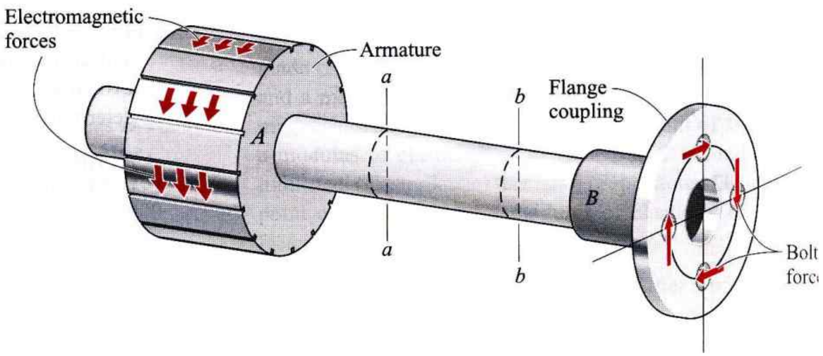
6. Shafts: Torsional Loading

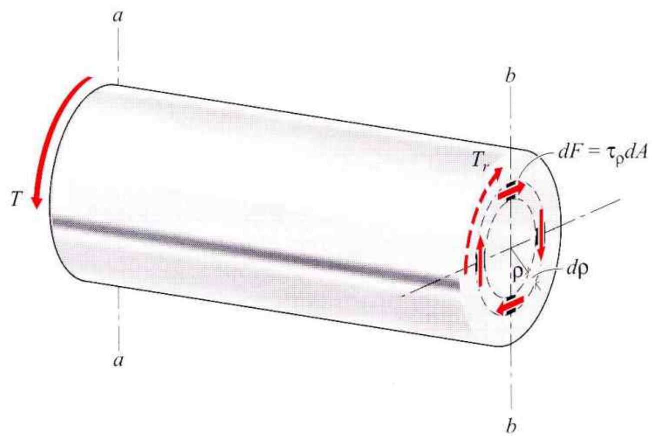
6.1 Introduction

• Torque and shear stress in a shaft cross-section

- A statically indeterminate problem can be solved by considering geometrical information of shaft deformation.

$$T_r = \int_{area} \rho \, dF = \int_{area} \rho \, \tau_\rho \, dA$$

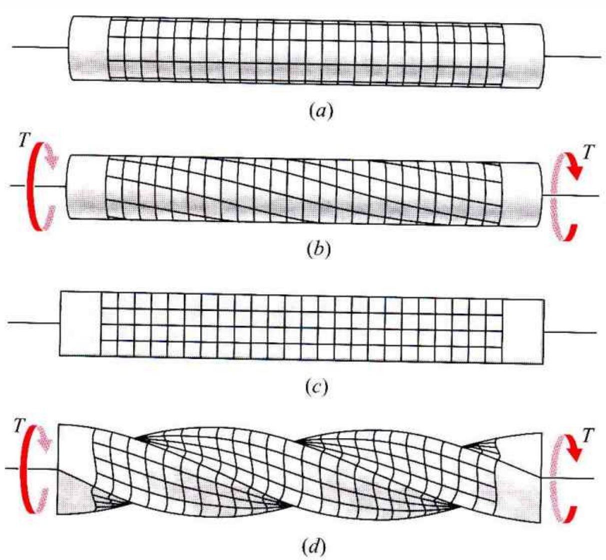




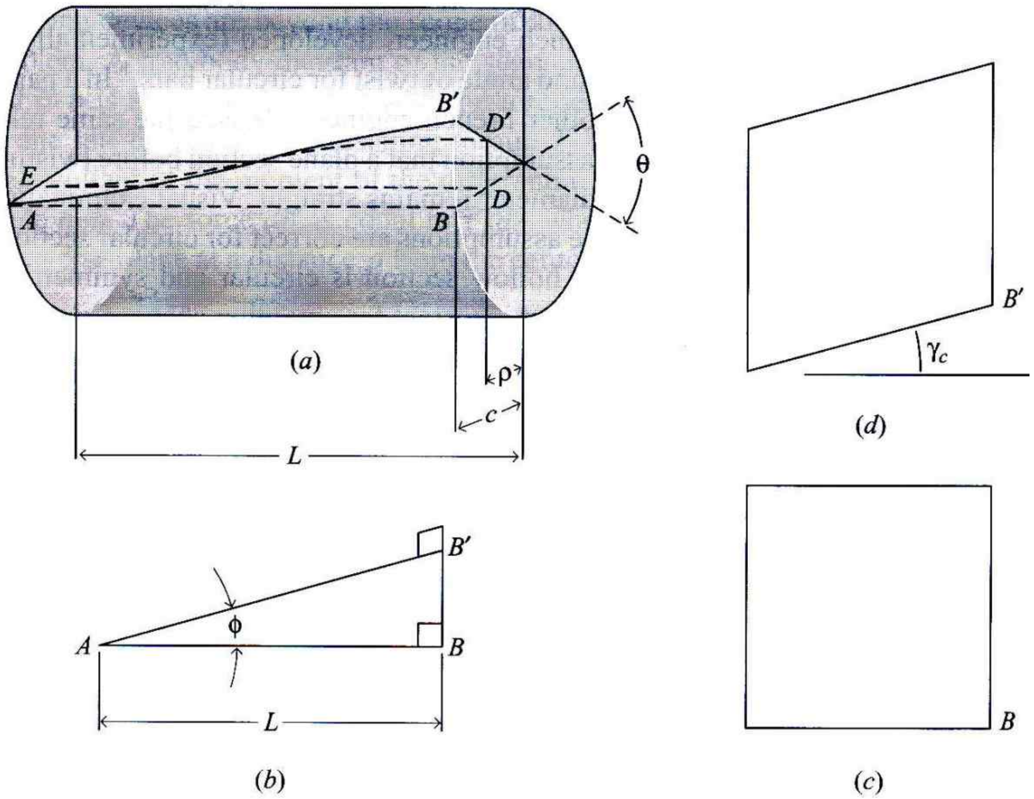
6.2 Torsional shearing stress

• Geometrical feature of a circular shaft twisted by torque

- A plane section before twisting remains plane after twisting.
- A diameter remains straight after the twisting.



