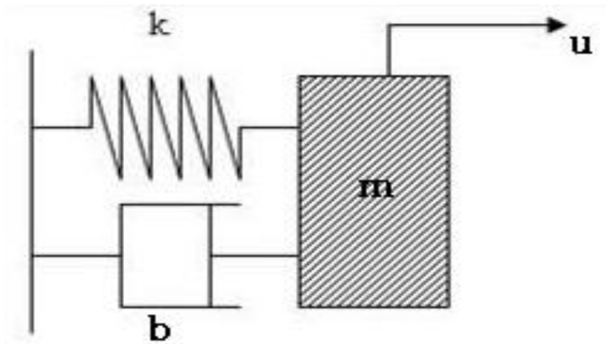
Mathematical Modeling of Dynamic Systems in State Space

A Simple Problem



Consider the system equation :

Laplace transform : $(ms^2 + bs + k)Y(s) = U(s)$

$$Transfer\ \ function: \frac{Y(s)}{U(s)} =$$

Consider the system equation again : $\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u$

Let's change to multiple single order equations.

Choose variables : $x_1 = y, \qquad x_2 = \dot{y}$

Then we get : $\dot{y} = \dot{x}_1 = x_2 \qquad \ddot{y} = \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

Simplify the matrix : $\dot{x} = Ax + Bu$

Why State Space Modeling?

Transfer function Approach

Uses single, higher-order equation

Classical, Frequency Domain Approach.

Replace differential equations with algebraic equations → simplifies representation of individual subsystems, and modeling of interconnected subsystems.

Disadvantage: **Limited applicability**. Only can be applied to linear, time-invariant systems.

Advantage: Rapidly provide stability and transient response information. Can immediately see the effects of varying system parameters until an acceptable design is met.

State Space Approach

Consist of coupled first-order differential equations.
Uses vector and matrix notation.

Modern, time domain approach.

Unified approach for modeling, analyzing and designing **a wide range of systems**.

Can be used to represent **nonlinear systems** that have **backlash, saturation** and **dead zone**.

Can handle systems with **non-zero initial conditions** and **time-varying systems**.

Multi input and multi output systems can be compactly represented with a complexity similar to that of SISO system.

Numerous state space **software packages** available.

System analysis can be done using linear algebra. (Matrix form)

Disadvantage: It is **not as intuitive** as the classical approach.

State Space Modeling

If the system is linear time-invariant system, the system can be presented as *n* state variables, *r* input variables, and *m* output variables.

State equation :

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1r}u_r \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2r}u_r \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nr}u_r
\end{aligned}$$

Output equation :

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

$$\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1r}u_r \\
y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2r}u_r \\
&\vdots \\
y_m &= c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n + d_{m1}u_1 + d_{m2}u_2 + \cdots + d_{mr}u_r
\end{aligned}$$

State Space Modeling

State of a dynamic system : The smallest set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$. And the variables are called *state variables.*

State vector : *n* state variables which is needed to completely describe the behavior of a given system can be considered the *n* components of a vector ***x***. Such a vector is called a *state vector.*

State space : The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis is called a *state space*.

$\boxed{\begin{array}{l}\text{Minimum number of state variables} \\ = \text{number of energy storage elements} \\ = \text{order of differential equation}\end{array}}$

State Space Modeling

State space equation :

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}}$$

→State Equation

→Output Equation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix}$$

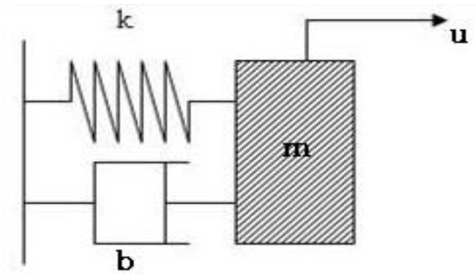
A : State matrix (System Matrix, Transition matrix)

B : Input matrix

C : State output matrix

D : Control output matrix

A Simple Problem



$$m\ddot{y} + b\dot{y} + ky = u$$

$$\ddot{y} = -$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

State vector: Choice should result in a set of 1 st order eqns.

$$x_1 = y(displacement), \qquad x_2 = \dot{y}(speed) \qquad \dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u$$

Output Vector: What are you interested in?

