Aircraft Structures

CHAPER 12.
Variational and Energy Principles

Prof. SangJoon Shin

12. Variational and Energy Principles

- Chap. 9 … PVW, PCW for particles, systems of particles and trusses

  no attempt is made for 3-D solids

- Chap. 10 … PMTPE, PMCE for mechanical systems and trusses basic concepts of FEM

  applied to truss

- Chap. 11 … development of approximate solution for beam problems

  recast the DE of equilibrium into integral forms

  equivalence between $\left\{ \begin{array}{l} \text{Weak statement of equilibrium} \\ \text{PVW} \end{array} \right.$

  basic concepts of FEM applied to beam

- Chap. 12 … stationary values of functionals (functions of functions) <- "calculus of variation"

  $\left. \begin{array}{l} \text{PVW, PCW} \\ \text{PMTPE, PMCE} \\ \text{Hu-Washizu principle} \\ \text{Hellinger-Reissner principle} \end{array} \right\}$ 3-D solis

12.1 Mathematical preliminaries

- Basic equations of elasticity in Chap. 1… differential calculus, PDE
- Calculus of variation … in this section

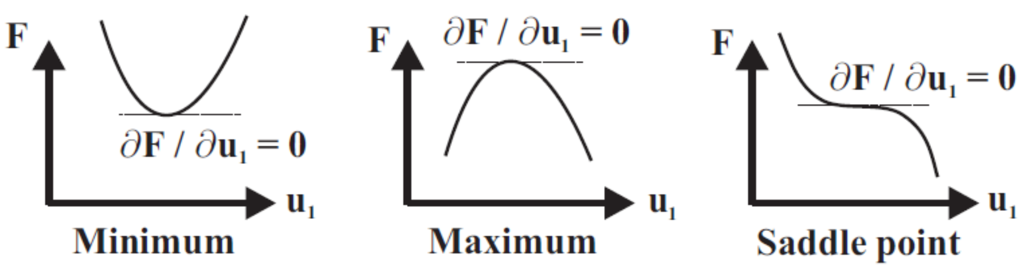
12.1.1 Stationary point of a function

- Function of n variables, $F = F(u_1, u_2, \ldots, u_n)$

  stationary points of this function is defined as

$$\frac{\partial F}{\partial u_i} = 0, \quad i = 1, 2, \ldots, n \tag{12.1}$$

… for a function of a single variable, corresponds to a horizontal tangent to the curve



- Minimum
- Maximum
- Saddle point

Fig. 12.1. Stationary points of a function.

12.1 Mathematical preliminaries

- Stationary at a point, Eq. (12.1) hold and the following statment

$$\frac{\partial F}{\partial u_1} w_1 + \frac{\partial F}{\partial u_2} w_2 + \ldots + \frac{\partial F}{\partial u_n} w_n = 0$$

$w_1, w_2, \ldots, w_n$: arbitrary quantities

convenient to use a special notation, $w_i = \delta u_i$, "virtual changes in u_i

-> $$\frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_2} \delta u_2 + \ldots + \frac{\partial F}{\partial u_n} \delta u_n = 0$$

- Virtual change operator, "δ", behaves in a manner similar to the differential operator, "d"
- "variation in F,δF" definition

$$\delta F = \frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_2} \delta u_2 + \ldots + \frac{\partial F}{\partial u_n} \delta u_n \qquad (12.2)$$

$$\delta F = 0 \qquad (12.3)$$

12.1 Mathematical preliminaries

- Differential condition, Eq.(12.1)
  variational condition, Eq.(12.3) } Both must hold at a stationary point

  Eq.(12.1) implies Eq.(12.3)
  Eq.(12.3) implies Eq.(12.1) } -> two conditions are entirely equivalent

- Determine whether a stationary point is { Minimum / Maximum / Saddle point

  -> it is necessary to consider the second derivatives

$$\sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j} \mathrm{d}u_i \mathrm{d}u_j > 0 \quad \text{-> minimum} \tag{12.4}$$

$$\sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j} \mathrm{d}u_i \mathrm{d}u_j < 0 \quad \text{-> maximum}$$

can be positive or negative depending on the choice of the increments -> saddle point

12.1 Mathematical preliminaries

- Second variation of function F

$$\delta^2 F = \sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j} \delta u_i \delta u_j$$

stationary point is a minimum if $\delta^2 F > 0$ (12.5)

stationary point is a maximum if $\delta^2 F < 0$

stationary point is a saddle point if the sign of $\delta^2 F$ depends on the choice of the variation of the independent variables

12.1.2 Lagrange multiplier method

- Problem of determining a stationary point of a function of several variables.

$F = F(u_1, u_2, \ldots, u_n)$, where the variables are not independent.

-> subjected to a constraint

$$f(u_1, u_2, \ldots, u_n) = 0. \qquad (12.6)$$

12.1 Mathematical preliminaries

- Constraint can be used to express one variable, say u_n , in terms of the others.
  -> u_n can be eliminated from $F = F(u_1, u_2, \ldots, u_{n-1})$
- However, it might be cumbersome, or even impossible, to completely eliminate one variable
- Alternative approach to avoid this elimination-of-variable process
  at stationary point

$$\delta F = \frac{\partial F}{\partial u_1}\delta u_1 + \frac{\partial F}{\partial u_2}\delta u_2 + \ldots + \frac{\partial F}{\partial u_n}\delta u_n = 0 \qquad (12.7)$$

-> however, does NOT imply $\frac{\partial F}{\partial u_i} = 0$ for $i = 1, 2, \ldots, n$
because δu_i CANNOT be chosen arbitrarily since they must satisfy the constraint, Eq.(12.6)

- Variation of a constraint

$$\delta f = \frac{\partial f}{\partial u_1}\delta u_1 + \frac{\partial f}{\partial u_2}\delta u_2 + \ldots + \frac{\partial f}{\partial u_n}\delta u_n = 0 \qquad (12.8)$$

12.1 Mathematical preliminaries

- Linear combination of Eqs.(12.7) and (12.8)

$$\frac{\partial F}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial F}{\partial u_n}\,\delta u_n + \lambda\left[\frac{\partial f}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial f}{\partial u_n}\,\delta u_n\right] = 0$$

λ : arbitrary function of $u_1, u_2, \ldots, u_n$ "Lagrange multiplier"

regrouping

$$\sum_{i=1}^{n}\left[\frac{\partial F}{\partial u_i} + \lambda\,\frac{\partial f}{\partial u_i}\right]\,\delta u_i = 0 \qquad (12.9)$$

… δu_n could now be expressed in term of the (n-1) other variations, δu_i

- To avoid this cumbersome algebraic step, the arbitrary Lagrange multiplier is chosen such that

$$\frac{\partial F}{\partial u_n} + \lambda\,\frac{\partial f}{\partial u_n} = 0$$

… with this choice, the last term in Eq.(12.9) vanishes for all δu_n

Eq. (12.9) ->

$$\frac{\partial F}{\partial u_i} + \lambda\,\frac{\partial f}{\partial u_i} = 0 \qquad i = 1, 2, \ldots, n-1 \qquad (12.10)$$

12.1 Mathematical preliminaries

- Combining the last two equations

$$\delta F + \lambda \delta f = 0$$

where all variations, $\delta u_i, i = 1, 2, \ldots, n$ are "independent"

- Eq. (12.6) 0> $f \delta \lambda = 0$ for any arbitrary $\delta \lambda$

stationary condition -> $\delta F + \lambda \delta f = \delta F + \lambda \delta f + f \delta \lambda = \delta(\underbrace{F + \lambda f}_{\text{Modified function } F^+})$