6.2 Torsional shearing stress



$$\tan\gamma_c = \frac{BB'}{AB} = \frac{c\theta}{L} \approx \gamma_c$$

$$\tan\gamma_\rho = \frac{DD'}{ED} = \frac{\rho\theta}{L} \approx \gamma_\rho$$

$$\theta = \frac{\gamma_c L}{c} = \frac{\gamma_\rho L}{\rho}$$

$$\rightarrow \gamma_\rho = \frac{\gamma_c}{c}\rho$$

: valid for elastic or inelastic and for homogeneous or heterogeneous materials

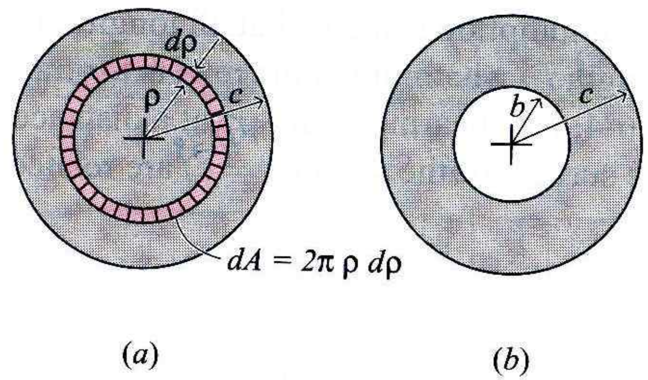
6.3 Torsional shearing stress-The elastic torsion formula

- Applied for the cases below the proportional limit of the material

$$\gamma_\rho = \frac{\gamma_c}{c}\rho \rightarrow \tau_\rho = \frac{\tau_c}{c}\rho$$

$$T_r = \int \rho \tau_\rho dA = \frac{\tau_c}{c}\int \rho^2 dA = \frac{\tau_\rho}{\rho}\int \rho^2 dA$$

- *Polar* second *moment of area*, J



For a circular shaft: $J = \int \rho^2 dA = \int_0^c \rho^2 \left(2\pi\rho \, d\rho\right) = \dfrac{\pi c^4}{2}$

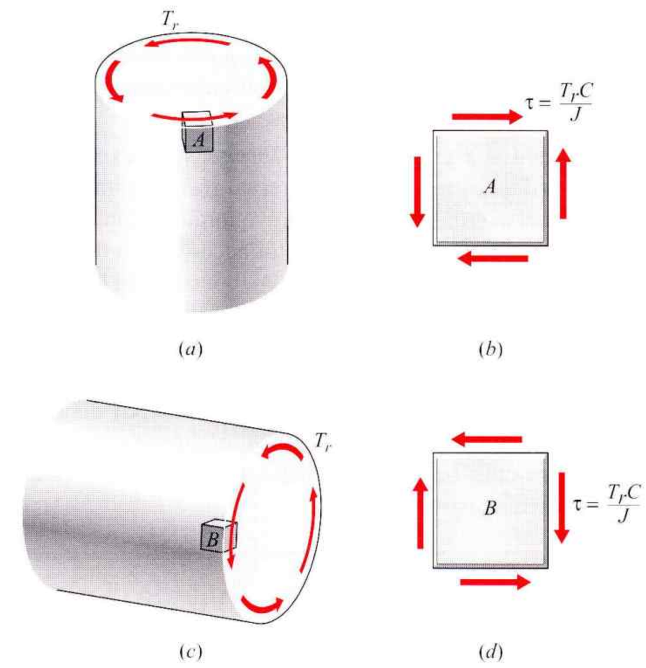
For a circular annulus: $J = \int \rho^2 dA = \int_{r_i}^{r_o} \rho^2 \left(2\pi\rho \, d\rho\right) = \dfrac{\pi}{2}\left(r_o^{\,4} - r_i^{\,4}\right)$

$$T_r = \frac{\tau_c J}{c} = \frac{\tau_\rho J}{\rho}$$

$$\tau_\rho = \frac{T_r \rho}{J} \quad and \quad \tau_c = \frac{T_r c}{J} \quad : \quad elastic\ torsion\ formula$$

6.4 Torsional Displacements

$$\theta = \frac{\gamma_\rho L}{\rho} = \frac{\tau_\rho L}{G\rho} = \frac{T_r L}{GJ} \quad \rightarrow \quad \theta = \sum_{i=1}^{n} \frac{T_{ri} L_i}{G_i J_i} \quad \text{or} \quad \theta = \int_0^L \frac{T_r}{GJ} dx$$



- Sign convention

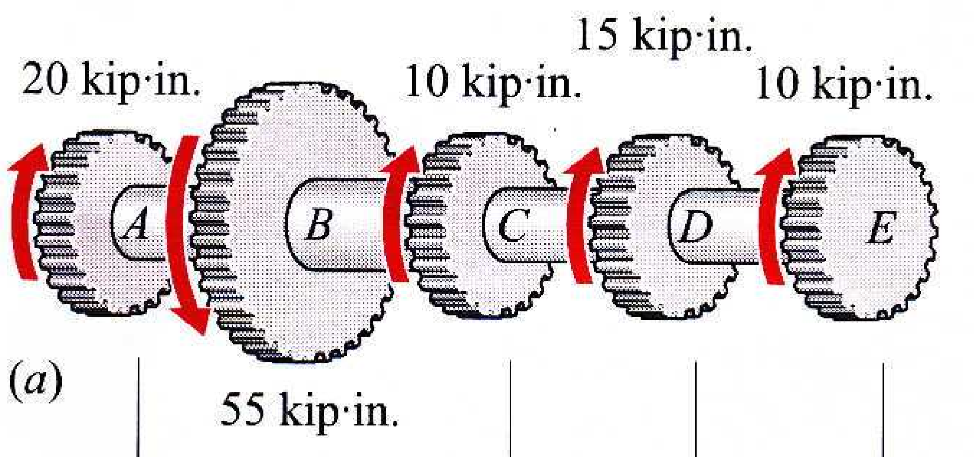
The internal torque and the angle of twist are considered positive when the vectors representing them point outward from the internal section.

c.f. sign convention of shear stress

- Example Problem 6-1

The torque is input at gear B and is removed at gears A, C, D, and E.

