Engineering Economic Analysis

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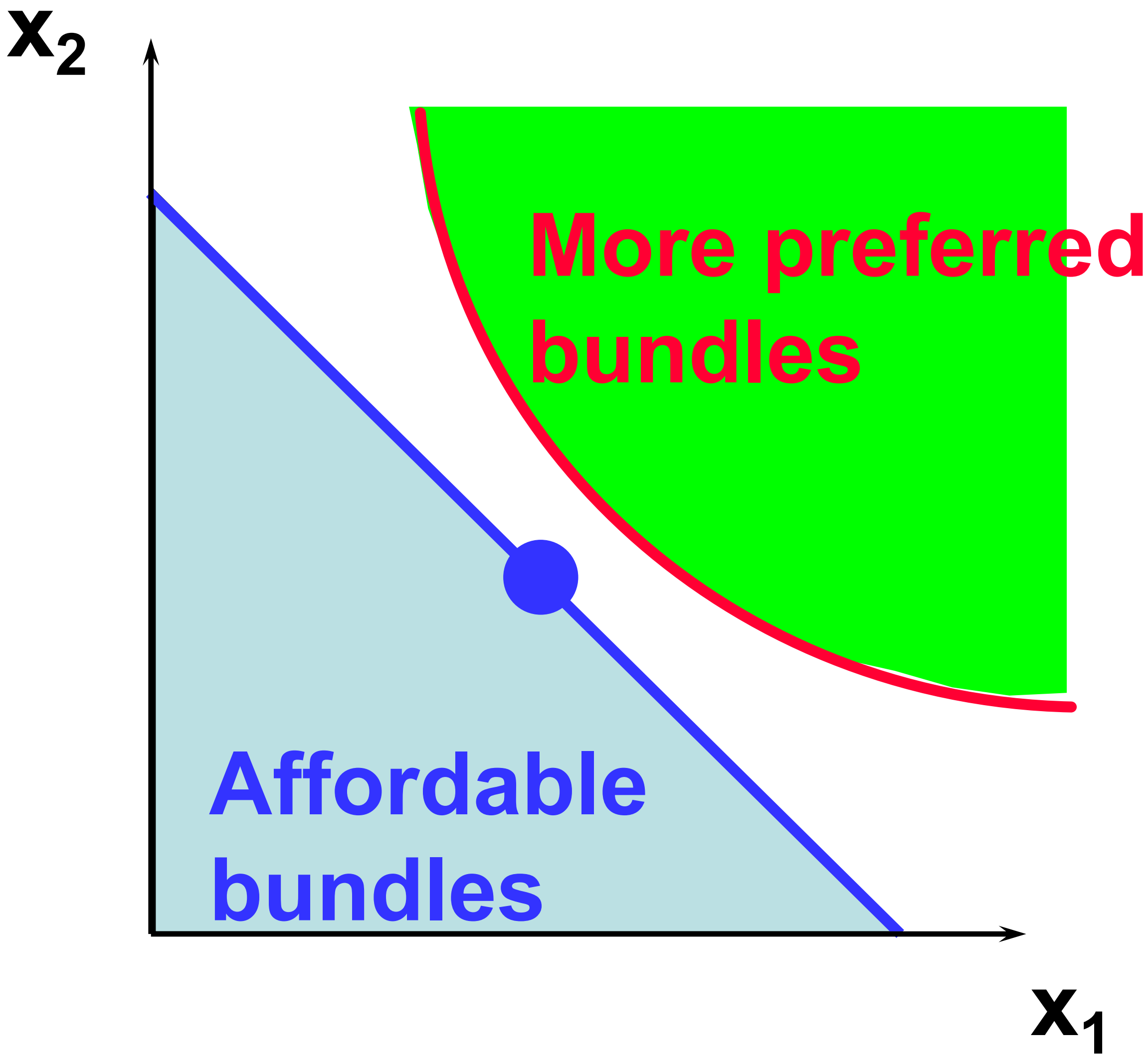
Chap. 5