… variation in $F^+ = 0$ for all arbitrary variations $\delta u_i,\ i = 1, 2, \ldots, n, and\ \delta \lambda$

- Summary … initial constrained problem -> "unconstrained problem"

$$\delta F^+ = 0, \quad \text{where} \quad F^+ = F + \lambda f \qquad (12.11)$$

modified function F^+ involves (n+1) variables, $u_i, i = 1, 2, \ldots, n$ and λ

$$\sum_{i=1}^{n} \left[\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} \right] \delta u_i + f\, \delta \lambda = 0$$

because $\delta u_i,\ i = 1, 2, \ldots, n$ and $\delta \lambda$ are all independent, arbitrary

$$\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} = 0, \quad i = 1, 2, \ldots, n; \quad \text{and} \quad f = 0$$

-> (n+1) equations to be solved for (n+1) unknowns

12.1 Mathematical preliminaries

- Lagrange multiplier method … "unconstrained problem" but increase number of unknowns from n to (n+1), additional unknown is the Lagrange multiplier.
- Multiple constraints, $f_i = 0, i = 1, 2, \ldots, m$

  -> m Lagrange multipliers $\lambda_i, i = 1, 2, \ldots, m$

$$F^+ = F + \sum_{i=1}^{m} \lambda_i f_i \qquad (12.12)$$

12.1 Mathematical preliminaries

12.1.3 Stationary point of a definite integral

- Definite integral

$$I = \int_a^b F(y,\, y',\, x)\, \mathrm{d}x \qquad (12.13)$$

$(\cdot)'$: derivative with respect to x.

$y(x)$: unknown function of x subject to BC's, $\begin{cases} y(a) = \alpha \\ y(b) = \beta \end{cases}$

I : "functional, function of a function"

… the value of the definite integral I depends on the choice of the unknown function $y(x)$

There are an infinite number of y between a and b

-> I is equivalent to a function of an infinite number of variables

12.1 Mathematical preliminaries

- Variational formalism (sec. 12.1.1)

  "variation of a function" -> δf

  Fig. 12.2 … two functions, $f(x)$, $\hat{f}(x)$

$$\delta f = \hat{f}(x) - f(x) = \psi(x)$$

$\psi(x)$: continuous and differentiable, but otherwise arbitrary function,

$$\psi(a) = \psi(b) = 0$$

δf : virtual change that bring the function $f(x)$ to a new, arbitrary function $\hat{f}(x)$

$\delta f(a) = \delta f(b) = 0$ -> δf does not violate BC's of the problem

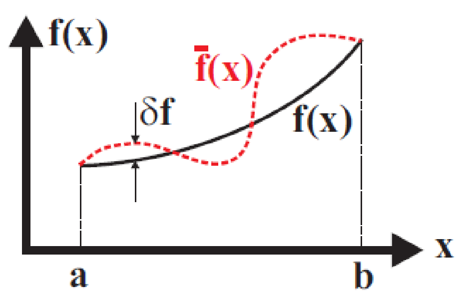


Fig. 12.2. The concept of variation of a function.

- Stationarity of functional I

$$\delta I = \delta \int_a^b F(y,\, y',\, x)\,\mathrm{d}x = \int_a^b \delta F(y,\, y',\, x)\,\mathrm{d}x = 0$$

Eq.(12.2) and treating δ as a differential

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\,\delta y'\right]\,\mathrm{d}x = 0$$

Integration by parts on the second term

$$\int_a^b \frac{\partial F}{\partial y'}\delta\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x = \int_a^b \frac{\partial F}{\partial y'}\frac{\mathrm{d}}{\mathrm{d}x}\,(\delta y)\,\mathrm{d}x = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\delta y\,\mathrm{d}x + \left[\frac{\partial F}{\partial y'}\delta y\right]_a^b$$

Boundary term vanishes because $\delta y(a) = \delta y(b) = 0$ ->

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,\mathrm{d}x = 0 \qquad (12.14)$$

… Euler-Lagrange equation for the problem

-> the necessary and sufficient condition for the definite integral to be at a stationary point

12.1 Mathematical preliminaries

- Equation of elasticity … can be viewed as the Euler-Lagrange equations associated with the stationary condition of definite integrals
- Crucial difference between on $\begin{cases} \text{Increment} \\ \text{Variation } \delta f \end{cases}$ (Fig.12.3)

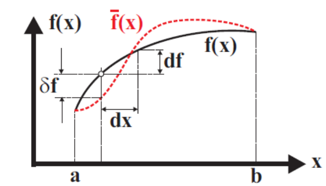


Fig. 12.3. The difference between an increment $\mathrm{d}f$ and a variation δf.

① differential, $\mathrm{d}f$… an infinitesimal change in $f(x)$ resulting from an infinitesimal change, dx, in the independent variable

df/dx … the rate of change or tangent at the point

② δf … arbitrary virtual change that brings $f(x)$ to $\hat{f}(x)$

-> $\mathrm{d}f$ and δf are clearly unrelated

- Manipulations of the two symbols are quite similar

… the order of application of the two operations can be interchanged.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\delta f) = \frac{\mathrm{d}}{\mathrm{d}x}(\hat{f} - f) = \frac{\mathrm{d}\hat{f}}{\mathrm{d}x} - \frac{\mathrm{d}f}{\mathrm{d}x} = \delta\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \tag{12.15}$$

The order of the integration and variational operations commute

$$\delta \int_a^b F \,\mathrm{d}x = \int_a^b \hat{F} \,\mathrm{d}x - \int_a^b F \,\mathrm{d}x = \int_a^b (\hat{F} - F) \,\mathrm{d}x = \int_a^b \delta F \,\mathrm{d}x \tag{12.16}$$

12.1.4 Variational and energy principles

◦ Fig. 12.14 … elastic body of arbitrary shape subjected to surface tractions and body forces as well as geometric BC's
- Prescribed displacements at point
- Prescribed displacement over a portion of outer surface

$\mathcal{V}$: volume of the body

$\mathcal{S}$: outer surface

$\bar{n}$: unit vector, outer normal to

$\mathcal{S}_1$: portions of the outer surface where prescribed tractions $\underline{\hat{t}}$ are applied

$\mathcal{S}_2$: portions of the outer surface where prescribed displacements $\underline{\hat{u}}$ are applied

$\mathcal{S}_1$ and $\mathcal{S}_2$ share no common points -> $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$

a point of the outer surface that is traction free belong to $\mathcal{S}$

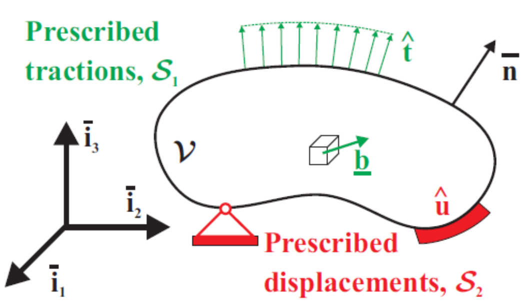


Fig. 12.4. General elasticity problem.