Total Force: $y_1 = u - kx - b\dot{x}$ Momentum: $y_2 = m\dot{x}$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u$$

Transformation of System Models

Step Response : sys=sys(A,B,C,D)

we use 'step(sys)' or 'step(A,B,C,D)' in MATLAB

Transfer Matrix : r inputs, $u_1, u_2, \cdots, u_r$ and m outputs, $y_1, y_2, \cdots, y_m$

We define those vectors.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Then, the transfer matrix G(s) expresses the relationship between Y(s) and U(s)

$$Y(s) = G(s)U(s)$$

Transformation of System Models

System equation : $\dot{x} = Ax + Bu$

$$y = Cx + Du$$

Laplace Transformation :

Assume, $x(0)=0$: $X(s) = \boxed{} U(s)$

$$Y(s) = \boxed{} U(s) \rightarrow \therefore G(s) = C(sI - A)^{-1}B + D$$

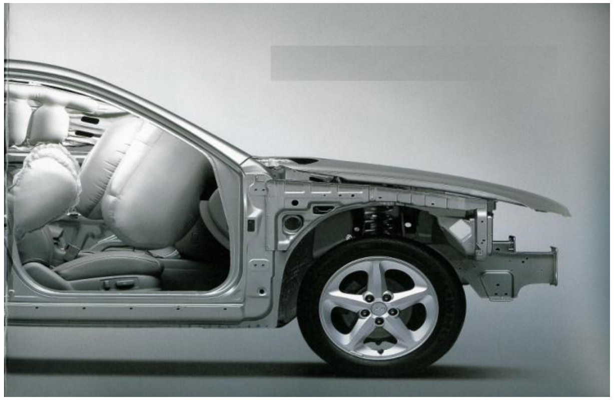
ex)

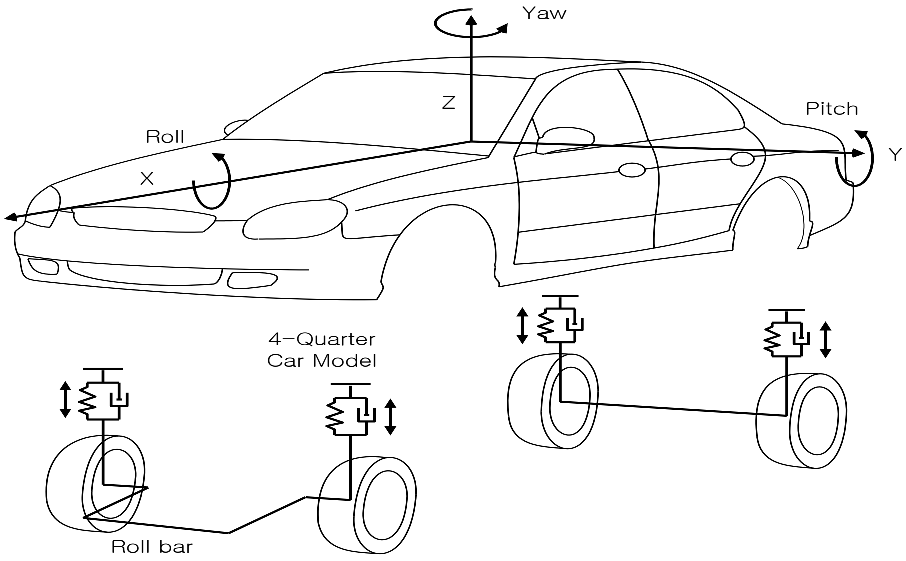
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$G(s) = \boxed{}$

