Fusion Reactor Technology I

(459.760, 3 Credits)

Prof. Dr. Yong-Su Na

(32-206, Tel. 880-7204)

Contents

Contents

Plasmas as Fluids

- Plasma: electrically conducting fluid
- Introducing effects specific to an electrically charged current-carrying fluid, such as electric and magnetic forces

  - Continuity equation

$$\frac{\partial n}{\partial t} d^3x = [n\langle v_1\rangle dx_2 dx_3]_{x_1} - [n\langle v_1\rangle dx_2 dx_3]_{x_1+dx_1} + \cdots$$

$$\frac{\partial n}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(n\langle v_i\rangle)$$

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = S$$

u for the average velocity <**v**>
S: ionization for a source, recombination for a sink term

Plasmas as Fluids

- Momentum balance equation: rate of change of momentum density in a differential element of volume

$$\vec{F} = nq(\vec{E} + \vec{u} \times \vec{B})$$

$$\frac{\partial(nm\vec{u})}{\partial t} = \vec{F} = nq(\vec{E} + \vec{u} \times \vec{B})$$

Lorentz force extended to all the particles of a given species per unit volume: a local source rate of momentum density

$$\frac{\partial(nmu_2)}{\partial t} = -\frac{\partial}{\partial x_1}(mn\langle v_2 v_1\rangle) - \frac{\partial}{\partial x_2}(mn\langle v_2 v_2\rangle) - \frac{\partial}{\partial x_3}(mn\langle v_2 v_3\rangle)$$

The rate of change of x_2-directed momentum, averaged over all of the particles

$$\frac{\int mv_2 f d^3v d^3x}{dt} = \int mv_2 v_1 f d^3v dx_2 dx_1$$

Plasmas as Fluids

Pressure tensor: momentum flux

$$P_{ij} \equiv mn\left\langle (v_i - u_i)(v_j - u_j) \right\rangle = mn\left(\left\langle v_i v_j \right\rangle - u_i u_j \right)$$

i direction of *j*-directed momentum

$$= \begin{bmatrix} nT_\perp & 0 & 0 \\ 0 & nT_\perp & 0 \\ 0 & 0 & nT_\parallel \end{bmatrix}$$

Off-diagonal elements: viscosity

Magnetic field direction

$$\frac{\partial (nmu_2)}{\partial t} = -\frac{\partial P_{12}}{\partial x_1} - \frac{\partial P_{22}}{\partial x_2} - \frac{\partial P_{32}}{\partial x_3} - m\left(\frac{\partial (nu_1u_2)}{\partial x_1} + \frac{\partial (nu_2u_2)}{\partial x_2} + \frac{\partial (nu_3u_2)}{\partial x_3} \right)$$

$$\frac{\partial (nmu_j)}{\partial t} = -\sum_i \frac{\partial P_{ij}}{\partial x_i} - m\sum_i \frac{\partial}{\partial x_i}(nu_iu_j)$$

Plasmas as Fluids

Momentum balance equation

$$\frac{\partial(mn\vec{u})}{\partial t} = nq(\vec{E}+\vec{u}\times\vec{B}) - \nabla\cdot\overleftrightarrow{P} - \nabla\cdot(mn\vec{u}\vec{u})$$

$$m\nabla\cdot(n\vec{u}\vec{u}) = m\vec{u}(\nabla\cdot n\vec{u}) + mn(\vec{u}\cdot\nabla)\vec{u}$$

$$m\sum_i \frac{\partial}{\partial x_i}(nu_i u_j) = mu_j \sum_i \frac{\partial}{\partial x_i}(nu_i) + mn\sum_i u_i \frac{\partial u_j}{\partial x_i}$$

Combining with the continuity equation

$$mn\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) = nq(\vec{E}+\vec{u}\times\vec{B}) - \nabla\cdot\overleftrightarrow{P} - mS\vec{u}$$

$$mn\frac{d\vec{u}}{dt} = nq(\vec{E}+\vec{u}\times\vec{B}) - \nabla\cdot\overleftrightarrow{P}$$

In the case, where a plasma is nearly Maxwellian (or at least nearly isotropic), the pressure tensor term can be replaced by the gradient of a scalar pressure, ∇p

Plasmas as Fluids

- Two-fluid equations

The continuity equation will apply separately to each of the different species. The momentum balance equation must consider the fact that particles of one species can collide with particles of another species, thereby transferring momentum between the different species.

$$R_{\alpha\beta} = -m_\alpha n_\alpha \nu_{\alpha\beta}(\vec{u}_\alpha - \vec{u}_\beta)$$

The rate at which momentum per unit volume is gained by species α due to collisions with species β.