12.1 Mathematical preliminaries

- Body forces … might also be applied over the entire volume

  ex) gravity forces, electronic or magnetic fields

  internal forces in accordance with D'Alembert's principle

- Basic equations of elasticity in Chap.1 -> form a set of PDE's that can be solved to find the displacements, strain, and stress fields at all points in $\mathcal{V}$

  subsequent sections … variational and energy principles presented to provide an alternative formalism

12.2.1 Review of the equations of linear elasticity

◦ Fig. 3.1 (Page 101) … 3 groups of the equations of elasticity

solutions of an elasticity problem involves

① a statically admissible stress field

② a kinematically admissible displacement field and the corresponding strain field

③ a constitutive law satisfied at all points in volume $\mathcal{V}$

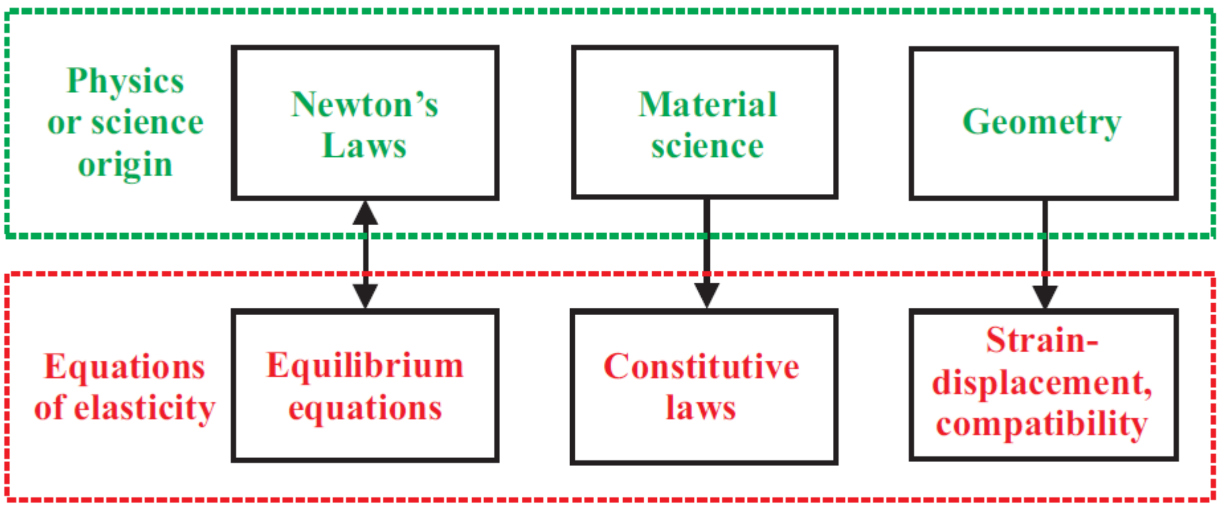


Fig. 3.1. The elasticity equations separated into three groups.

12.2 Variational and energy principles

- Equilibrium equations

  … most fundamental equations, Sec 1.1.2 and 1.1.3

  derived from Newton's law stating that the sum of all the forces acting on a differential element should vanish.

- Equilibrium equations for a differential element of a body, Fig. 1.4

$$\begin{aligned}
\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 = 0 \\
\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 = 0 \\
\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 = 0
\end{aligned} \tag{12.17}$$

must be satisfied at all points of volume $\mathcal{V}$

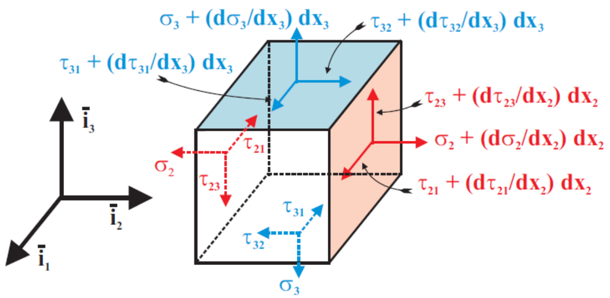


Fig. 1.4. Stress components acting on a differential element of volume. For clarity of the figure, the stress components acting on the faces normal to $\bar{\imath}_1$ are not shown.

12.2 Variational and energy principles

- Traction equilibrium equations

$$t_1 = \hat{t}_1, \quad t_2 = \hat{t}_2, \quad t_3 = \hat{t}_3 \tag{12.18}$$

definition of the surface tractions -> Eq. (1.9)

surface equilibrium equations -> "force or natural BC's"

compact stress array, $\underline{\sigma}$-> defined in Eq. (2.11b)

$$\underline{\sigma} = \left\{\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}\right\}^T \tag{2.11b}$$

Definition 12.1 A stress field, $\underline{\sigma}$, is said to be statically admissible if it satisfies the equilibrium equations, Eq.(12.17), at all points of volume $\mathcal{V}$ and surface equilibrium equations, Eq.(12.18) at all points of surface $\mathcal{S}_1$

- Strain-displacement relationships

  … merely define the strain components that are used for the characterization of the deformation of the body

- When the displacements are small, it is convenient to use the engineering strain components to measure that deformation at a point.

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3} \qquad \text{"axial strain"}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}, \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{12.19}$$

12.2 Variational and energy principles

- To compute strain components, the displacement field must be continuous and differentiable

  must be equal to the prescribed displacements over surface S_2

$$u_1 = \hat{u}_1, \quad u_2 = \hat{u}_2, \quad u_3 = \hat{u}_3 \tag{12.20}$$

  … geometric BC's

  compact strain array, $\underline{\epsilon}$, defined in Eq.(2.11a)

$$\underline{\epsilon} = \left\{\epsilon_1, \epsilon_2, \epsilon_3, \gamma_{23}, \gamma_{13}, \gamma_{12}\right\}^T \tag{2.11a}$$

Definition 12.2 A displacement field, $\underline{u}$, is said to be kinematically admissible if it is continuous and differentiable at all points in $\mathcal{V}$ and satisfies geometric BC's, Eq.(12.20), at all points on surface $\mathcal{S}_2$

Definition 12.3 A strain field, $\underline{\epsilon}$, is said to be compatible if it is derived from a kinematically admissible displacement field through the strain-displacement relationship, Eq.(12.19)

12.2 Variational and energy principles

- Constitutive laws … relates the stress and strain components
  mathematical idealization of the experimentally observed behavior

Sec. 2.1.1 … homogeneous, isotropic, linearly elastic material behavior

-> frequently used highly idealized constitutive law

Many materials -> anisotropy, plasticity, visco-elasticity, or creep

- Hooke's law, Eq.(2.10) … simple linear relationship between the stress and strain fields

$$\underline{\epsilon} = \underline{\underline{S}}\, \underline{\sigma}. \qquad (2.10)$$

positive-definite, symmetric stiffness matrix, $\underline{\underline{C}}$ … Eq.(2.12)

positive-definite, compliance matrix, $\underline{\underline{S}}$ … Eq.(2.14)

$$\underline{\underline{S}} = \frac{1}{E}\left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{array}\right]$$

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{array}\right]$$

12.2 Variational and energy principles

12.2.2 The principle of virtual work

- Elastic body in equilibrium under applied body forces and surface tractions

  -> the stress field is statically admissible

  -> equilibrium equations, Eq.(12.17), are satisfied at all points in $\mathcal{V}$ and the surface equilibrium equations, Eq.(12.18), at all points on

=>
$$\int_{\mathcal{V}} \left\{ \left[\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right] \delta u_1 + \left[\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right] \delta u_2 + \left[\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right] \delta u_3 \right\} \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \left[\underline{t} - \underline{\hat{t}} \right]^T \delta \underline{u} \, \mathrm{d}\mathcal{S} = 0. \quad (12.21)$$

3 equilibrium equations X an arbitrary, virtual change in displacement then integrated over the range of validity of the equation, volume $\mathcal{V}$

3 surface equilibrium equations X arbitrary, virtual change in displacement, then integrated over the range of validity of the equation, surface $\mathcal{S}_1$

stress field is statically admissible -> tracked term vanished -> multiplication by an arbitrary quantity results in a vanishing product.