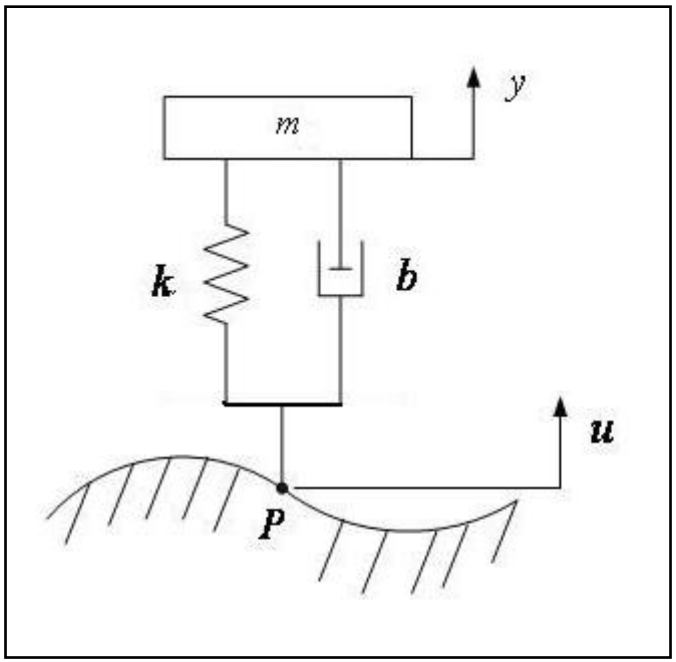
Vehicle Suspension Problem



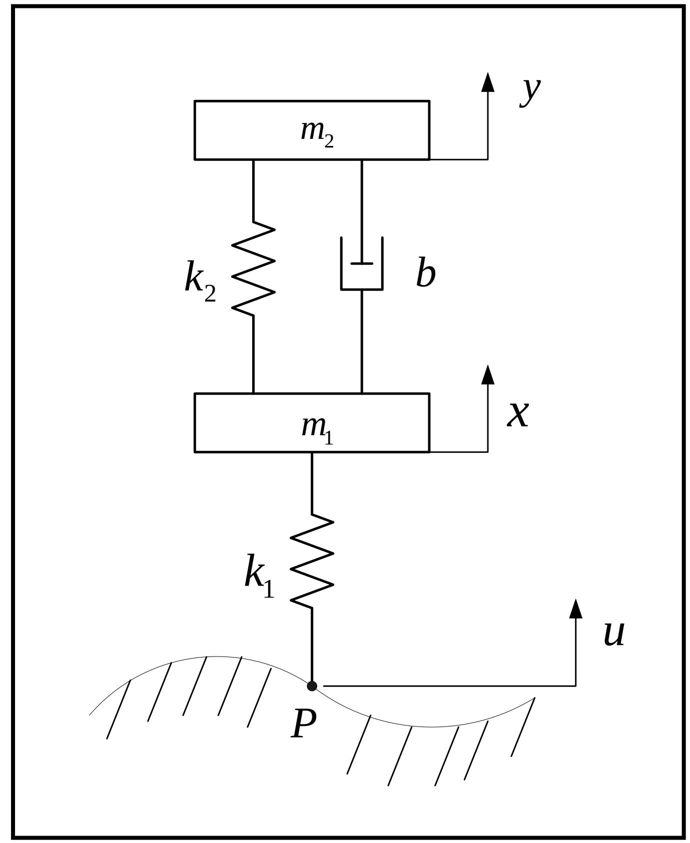


Vehicle Suspension Problem

ex1) Spring, damper, mass system

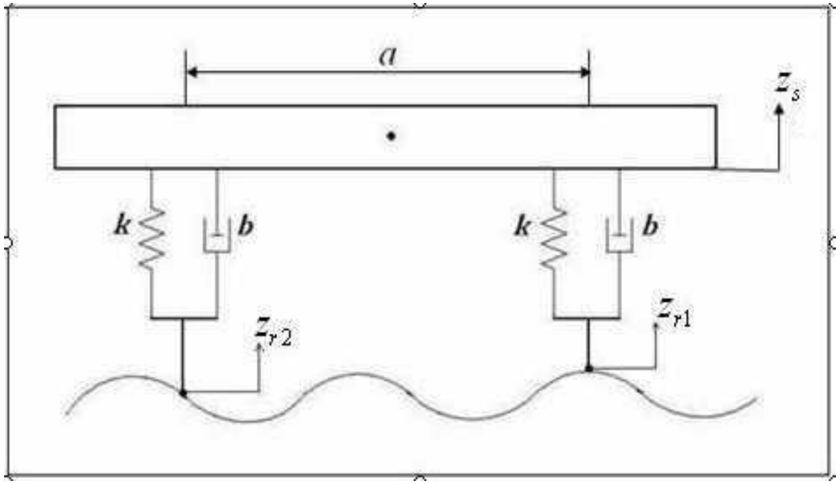


ex2) Body and tire model

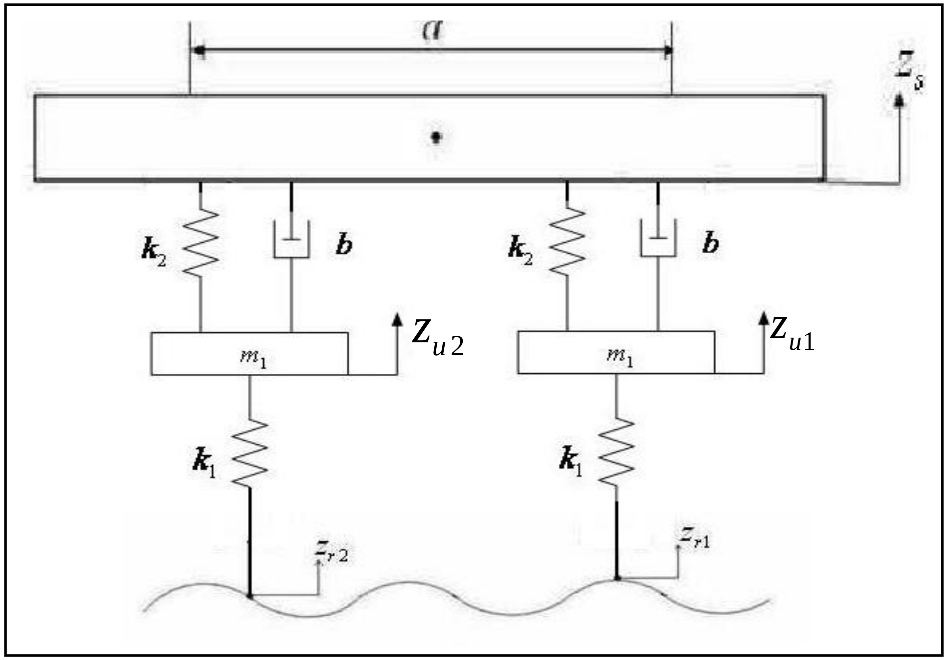


Vehicle Suspension Problem

ex3) Two inputs



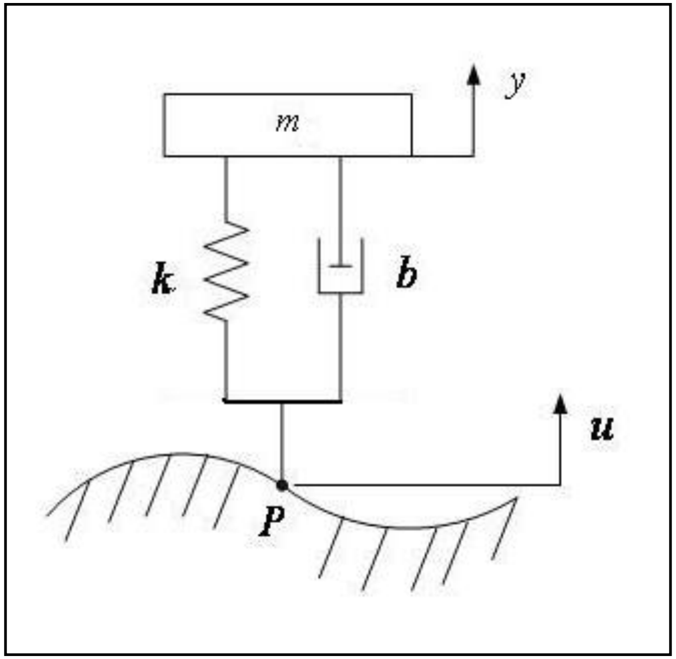
ex4) Two inputs, Body and Tire Model



Two inputs : $z_{r1},\, z_{r2} \qquad z_{r2}(t) = z_{r1}(t-\tau)$

Vehicle Suspension Problem

ex1) Spring, damper, mass system



- Design Considerations

  1. Ride Quality

  $\rightarrow Sprung\ mass\ acceleration\ :\ \ddot{y}$