- Torques in intervals AB, BC, CD, and DE of the shaft



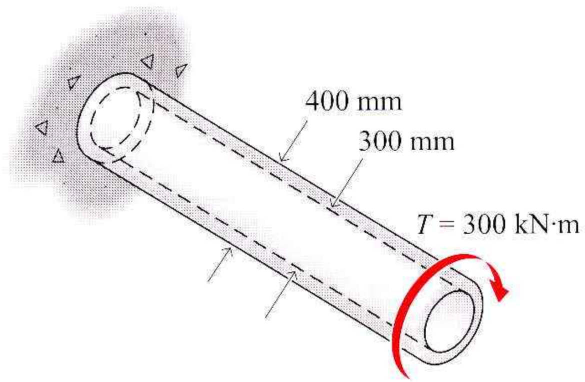
- Example Problem 6-2

A hollow shaft whose outside and inside diameters are 400 mm and 300 mm, respectively is subjected to a torque of 300 kNm. The shear modulus of the shaft is 80 GPa.

- The max. shearing stress in the shaft

- The shearing stress on a cross section at the inside surface of the shaft

- The magnitude of the angle of twist in a 2-m length



6.4 Torsional Displacements

- Example Problem 6-3

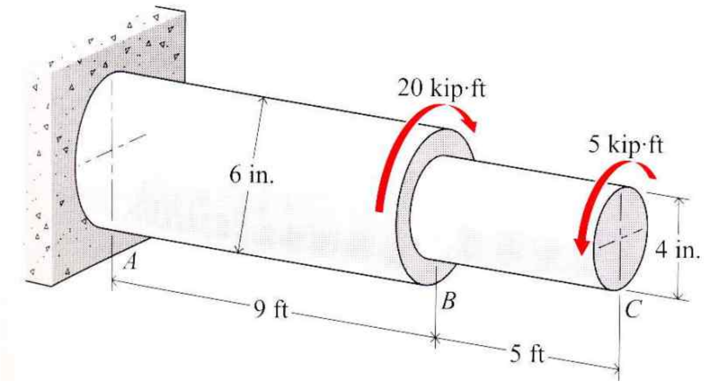
A solid steel shaft shown below is in equilibrium when subjected to the two torques. The shear modulus is 12,000 ksi.

- The max. shearing stress in the shaft

- The rotation of end B with respect to end A

- The rotation of end C with respect to end B

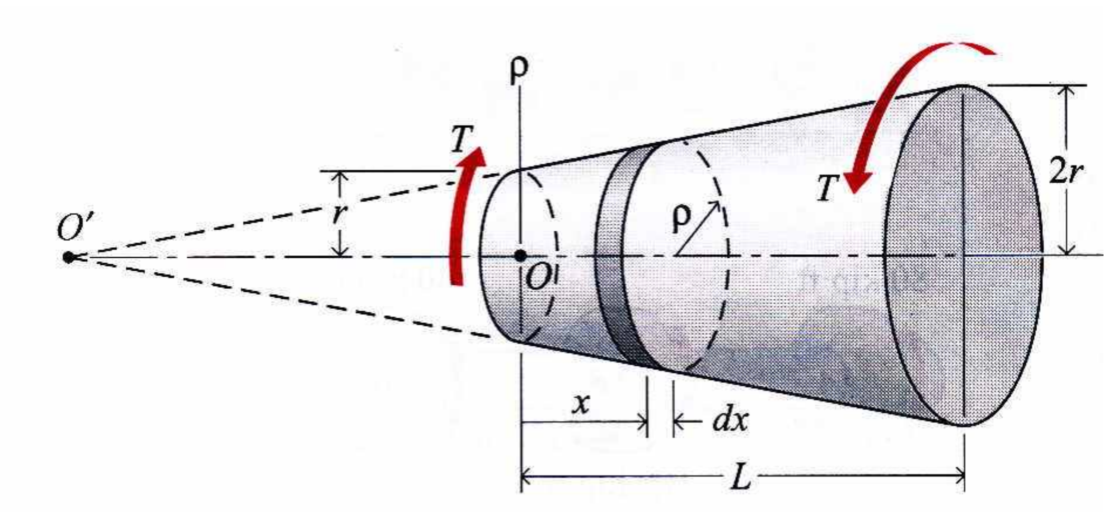
- The rotation of end C with respect to end A



• Example Problem 6-5

A solid steel circular tapered shaft is subjected to end torques in transverse planes.

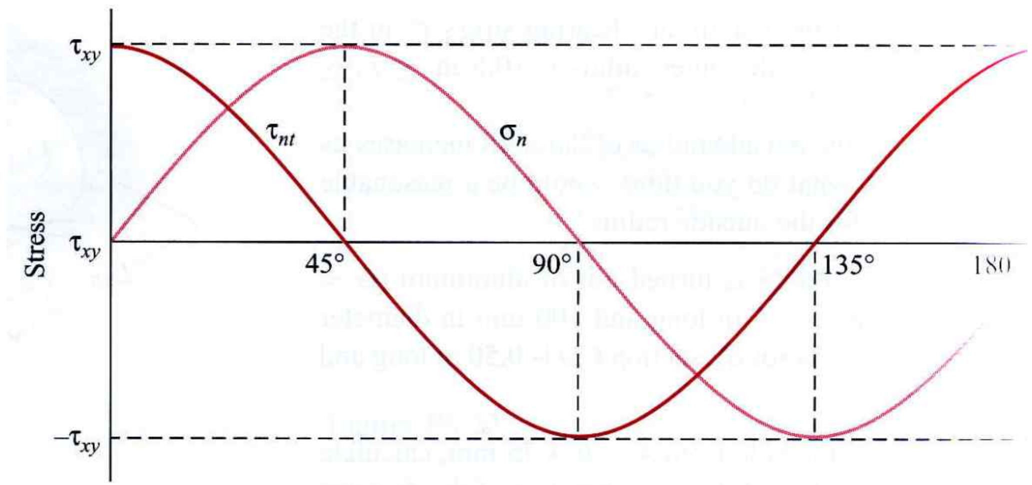
- The magnitude of the twisting angle in terms of T, L, G, and r

