) $\mathbf{u} = \mathbf{n} \times \mathbf{v} = \dfrac{\mathbf{n} \times (\mathbf{P} - \mathbf{Q})}{\mathrm{r}}$

Rotation about arbitrary axis -continued

- Substitute (b), (c) into (a)

$$\mathbf{P}' = \mathbf{Q} + (\mathbf{P} - \mathbf{Q})\cos\theta + r\sin\theta\,\frac{\mathbf{n}\times(\mathbf{P}-\mathbf{Q})}{r} \qquad (\mathbf{n}\times\mathbf{Q} = \mathbf{0})$$

$$= \mathbf{Q}(1-\cos\theta) + \mathbf{P}\cos\theta + (\mathbf{n}\times\mathbf{P})\sin\theta \qquad \textbf{(d)}$$

- Substitute Q=(P · n)n into (d)

$$\mathbf{P}' = (\mathbf{P}\cdot\mathbf{n})\mathbf{n}(1-\cos\theta) + \mathbf{P}\sin\theta + (\mathbf{n}\times\mathbf{P})\sin\theta$$

Rotation about arbitrary axis -continued

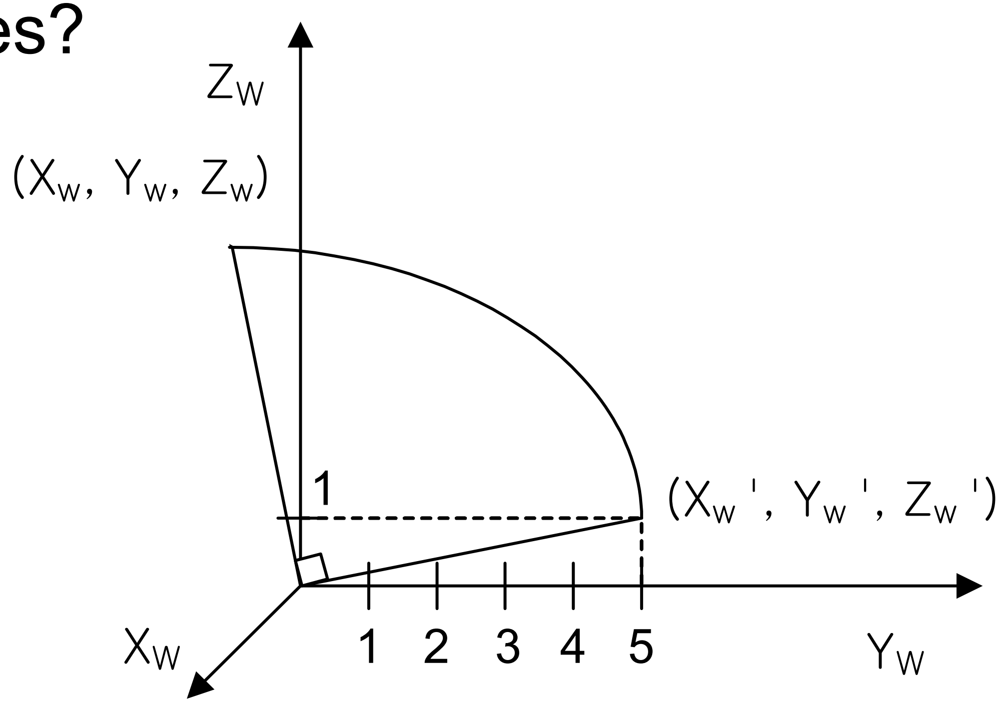
$$\begin{bmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{bmatrix} = \mathrm{Rot}(n,\theta)\cdot\begin{bmatrix} X_M \\ Y_M \\ Z_M \\ 1 \end{bmatrix}$$