12.2 Variational and energy principles

- Integration by parts … Green's theorem, first term of the volume integral

$$-\int_{\mathcal{V}} \underline{\sigma}^T \delta\underline{\epsilon}\, \mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \underline{b}^T \delta\underline{u}\, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}} \underline{t}^T \delta\underline{u}\, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_1} (\underline{t} - \underline{\hat{t}})^T \delta\underline{u}\, \mathrm{d}\mathcal{S} = 0 \tag{12.22}$$

n_1 : component of the outward unit normal along $\bar{\imath}_1$, (Fig. 12.4)

=> $$-\int_{\mathcal{V}} \underline{\sigma}^T \delta\underline{\epsilon}\, \mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \underline{b}^T \delta\underline{u}\, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}} \underline{t}^T \delta\underline{u}\, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_1} (\underline{t} - \underline{\hat{t}})^T \delta\underline{u}\, \mathrm{d}\mathcal{S} = 0 \tag{12.23}$$

$\delta\underline{\epsilon}$: virtual, compatible strain field

$$\begin{gathered}\delta\epsilon_1 = \frac{\partial \delta u_1}{\partial x_1},\ \delta\epsilon_2 = \frac{\partial \delta u_2}{\partial x_2},\ \delta\epsilon_3 = \frac{\partial \delta u_3}{\partial x_3},\\ \delta\gamma_{23} = \frac{\partial \delta u_2}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_2},\ \delta\gamma_{13} = \frac{\partial \delta u_1}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_1},\ \delta\gamma_{12} = \frac{\partial \delta u_1}{\partial x_2} + \frac{\partial \delta u_2}{\partial x_1}\end{gathered} \tag{12.24}$$

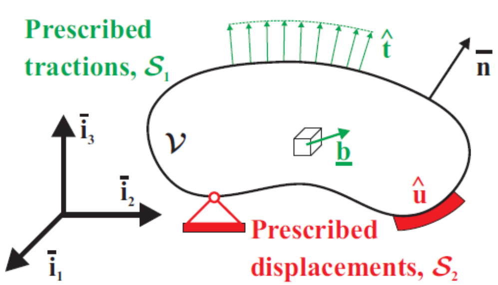


Fig. 12.4. General elasticity problem.

12.2 Variational and energy principles

- Virtual displacements are now chosen to be kinematically admissible
  -> $\delta \underline{u} = 0 \text{ on } \mathcal{S}_2$ -> Eq. (12.23)

$$\underbrace{-\int_{\mathcal{V}} \underline{\sigma}^T \delta \underline{\epsilon} \, \mathrm{d}\mathcal{V}}_{\text{Virtual work done by the internal stresses, } \delta W_I \text{, Eq.(9.77a)}} + \underbrace{\int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S}}_{\text{Virtual work done by the externally applied body forces and surface tractions}} = 0 \qquad (12.25)$$

$$\delta W_I = -\int_{\mathcal{V}} \underline{\sigma}^T \delta \underline{\epsilon} \, \mathrm{d}\mathcal{V}$$

-> $\delta W_I + \delta W_E = 0$

Reverse direction also holds. Eq.(12.25) -> Eq.(12.21)

=> Principle of Virtual Work

12.2 Variational and energy principles

Principle 15 (PVW) A body is in equilibrium if and only if the sum of the internal and external virtual work vanishes for all arbitrary kinematically admissible virtual displacement fields and corresponding compatible strain fields

equation of equilibrium, Eqs. (12.17), (12.18)
Principle of virtual work
<- 2 entirely equivalent statement PVW

However, for the solution of the specific elasticity problems, it must be complemented with

- stress-strain relationships
- strain-displacement relationships

- Comparison between
  - the present statement of PVW
  - that derived for beams under axial and transverse loads, Eqs.(11.42), (11.44)

  … different, but physical interpretation is identical

12.2.3 The principle of complementary virtual work

- Elastic body undergoing kinematically admissible displacements and compatible strains

  -> the strain-displacement relationship, Eq.(12.19), are satisfied at all points in volume $\mathcal{V}$

  and the geometric BC's, Eq.(12.20), are satisfied at all points on surface $\mathcal{S}_1$

=>

$$
\begin{aligned}
-\int_{\mathcal{V}} \Bigg\{ & \left[\epsilon_1 - \frac{\partial u_1}{\partial x_1}\right] \delta\sigma_1 + \left[\epsilon_2 - \frac{\partial u_2}{\partial x_2}\right] \delta\sigma_2 + \left[\epsilon_3 - \frac{\partial u_3}{\partial x_3}\right] \delta\sigma_3 \\
& + \left[\gamma_{23} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right] \delta\tau_{23} + \left[\gamma_{13} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right] \delta\tau_{13} \\
& + \left[\gamma_{12} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right] \delta\tau_{12} \Bigg\} \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_2} [\underline{u} - \underline{\hat{u}}]^T \, \delta\underline{t} \, \mathrm{d}\mathcal{S} = 0
\end{aligned}
\tag{12.26}
$$

6 strain-displacement relationships X arbitrary, virtual changes in stress, then integrated

Over the range of validity of the equations, volume $\mathcal{V}$

3 geometric BC's X arbitrary, virtual changes in surface traction then integrated over the range of validity of the equations, surface $\mathcal{S}_2$

Strain field is compatible, displacement field is kinematically admissible

-> bracket term vanishes -> multiplication by an arbitrary quantity results in a vanishing product

12.2 Variational and energy principles

- Integration by parts … by Green's theorem, first term of the volume integral

$$\int_{\mathcal{V}} \frac{\partial u_1}{\partial x_1} \delta\sigma_1 \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} u_1 \frac{\partial \delta\sigma_1}{\partial x_1} \,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}} u_1 n_1 \delta\sigma_1 \,\mathrm{d}\mathcal{S} \qquad (12.27)$$

n_1 : component of the outward unit normal along $\bar{\imath}_1$, (Fig. 12.4)

->

$$\begin{aligned} &-\int_{\mathcal{V}} \underline{\epsilon}^T \delta\underline{\sigma} \,\mathrm{d}\mathcal{V} - \int_{\mathcal{V}} \left[\left(\frac{\partial \delta\sigma_1}{\partial x_1} + \frac{\partial \delta\tau_{21}}{\partial x_2} + \frac{\partial \delta\tau_{31}}{\partial x_3} \right) u_1 \right. \\ &\left. + \left(\frac{\partial \delta\tau_{12}}{\partial x_1} + \frac{\partial \delta\sigma_2}{\partial x_2} + \frac{\partial \delta\tau_{32}}{\partial x_3} \right) u_2 + \left(\frac{\partial \delta\tau_{13}}{\partial x_1} + \frac{\partial \delta\tau_{23}}{\partial x_2} + \frac{\partial \delta\sigma_3}{\partial x_3} \right) u_3 \right] \mathrm{d}\mathcal{V} \qquad (12.28) \\ &+ \int_{\mathcal{S}} \underline{u}^T \delta\underline{t} \,\mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} (\underline{u} - \underline{\hat{u}})^T \delta\underline{t} \,\mathrm{d}\mathcal{S} = 0. \quad (12. \end{aligned}$$

- "statically admissible virtual stress field" … virtual stress field that satisfies equilibrium equations in volume $\mathcal{V}$

$$\begin{aligned} \frac{\partial \delta\sigma_1}{\partial x_1} + \frac{\partial \delta\tau_{21}}{\partial x_2} + \frac{\partial \delta\tau_{31}}{\partial x_3} &= 0 \\ \frac{\partial \delta\tau_{12}}{\partial x_1} + \frac{\partial \delta\sigma_2}{\partial x_2} + \frac{\partial \delta\tau_{32}}{\partial x_3} &= 0 \\ \frac{\partial \delta\tau_{13}}{\partial x_1} + \frac{\partial \delta\tau_{23}}{\partial x_2} + \frac{\partial \delta\sigma_3}{\partial x_3} &= 0 \end{aligned} \qquad (12.29)$$