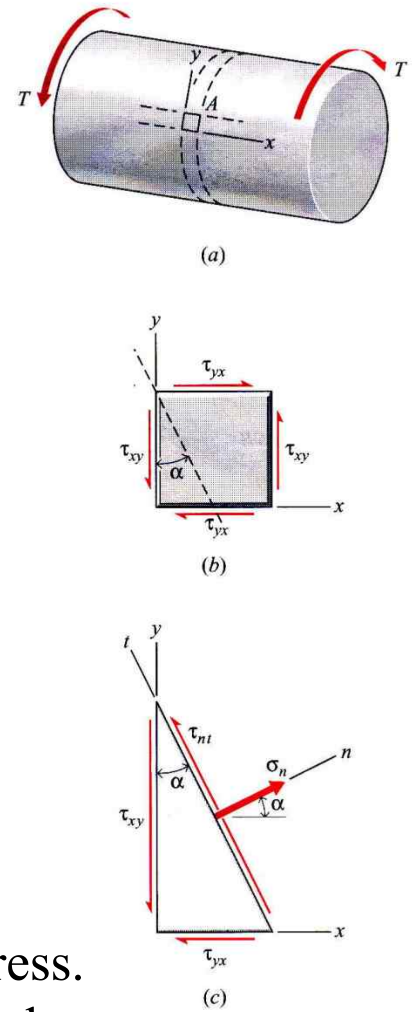


6.5 Stresses on oblique planes

$$\sigma_n = \tau_{xy} \sin 2\alpha \quad \left(\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \right)$$

$$\tau_{nt} = \tau_{xy} \cos 2\alpha \quad \left(\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \right)$$

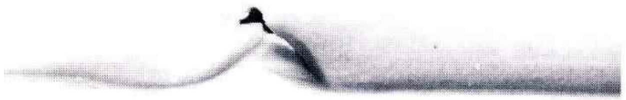




- The max. normal stress is 45˚ apart from the max. shear stress.
- The max. normal and shear stresses have the same magnitude.

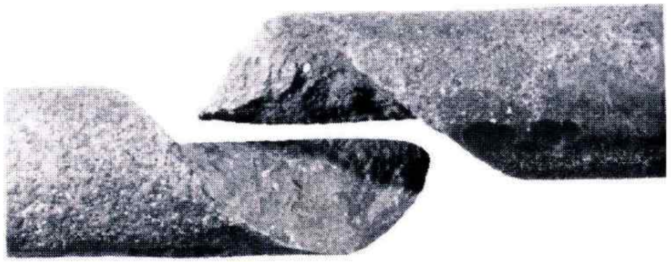
6.5 Stresses on oblique planes

(a) Steel rear axle of a truck: longitudinal split



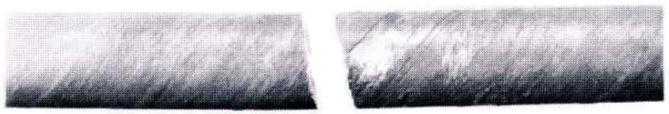
(b)

Aluminum alloy tube (brittle): compressive failure



(c)

Gray cast iron bar (brittle): tensile failure



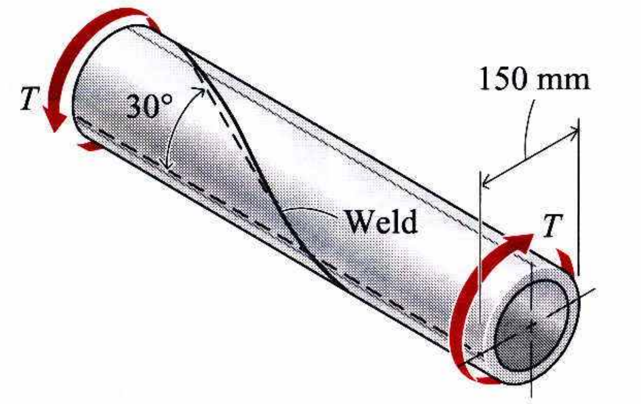
(d)

Low-carbon steel bar (ductile): shear failure

- Example Problem 6-6

A cylindrical tube is fabricated by butt-welding a 6-mm-thick steel plate. If the compressive strength of the tube is 80 MPa, determine

- The max. torque *T* that can be applied to the tube
- The factor of safety when 12 kNm torque is applied, if ultimate shear and tension strength are 205 MPa and 345 MPa, respectively.



6.6 Power transmission

$$Work : W_k = T\phi$$

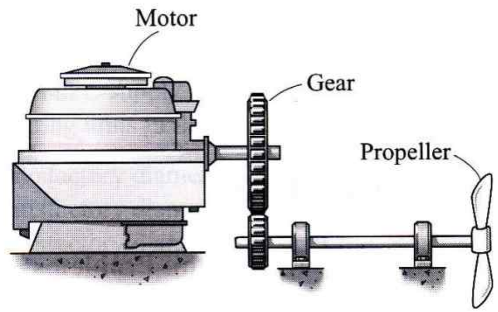
$$Power = \frac{dW_k}{dt} = T\frac{d\phi}{dt} = T\omega \quad (N \cdot m / \text{s}:\ watts)$$

$$1\ hp = 33,000\ lb \cdot ft / \text{min}$$

- Example Problem 6-7

A diesel engine for a boat operates at 200 rpm and delivers 800 hp through 4:1 gear to the propeller. The shaft is of heat-treated alloy steel whose modulus of rigidity is 12,000 ksi. The allowable shearing stress of the 10-ft – long shaft is 20 ksi and the angle of twist is not to exceed 4˚.