$$Rot(n,\theta)$$

$$= \begin{bmatrix} \mathrm{n}_\mathrm{x}^2\left(1-\mathrm{n}_\mathrm{x}^2\right)+\cos\theta & \mathrm{n_x n_y}(1-\cos\theta)+\mathrm{n_z}\sin\theta & \mathrm{n_x n_z}+(1-\cos\theta)-\mathrm{n_y}\sin\theta & 0 \\ \mathrm{n_x n_y}(1-\cos\theta)-\mathrm{n_z}\sin\theta & \mathrm{n}_\mathrm{y}^2(1-\cos\theta)+\cos\theta & \mathrm{n_y n_z}(1-\cos\theta)+\mathrm{n_x}\sin\theta & 0 \\ \mathrm{n_x n_z}(1-\cos\theta)+\mathrm{n_y}\sin\theta & \mathrm{n_y n_z}(1-\cos\theta)-\mathrm{n_x}\sin\theta & \mathrm{n}_\mathrm{z}^2(1-\cos\theta)+\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example1

- An object is translated 5 units in y direction of world coordinate system and then rotated 90 degrees about x-axis of world coordinate system.
- What are the world coordinates of point (0, 0, 1) in model coordinates?



Example1(cont')

- Translation

$$\begin{bmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{bmatrix} = \text{Trans}(0,5,0) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 1 \end{bmatrix} \qquad \text{(a)}$$

- Rotation

$$\begin{bmatrix} X'_W \\ Y'_W \\ Z'_W \\ 1 \end{bmatrix} = Rot(x,90°) \cdot \begin{bmatrix} 0 \\ 5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° & 0 \\ 0 & \sin 90° & \cos 90° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 1 \end{bmatrix} \qquad (b)$$