CHOICE

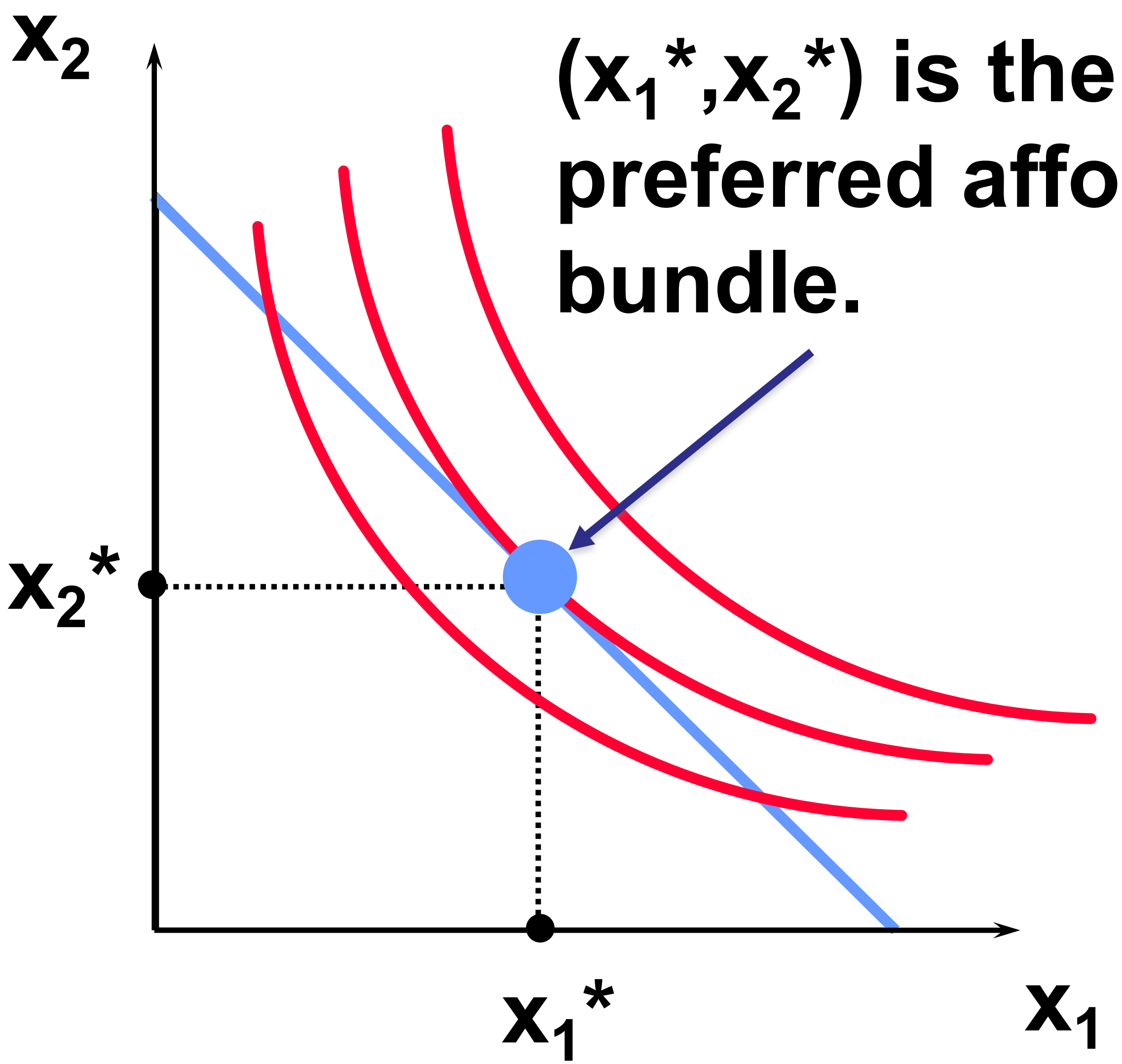
Economic Rationality

- The principal behavioral postulate is that a decision maker chooses its most preferred alternative from those available to it.
- Utility maximization with budget constraint