And the surface traction equilibrium equations, $\delta\underline{t} = 0$ on surface S_1

12.2 Variational and energy principles

- Because the virtual stresses are arbitrary, they can be chosen to be statically admissible -> Eq.(12.28)

$$-\int_{\mathcal{V}} \underline{\epsilon}^T \delta\underline{\sigma} \,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta\underline{t} \,\mathrm{d}\mathcal{S} = 0. \qquad (12.30)$$

Complementary virtual work done by the internal stresses $\delta W_I'$

Eq.(9.77b)

$$\delta W_I' = -\int_{\mathcal{V}} \underline{\epsilon}^T \delta\underline{\sigma} \,\mathrm{d}\mathcal{V}.$$

Complementary virtual work done by the prescribed displacements $\delta W_E'$

-> $\delta W_I' + \delta W_E' = 0$

reverse direction also holds -> principle of virtual work

12.2 Variational and energy principles

Principle 16 (PCW) A body is undergoing kinematically admissible displacements and compatible strains if and only if the sum of the internal and external CVW vanishes for all statically admissible virtual stress fields

- strain-displacement relationships, Eq.(12.19)
  Geometric BC's, Eq.(12.20)

Entirely equivalent ⟷ PCW

Eq.(12.30) with Principle 7 in Chap.9 … Principle 16 is simply a more general statement

Principle 7 (Principle of complementary virtual work) *A truss undergoes compatible deformations if and only if the sum of the internal and external complementary virtual work vanishes for all statically admissible virtual forces.*

12.2.4 strain and complementary strain E density functions

- Sec. 10.5(Page 519) … $\left\{ \begin{array}{c} \text{strain energy density function} \\ \text{complementary strain energy density function} \end{array} \right\}$

developed for a linearly elastic, isotropic material -> Eqs.(10.47), (10.50)

$$a(\underline{\epsilon}) = \frac{1}{2}\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2)+2\nu(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3)\right. \\ \left.+\frac{1-2\nu}{2}(\gamma_{23}^2+\gamma_{31}^2+\gamma_{12}^2)\right]. \tag{10.47}$$

$$a'(\underline{\sigma}) = \frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu\left(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3\right)\right. \\ \left.+2(1+\nu)\left(\tau_{12}^2+\tau_{23}^2+\tau_{31}^2\right)\right]. \tag{10.50}$$

12.2 Variational and energy principles

- If the internal forces in the solid are assumed to be conservative

  -> can be derived from a potential

  internal forces … the components of stress

  potential … the strain energy density function

- Stress in solid … derived from a strain energy density function, $a(\underline{\epsilon})$

$$\underline{\sigma} = \frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}} \tag{12.31}$$

  -> material is said to be "elastic material"

$$\left\{\begin{array}{c} \text{assumption of elastic material} \\ \text{assumption of existence of a strain energy density function} \end{array}\right\} \text{"Two equivalent assumption"}$$

- If elastic material, work done by the internal stresses

  when the system is brought from one state of deformation to another

  -> only depends on the two states of deformations

  but not on the specific path that the system followed from one deformation state to the other

  => This restricts the types of material constitutive laws that can be expressed in terms of a strain energy density function

12.2 Variational and energy principles

Ex) plastic range … the work of deformation will depend on the specific deformation history

-> no strain energy density function that describes material behavior when plastic deformations are involved.

- Complementary strain energy … its concept is first introduced for springs in Sec. 10.3.1

For nonlinearly elastic material

$$a(\underline{\epsilon}) + a'(\underline{\sigma}) = \underline{\epsilon}^T \underline{\sigma} \qquad (12.32)$$

taking differential

$$\left(\underbrace{\frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}} - \underline{\sigma}}_{=0 \text{ due to Eq.(12.31)}} \right)^T \mathrm{d}\underline{\epsilon} + \left(\frac{a'(\underline{\sigma})}{\partial \underline{\sigma}} - \underline{\epsilon} \right)^T \mathrm{d}\underline{\sigma} = 0$$

Then, the second parenthesis must vanish

$$\underline{\epsilon} = \frac{a'(\underline{\sigma})}{\partial \underline{\sigma}} \qquad (12.33)$$

existence of strain energy density function => existence of the complementary strain energy density function

12.2 Variational and energy principles

Eq.(12.31) … definition of the stresses by means of the strain energy density function, also constitutive laws for the elastic materials

Eq.(12.33) … definition of the strains by means of the complementary strain energy density function, also constitutive laws for the elastic materials

strain energy density function

complementary strain energy density function

-> define the constitutive laws for the elastic materials

Eq.(12.31) … stiffness form of the constitutive laws <- strain energy density function

Eq.(12.33) … compliance form of the constitutive laws <- complementary strain energy density function

$$\underline{\sigma} = \frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}} \tag{12.31}$$

$$\underline{\epsilon} = \frac{a'(\underline{\sigma})}{\partial \underline{\sigma}} \tag{12.33}$$

12.2.5 PMTPE

- General elastic body in equilibrium under applied body forces and surface tractions -> PVW, Eq.(12.25), must apply

$$-\int_{\mathcal{V}} \underline{\sigma}^T \delta\underline{\epsilon}\,\mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \underline{b}^T \delta\underline{u}\,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta\underline{u}\,\mathrm{d}\mathcal{S} = 0. \qquad (12.25)$$

- Now assuming that the constitutive law for the material can be expressed in terms of a strain energy density function, Eq.(12.31)

$$\underline{\sigma} = \frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}} \qquad (12.31)$$

-> VW done by the internal stresses can be

$$-\int_{\mathcal{V}} \delta\underline{\epsilon}^T \underline{\sigma}\,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta\underline{\epsilon}^T \frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}}\,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta a(\underline{u})\,\mathrm{d}\mathcal{V} = \delta \int_{\mathcal{V}} a(\underline{u})\,\mathrm{d}\mathcal{V} = \delta A(\underline{u})$$

where the chain rule for derivatives is used.

12.2 Variational and energy principles

- Strain energy density
  total energy E, $A = \int_{\mathcal{V}} a \, \mathrm{d}\mathcal{V}$

  Must be expressed in terms of the displacement field $\underline{u}$ using the strain-displacement relationship because PVW requires a compatible strain field

- PVW, Eq.(12.25) ->

$$-\delta A(\underline{u}) + \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} = 0 \qquad (12.34)$$

- body forces
  surface tractions

  are assumed to be derivable from potential functions

$$\underline{b} = -\frac{\partial \phi}{\partial \underline{u}}; \quad \underline{\hat{t}} = -\frac{\partial \psi}{\partial \underline{u}}$$

ϕ : potential of the body forces

ψ : potential of the surface tractions

12.2 Variational and energy principles

- 2nd and 3rd terms in Eq.(12.34)

$$-\delta A(\underline{u}) + \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} = 0 \qquad (12.34)$$

$$\int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{V}} \frac{\partial \phi}{\partial \underline{u}}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \frac{\partial \psi}{\partial \underline{u}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S}$$
$$= -\int_{\mathcal{V}} \delta\phi(\underline{u}) \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \delta\psi(\underline{u}) \, \mathrm{d}\mathcal{S} = -\delta \int_{\mathcal{V}} \phi(\underline{u}) \, \mathrm{d}\mathcal{V} - \delta \int_{\mathcal{S}_1} \psi(\underline{u}) \, \mathrm{d}\mathcal{S}$$
$$= -\delta \Phi(\underline{u}),$$

$\Phi(\underline{u}) = \int_{\mathcal{V}} \phi(\underline{u}) \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \psi(\underline{u})$ Total potential of the externally applied loads