$\nu_{\alpha\beta}$: collision frequency of α on β

$$m_\alpha n_\alpha \left(\frac{\partial \vec{u}_\alpha}{\partial t} + (\vec{u}_\alpha \cdot \nabla)\vec{u}_\alpha \right) = n_\alpha q_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) - \nabla \cdot \overleftrightarrow{P}_\alpha + \sum_\beta R_{\alpha\beta}$$

$$R_{\beta\alpha} = -R_{\alpha\beta}$$

The rate at which momentum per unit volume is gained by species β due to collisions with species α.

$$m_\alpha n_\alpha \nu_{\alpha\beta} = m_\beta n_\beta \nu_{\beta\alpha}$$

Plasmas as Fluids

- Plasma resistivity

The acceleration of electrons by an electric field applied along (or in the absence of) a magnetic field is impeded by collisions with non-accelerated particles, in particular the ions, which, because of their much larger mass, are relatively unresponsive to the applied electric field. Collisions between electrons and ions, acting in this way to limit the current that can be driven by an electric field, give rise to an important plasma quantity, namely its electrical resistivity, η

$$R_{ei} = -m_e n_e \nu_{ei} (\vec{u}_e - \vec{u}_i) \qquad 0 = -n_e e\vec{E}_\parallel + R_{ei\parallel}$$

Homogeneous (neglecting the electron pressure and velocity gradients along B)

$$j_\parallel = -n_e e(\vec{u}_{e\parallel} - \vec{u}_{i\parallel})$$

$$E_\parallel = -\frac{m_e \nu_{ei}}{e}(\vec{u}_{e\parallel} - \vec{u}_{i\parallel}) = \frac{m_e \nu_{ei}}{n_e e^2} j_\parallel = \eta j_\parallel$$

Simplified Ohm's law

$$\eta = \frac{m_e \langle \nu_{ei} \rangle}{n_e e^2}$$

Momentum gained by electrons due to collisions with ions

$$R_{ei} = -m_e n_e \langle \nu_{ei} \rangle (\vec{u}_e - \vec{u}_i) = -\eta n_e^2 e^2 (\vec{u}_e - \vec{u}_i) = \eta n_e e\vec{j}$$

Plasmas as Fluids

- Single-fluid magnetohydrodynamics (MHDs)

A single-fluid model of a fully ionized plasma, in which the plasma is treated as a single hydrogynamic fluid acted upon by electric and magnetic forces.

The magnetohydrodynamic (MHD) equation

Hydrogen plasma, charge neutrality assumed

$$\rho = n_i M + n_e m \approx n(M+m) \approx nM$$ mass density

$$\sigma = (n_i - n_e)e$$ charge density

$$\vec{u} = (n_i M \vec{u}_i + n_e m \vec{u}_e)/\rho \approx (M\vec{u}_i + m\vec{u}_e)/(M+m) \approx \vec{u}_i + (m/M)\vec{u}_e$$ mass velocity

$$\vec{j} = e(n_i \vec{u}_i - n_e \vec{u}_e) \approx ne(\vec{u}_i - \vec{u}_e)$$

$$\vec{u}_i \approx \vec{u} + \frac{m}{M}\frac{\vec{j}}{ne}, \quad \vec{u}_e \approx \vec{u} - \frac{\vec{j}}{ne}$$

Plasmas as Fluids

- The magnetohydrodynamic (MHD) equation

$$\frac{\partial n_{i,e}}{\partial t} + \nabla \cdot (n_{i,e}\vec{u}_{i,e}) = 0$$

multiplied by *M* and *m*, respectively and added together

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Mass continuity equation

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot j = 0$$

Charge continuity equation

$$Mn_i \frac{d\vec{u}_i}{dt} = en_i(\vec{E} + \vec{u}_i \times \vec{B}) - \nabla p_i + R_{ie}$$

$$mn_e \frac{d\vec{u}_e}{dt} = -en_e(\vec{E} + \vec{u}_e \times \vec{B}) - \nabla p_e + R_{ei}$$

Equations of motion: isotropic pressure assumed

$$\rho \frac{d\vec{u}}{dt} = \rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \sigma \vec{E} + \vec{j} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

Plasmas as Fluids

$$R_{ei} = mn\langle \nu_{ei} \rangle (\vec{u}_i - \vec{u}_e) = \eta n^2 e^2 (\vec{u}_i - \vec{u}_e) = \eta n e \vec{j}$$

$$\vec{E} + \vec{u}_e \times \vec{B} = \eta \vec{j} - \frac{\nabla p_e}{ne}$$

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} + \frac{\vec{j} \times \vec{B} - \nabla p_e}{ne}$$

Neglect electron inertia entirely: valid for phenomena that are sufficiently slow that electrons have time to reach dynamical equilibrium in regard to their motion along the magnetic field

Generalized Ohm's law

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\varepsilon_0 \vec{E}) = \sigma$$

Maxwell equation