Optimal Choice

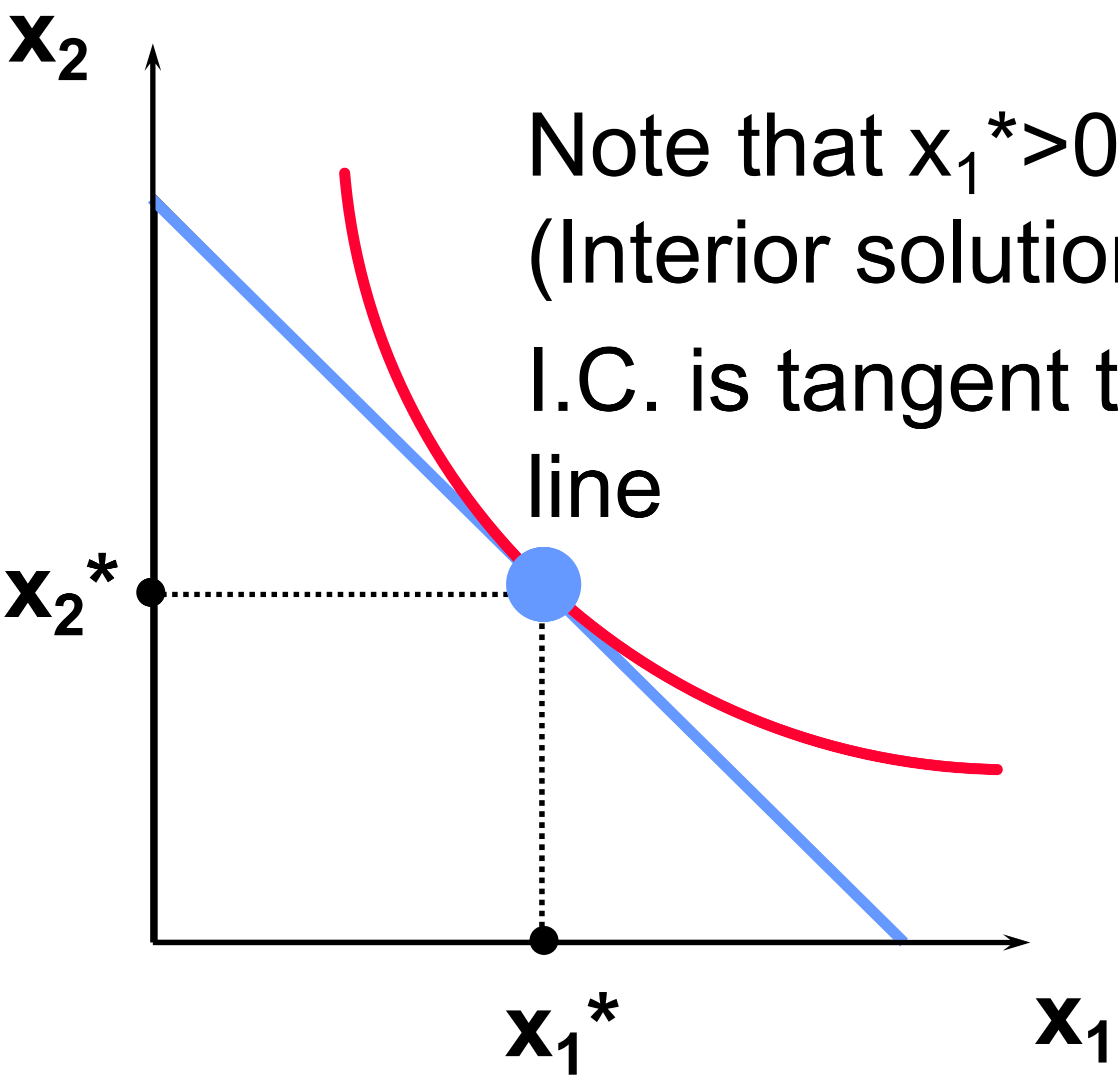


Optimal Choice



(x_1^*, x_2^*) is the most preferred affordable bundle.

Optimal Choice



Note that $x_1^*>0$, $x_2^*>0$ (Interior solution)

I.C. is tangent to budget line

Optimal Choice

- (x_1^*, x_2^*) satisfies two conditions:
  - the budget is exhausted;

    $p_1x_1^* + p_2x_2^* = m$
  - the slope of the budget constraint, $-p_1/p_2$, and the slope of the indifference curve containing (x_1^*, x_2^*) are equal at (x_1^*, x_2^*).

$$MRS = \frac{dx_2}{dx_1} = \frac{MU_1}{MU_2} = \frac{p_1}{p_2} \qquad \text{at } (x_1^*, x_2^*)$$

- Are these conditions always hold at the optimal choice? (Necessary & sufficient condition?)

Optimal Choice

- Kinky tastes

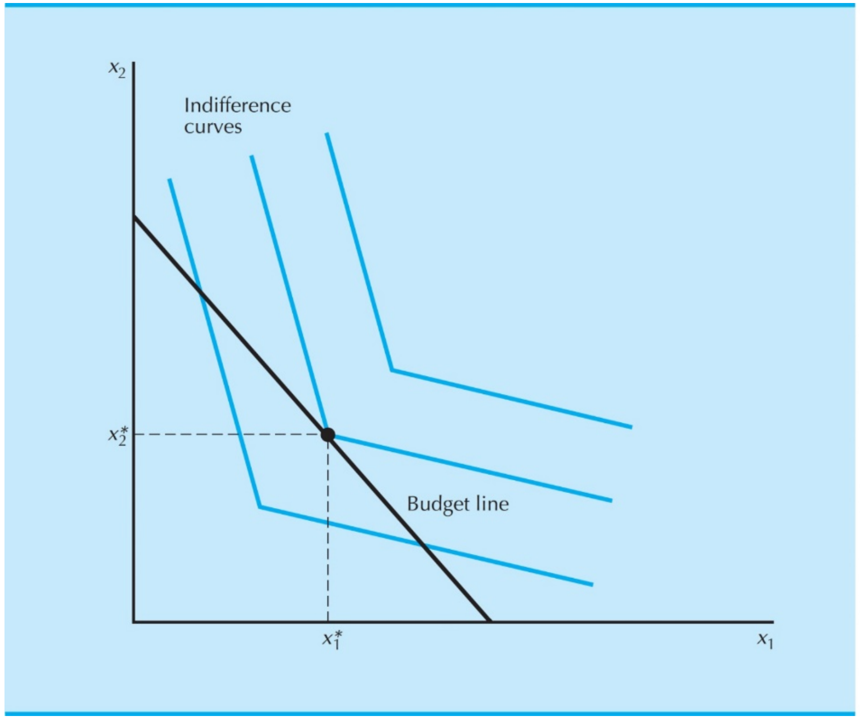


Figure 5.2

- I.C. has a kink at (x_1*,x_2*), there is no tangency!

Optimal Choice

- Boundary optimum (corner solution): optimal point occurs where some x_i*=0

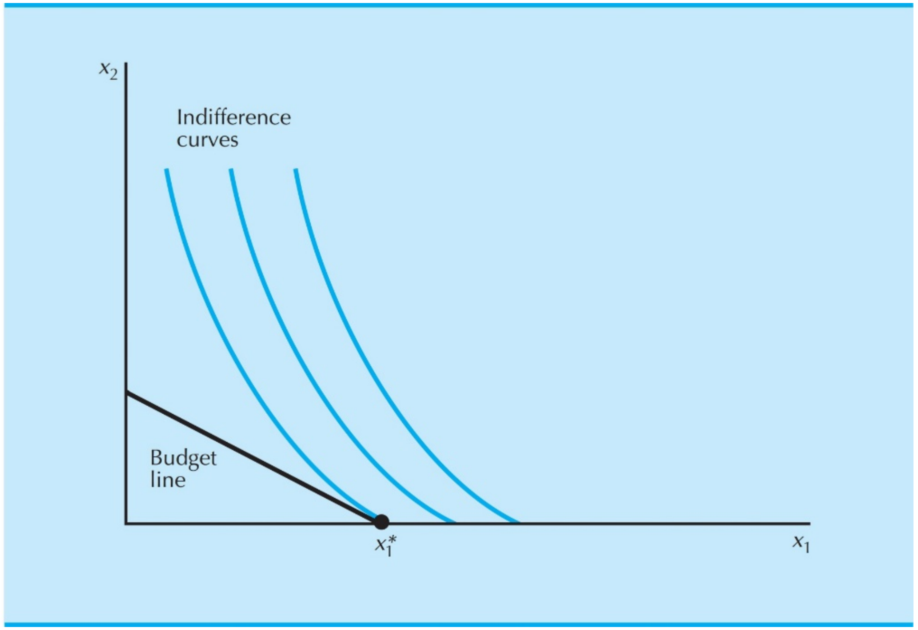


Figure 5.3

- No tangency since $MRS \neq \dfrac{p_1}{p_2}$ at (x_1^*, x_2^*)