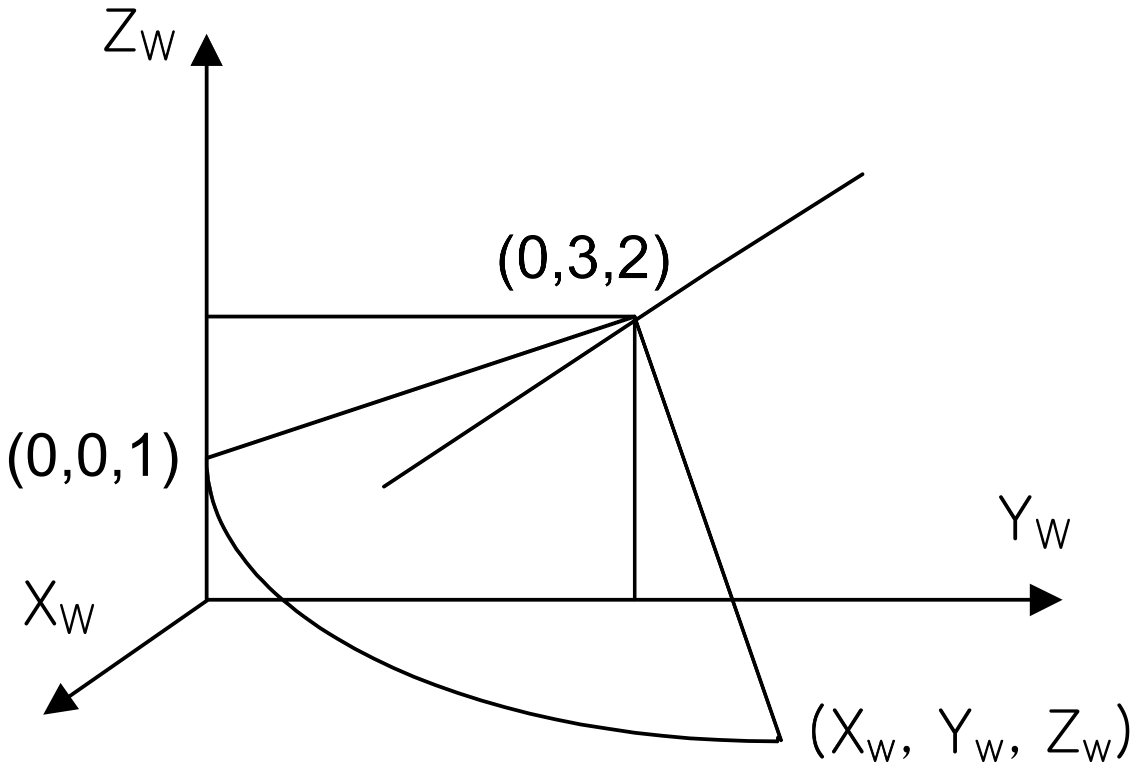
Example1

- From (a) and (b)

$$\begin{bmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{bmatrix} = \underbrace{Rot(x,90°)\cdot Trans(0,5,0)}_{\text{concatenation}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Example2

- Point (0, 0, 1) is rotated 90 degrees about an axis passing (0, 3, 2) in x-axis direction



Example2

$$\begin{bmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{bmatrix} = Trans(0,3,2) \cdot Rot(x,90°) \cdot Trans(0,-3,-2) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° & 0 \\ 0 & \sin 90° & \cos 90° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

Scaling Transformation Matrix

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} S_X & 0 & 0 & 0 \\ 0 & S_Y & 0 & 0 \\ 0 & 0 & S_Z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Scale(S_X, S_Y, S_Z)} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Scaling w.r.t. (X_P, Y_P, Z_P)

$$Trans\,(X_P, Y_P, Z_P) \cdot Scale\,(S_P, S_P, S_P) \cdot Trans\,(-X_P, -Y_P, -Z_P) \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Mirror Reflection

- w.r.t. x-y plane

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Reflections w.r.t. y-z plane and x-z plane are similarly derived

Modeling Transformation in OpenGL

- void glMatrixMode(GL_MODELVIEW)
  - Select matrix to be transformed

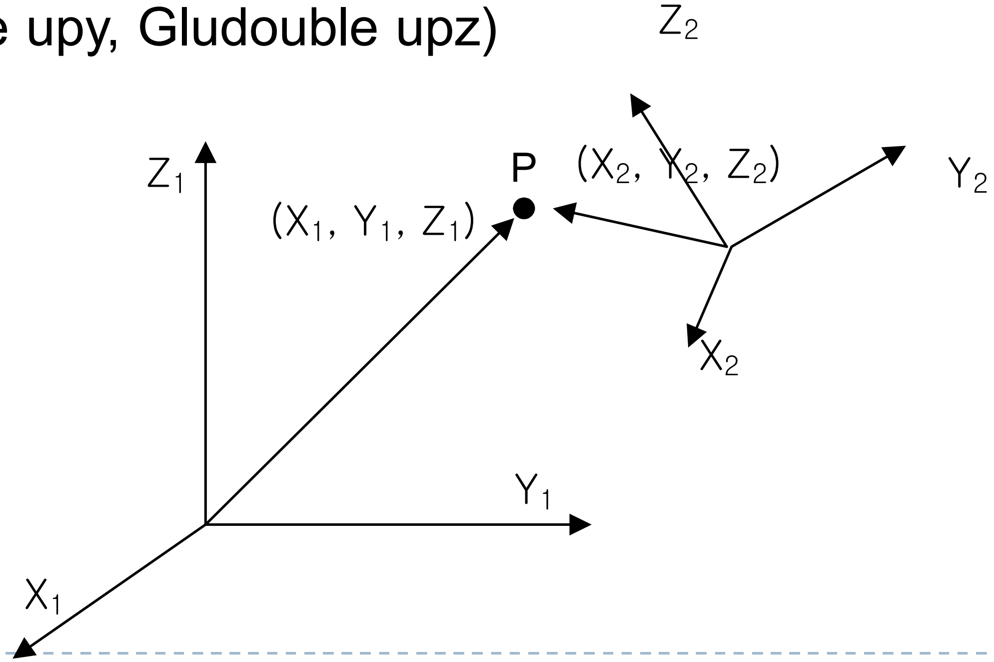
- void glTranslatef(Type x, Type y, Type z)
  - Translate model by x, y, z