- Introducing the result into PVW in Eq.(12.34)

$$-\delta A(\underline{u}) - \delta \Phi(\underline{u}) = 0, \text{ or } \quad \delta \left(A(\underline{u}) + \Phi(\underline{u}) \right) = 0 \qquad (12.35)$$

- Total potential energy of the body

$$\Pi(\underline{u}) = A(\underline{u}) + \Phi(\underline{u}) \qquad (12.36)$$

-> $\delta \Pi(\underline{u}) = 0$ (12.37)

… total potential energy must assume a stationary value w.r.t the compatible deformations when the body is in equilibrium

12.2 Variational and energy principles

- 1st variation of Π

$$\delta\Pi(\underline{u}) = \int_{\mathcal{V}} \left(\frac{\partial a}{\partial \underline{\epsilon}}\right)^T \delta\underline{\epsilon}\,\mathrm{d}\mathcal{V} - \int_{\mathcal{V}} \underline{b}^T \delta\underline{u}\,\mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta\underline{u}\,\mathrm{d}\mathcal{S}$$

2nd variation

$$\delta^2\Pi(\underline{u}) = \int_{\mathcal{V}} \delta\underline{\epsilon}^T \frac{\partial^2 a}{\partial\underline{\epsilon}\partial\underline{\epsilon}} \delta\underline{\epsilon}\,\mathrm{d}\mathcal{V}$$

… strain energy density function must be a positive-definite function

-> $\delta\underline{\epsilon}^T \partial^2 a/(\partial\underline{\epsilon}\partial\underline{\epsilon})\,\delta\underline{\epsilon} \geq 0$ for all $\delta\underline{\epsilon}$.

if the strain energy density function is NOT positive-definite, strain state will generate a (-) strain energy -> the elastic body will generate energy under deformation

… physically impossible

Thus $\delta^2\Pi \geq 0$ -> Π presents an absolute minimum at its stationary points

12.2 Variational and energy principles

Principle 17 (PMTPE)

Among all kinematically admissible displacement fields, the actual displacement field that corresponds to the equilibrium configuration of the body makes the total potential energy an absolute minimum.

- Reverse direction also holds

  Also, these equations are the Euler-Lagrange equations arising from the stationarity condition for the total potential energy

- PMTPE -> PVW

  PVW -> PMTPE

  under restrictive assumptions on existences of { strain energy density function
  potential of the body forces
  of the surface tractions }

… PVW is more general statement but possibly less useful statement.

12.2.6 PMCE(The Principle of Minimum Complementary Energy)

- Elastic body undergoing kinematically admissible displacements and compatible strains -> PCVW, Eq.(12.30), must apply

$$-\int_{\mathcal{V}} \underline{\epsilon}^T \delta\underline{\sigma}\,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta\underline{t}\,\mathrm{d}\mathcal{S} = 0. \qquad (12.30)$$

- Now assuming that the constitutive law for the material can be expressed in terms of a stress energy density function, Eq.(12.33) $\underline{\epsilon} = \frac{a'(\underline{\sigma})}{\partial\underline{\sigma}}$
- VW done by the internal strain in Eq.(12.30)

$$\text{->} \quad \int_{\mathcal{V}} \delta\underline{\sigma}^T \underline{\epsilon}\,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta\underline{\sigma}^T \frac{\partial b(\underline{\sigma})}{\partial\underline{\sigma}}\,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta b(\underline{\sigma})\,\mathrm{d}\mathcal{V} = \delta\int_{\mathcal{V}} b(\underline{\sigma})\,\mathrm{d}\mathcal{V} = \delta A'(\underline{\sigma})$$

$A'(\underline{\sigma})$: total stress energy in the body

- PCVW, Eq.(12.30)

$$\text{->} \qquad -\delta A'(\underline{\sigma}) + \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta\underline{t}\,\mathrm{d}\mathcal{S} = 0 \qquad (12.38)$$

12.2 Variational and energy principles

- Prescribed displacements are assumed to be derivable from a potential function

$$\underline{\hat{u}} = -\frac{\partial \chi(\underline{t})}{\partial \underline{t}}$$

$\chi(\underline{t})$: "potential of the prescribed displacement"

Ex) simply $\chi = -\underline{\hat{u}}^T \underline{t}$, but potential functions do NOT exist for all types of prescribed displacements

- 2nd term in Eq.(12.38) ->

$$\int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}_2} \frac{\partial \chi}{\partial \underline{t}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}_2} \delta \chi(\underline{t}) \, \mathrm{d}\mathcal{S} = -\delta \int_{\mathcal{S}_2} \chi(\underline{t}) \, \mathrm{d}\mathcal{S} = -\delta \Phi'$$

$\Phi'(\underline{t}) = \int_{\mathcal{S}_2} \chi(\underline{t})$ … total potential of the prescribed displacements

- Introducing this result into Eq.(12.38)

$$-\delta A'(\underline{\sigma}) - \delta \Phi'(\underline{t}), \text{ or } \quad \delta\left[A'(\underline{\sigma}) + \Phi'(\underline{t})\right] = 0 \qquad (12.39)$$

- Total complementary energy of the body

$$\Pi'(\underline{\sigma}) = A'(\underline{\sigma}) + \Phi'(\underline{t}) \qquad (12.40)$$

$$\rightarrow \quad \delta \Pi'(\underline{\sigma}) = 0 \qquad (12.41)$$

12.2 Variational and energy principles

Principle 18 (PMCE)

Among all statically admissible stress fields, the actual stress field that corresponds to the compatible deformation of the body makes the total complementary energy on absolute minimum

- 1st variation of Π'

$$\delta\Pi'(\underline{\sigma}) = \int_{\mathcal{V}} \sum_{i=1}^{6} \frac{\partial a'}{\partial \sigma_i} \delta\sigma_i \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta\underline{t} \, \mathrm{d}\mathcal{S} \qquad (12.42)$$

- 2nd variation

$$\delta^2\Pi'(\underline{\sigma}) = \int_{\mathcal{V}} \sum_{i,j=1}^{6} \frac{\partial^2 a'}{\partial \sigma_i \partial \sigma_j} \delta\sigma_i \, \delta\sigma_j \, \mathrm{d}\mathcal{V} \qquad (12.43)$$

stress energy density function must be a positive-definite function of the stress components

-> $\sum_{i,j=1}^{6} \frac{\partial^2 a'}{\partial \sigma_i \partial \sigma_j} \delta\sigma_i \, \delta\sigma_j$

if NOT positive-definite, stress states will exist that generate a (-) stress energy

-> elastic body will generate energy under stress -> physically impossible

12.2 Variational and energy principles

- Reverse direction also holds

  There equations are the Euler-Lagrange equations arising from the stationary condition for the complementary energy.