Optimal Choice

- By ruling out the kinky case (non-differentiable case),
- Necessary condition of the optimal choice: *If the optimal choice is an interior point, then necessarily the I.C. will be tangent to the budget line*

- *Sufficiency?*

Optimal Choice

- No convex case

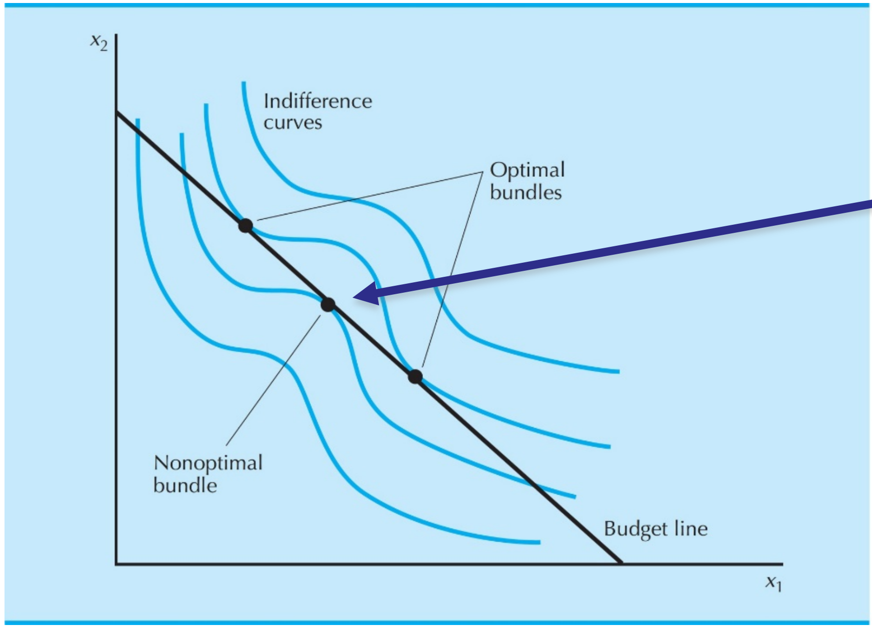


Figure 5.4

Tangent but not optimal !

- In general, the tangency condition is only a necessary condition for optimality, not a sufficient one

Optimal Choice

- However, the convex preference is the case where the tangency condition is sufficient
- Uniqueness?
- If the I.C.s are strictly convex, then there will be only one optimal choice on each budget line

Optimal Choice

- Economic meaning of tangency condition

$$MRS = \frac{dx_2}{dx_1} = \frac{MU_1}{MU_2} = \frac{p_1}{p_2} \qquad \text{at } (x_1^*, x_2^*)$$

  - MRS = the rate of change at which the consumer is just willing to substitute
  - p_1/p_2 = the rate of change the consumer can do in the market
  - If MRS> $p_1/p_2 \rightarrow p_2dx_2 > p_1dx_1 \rightarrow$ Buy x_1 more!
    and vice versa
  - Thus at MRS = p_1/p_2, there will be no more exchange
  - Consumer equilibrium condition

Utility maximization & Demand function

- Utility maximization problem

$$\begin{aligned} &Max \quad u(x) \\ &s.t. \quad p \cdot x \le m \\ &\qquad\quad x \in X, p \in R_+^n \end{aligned}$$

- Demand function
  - the solution of ‘Utility maximization problem’
  - The function that relates the optimal choice to the different values of prices and income

$$x_j^*(p_1, \ldots, p_n, m) \qquad \text{for } j = 1, \ldots, n$$

Utility maximization & Demand function

- Two-good case with equality constraint

$$\max_{x_1, x_2} U(x_1, x_2)$$

$$s.t.\ p_1 x_1 + p_2 x_2 = m$$

  - Lagrangian function

$$L = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m)$$

  - First-order conditions (F.O.C.)

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = \dfrac{\partial u(x_1^*, x_2^*)}{\partial x_1} - \lambda p_1 = 0 & (1) \\ \dfrac{\partial L}{\partial x_2} = \dfrac{\partial u(x_1^*, x_2^*)}{\partial x_2} - \lambda p_2 = 0 & (2) \\ \dfrac{\partial L}{\partial \lambda} = m - p_1 x_1^* - p_2 x_2^* = 0 & (3) \end{cases} \Rightarrow \begin{cases} x_1^* = x_1(p_1, p_2, m) \\ x_2^* = x_2(p_1, p_2, m) \end{cases}$$

  - Optimal choice: demand function

Utility maximization & Demand function

- Consumer equilibrium condition
  - By Eq. (1) & (2),

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

$$\therefore \frac{MU_1}{MU_2} = \frac{p_1}{p_2} = MRS$$

Utility maximization & Demand function

- Second-order (sufficient) condition
  - Bordered Hessian matrix should be negative definite (ND) (positive definite (PD) when min. problem)
  - Bordered Hessian: matrix of second derivatives of the Lagrangian

$$\bar{\mathbf{H}} = \mathbf{D}^2 L(\lambda, x_1, x_2) = \begin{pmatrix} \frac{\partial^2 L}{\partial \lambda^2} & \frac{\partial^2 L}{\partial \lambda \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_2} \\ \frac{\partial^2 L}{\partial x_1 \partial \lambda} & \frac{\partial^2 L}{\partial x_1{}^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial \lambda} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2{}^2} \end{pmatrix} = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{pmatrix}$$