  2. Rattle space

  $\rightarrow Suspension\ Deflection\ :\ y$

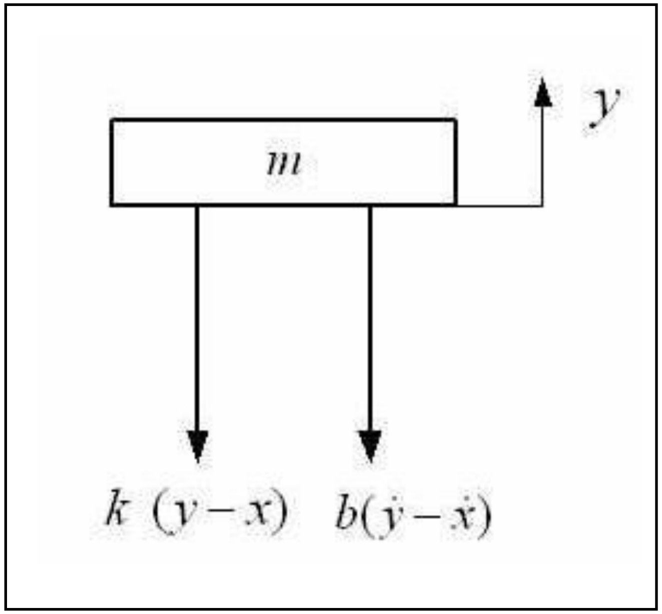
- Suspension Design Parameters

  $\rightarrow Spring\ Stiffness\ :\ k$

  $\rightarrow Damping\ Ratio\ :\ b$

Vehicle Suspension Problem

- **Free Body Diagram**



- **Dynamic Equations**

$$m\ddot{y}+b(\dot{y}-\dot{u})+k(y-u)=0$$

$$m\ddot{y}+b\dot{y}+ky=b\dot{u}+ku$$

- **Laplace Transform**

$$(ms^2+bs+k)Y(s)=(bs+k)X(s)$$

$$Transfer\ \ function:\ \frac{Y(s)}{U(s)}=\frac{bs+k}{ms^2+bs+k}$$

Vehicle Suspension Problem

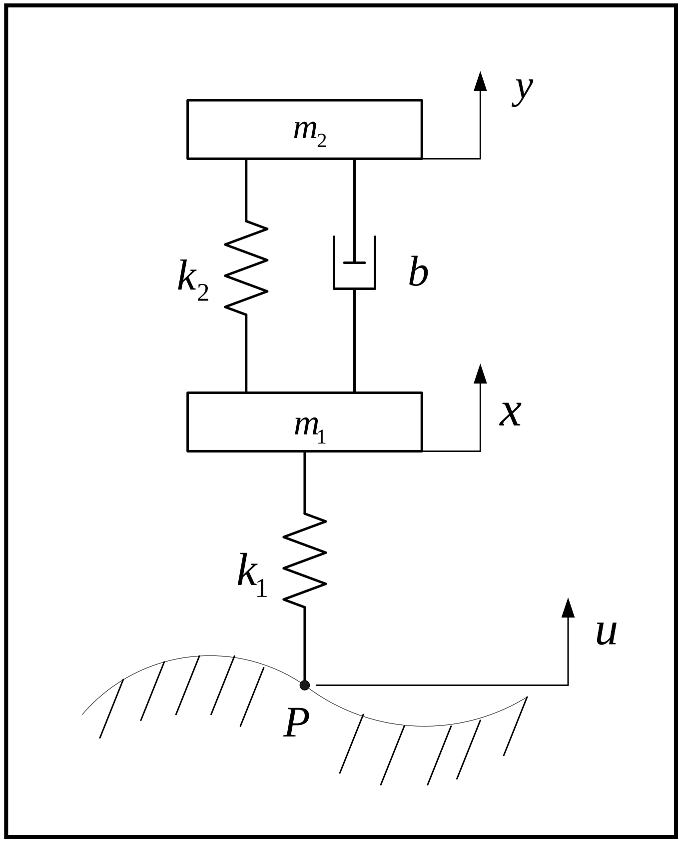
- **General Form of State Equation** $\dot{x} = Ax + Bu$
- **The State variables** $(\mathrm{u} = z_r)$

$$x_1 = z_s - z_r \quad : \textit{Suspension Deflection}$$
$$x_2 = \dot{z}_s \quad : \textit{absolute velocity of body}$$
$$\dot{x}_1 = \dot{z}_s - \dot{z}_r = x_2 - \dot{z}_r ,$$
$$\dot{x}_2 = \ddot{z}_s = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{z}_r \quad : \textit{acceleration of body}$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ \dfrac{b}{m} \end{bmatrix} \mathrm{u}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \ddot{z}_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{b}{m} \end{bmatrix} \mathrm{u}$$

