# 2. Orthogonal Coordinate Systems

In view of Eq. (2) for the expression of an arbitrary vector in the n-D vector space, a scalar or vector field at a certain point in space needs the description of the location of this point in an appropriate **curvilinear** (orthogonal or nonorthogonal) coordinate system.

In a 3-D space, a vector $A = A_1 a_1 + A_2 a_2 + A_3 a_3$ can be expressed by three mutually perpendicular unit vectors (**base vectors**: $a_1, a_2, a_3$), and its **position** can be located as the **intersection of three constant coordinate surfaces** ($u_1$=constant, $u_2$=constant, $u_3$=constant) mutually perpendicular to one another in an **orthogonal coordinate system** ($u_1, u_2, u_3$).

*(e.g.)* $(u_1, u_2, u_3) \Rightarrow$ $(x, y, z)$: Cartesian coordinates
$(r, \phi, z)$: Cylindrical coordinates
$(R, \theta, \phi)$: Spherical coordinates

## *A. Cartesian (or Rectangular) Coordinates* $(x, y, z)$

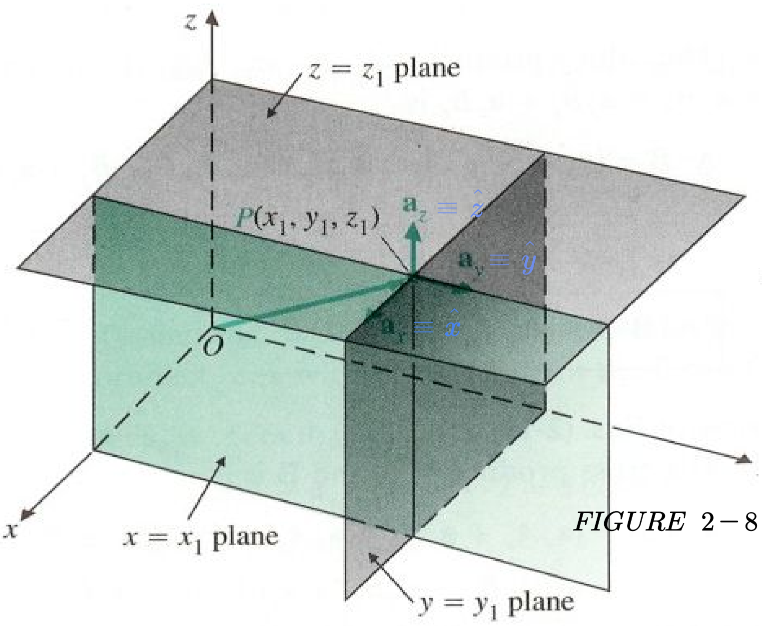
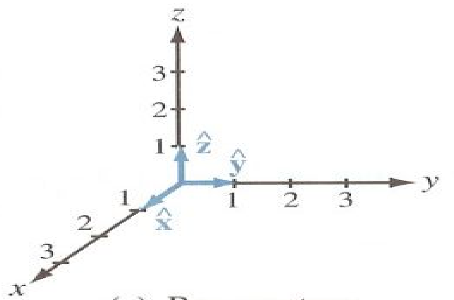
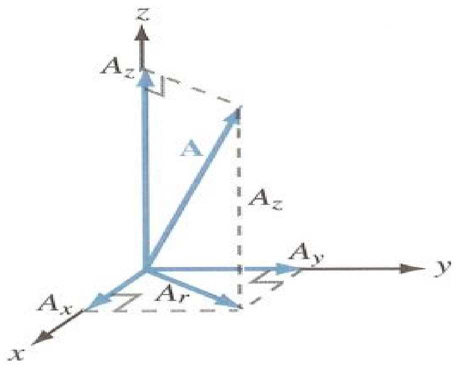



*FIGURE* 2-8




(a) Base vectors




(b) Components of **A**

### 1) Variables and Vector Relations

Coordinate variables:

$$(u_1, u_2, u_3) = (x, y, z), \qquad -\infty < x, y, z < +\infty \tag{7}$$

Base vectors: mutually perpendicular unit vectors

$$(a_x, a_y, a_z) \equiv (\hat{x}, \hat{y}, \hat{z}) \tag{8}$$

which have the following properties from (2-6) and (2-12) for the definitions of scalar and vector products:

i) orthogonal & orthonormal relations

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{x} \cdot \hat{z} = 0 \;\&\; \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \quad \text{(2-19, 20)}$$

ii) right-handed cyclic relation

$$\hat{x} \times \hat{y} = \hat{z}, \qquad \hat{y} \times \hat{z} = \hat{x}, \qquad \hat{z} \times \hat{x} = \hat{y} \qquad \text{(2-18)}$$

Position vector to point $P(x_1, y_1, z_1)$:

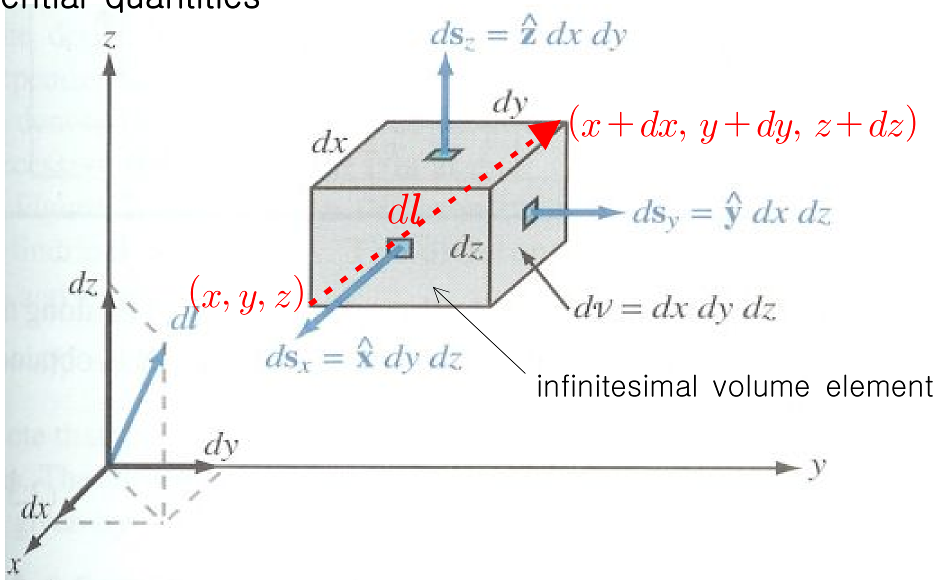
$$\overrightarrow{OP} = \hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1 \tag{2-21}$$