Utility maximization & Demand function

- ND: naturally ordered principal minors must alternate in sign starting from (-) to (+) to (-) …..

$$\det(\bar{\mathbf{H}}) = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{21} & U_{22} \end{vmatrix} > 0$$

- PD: naturally ordered principal minors must have the same sign of $(-1)^k$, where k is the number of constraints

Examples: Cobb-Douglas

$$u(x_1, x_2) = x_1^c x_2^d$$

- By monotonic transformation,

$$\ln u(x_1, x_2) = c \ln x_1 + d \ln x_2$$

- Utility max. problem; $\begin{array}{ll} \max & c \ln x_1 + d \ln x_2 \\ s.t. & p_1 x_1 + p_2 x_2 = m \end{array}$

- Lagrangian;

$$L = c \ln x_1 + d \ln x_2 - \lambda(p_1 x_1 + p_2 x_2 - m)$$

- F.O.C. $\begin{cases} \dfrac{\partial L}{\partial x_1} = \dfrac{c}{x_1} - \lambda p_1 = 0 & (1) \\ \dfrac{\partial L}{\partial x_2} = \dfrac{d}{x_2} - \lambda p_2 = 0 & (2) \\ \dfrac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0 & (3) \end{cases}$

Examples: Cobb-Douglas

- Demand function

$$\begin{cases} x_1^*(p_1, p_2, m) = \dfrac{c}{c+d} \cdot \dfrac{m}{p_1} \\ x_2^*(p_1, p_2, m) = \dfrac{d}{c+d} \cdot \dfrac{m}{p_2} \end{cases}$$

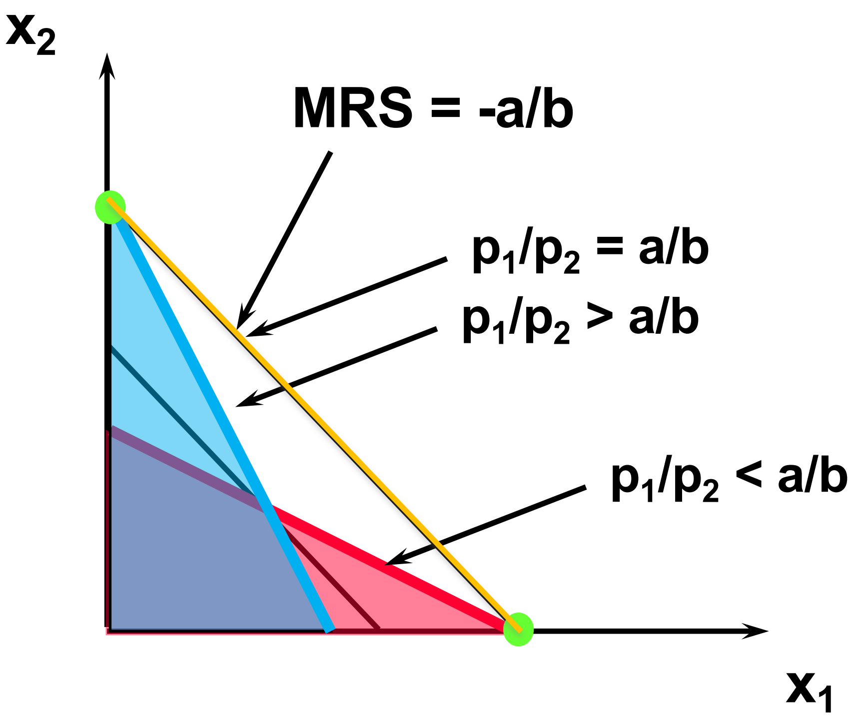
- To check S.O.C.

$$\bar{\mathbf{H}} = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -c/x_1^2 & 0 \\ -p_2 & 0 & -d/x_2^2 \end{pmatrix} \Rightarrow \left|\bar{\mathbf{H}}\right| = c(p_2/x_1)^2 + d(p_1/x_2)^2 > 0$$

ND !!