Vehicle Suspension Problem

ex2) Body and tire model



- Design Considerations

  1. Ride Quality

  $\rightarrow$ *Sprung mass acceleration* : $\ddot{y}$

  2. Rattle space

  $\rightarrow$ *Suspension Deflection* : $y - x$

  3. Tire Force Vibration

  $\rightarrow$ *Tire Deflection* : $x - u$

- Suspension Design Parameters

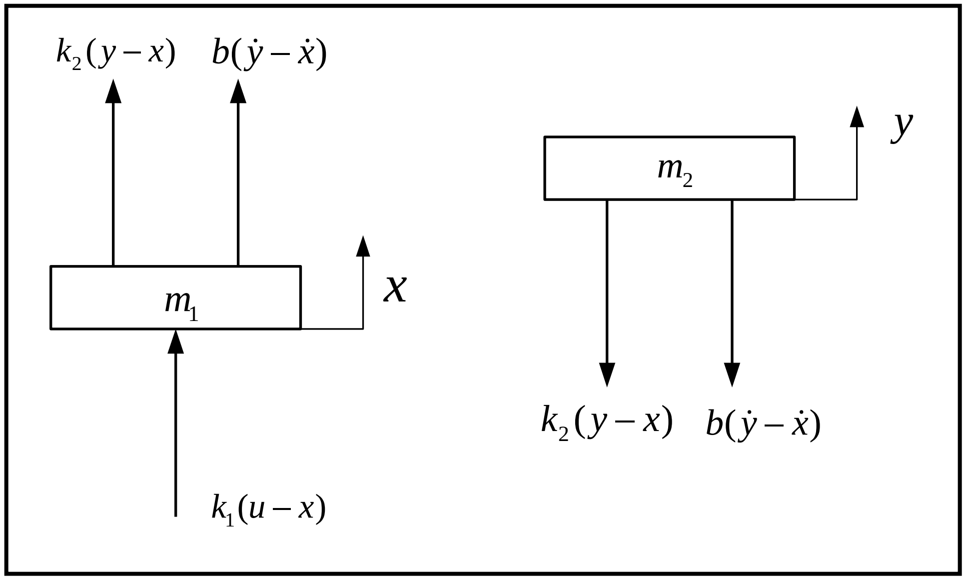
  $\rightarrow$ *Spring Stiffness* : k_2

  $\rightarrow$ *Damping Ratio* : b

  $\rightarrow$ *Tire Stiffness* : k_1

Vehicle Suspension Problem

- Free Body Diagram



- Dynamic Equations

$$m_1\ddot{x} = k_2(y-x) + b(\dot{y}-\dot{x}) + k_1(u-x)$$

$$m_2\ddot{y} = -k_2(y-x) - b(\dot{y}-\dot{x})$$

- Laplace Transform

$$[m_1 s^2 + bs + (k_1 + k_2)]X(s)$$

$$= (bs + k_2)Y(s) + k_1 U(s)$$

$$[m_2 s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$$

Vehicle Suspension Problem

- General Form of State Equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Dynamic Equations

$$m_1 \ddot{z}_u = k_2 (z_s - z_u) + b(\dot{z}_s - \dot{z}_u) + k_1 (u - z_u)$$

$$m_2 \ddot{z}_s = -k_2 (z_s - z_u) - b(\dot{z}_s - \dot{z}_u)$$

- The State variables $(x = z_u, \quad y = z_s)$

$$x_1 = z_s - z_u \quad : Suspension\ Deflection$$
$$x_2 = \dot{z}_s \quad : absolute\ velocity\ of\ sprung\ mass$$
$$x_3 = z_u - u \quad : Tire\ Deflection$$
$$x_4 = \dot{z}_u \quad : absolute\ velocity\ of\ unsprung\ mass$$