Vector representation of $\boldsymbol{A}$:

$$\boldsymbol{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \tag{2-22}$$

$$A = |\boldsymbol{A}| = \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}} = \sqrt{A_x^2 + A_y^2 + A_x^2}$$ magnitude of $\boldsymbol{A}$ by (2-9)

## 2) Differential quantities




Vector differential length $\boldsymbol{dl}$:

In arbitrary orthogonal coordinates $(u_1, u_2, u_3)$,

$$\boldsymbol{dl} = \hat{u}_1 dl_{u_1} + \hat{u}_2 dl_{u_2} + \hat{u}_3 dl_{u_3} = \hat{u}_1 h_1 du_1 + \hat{u}_2 h_2 du_2 + \hat{u}_3 u_3 dl_3 \tag{9}$$

$$(dl)^2 = (dl_{u_1})^2 + (dl_{u_2})^2 + (dl_{u_3})^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2 \quad (9)'$$

where $h_1, h_2, h_3$ are the metric coefficients to convert a differential coordinate variable change $du_i$ to a differential length change $dl_{u_i}$

In Cartesian coordinates $(x, y, z)$,

$$\boldsymbol{dl} = \hat{x}dl_x + \hat{y}dl_y + \hat{z}dl_z = \hat{x}dx + \hat{y}dy + \hat{z}dz \tag{2-23}$$

Then, the metric coefficients in this case are

$$h_1 = 1,\ h_2 = 1,\ h_3 = 1 \tag{10}$$

Vector differential surface areas $\boldsymbol{ds}_i$:

$\boldsymbol{ds}_x = \hat{x}\,dy\,dz$ on the y-z plane with an area $dy\,dz$

directing to $\hat{x}$ (outward normal to the plane) (11)

Likewise, $\boldsymbol{ds}_y = \hat{y}\,dx\,dz$ (x-z plane)

$\boldsymbol{ds}_z = \hat{z}\,dx\,dy$ (x-y plane)

Differential volume $dv$:

$$dv = dx\, dy\, dz \qquad (2\text{-}24)$$

## 3) Scalar and Vector Products

### a) Scalar product

By using the orthogonal and orthonormal relations (2-19, 20),

$$\boldsymbol{A} \cdot \boldsymbol{B} = (\hat{\boldsymbol{x}} A_x + \hat{\boldsymbol{y}} A_y + \hat{\boldsymbol{z}} A_z) \cdot (\hat{\boldsymbol{x}} B_x + \hat{\boldsymbol{y}} B_y + \hat{\boldsymbol{z}} B_z)$$

$$= A_x B_x + A_y B_y + A_z B_z = \sum_{i=x,y,z} A_i B_i \qquad (2\text{-}25)$$

### b) Vector product

By using the right-handed cyclic relation (2-18),

$$\boldsymbol{A} \times \boldsymbol{B} = (\hat{\boldsymbol{x}} A_x + \hat{\boldsymbol{y}} A_y + \hat{\boldsymbol{z}} A_z) \times (\hat{\boldsymbol{x}} B_x + \hat{\boldsymbol{y}} B_y + \hat{\boldsymbol{z}} B_z)$$

$$= \hat{\boldsymbol{x}} (A_y B_z - A_z B_y) + \hat{\boldsymbol{y}} (A_z B_x - A_x B_z) + \hat{\boldsymbol{z}} (A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (2\text{-}27)$$

*(cf.)* Calculation of Determinant A

$$\det A \equiv \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$= A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} - A_{31}A_{22}A_{13} - A_{21}A_{12}A_{33} - A_{11}A_{32}A_{23}$$

$$= \sum_{i,j,k=1}^{3} (\pm) A_{i1} A_{j2} A_{k3} \qquad (12)$$

Sum ranges over i, j, k of all permutations of 1, 2, 3.
Use + when permutation is even and
− when permutation is odd.

*Notes) Some useful mathematical notations and symbols*

i) Summation convention

Let dummy indices (*i, j, k*) designate (*x, y, z*),

then $A_i = B_i$ means $A_x = B_x,\ A_y = B_y,\ A_z = B_z$

If indices are repeated twice in a product,

a sum on them is understood as

$$A_i B_i \equiv \sum_{i=x,y,z} A_i B_i = A_x B_x + A_y B_y + A_z B_z = \boldsymbol{A} \cdot \boldsymbol{B} \qquad (13)$$

$\Longrightarrow$ (2-25)*

ii) Kronecker delta $\delta_{ij}$

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{14}$$

*(e.g.)* ① $A_i \delta_{ij} = A_j$

② Orthogonal & orthonormal relations : $\hat{i} \cdot \hat{j} = \delta_{ij}$ (2-19, 20)*

iii) Levi-Civita symbol $\epsilon_{ijk}$

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ are even parmutation of } x,\ y,\ z \\ -1 & \text{if } ijk \text{ are odd parmutation of } x,\ y,\ z \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

*(e.g.)* $\epsilon_{xyz} = 1, \quad \epsilon_{zyx} = -1, \quad \epsilon_{yxz} = -1,$

$\epsilon_{zxy} = 1, \quad \epsilon_{xxy} = 0, \quad \epsilon_{yyz} = 0$

*Note)* $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ (16)