- PMCE -> PCVW

  PCVW -> PMCE under restrictive assumptions on existence of

  stress energy density function

  potential for the prescribed displacements

12.2 Variational and energy principles

12.2.7 Energy theorems

Sec.10.9 … energy theorems -> corollaries of the fundamental energy principles

$\left.\begin{array}{l}\text{Clayperon's Theorem (Theorem 10.1)} \\ \text{Castigaliano's 1}^{\text{st}}\text{ Theorem (Theorem 10.2)}\end{array}\right\}$ corollaries of PMTPE

Theorem 10.1 (Clapeyron's theorem). *The strain energy stored in a linearly elastic structure equals the sum of the half product of the applied loads by the displacements of their respective points of applications projected along their lines of action.*

Theorem 10.2 (Castigliano's first theorem). *For an elastic system, the magnitude of the load applied at a point is equal to the partial derivative of the strain energy with respect to the projected load's displacement.*

12.2 Variational and energy principles

Principle of Least work (Principle 14)
Crotti-Engesser Theorem (Theorem 10.3)
Castigaliano's 2nd Theorem (Theorem 10.4)

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ Corollaries of PMCE

Principle 14 (Principle of least work) *In the absence of prescribed displacements, a linearly elastic system undergoes compatible deformations if and only if the strain energy is a minimum with respect to arbitrary changes in statically admissible forces.*

Theorem 10.3 (Crotti-Engesser theorem). *For an elastic structure, the prescribed deflection at a point is given by the partial derivative of the complementary energy with respect to the driving force.*

Theorem 10.4 (Castigliano's second theorem). *For a linearly elastic structure, the prescribed deflection at a point is given by the partial derivative of the strain energy with respect to the driving force.*

12.2 Variational and energy principles

Reciprocity theorem of Betti & Maxwell (Theorem 10.5/10.6) <- direct sequence

=> All theorems are now also valid for general, 3-D structures

Theorem 10.5 (Reciprocity theorem or Betti's theorem). *A linearly elastic body is subjected to two loading states characterized by loads of different magnitudes but identical points of applications and lines of action. The sum of the product of the loads in one state by the projected displacements of the other is identical to that obtained when the two states are interchanged.*

Theorem 10.6 (Maxwell's theorem). *For a linearly elastic structure, the influence coefficient of point* **1** *on point* **2** *equals that of point* **2** *on point* **1**, *for any choice of points* **1** *and* **2**.

12.2 Variational and energy principles

PVW … entirely equivalent to the equation of equilibrium of a 3-D solid, Eq. (12.17), (12.18)

but must be complemented with
- stress-strain relationship
- constitutive law

in order to solve specific elasticity problems

PCVW … entirely equivalent to the strain-displacement relationships and geometric BC's, Eq. (12.19), (12.20)

but must be complemented with
- equilibrium equations
- constitutive law

Hu-Washizu's principle … remedies this shortcoming, equivalent to the complete set of equations required to solve elasticity problems

12.2 Variational and energy principles

- Elastic body in equilibrium under the applied body forces and surface tractions undergoing compatible strain s whose displacement field is kinematically admissible, and for which the stress and strain fields satisfy the material constitutive laws
  -> stress fields are statically admissible
  the equilibrium equations, Eq.(12.17), are satisfied at all points in $\mathcal{V}$
  surface equilibrium equations, Eq.(12.18), at all points on $\mathcal{S}_1$
  strain-displacement relationships, Eq.(12.19), are satisfied at all points in $\mathcal{V}$
  geometric BC's, Eq.(12.20), at all points on $\mathcal{S}_2$
  constitutive laws, expressed in terms of a strain energy density function, Eq.(12.31), must hold at all points in $\mathcal{V}$

$$\boxed{\begin{aligned}\frac{\partial \sigma_1}{\partial x_1}+\frac{\partial \tau_{21}}{\partial x_2}+\frac{\partial \tau_{31}}{\partial x_3}+b_1=0\\ \frac{\partial \tau_{12}}{\partial x_1}+\frac{\partial \sigma_2}{\partial x_2}+\frac{\partial \tau_{32}}{\partial x_3}+b_2=0\\ \frac{\partial \tau_{13}}{\partial x_1}+\frac{\partial \tau_{23}}{\partial x_2}+\frac{\partial \sigma_3}{\partial x_3}+b_3=0\end{aligned}}$$

(12.17)

$$\boxed{t_1=\hat{t}_1, \quad t_2=\hat{t}_2, \quad t_3=\hat{t}_3} \quad (12.18)$$

$$\boxed{\epsilon_1=\frac{\partial u_1}{\partial x_1}, \quad \epsilon_2=\frac{\partial u_2}{\partial x_2}, \quad \epsilon_3=\frac{\partial u_3}{\partial x_3}}$$ "axial strain"

$$\boxed{\gamma_{23}=\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}, \quad \gamma_{13}=\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}, \quad \gamma_{12}=\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}} \quad (12.19)$$

$$\boxed{u_1=\hat{u}_1, \quad u_2=\hat{u}_2, \quad u_3=\hat{u}_3} \quad (12.20)$$

- Combining Eqs. (12.21), (12.26), (12.31) into a single integral equation

$$\begin{aligned}
&\int_{\mathcal{V}} \left\{ \left[\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right] \delta u_1 + \left[\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right] \delta u_2 \right. \\
&\left. + \left[\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right] \delta u_3 \right\} \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \left[\underline{t} - \underline{\hat{t}} \right]^T \delta \underline{u} \, \mathrm{d}\mathcal{S} \\
&- \int_{\mathcal{V}} \left\{ \left[\epsilon_1 - \frac{\partial u_1}{\partial x_1} \right] \delta \sigma_1 + \left[\epsilon_2 - \frac{\partial u_2}{\partial x_2} \right] \delta \sigma_2 + \left[\epsilon_3 - \frac{\partial u_3}{\partial x_3} \right] \delta \sigma_3 \right. \\
&+ \left[\gamma_{23} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right] \delta \tau_{23} + \left[\gamma_{13} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right] \delta \tau_{13} \\
&\left. + \left[\gamma_{12} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right] \delta \tau_{12} \right\} \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_2} \left[\underline{u} - \underline{\hat{u}} \right]^T \delta \underline{t} \, \mathrm{d}\mathcal{S} \\
&+ \int_{\mathcal{V}} \left\{ \left[\frac{\partial a}{\partial \epsilon_1} - \sigma_1 \right] \delta \epsilon_1 + \left[\frac{\partial a}{\partial \epsilon_2} - \sigma_2 \right] \delta \epsilon_2 + \left[\frac{\partial a}{\partial \epsilon_3} - \sigma_3 \right] \delta \epsilon_3 \right. \\
&\left. + \left[\frac{\partial a}{\partial \gamma_{23}} - \tau_{23} \right] \delta \gamma_{23} + \left[\frac{\partial a}{\partial \gamma_{13}} - \tau_{13} \right] \delta \gamma_{13} + \left[\frac{\partial a}{\partial \gamma_{12}} - \tau_{12} \right] \delta \gamma_{12} \right\} \mathrm{d}\mathcal{V} = 0
\end{aligned} \tag{12.24}$$

… can be manipulated in several ways

① terms appearing in the equilibrium equations could be integrated by parts(as is done for PVW)

② terms appearing in the strain-displacement relationships could be integrated by parts(as is done for PCVW)

③ both integrations by parts could be carried out

-> three different statements of Hu-Washizu's principle

12.2 Variational and energy principles

- 1st statement of Hu-Washizu's principle

  integrating by parts using Eq. (12.22) $$\int_{\mathcal{V}} \frac{\partial \sigma_1}{\partial x_1} \delta u_1 \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} \sigma_1 \frac{\partial \delta u_1}{\partial x_1} \,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}} n_1 \sigma_1 \,\delta u_1 \,\mathrm{d}\mathcal{S}$$

$$\begin{aligned}
\delta \int_{\mathcal{V}} \Big[& a(\underline{\epsilon}) - \left(\epsilon_1 - \frac{\partial u_1}{\partial x_1}\right) \sigma_1 - \left(\epsilon_2 - \frac{\partial u_2}{\partial x_2}\right) \sigma_2 - \left(\epsilon_3 - \frac{\partial u_3}{\partial x_3}\right) \sigma_3 \\
& - \left(\gamma_{23} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \tau_{23} - \left(\gamma_{13} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \tau_{13} \\
& - \left(\gamma_{12} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \tau_{12} \Big] \,\mathrm{d}\mathcal{V} \\
& - \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \,\mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \,\mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} \left(\underline{u} - \underline{\hat{u}}\right)^T \delta \underline{t} \,\mathrm{d}\mathcal{S} = 0.
\end{aligned} \qquad (12.45)$$