- The min. permissible diameters for the two shafts

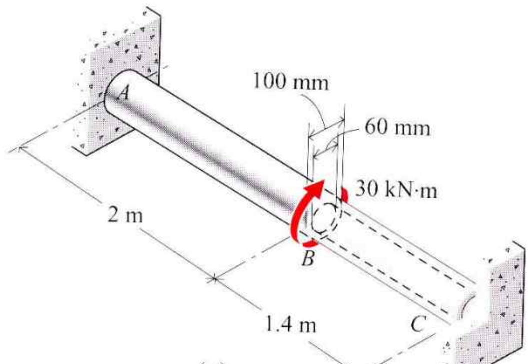


6.7 Statistically indeterminate members

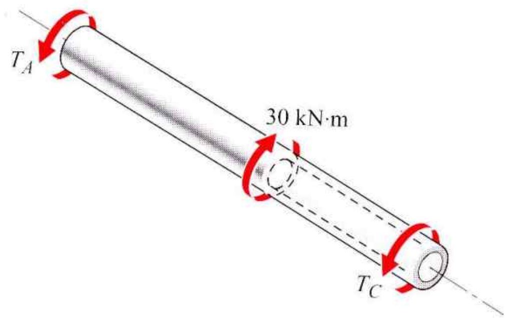
• Example Problem 6-8

The solid section AB is made of annealed bronze (G_{AB} = 45 GPa) and the hollow section BC is made of aluminum alloy (G_{BC} = 28 GPa). There is no stress in the shaft before the 30-kN·m torque is applied.

- The max. shearing stress in both the bronze and aluminum portion



(a)



(b)

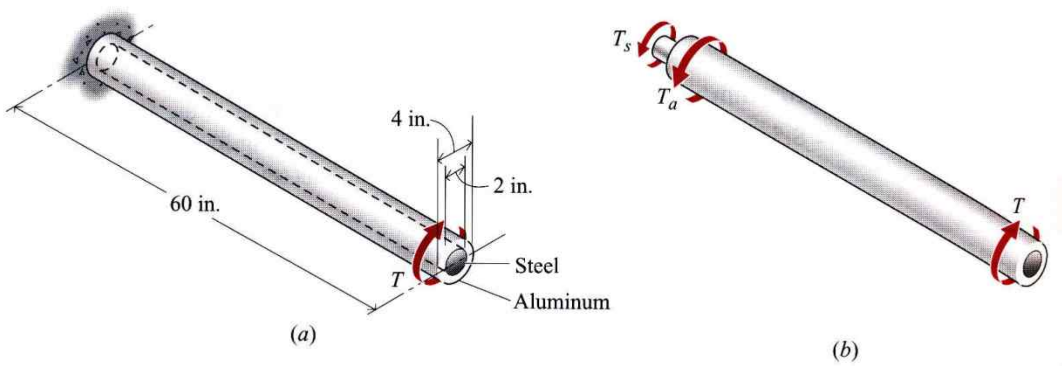
6.7 Statistically indeterminate members

- Example Problem 6-9

A hollow circular aluminum alloy (G_a = 4000 ksi) cylinder has a steel (G_s = 11,600 ksi) core. The allowable stresses in the steel and aluminum are 14 ksi and 10 ksi, respectively.

- The max. torque *T* that can be applied to the right end of the shaft

- The magnitude of the rotation of the right end

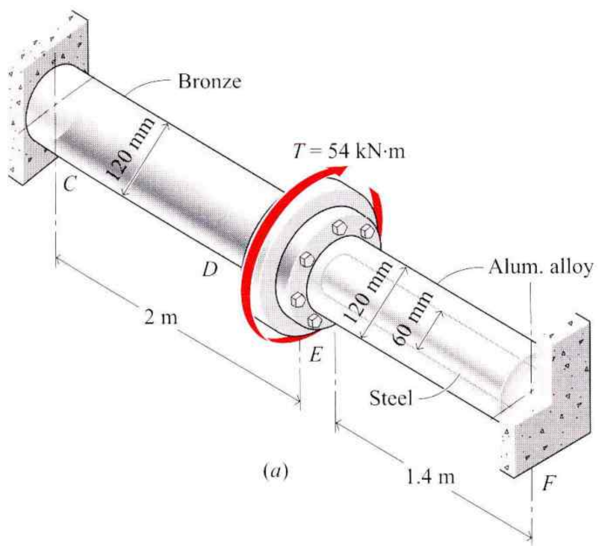


6.7 Statistically indeterminate members

- Example Problem 6-10

An torsional assembly consists of a solid bronze shaft *CD* (G_B = 45 GPa), a hollow aluminum alloy *EF* (G_A = 28 GPa) and its steel core (G_S = 80 GPa). The bolt clearance permits flange D to rotate through 0.03 rad before EF carries any of the load.

- The max. shearing stress in each shaft materials when torque T = 54 kN·m



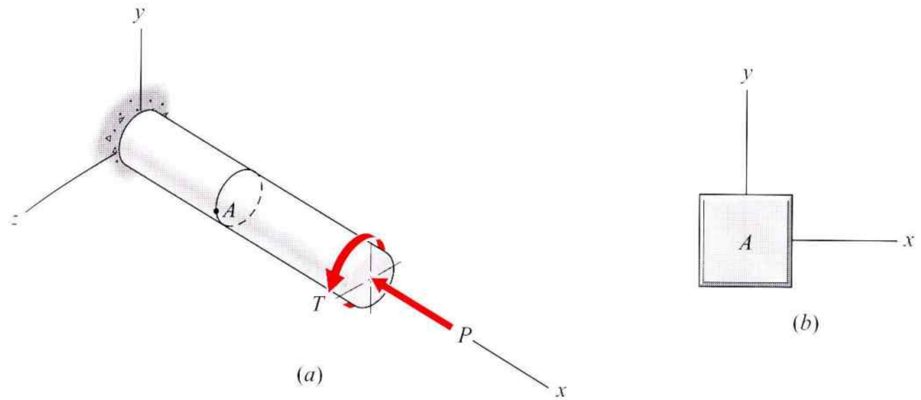
6.8 Combined loading-axial, torsional, and pressure vessel

As long as the strains are small, the stresses can be computed separately and superimposed on an element subjected to a combined loading.