(proof) If $j = k$ or $l = m$, then LHS = RHS = 0

If $j \neq k$ and $l \neq m$,

(a) LHS = RHS = 0 for $j \neq l$ & $j \neq m$

(b) LHS = RHS = 0 for $k \neq l$ & $k \neq m$

(c) LHS = RHS = -1 for $j = m$ & $k = l$

(d) LHS = RHS = 1 for $j = l$ & $k = m$

iv) Application of summation convention and Levi-Civita symbol

① Vector product (2-27):

$$\boldsymbol{A} \times \boldsymbol{B} = \hat{\boldsymbol{x}}(A_y B_z - A_z B_y) + \hat{\boldsymbol{y}}(A_z B_x - A_x B_z) + \hat{\boldsymbol{z}}(A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \epsilon_{ijk}\, \hat{\imath}\, A_j B_k = \hat{\imath}\, \epsilon_{ijk} A_j B_k \tag{2-27*}$$

$$(\boldsymbol{A} \times \boldsymbol{B})_i = \epsilon_{ijk} A_j B_k \tag{17}$$

② (12) ⇒ $\det A = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$ (12)*

③ P.2-9: Proof of vector triple product (6)

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{C} \cdot \boldsymbol{A}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

*(proof)*
$$\begin{aligned} [\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})]_i &= \epsilon_{ijk} A_j (\boldsymbol{B} \times \boldsymbol{C})_k \\ &= \epsilon_{kij}\epsilon_{klm} A_j B_l C_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m \\ &= B_i A_j C_j - C_i A_j B_j \\ &= B_i (\boldsymbol{C} \cdot \boldsymbol{A}) - C_i (\boldsymbol{A} \cdot \boldsymbol{B}) \end{aligned}$$ (6) or (2-113)

## B. *Cylindrical Coordinates* $(r, \phi, z)$

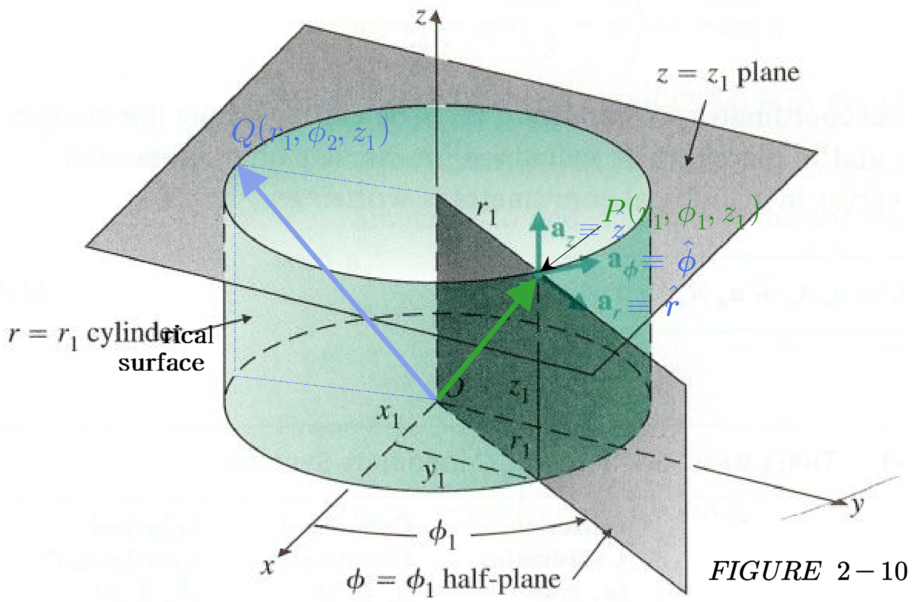



*FIGURE* 2−10

### 1) Variables and Vector Relations

Coordinate variables:

$$(u_1, u_2, u_3) = (r, \phi, z), \qquad 0 \le r < \infty,\ 0 \le \phi < 2\pi,\ -\infty < z < +\infty \tag{18}$$

Base vectors: mutually perpendicular unit vectors

$$(\boldsymbol{a}_r, \boldsymbol{a}_\phi, \boldsymbol{a}_z) \equiv (\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}) \tag{19}$$

which have the following properties:

i) orthogonal & orthonormal relations

$$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}} = 0 \ \& \ \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1$$

i.e., $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \delta_{ij}$ for $i$ and $j = r, \phi, z$ (20)

ii) right-handed cyclic relation

$$\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}, \qquad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}, \qquad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}} \tag{2-28}$$

Position vector to point $P(r_1, \phi_1, z_1)$:

$$\overrightarrow{OP} = \hat{\boldsymbol{r}}\, r_1 + \hat{\boldsymbol{z}}\, z_1 \tag{21}$$

*Note) For* $Q(r_1, \phi_2, z_1)$, $\overrightarrow{OQ} = \hat{\boldsymbol{r}}\, r_1 + \hat{\boldsymbol{z}}\, z_1$

$\overline{OQ} = \overline{OP}$ but $\overrightarrow{OQ} \neq \overrightarrow{OP}$ due to their different directions

Vector representation of $\boldsymbol{A}$:

$$\boldsymbol{A} = \hat{\boldsymbol{r}} A_r + \hat{\boldsymbol{\phi}} A_\phi + \hat{\boldsymbol{z}} A_z \tag{2-31}$$

$A = |\boldsymbol{A}| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}} = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$ magnitude of $\boldsymbol{A}$ by (2−9)

2) Differential quantities

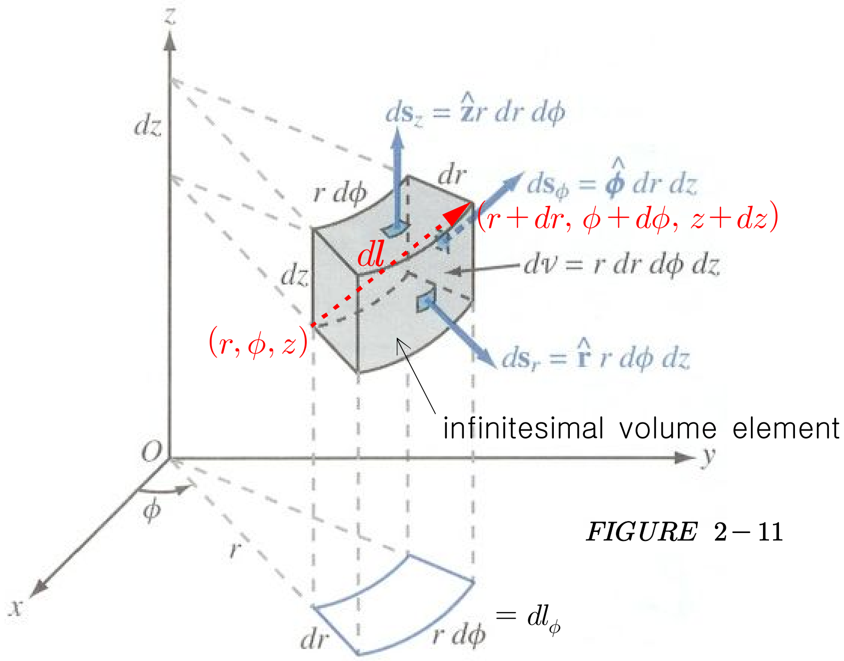



*FIGURE* 2-11

Vector differential length $\boldsymbol{dl}$:

$$\boldsymbol{dl} = \hat{\boldsymbol{r}}\, dl_r + \hat{\boldsymbol{\phi}}\, \underline{dl_\phi} + \hat{\boldsymbol{z}}\, dl_z = \hat{\boldsymbol{r}}\, h_1 dr + \hat{\boldsymbol{\phi}}\, \underline{h_2 d\phi} + \hat{\boldsymbol{z}}\, h_3 dz$$

$$= \hat{\boldsymbol{r}}\, dr + \hat{\boldsymbol{\phi}}\, \underline{r d\phi} + \hat{\boldsymbol{z}}\, dz \qquad (2\text{-}29)$$

Then, the metric coefficients in cylindrical coord. are

$$h_1 = 1,\ h_2 = r,\ h_3 = 1 \qquad (22)$$

Vector differential surface areas $\boldsymbol{ds}_i$:

$$\boldsymbol{ds}_r = dl_\phi dl_z = \hat{\boldsymbol{r}}\, r d\phi\, dz \qquad (\phi - z \text{ cylindrical surface})$$

$$\boldsymbol{ds}_\phi = \hat{\boldsymbol{\phi}}\, dr\, dz \qquad (\text{r-z plane}) \qquad (23)$$

$$\boldsymbol{ds}_z = \hat{\boldsymbol{z}}\, r dr\, d\phi \qquad (r - \phi \text{ plane})$$

Differential volume $dv$:

$$dv = dl_r dl_\phi dl_z = r dr\, d\phi\, dz \qquad (2\text{-}30)$$

## C. Spherical Coordinates $(R, \theta, \phi)$

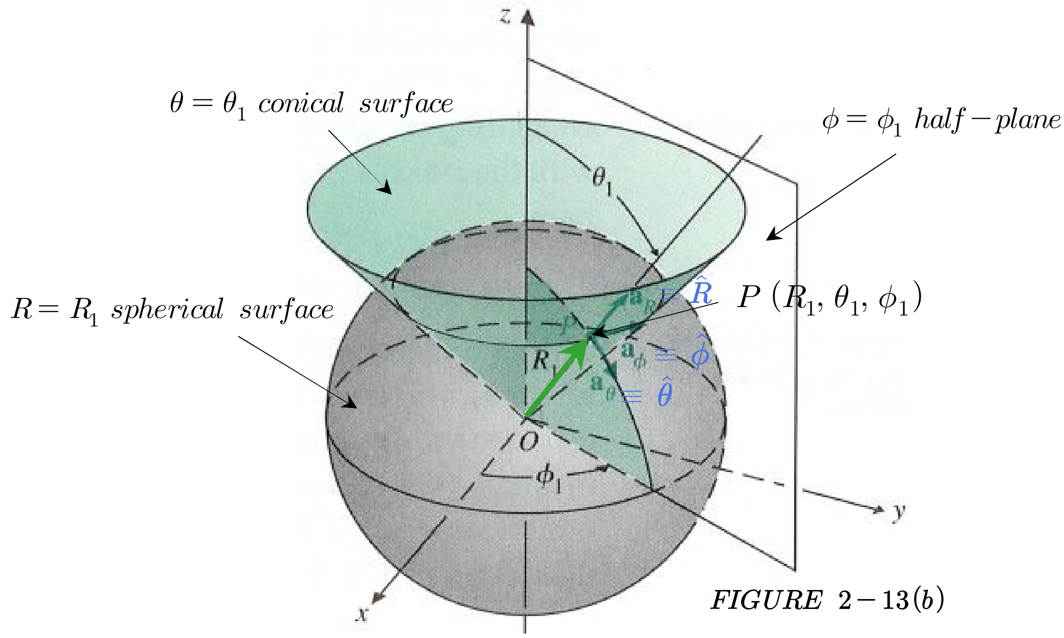



FIGURE 2-13(b)

### 1) Variables and Vector Relations

Coordinate variables:

$$(u_1, u_2, u_3) = (R, \theta, \phi), \quad 0 \le R < \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi \tag{24}$$

Base vectors: mutually perpendicular unit vectors

$$(\boldsymbol{a}_R, \boldsymbol{a}_\theta, \boldsymbol{a}_\phi) \equiv (\hat{\boldsymbol{R}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}) \tag{25}$$

which have the following properties:

i) orthogonal & orthonormal relations

$$\hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{\phi}} = 0 \ \&\ \hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$$

i.e., $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \delta_{ij}$ for $i$ and $j = R, \theta, \phi$ (26)

ii) right-handed cyclic relation

$$\hat{\boldsymbol{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{R}} = \hat{\boldsymbol{\theta}} \tag{2-41}$$

Position vector to point $P(R_1, \theta_1, \phi_1)$:

$$\overrightarrow{OP} = \hat{\boldsymbol{R}} R_1 \tag{27}$$

Vector representation of $\boldsymbol{A}$:

$$\boldsymbol{A} = \hat{\boldsymbol{R}} A_R + \hat{\boldsymbol{\theta}} A_\theta + \hat{\boldsymbol{\phi}} A_\phi \tag{2-42}$$

$A = |\boldsymbol{A}| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}} = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$ magnitude of $\boldsymbol{A}$ by (2-9)