Examples: Perfect substitutes

$$u(x_1, x_2) = ax_1 + bx_2$$

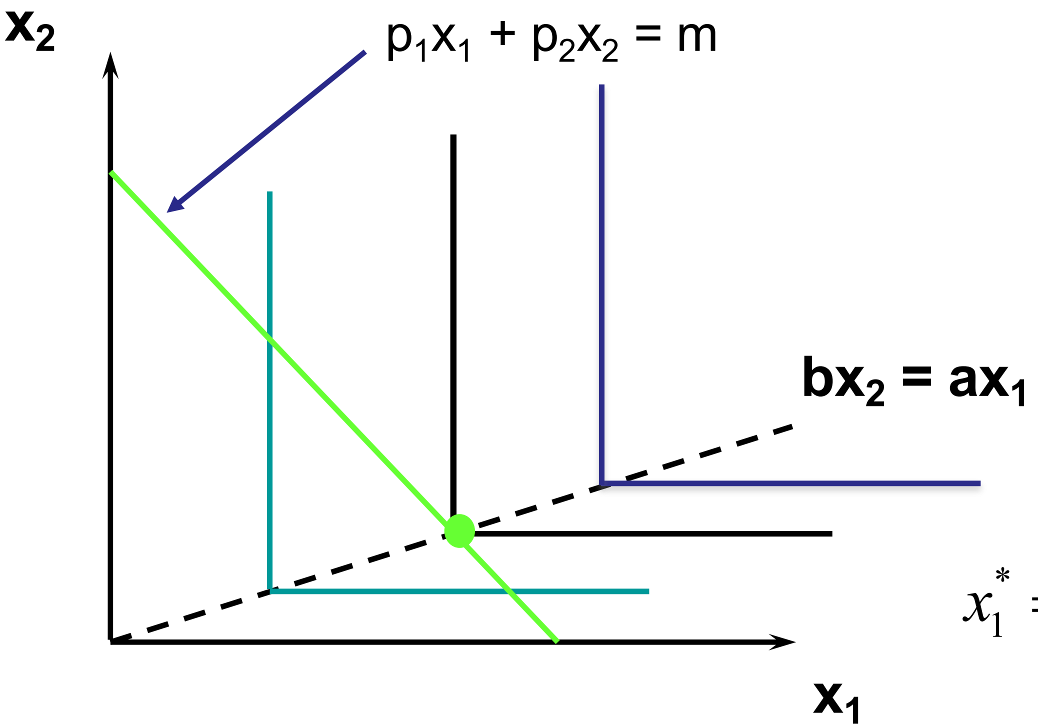


$$\begin{cases} x_1^* = \dfrac{m}{p_1}, \quad x_2^* = 0 & \text{if } \dfrac{a}{b} > \dfrac{p_1}{p_2} \\ x_1^* = 0, \quad x_2^* = \dfrac{m}{p_2} & \text{if } \dfrac{a}{b} < \dfrac{p_1}{p_2} \\ p_1 x_1^* + p_2 x_2^* = m & \text{if } \dfrac{a}{b} = \dfrac{p_1}{p_2} \end{cases}$$

- Boundary solution case

Examples: Perfect complements

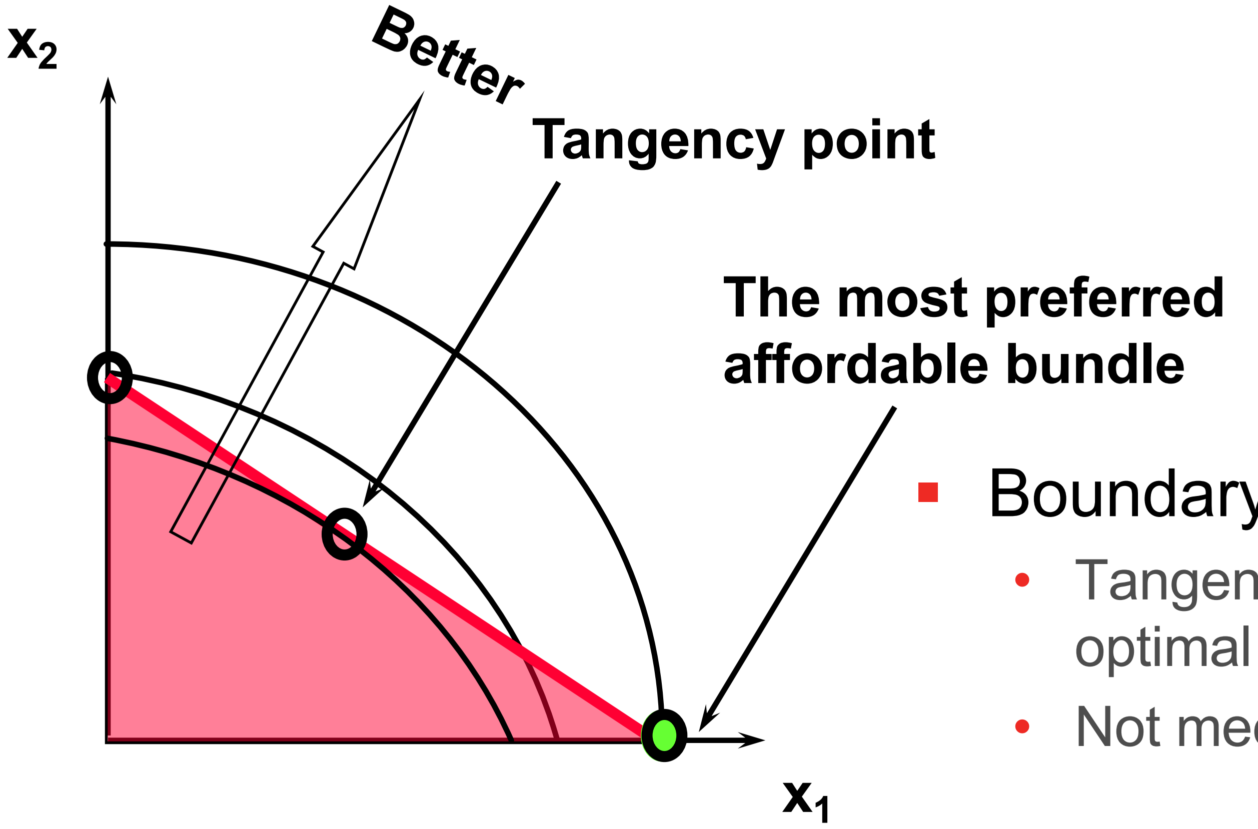
$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$



$$x_1^* = \frac{mb}{bp_1 + ap_2}, \quad x_2^* = \frac{ma}{bp_1 + ap_2}$$

Examples: Concave preference

$$u(x_1, x_2) = x_1^{\ 2} + x_2^{\ 2}$$



- Boundary solution case
  - Tangency point is not optimal
  - Not meet S.O.C.