- Example Problem 6-11

The solid 100-mm-diameter shaft is subjected to an axial compressive force P = 200 kN and a torque T = 30 kN·m. For point A, determine follows.

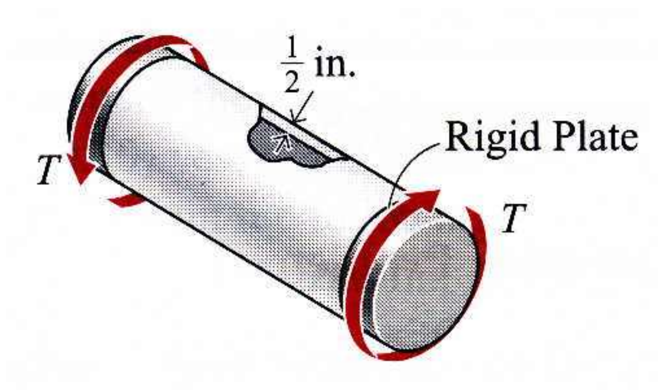
- The x- and y-components of stress
- The principal stresses and the max. shearing stress



- Example Problem 6-12

The thin walled cylindrical pressure vessel whose inside diameter is 24 in. and thickness of a wall is of 0.5 in. The internal pressure is 250 psi and the torque is 150 kip·ft.

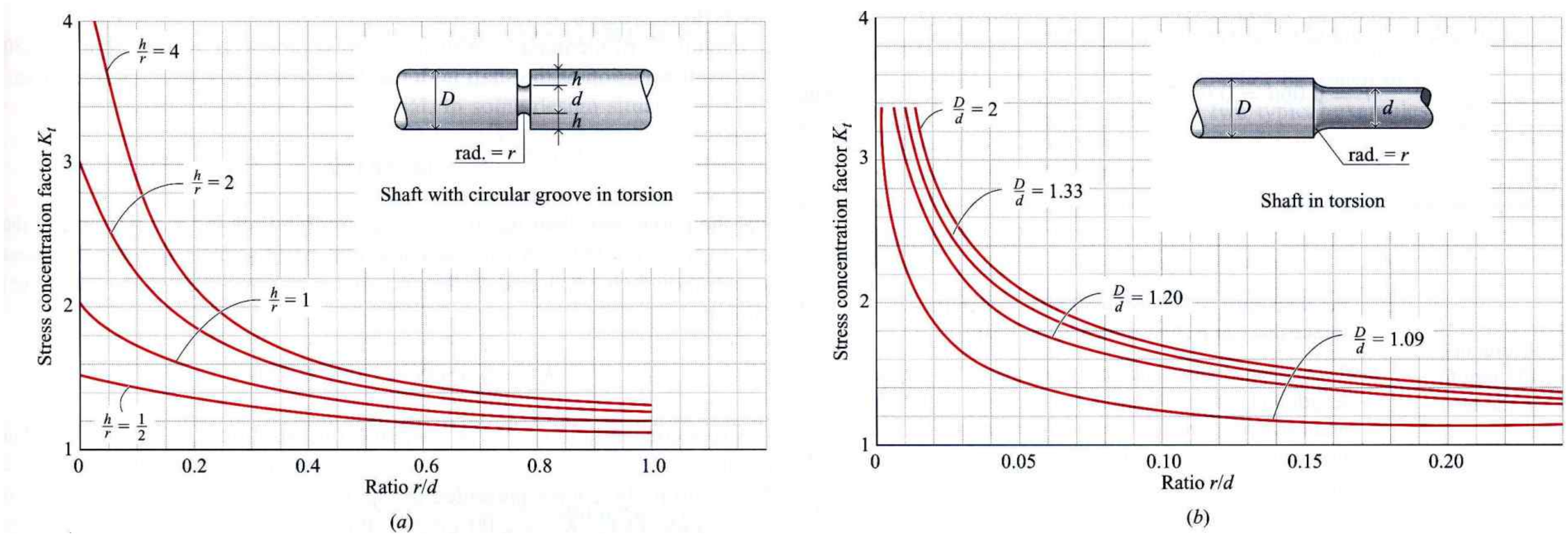
- The max. normal and shearing stresses at a point outside surface



6.9 Stress concentrations in circular shafts under torsional loadings

The stress concentration in a shaft caused by an abrupt change in diameter can be reduced by using a fillet between the parts.
The max. shearing stress in the fillet can be expressed in terms of a *stress concentration factor* K as

$$\tau_{\max} = K_t \tau_c = K_t \frac{T_r c}{J}$$



• Example Problem 6-13

A stepped shaft has a 4-in. dia. for one-half of its length and a 2-in. diameter for the other half. The max. shearing stress should not be over 8 ksi when the shaft is subjected to a torque of 6280 lb·in.