## 2) Differential quantities

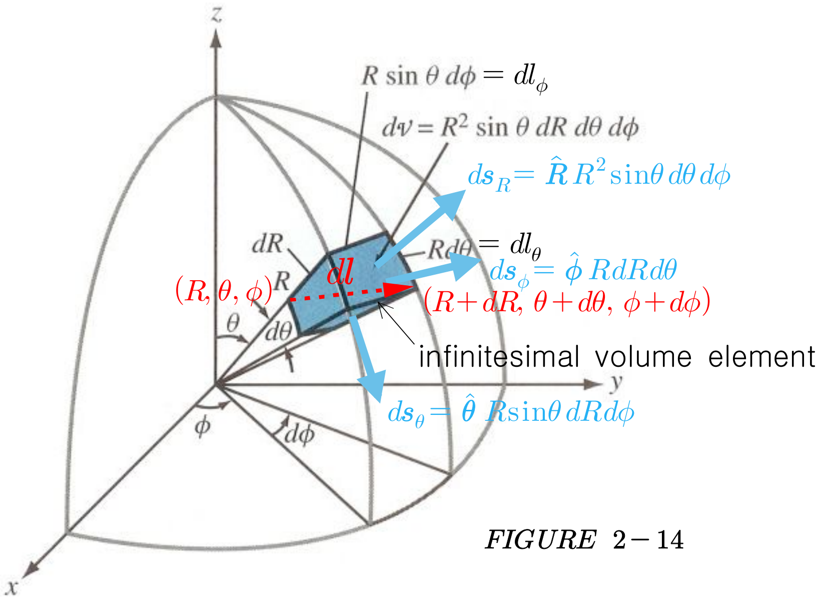



*FIGURE* 2−14

Vector differential length $d\mathbf{l}$:

$$d\mathbf{l} = \hat{\mathbf{R}} dl_R + \hat{\boldsymbol{\theta}}\, \underline{dl_\theta} + \hat{\boldsymbol{\phi}}\, \underline{dl_\phi} = \hat{\mathbf{r}}\, h_1 dr + \hat{\boldsymbol{\theta}}\ \underline{h_2 d\theta} + \hat{\boldsymbol{\phi}}\, \underline{h_3 d\phi}$$

$$= \hat{\mathbf{R}}\, dR + \hat{\boldsymbol{\theta}}\ \underline{R d\theta} + \hat{\boldsymbol{\phi}}\ \underline{R \sin\theta\, d\phi} \qquad (2\text{-}43)$$

Then, the metric coefficients in spherical coord. are

$$h_1 = 1,\ \ h_2 = R,\ \ h_3 = R\sin\theta \qquad (28)$$

Vector differential surface areas $d\mathbf{s}_i$:

$$d\mathbf{s}_R = dl_\theta\, dl_\phi = \hat{\mathbf{R}}\, R^2 \sin\theta\, d\theta\, d\phi \qquad (\theta - \phi \text{ spherical surface })$$

$$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\ R \sin\theta\, dR\, d\phi \qquad (\text{r}-\phi \text{ conical surface}) \qquad (29)$$

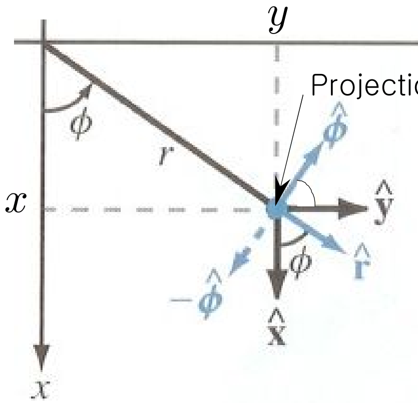
$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, R\, dR\ d\theta \qquad (r - \theta \text{ plane})$$

Differential volume $dv$:

$$dv = dl_R\, dl_\theta\, dl_\phi = R^2 \sin\theta\, dR\, d\theta\, d\phi \qquad (2\text{-}44)$$

## D. *Coordinate Transformations*

### 1) Transformation between Cartesian and Cylindrical Coordinates




Projection of $\forall$ point $P(r,\phi,z)$ on x-y plane
from FIGURE 2-10

*FIGURE* 2−12
Relations between $(\hat{x},\, \hat{y})$ and $(\hat{r},\, \hat{\phi})$

a) Transformation of variables

Cylindrical $\rightarrow$ Cartesian :

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z \qquad \text{(2-40)}$$

Cartesian $\rightarrow$ Cylindrical :

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right), \qquad z = z \qquad \text{(30)}$$

b) Scalar products of base vectors

| *TABLE* 1 | $\hat{x}$ | $\hat{y}$ | $\hat{z}$ | |
|---|---|---|---|---|
| $\hat{r}\,\cdot$ | $\cos\phi$ | $\sin\phi$ | 0 | (2-33), (2-36) |
| $\hat{\phi}\,\cdot$ | $-\sin\phi$ | $\cos\phi$ | 0 | (2-34), (2-37) |
| $\hat{z}\,\cdot$ | 0 | 0 | 1 | (2-19, 20) |

c) Transformation of vector

A vector $\boldsymbol{A}$ at point P:

$$\boldsymbol{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \quad \text{in Cartesian coordinates} \qquad \text{(2-22)}$$

$$\boldsymbol{A} = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z \quad \text{in cylindrical coordinates} \qquad \text{(2-31)}$$

Cylindrical $\rightarrow$ Cartesian :

(2-31) ↘      *TABLE* 1 ↘

$$\begin{cases} \underline{A_x} = \boldsymbol{A}\cdot\hat{x} = A_r\hat{r}\cdot\hat{x} + A_\phi\hat{\phi}\cdot\hat{x} + A_z\hat{z}\cdot\hat{x} = \underline{A_r\cos\phi - A_\phi\sin\phi + 0} & \text{(2-35)} \\ \underline{A_y} = \boldsymbol{A}\cdot\hat{y} = A_r\hat{r}\cdot\hat{y} + A_\phi\hat{\phi}\cdot\hat{y} + A_z\hat{z}\cdot\hat{y} = \underline{A_r\sin\phi + A_\phi\cos\phi + 0} & \text{(2-38)} \\ \underline{A_z} = \boldsymbol{A}\cdot\hat{z} = A_r\hat{r}\cdot\hat{z} + A_\phi\hat{\phi}\cdot\hat{z} + A_z\hat{z}\cdot\hat{z} = \underline{0 \quad + \quad 0 \quad + 1} & \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} \qquad \text{(2-39)}$$

$$\Rightarrow \quad A_i = \sum_{j=r,\phi,z} M_{ij}A_j \quad \text{for } i = x, y, z \qquad (\text{or } A_i = M_{ij}A_j) \qquad \text{(2-39)*}$$