- void glRotatef(Type angle, Type x, Type y, Type z)
  - Rotate model by angle(degree) about Vector

- void glScalef(Type x, Type y, Type z)
  - scaling by ratio of x, y, and z in each axis direction

Mapping

- Convert coordinates in World Coordinate System into those in Viewing Coordinate System
- void glMatrixMode(GL_PROJECTION)
- void gluLookAt(Gldouble eyex, Gldouble eyey, Gldouble eyez, Gldouble centerx, Gludouble centery, Gludouble centerz, Gludouble upx, Gludouble upy, Gludouble upz)



Mapping (cont')

- Let $T_{1\text{-}2}$ to be a transformation matrix to derive coordinate values w.r.t (x_2 y_2 z_2)coordinate system from those w.r.t (x_1 y_1 z_1) coordinate system

$$(X_2 \;\; Y_2 \;\; Z_2 \;\; 1)^T = T_{1\text{-}2}(X_1 \; Y_1 \; Z_1 \; 1)^T \qquad \text{(a)}$$

- Derive $T_{1\text{-}2}$ from (a) after expressing Eq. (a) as follows

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix} \qquad \text{(b)}$$

Mapping (cont')

- Substituting X_1=0, Y_1=0, Z_1=0 into (b) gives

$$X_2 = P_x, \; Y_2 = P_y, \; Z_2 = P_z \qquad (c)$$

  - Px , Py , Pz are {x2 y2 z2} coordinates of the origin of {x1 y1 z1} frame

- Substituting X_1=1, Y_1=0, Z_1=0 into (b) gives

$$X_2 = n_x + p_x, \; Y_2 = n_y + py, \; Z_2 = n_z + p_z \quad (d)$$

  - n_x , n_y, n_z are X_2 , Y_2 , Z_2 components of the unit vector along x_1 axis
  - o_x , o_y, o_z are X_2 , Y_2 , Z_2 components of the unit vector along y_1 axis
  - a_x , a_y, a_z are X_2 , Y_2 , Z_2 components of the unit vector along z_1 axis

Example1 (cont')

- View point (-10, 0, 1), View site (0, 0, 1), up vector (0, 0, 1)

- $T_{W\text{-}V}$?
  What are X_V, Y_V, Z_V of (5, 0, 1) in world coordinates?

Example1 (cont')

$$T_{W-V} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- n_x , n_y , n_z are x_v , y_v , z_v components of x_w axis ➔ (0, 0, -1)
- o_x , o_y , o_z are x_v , y_v , z_v components of y_w axis ➔ (-1, 0, 0)
- a_x , a_y , a_z are x_v , y_v , z_v components of z_w axis ➔ (0, 1, 0)
- p_x , p_y , p_z are x_v , y_v , z_v coordinates of the origin of $\{x_w \;\; y_w \;\; z_w\}$ ➔ (0, -1, 0)