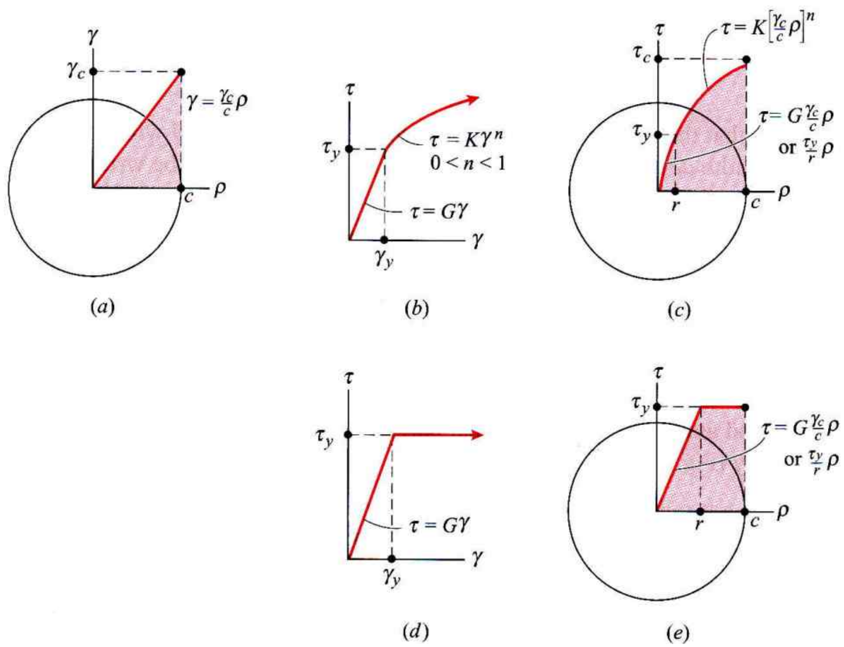
- The min. fillet radius needed at the junction between the two portions of the shaft.

6.10 Inelastic behavior of torsional members

Even with inelastic materials, a plane section remains plane and a diameter remains straight for circular sections under pure torsion, if the strains are not too large. Therefore following relationship can be applied to inelastic members.

$$\gamma_\rho = \frac{\gamma_c}{c}\rho$$

The shear stress-strain relationship of an inelastic member determines the shear stress at a certain distance from the axis.



6.10 Inelastic behavior of torsional members

- Example Problem 6-14

A solid circular steel shaft of 4-in. diameter is subjected to a pure torque of 7π kip·ft. The steel is elastoplastic, having a yield point τ_y in shear of 18 ksi and a modulus of rigidity G of 12,000 ksi.

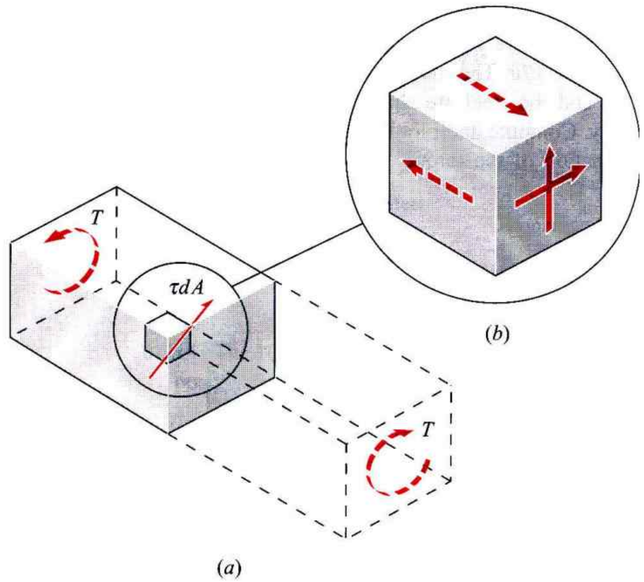
- The max. shearing stress and the magnitude of the angle of twist in a 10-ft length

6.11 Torsion of noncircular sections

- The shearing stress at the corners of the rectangular bar is zero.

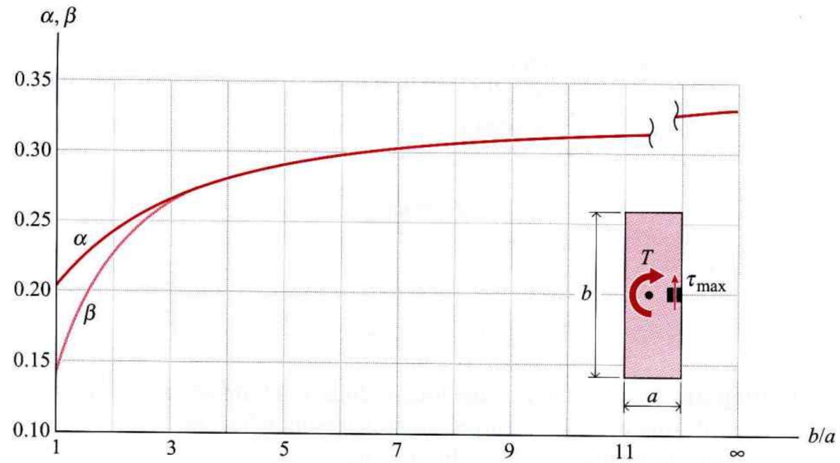
-The results of Saint Venant's analysis indicate that, in general, except for the circular cross sections, every section will warp (not remain plane) when the bar is twisted.

- The distortion of the small squares of a rectangular bar is maximum at the midpoint of a side of the cross section and disappears at the corners.



$$\tau_{\max} = \frac{T}{\alpha a^2 b}$$

$$\theta = \frac{TL}{\beta a^3 b G}$$

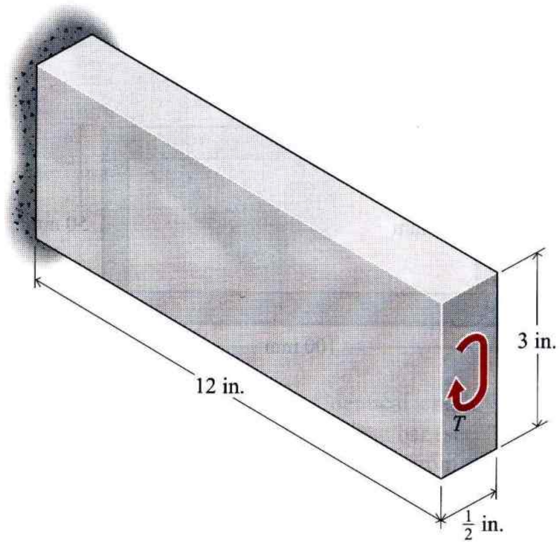


6.11 Torsion of noncircular sections

- Example Problem 6-16

The aluminum alloy (G=4000 ksi) bar is subjected to a torque T = 2500 lb·in.