  … 3 independent fields: strain, stress, displacement field -> "three field principle"

  PMTPE, PMCE … single field principle, involving only displacement/stress closely related to PMTPE, Eq. (12.34)

- 1st statement of Hu-Washizu's principle

  PVW(statement of equilibrium(+ constitutive laws, but no strain-displacement relationship

  -> constrained minimization problem that yields all equations of elasticity

  -> unconstrained minimization problem using the Lagrange multiplier

  Eq. (12.45) … λ : stress components used to enforce the corresponding compatibility equations

12.2 Variational and energy principles

- 2nd statement of Hu-Washizu's principle

  strain-displacement relationship in Eq. (12.44) are integrated by parts using Eq. (12.27)

$$\int_{\mathcal{V}} \frac{\partial u_1}{\partial x_1}\, \delta\sigma_1 \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} u_1 \frac{\partial \delta\sigma_1}{\partial x_1} \,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}} u_1 n_1 \delta\sigma_1 \,\mathrm{d}\mathcal{S} \qquad (12.27)$$

$$\begin{aligned} &\delta \int_{\mathcal{V}} \left[\left(a(\underline{\epsilon}) - \underline{\epsilon}^T \underline{\sigma} \right) + \left(\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right) u_1 \right. \\ &\left. + \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right) u_2 + \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right) u_3 \right] \mathrm{d}\mathcal{V} \qquad (12.46) \\ &- \int_{\mathcal{S}_1} (\underline{t} - \hat{\underline{t}})^T \delta \underline{u} \,\mathrm{d}S - \int_{\mathcal{S}_2} \hat{\underline{u}}^T \delta \underline{t} \,\mathrm{d}\mathcal{S} = 0. \end{aligned}$$

… closely related to PMCE, Eq. (12.38)

- 2nd statement of Hu-Washizu's principle

  PMCE(statement of compatibility)+constitutive laws, but no equilibrium equations

  -> constrained minimization problem that yields all equations

  -> unconstrained minimization problem that yields all equations

  Eq. (12.46) … λ : displacement components used to enforce the corresponding equilibrium equations

12.2 Variational and energy principles

- 3rd statement of Hu-Washizu's principle

both { Equations of equilibrium / Strain-displacement relationship } in Eq. (12.44) are integrated by parts using Eq.(12.22) and (12.27)

$$
\begin{aligned}
&\int_{\mathcal{V}} \Bigg\{ \delta\left[a(\underline{\epsilon}) - \underline{\epsilon}^T \underline{\sigma}\right] + \sigma_1 \frac{\partial \delta u_1}{\partial x_1} + \sigma_2 \frac{\partial \delta u_2}{\partial x_2} + \sigma_3 \frac{\partial \delta u_3}{\partial x_3} + \tau_{23}\left[\frac{\partial \delta u_2}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_2}\right] \\
&+\tau_{13}\left[\frac{\partial \delta u_1}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_1}\right] + \tau_{12}\left[\frac{\partial \delta u_1}{\partial x_2} + \frac{\partial \delta u_3}{\partial x_2}\right] - u_1\left[\frac{\partial \delta \sigma_1}{\partial x_1} + \frac{\partial \delta \tau_{12}}{\partial x_2} + \frac{\partial \delta \tau_{13}}{\partial x_3}\right] \\
&- u_2\left[\frac{\partial \delta \tau_{12}}{\partial x_1} + \frac{\partial \delta \sigma_2}{\partial x_2} + \frac{\partial \delta \tau_{23}}{\partial x_3}\right] - u_3\left[\frac{\partial \delta \tau_{13}}{\partial x_1} + \frac{\partial \delta \tau_{23}}{\partial x_2} + \frac{\partial \delta \sigma_3}{\partial x_3}\right] \Bigg\} \,\mathrm{d}\mathcal{V} \\
&- \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \,\mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \,\mathrm{d}\mathcal{S} + \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \,\mathrm{d}\mathcal{S} = 0.
\end{aligned}
\tag{12.47}
$$

… main advantage is that no derivatives of the 3 unknown fields present

-> important implications on the way in which the unknown fields can be approximated, because minimal continuity requirements are imposed

12.2.9 Hellinger-Reissner's principle

- Complexity of 3-field Hu-Washizu's principle -> simpler, 2-field principle eliminating the strain field in H-W principle -> Hellinger-Reissner's principle

- 1st statement of H-W principle, Eq.(12.45)

… Eq. (12.32) is used to eliminate the strain field

$$\delta[a(\underline{\epsilon}) - \underline{\epsilon}^T \underline{\sigma}] = -\delta a'(\underline{\sigma})$$

-> 1st statement of Hellinger-Reissner's principle

$$\begin{aligned}
&\delta \int_{\mathcal{V}} \left[\frac{\partial u_1}{\partial x_1}\,\sigma_1 + \frac{\partial u_2}{\partial x_2}\,\sigma_2 + \frac{\partial u_3}{\partial x_3}\,\sigma_3 + \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \tau_{23} \right. \\
&\quad \left. + \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \tau_{13} + \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \tau_{12} - a'(\underline{\sigma}) \right] \mathrm{d}\mathcal{V} \\
&\quad - \int_{\mathcal{V}} \underline{b}^T \delta \underline{u}\, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u}\, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} (\underline{u} - \underline{\hat{u}})^T \delta \underline{t}\, \mathrm{d}\mathcal{S} = 0.
\end{aligned} \tag{12.48}$$

- 2nd statement of H-W principle, Eq.(12.46)

  … strain field is eliminated in a similar manner

  -> 2nd statement of Hellinger-Reissner's principle

$$\delta \int_{\mathcal{V}} \left[\left(\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right) u_1 + \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right) u_2 + \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right) u_3 - a'(\underline{\sigma}) \right] \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} (\underline{t} - \underline{\hat{t}})^T \delta \underline{u} \, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = 0. \tag{12.49}$$

- 3rd statement of H-W principle, Eq.(12.47)

  -> 3rd statement of Hellinger-Reissner's principle

$$\begin{aligned} \int_{\mathcal{V}} \Big[\delta a'(\underline{\sigma}) &+ u_1 \left(\frac{\partial \delta \sigma_1}{\partial x_1} + \frac{\partial \delta \tau_{12}}{\partial x_2} + \frac{\partial \delta \tau_{13}}{\partial x_3} \right) + u_2 \left(\frac{\partial \delta \tau_{12}}{\partial x_1} + \frac{\partial \delta \sigma_2}{\partial x_2} + \frac{\partial \delta \tau_{23}}{\partial x_3} \right) \\ &+ u_3 \left(\frac{\partial \delta \tau_{13}}{\partial x_1} + \frac{\partial \delta \tau_{23}}{\partial x_2} + \frac{\partial \delta \sigma_3}{\partial x_3} \right) - \sigma_1 \frac{\partial \delta u_1}{\partial x_1} - \sigma_2 \frac{\partial \delta u_2}{\partial x_2} - \sigma_3 \frac{\partial \delta u_3}{\partial x_3} \\ &- \tau_{23} \left(\frac{\partial \delta u_2}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_2} \right) - \tau_{13} \left(\frac{\partial \delta u_1}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_1} \right) - \tau_{12} \left(\frac{\partial \delta u_1}{\partial x_2} + \frac{\partial \delta u_3}{\partial x_2} \right) \Big] \mathrm{d}\mathcal{V} \\ &+ \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = 0. \end{aligned} \tag{12.50}$$

12.2 Variational and energy principles

12.2 Variational and energy principles

12.2 Variational and energy principles

12.2 Variational and energy principles