Cartesian $\rightarrow$ Cylindrical :

Conversely in a similar manner,

(2-22) ↘ TABLE 1 ↘

$$\begin{cases} \underline{A_r} = \boldsymbol{A}\cdot\hat{\boldsymbol{r}} = A_x\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{r}} + A_y\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{r}} + A_z\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}} = \underline{A_x\cos\phi + A_y\sin\phi + 0} \\ \underline{A_\phi} = \boldsymbol{A}\cdot\hat{\boldsymbol{\phi}} = A_x\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{\phi}} + A_y\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\phi}} + A_z\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\phi}} = \underline{-A_x\sin\phi + A_y\cos\phi + 0} \\ \underline{A_z} = \boldsymbol{A}\cdot\hat{\boldsymbol{z}} = A_x\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{z}} + A_y\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{z}} + A_z\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}} = \underline{0 \quad + \quad 0 \quad + 1} \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad (31)$$

$$\Rightarrow \quad A_j = \sum_{i=x,y,z} M_{ji}^T A_i \quad \text{for } j = r, \phi, z \qquad (\text{or } A_j = M_{ji}^T A_i) \qquad (31)^*$$

*Notes)*

i) Matrices:

matrix $[M] \equiv \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$, matrix $[M]^T \equiv \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[M]^T$ is the transpose of $[M]$,

which is obtained by interchanging the rows and columns of $[M]$.

$(M_{ij} = M_{ji}^T)$

ii) Vector and tensor representation:

$$(2\text{-}39) \quad \Rightarrow \quad \boldsymbol{A}_{Cart.} = \overleftrightarrow{M} \cdot \boldsymbol{A}_{cyl.} \qquad (2\text{-}39)^{**}$$

$$(31) \quad \Rightarrow \quad \boldsymbol{A}_{cyl.} = \overleftrightarrow{M^T} \cdot \boldsymbol{A}_{Cart.} \qquad (31)^{**}$$

iii) Matrix multiplication

$$[M][M]^T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\sum_{k=1}^{3} M_{ik} M_{kj}^T\right] = [P_{ij}]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv [\mathrm{I}]: \text{ Unit matrix} \qquad (32)$$

iv) Inverse matrix

$$[M]^T = [M]^{-1} : \text{ inverse of } [M] \text{ since } [M][M]^T = [I] = [M][M]^{-1} \qquad (33)$$

2) Transformation between Cartesian and Spherical Coordinates

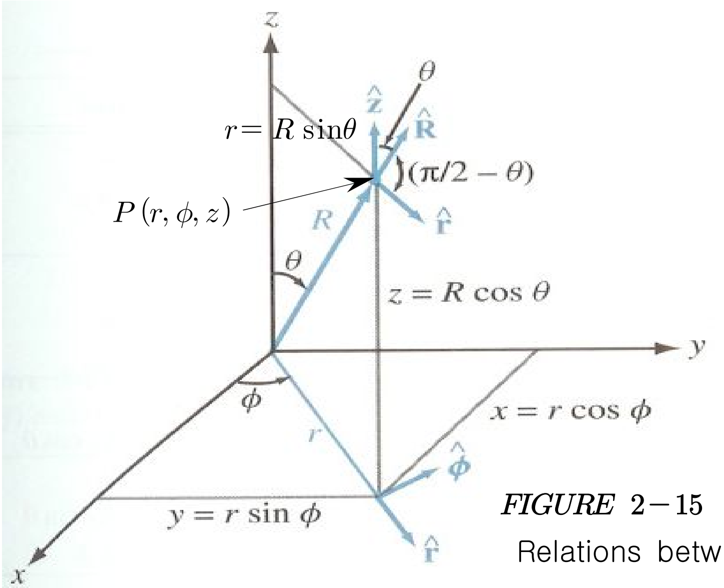



*FIGURE* 2−15
Relations between $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{r}, \hat{\theta}, \hat{\phi})$

a) Transformation of variables

Spherical → Cartesian :

$$x = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi, \qquad z = R\cos\theta \tag{2-45}$$

Cartesian → Spherical :

$$R = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{34}$$

b) Scalar products of base vectors

| *TABLE* 2 | $\hat{x}$ | $\hat{y}$ | $\hat{z}$ |
|---|---|---|---|
| $\hat{R}\cdot$ | $\sin\theta\cos\phi$ | $\sin\theta\sin\phi$ | $\cos\theta$ |
| $\hat{\theta}\cdot$ | $\cos\theta\cos\phi$ | $\cos\theta\sin\phi$ | $-\sin\theta$ |
| $\hat{\phi}\cdot$ | $-\sin\phi$ | $\cos\phi$ | 0 |

(35)

↑ (2−46)

c) Transformation of vector

A vector $\boldsymbol{A}$ at point P:

$$\boldsymbol{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \text{ in Cartesian coordinates} \tag{2-22}$$

$$\boldsymbol{A} = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi \text{ in spherical coordinates} \tag{2-42}$$

Spherical → Cartesian :

(2-42) ↘ *TABLE* 2 ↘

$$\begin{cases} \underline{A_x} = \boldsymbol{A}\cdot\hat{x} = A_R\hat{R}\cdot\hat{x} + A_\theta\hat{\theta}\cdot\hat{x} + A_\phi\hat{\phi}\cdot\hat{x} = \underline{A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi} \\ \underline{A_y} = \boldsymbol{A}\cdot\hat{y} = A_R\hat{R}\cdot\hat{y} + A_\theta\hat{\theta}\cdot\hat{y} + A_\phi\hat{\phi}\cdot\hat{y} = \underline{A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi} \\ \underline{A_z} = \boldsymbol{A}\cdot\hat{z} = A_R\hat{R}\cdot\hat{z} + A_\theta\hat{\theta}\cdot\hat{z} + A_\phi\hat{\phi}\cdot\hat{z} = \underline{A_R\cos\theta - A_\theta\sin\theta + A_\phi\, 0} \end{cases}$$

$$\Rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} \quad (36)$$

$$\Rightarrow A_i = \sum_{j=R,\theta,\phi} N_{ij} A_j \quad \text{for } i = x, y, z \qquad (\text{or } A_i = N_{ij} A_j) \quad (36)^*$$