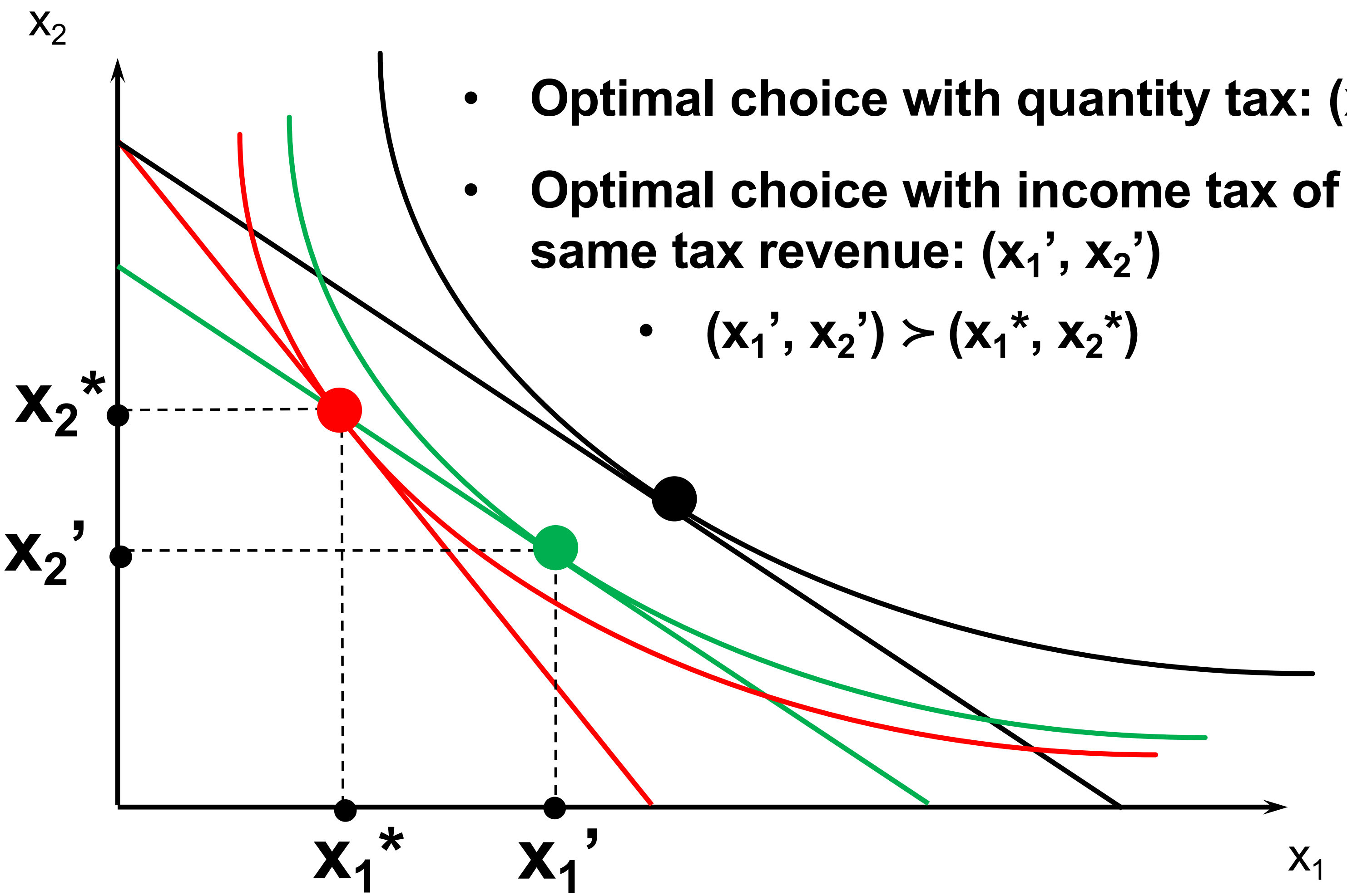
Choosing taxes

- If the government wants to raise a certain amount of revenue, is it better to raise it via quantity tax or an income tax?

- Imposition of quantity tax on good 1 with a rate t
  - Budget constraint changes with price increase from p_1 to (p_1 +t)
  - Let (x_1^*, x_2^*) be the optimal choice under the new budget set
  - Then we know that $(p_1+t)x_1^* + p_1x_2^* = m$ and tax revenue=tx_1^*

- Imposition of income tax which raises the same amount of tax revenue
  - Budget constraint changes with income decrease from m to m-tx_1^*

Choosing taxes



- **Optimal choice with quantity tax: (x_1^*, x_2^*)**
- **Optimal choice with income tax of the same tax revenue: (x_1', x_2')**
  - **$(x_1', x_2') \succ (x_1^*, x_2^*)$**

- Income tax is superior to the quantity tax !

Indirect utility function/ Expenditure function

- Local non-satiation preference

$$\text{Given any } x \text{ in } X \text{ and any } \varepsilon > 0,$$

$$\text{then there is some bundle } y \text{ in } X \text{ with } |x - y| < \varepsilon \text{ such that } y \quad x$$

  - Under the local non-satiation assumption, a utility-max. bundle must meet the budget constraint with equality.