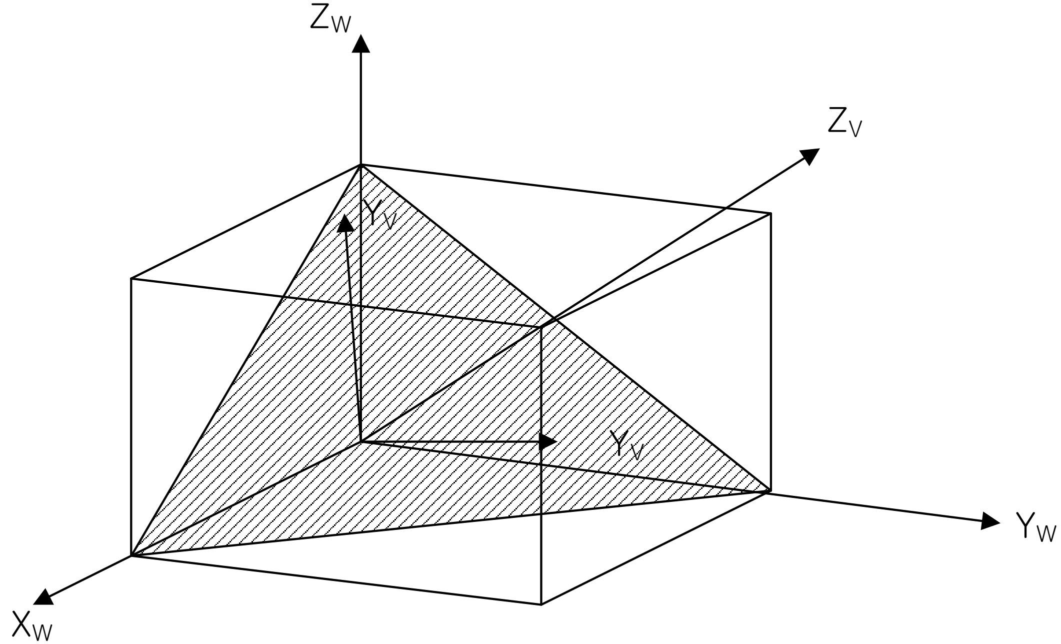
Example1

$$\therefore T_{W-V} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X_V \\ Y_V \\ Z_V \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

Example2 (cont')

- Isometric View: view point (5, 5, 5), view site (0, 0, 0), up vector (0, 0, 1)
- Show Viewing Coordinate System and derive $T_{W\text{-}V}$?
  X_V, Y_V, Z_V of (0, 0, 5)?



Example2 (cont')

$$\mathbf{k}_v = \frac{5\mathbf{i}+5\mathbf{j}+5\mathbf{k}-(0\mathbf{i}+0\mathbf{j}+0\mathbf{k})}{\left|5\mathbf{i}+5\mathbf{j}+5\mathbf{k}-(0\mathbf{i}+0\mathbf{j}+0\mathbf{k})\right|} = \frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$$

$$= -\frac{1}{\sqrt{6}}\mathbf{i}-\frac{1}{\sqrt{6}}\mathbf{j}+\frac{2}{\sqrt{6}}\mathbf{k}$$

$$\mathbf{j}_v = \frac{\mathbf{u}_p-(\mathbf{u}_p \bullet \mathbf{k}_v)\mathbf{k}_v}{\left|\mathbf{u}_p-(\mathbf{u}_p \bullet \mathbf{k}_v)\mathbf{k}_v\right|} = \frac{-\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}}{\left|-\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right|}$$

$$\mathrm{i}_v = \frac{\mathbf{j}_v \times \mathbf{k}_v}{\left|\mathbf{j}_v \times \mathbf{k}_v\right|} = \frac{1}{-\sqrt{2}}\mathbf{i}+\frac{1}{-\sqrt{2}}\mathbf{j}$$

Example2 (cont')

- X_V component of X_W axis

$$\mathbf{i} \cdot \mathbf{i}_V = \frac{1}{-\sqrt{2}} = n_x$$

- Y_V component of X_W axis

$$\mathbf{i} \cdot \mathbf{j}_V = \frac{1}{-\sqrt{6}} = n_Y$$

- Z_V component of X_W axis

$$\mathbf{i} \cdot \mathbf{k}_V = \frac{1}{\sqrt{3}} = n_Z$$

Example2

$$T_{w-v} = \begin{bmatrix} \frac{1}{-\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{-\sqrt{6}} & \frac{1}{-\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X_V \\ Y_V \\ Z_V \\ 1 \end{bmatrix} = T_{w-v} \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5\sqrt{6}}{3} \\ \frac{5\sqrt{3}}{3} \\ 1 \end{bmatrix}$$