Cartesian → Cylindrical :

Conversely in a similar way by using (2-22) and *TABLE* 2,

$$\begin{cases} \underline{A_R} = \boldsymbol{A} \cdot \hat{\boldsymbol{R}} = \underline{A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta} \\ \underline{A_\theta} = \boldsymbol{A} \cdot \hat{\theta} = \underline{A_x \cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta} \\ \underline{A_\phi} = \boldsymbol{A} \cdot \hat{\phi} = \underline{-A_x \sin\phi + A_y \cos\phi + A_z\, 0} \end{cases}$$

$$\Rightarrow \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (37)$$

$$\Rightarrow A_j = \sum_{i=x,y,z} N^T_{ji} A_i \quad \text{for } j = R, \theta, \phi \quad (\text{or } A_j = N^T_{ji} A_i) \quad (37)^*$$

*Notes)*

i) $[N] \equiv \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$,

$[N]^T \equiv \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$

ii) (36) $\Rightarrow \boldsymbol{A}_{Cart.} = \overleftrightarrow{N} \cdot \boldsymbol{A}_{sph.}$ (36)**

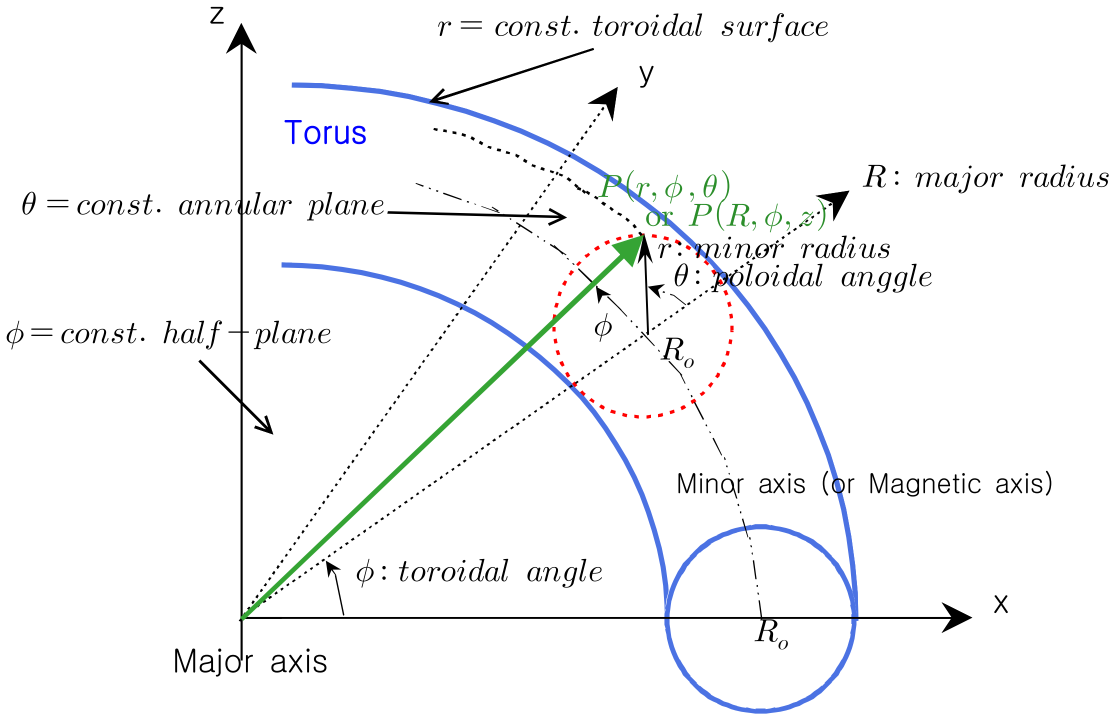
(37) $\Rightarrow \boldsymbol{A}_{sph.} = \overleftrightarrow{N^T} \cdot \boldsymbol{A}_{Cart.}$ (37)**

iii) $[N][N]^T = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix} = [\mathrm{I}] \quad (38)$$

iv) $[N]^T = [N]^{-1}$ since $[N][N]^T = [I] = [N][N]^{-1}$ (39)

## (cf.) Toroidal Coordinates $(r, \phi, \theta)$




### 1) Variables and Vector Relations

Coordinate variables:

$$(u_1, u_2, u_3) = (r, \phi, \theta), \quad 0 \le r < \infty,\ 0 \le \phi < 2\pi,\ 0 \le \theta < 2\pi \tag{40}$$

Base vectors: mutually perpendicular unit vectors

$$(\boldsymbol{a}_r, \boldsymbol{a}_\phi, \boldsymbol{a}_\theta) \equiv (\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}) \tag{41}$$

which have the following properties:

i) orthogonal & orthonormal relations

$$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \delta_{ij} \quad \text{for } i \text{ and } j = r, \phi, \theta \tag{42}$$

ii) right-handed cyclic relation

$$\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{r}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}} \tag{43}$$