- Utility maximization problem

$$\begin{aligned} &Max \quad u(x) \\ &s.t. \quad\ \ p \cdot x = m \\ &\qquad\quad\ \ x \in X, p \in R_+^n \end{aligned}$$

Indirect utility function/ Expenditure function

- Indirect utility function
  - The max. utility achievable at given prices and income

$$v(p,m) = Max \;\; u(x)$$
$$s.t. \quad p \cdot x = m$$

- Expenditure function
  - Inverse of indirect utility function w.r.t. income *m*

$$m = e(p,u)$$

  - the minimal amount of income necessary to achieve utility u at p

$$e(p,u) = \min p \cdot x$$
$$s.t. \quad u(x) \geq u$$

Hicksian demand function

- Hicksian demand function: $h_i(p,u)$
  - Expenditure-minimizing bundle necessary to achieve utility level *u* at prices p

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

Proof) Let h^* be a expenditure-minimizing bundle that gives utility u at prices p^*.
Then define the function,

$$g(p) = e(p,u) - p \cdot h^*$$

Since $e(p,u)$ is the cheapest way to achieve u, this function is always nonpositive.
At $p = p^*$, $g(p^*) = 0$. Since this is a maximum value of $g(p)$, its derivative must be zero by F.O.C.:

$$\frac{\partial g(p^*)}{\partial p_i} = \frac{\partial e(p^*,u)}{\partial p_i} - h_i^* = 0 \qquad i = 1,\ldots,n$$

Note that $x_i(p,m)$: Marshallian demand function

Some important identities

- Utility max.

$$\max_x u(x) \quad s.t.\ p \cdot x = m \;\Rightarrow\; x_i^*(p,m): \text{Marshallian demand function} \;\Rightarrow\; v(p,m) = u$$

- Expenditure min.

$$\min\ p \cdot x \quad s.t.\ u(x) \geq u \;\Rightarrow\; h_i^*(p,u): \text{Hicksian demand function} \;\Rightarrow\; e(p,u) = m$$

Some important identities

(1) $e(p, v(p,m)) \equiv m$

- the min expendicture necessary to reach utility $v(\tilde{p}, m)$ is m

(2) $v(p, e(p,m)) \equiv u$

- the max utility from income $e(\tilde{p}, u)$ is u

(3) $x_i(p,m) \equiv h_i(p, v(p,m))$

- the Marshallian demand at income m is the same as the Hicksian demand at utility $v(\tilde{p}, m)$

(4) $h_i(p,u) \equiv x_i(p, e(p,u))$

- the Hicksian demand at utility u is the same as the Marshallian demand at income $e(\tilde{p}, u)$

Roy's identity

- Utility max.

$$\max_x u(x) \quad s.t.\ p \cdot x = m$$

$x_i^*(p,m)$: Marshallian demand function

$v(p,m) = u$

Roy's identity

Inverse

- Expenditure min.

$$\min p \cdot x \quad s.t. u(x) \geq u$$

$h_i^*(p,u)$: Hicksian demand function

$e(p,u) = m$

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

Roy's identity

- Roy's identity

$$x_i\left(p,m\right) = -\frac{\partial v\left(p,m\right)/\partial p_i}{\partial v\left(p,m\right)/\partial m} \quad \text{when } p_i > 0, m > 0$$

  - Proof

The indirect utility funcion is given by $v\left(p,m\right) \equiv u\left(x\left(p,m\right)\right)$, where $x = (x_1,...,x_n)$

If we differnetiate this w.r.t p_j, we find $\dfrac{\partial v\left(p,m\right)}{\partial p_j} = \sum_{i=1}^{n} \dfrac{\partial u(x)}{\partial x_i} \cdot \dfrac{\partial x_i}{\partial p_j}$

Since $x\left(p,m\right)$ satisfies F.O.C. for utility max such that $\dfrac{\partial u(x)}{\partial x_i} - \lambda p_i = 0$,

$$\frac{\partial v\left(p,m\right)}{\partial p_j} = \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial p_j} \qquad (1)$$

And also $x\left(p,m\right)$ satisfies the budget constraint, $p \cdot x\left(p,m\right) \equiv m$

Differentiating this identity w.r.t. p_j gives $x_j\left(p,m\right) + \sum_{i=1}^{n} p_i \dfrac{\partial x_i}{\partial p_j} = 0 \qquad (2)$

Roy's identity

Substitute (2) into (1), $\frac{\partial v(p,m)}{\partial p_j} = -\lambda x_j(p,m)$

Now we differentiate $v(p,m) \equiv u\left(x_1(p,m),\ldots,x_n(p,m)\right)$ w.r.t. m to find

$$\frac{\partial v(p,m)}{\partial m} = \sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i} \cdot \frac{\partial x_i}{\partial m} = \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial m} \qquad (3)$$

Differnetiating $p \cdot x(p,m) \equiv m$ w.r.t. m, we have

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial m} = 1 \qquad (4)$$

Substituting (4) into (3) gives us

$$\frac{\partial v(p,m)}{\partial m} = \lambda$$

Finally, $x_j(p,m) = -\frac{\partial v(p,m)}{\partial p_j} / \lambda = -\frac{\partial v(p,m)/\partial p_j}{\partial v(p,m)/\partial m}$

Utility max. vs. Expenditure min.

- Utility max.

$$\max_x u(x) \quad s.t.\ p \cdot x = m \;\Rightarrow\; x_i^*(p,m): \text{Marshallian demand function} \;\Rightarrow\; v(p,m) = u$$

$$x_i(p,m) = -\frac{\partial v(p,m)/\partial p_i}{\partial v(p,m)/\partial m}$$

Inverse

- Expenditure min.

$$\min p \cdot x \quad s.t.\ u(x) \geq u \;\Rightarrow\; h_i^*(p,u): \text{Hicksian demand function} \;\Rightarrow\; e(p,u) = m$$

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

Examples

- Cobb-Douglas utility