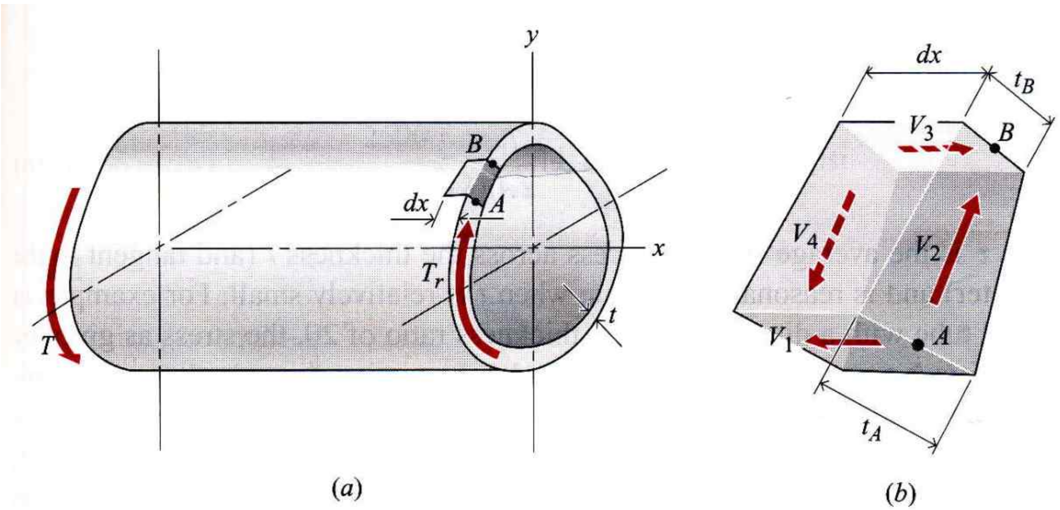
- The max. shearing stress and the angle of twist



6.12 Torsion of thin-walled tubes-shear flow

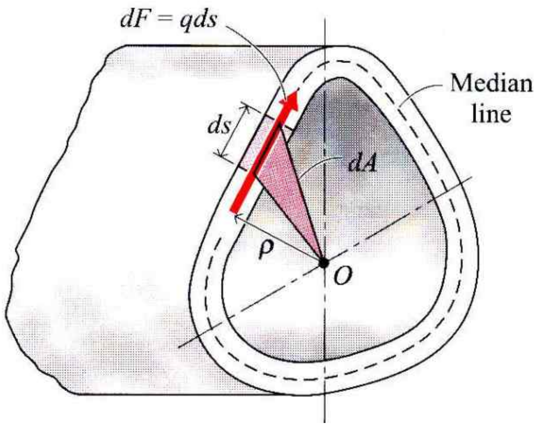
- The shear flow, q, is defined as the internal shearing force per unit length of a thin section: $q = \tau \cdot t$.

-The shear flow on a cross section is constant even though the thickness of the section wall varies.



$$V_1 = V_3 \rightarrow q_1 dx = q_3 dx$$
$$q_1 = q_3 \rightarrow \tau_1 t_A = \tau_3 t_B$$
$$\tau_1 = \tau_A \ , \quad \tau_3 = \tau_B$$
$$\rightarrow \tau_A t_A = \tau_B t_B$$
$$\therefore q_A = q_B$$

6.12 Torsion of thin-walled tubes-shear flow



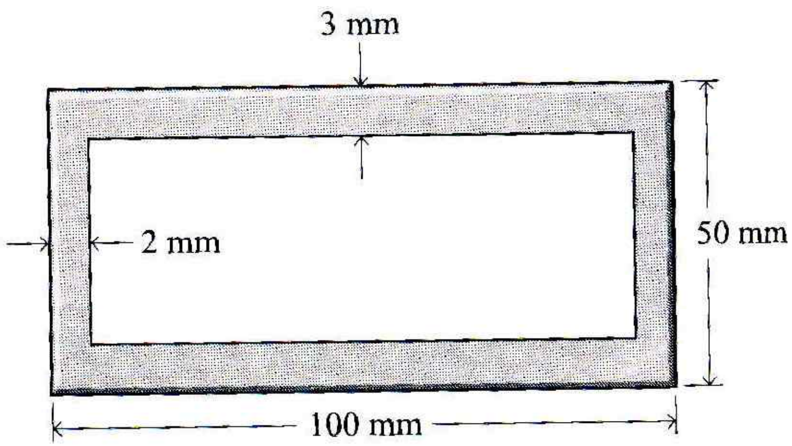
$$
\begin{aligned}
T_r &= \int \rho dF \\
&= \int \rho (q ds) \\
&= q \int \rho ds \\
&= q(2A)
\end{aligned}
$$

$$\tau = \frac{q}{t} = \frac{T_r}{2At}$$

- Example Problem 6-17

A rectangular box section of aluminum alloy has the thickness of 2 mm for 50-mm sides and 3 mm for 100-mm sides. If the max. shearing stress must be limited to 95 MPa,

- The max. torque that can be applied to the section. Neglect stress concentrations.



6.13 Design problems

- Example Problem 6-18

A solid circular shaft 4 ft long made of 2014-T4 wrought aluminum subjected to a torsional load of 10,000 lb·in. Failure is by yielding and a factor of safety (FS) of 2 is specified. The yield strength of 2014-T4 is 24 ksi.

- Suitable diameter for the shaft if 2014-T4 wrought aluminum bars are available with diameters in increments of 1/8 in.

6.13 Design problems

- Example Problem 6-19

A solid circular shaft 2 m long is to transmit 1000 kW at 600 rpm. Failure is by yielding and a factor of safety is 1.75. The shaft is made of structural steel of which yielding strength in shear is 125 MPa.

- Suitable diameter for the shaft if bars are available with diameters in increments of 10 mm.