### 2) Transformation of variables

Toroidal $(r, \phi, \theta)$ $\rightarrow$ Cylindrical $(R, \phi, z)$:

$$R = R_o + r\cos\theta = R_o\left(1 + \frac{r}{R_o}\cos\theta\right)$$

$$\phi = \phi,, \qquad z = r\sin\theta \tag{44}$$

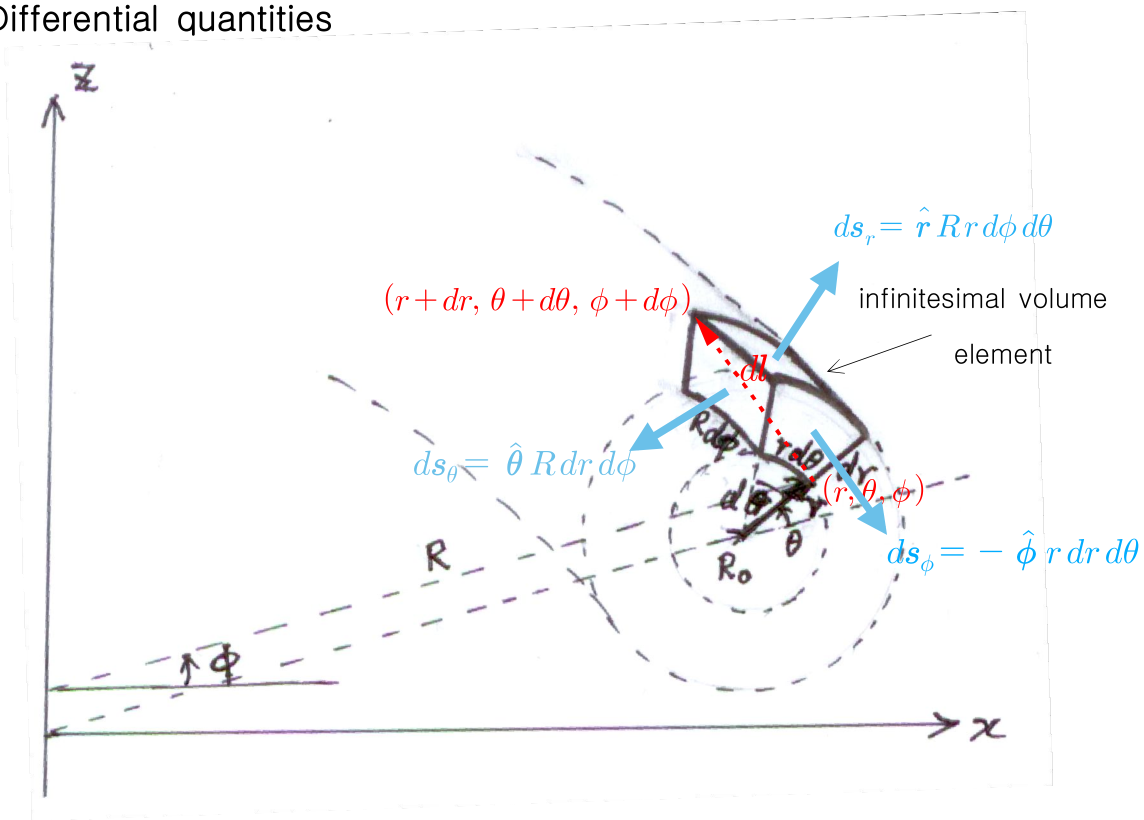
Toroidal $(r, \phi, \theta)$ $\rightarrow$ Cartesian $(x, y, z)$ :

$$x = R\cos\phi = (R_o + r\cos\theta)\cos\phi$$

$$y = R\sin\phi = (R_o + r\cos\theta)\sin\phi \tag{45}$$

$$z = r\sin\theta$$

2) Differential quantities




Vector differential length $\boldsymbol{dl}$:

$$\boldsymbol{dl} = \hat{\boldsymbol{r}}\, dl_r + \hat{\boldsymbol{\phi}}\, \underline{dl_\phi} + \hat{\boldsymbol{\theta}}\, \underline{dl_\theta} = \hat{\boldsymbol{r}}\, h_1 dr + \hat{\boldsymbol{\phi}}\, \underline{h_2 d\phi} + \hat{\boldsymbol{\theta}}\, \underline{h_3 d\theta}$$

$$= \hat{\boldsymbol{r}}\, dr + \hat{\boldsymbol{\phi}}\, \underline{R d\phi} + \hat{\boldsymbol{\theta}}\, \underline{r\, d\theta} \qquad (46)$$

Then, the metric coefficients in cylindrical coord. are

$$h_1 = 1,\ \ h_2 = R = R_o + r\cos\theta,\ \ h_3 = r \qquad (47)$$

*(cf.)* Non-graphical method from coordinate transformation:

Since the differential length $dl$ is invariant in any coordinate system,

$$(dl)^2 = (h_1 dr)^2 + (h_2 d\Phi)^2 + (h_3 d\Theta)^2 = (dx)^2 + (dy)^2 + (dz)^2 \qquad (48)$$

The metric coefficients are then determined by using (45) & (48)

$$h_1 = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right]^{1/2} = 1$$

$$h_2 = \left[\left(\frac{\partial x}{\partial \Phi}\right)^2 + \left(\frac{\partial y}{\partial \Phi}\right)^2 + \left(\frac{\partial z}{\partial \Phi}\right)^2\right]^{1/2} = R = R_o + r\cos\Theta \qquad (47)'$$

$$h_3 = \left[\left(\frac{\partial x}{\partial \Theta}\right)^2 + \left(\frac{\partial y}{\partial \Theta}\right)^2 + \left(\frac{\partial z}{\partial \Theta}\right)^2\right]^{1/2} = r$$

Vector differential surface areas $\boldsymbol{ds}_i$:

$$\boldsymbol{ds}_r = dl_\phi dl_\theta = \hat{\boldsymbol{r}}\, R r\, d\phi\, d\theta \qquad (\phi - \theta \text{ toroidal surface})$$

$$\boldsymbol{ds}_\phi = \hat{\boldsymbol{\phi}}\, r\, dr\, d\theta \qquad (r - \theta \text{ plane}) \qquad (49)$$

$$\boldsymbol{ds}_\theta = \hat{\boldsymbol{\theta}}\, R\, dr\, d\phi \qquad (r - \phi \text{ plane})$$

Differential volume $dv$:

$$dv = dl_r dl_\phi dl_\theta = R r\, dr\, d\phi\, d\theta \qquad (50)$$

# Homework Set 1

1) P.2-1
2) P.2-4
3) P.2-7
4) P.2-8
5) P.2-13
6) By using the representations of summation convention and Levi-Civita symbol, prove that the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ can be calculated by the following determinant in Cartesian coordinate system.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$