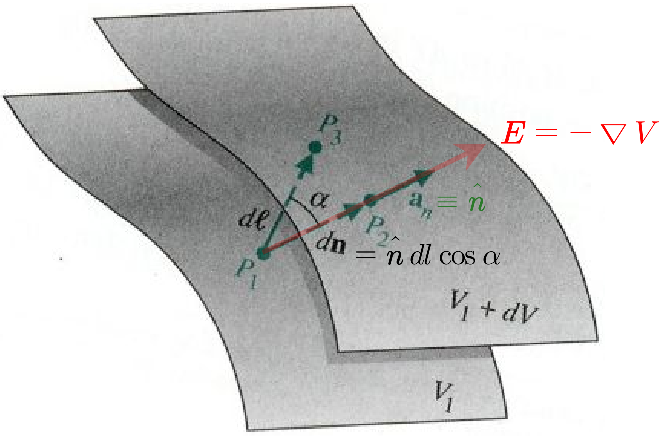
3. Vector Calculus

A. Gradient of Scalar Field



1) Definition of gradient

Physical definition:

Vector that has the maximum space change rate of physical quantity

Mathematical definition:

$$grad\, V \equiv \nabla V \triangleq \hat{n} \lim_{\Delta l \to 0} \left(\frac{\Delta V}{\Delta l} \right)_{\max} = \hat{n} \left(\frac{dV}{dl} \right)_{\max} = \hat{n} \frac{dV}{dn} \qquad \text{(2-48, 49)}$$

↘ maximum directional derivative

Note) ∇ is called the del or gradient operator

Directional derivative:

$$\frac{dV}{dl} = \frac{dV}{dn} \frac{dn}{dl} = \frac{dV}{dn} \cos\alpha = \frac{dV}{dn} \hat{n} \cdot \hat{l} = (\nabla V) \cdot \hat{l} \qquad \text{(2-50)}$$

↖ (2-49)

2) Calculation of gradient in orthogonal curvilinear coordinates

Space change rate of V:

$$\text{(2-50)} \Longrightarrow dV = (\nabla V) \cdot dl \gtreqless 0 \qquad \text{(2-51)}$$

↗ $dl = \hat{l}\, dl$

In orthogonal curvilinear coordinates (u_1, u_2, u_3),

$$dV = \frac{\partial V}{\partial l_{u_1}} dl_{u_1} + \frac{\partial V}{\partial l_{u_2}} dl_{u_2} + \frac{\partial V}{\partial l_{u_3}} dl_{u_3}$$

(9) ↘

$$= \frac{\partial V}{h_1 \partial u_1} dl_{u_1} + \frac{\partial V}{h_2 \partial u_2} dl_{u_2} + \frac{\partial V}{h_3 \partial u_3} dl_{u_3}$$

$$= \left[\left(\hat{u}_1 \frac{\partial}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial}{h_3 \partial u_3} \right) V \right] \cdot dl \qquad \text{(2-51)*}$$

By comparing RHS of (2-51) and (2-51)*, we can define ∇ operator:

$$\nabla \equiv \left(\hat{u}_1 \frac{\partial}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial}{h_3 \partial u_3} \right) \tag{2-57}$$

Then, $(\nabla V)_i = \frac{1}{h_i} \frac{\partial V}{\partial u_i}, \qquad (i = 1, 2, 3)$ (51)

(e.g)

In Cartesian coordinates $(u_1, u_2, u_3) = (x, y, z)$, $h_1 = h_2 = h_3 = 1$ (10) ;

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$ (2-57)$_{Cart.}$ =(2-56)

In cylindrical coordinates $(u_1, u_2, u_3) = (r, \phi, z)$, $h_1 = 1, h_2 = r, h_3 = 1$ (22) ;

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$$ (2-57)$_{cyl.}$

In spherical coordinates $(u_1, u_2, u_3) = (R, \theta, \phi)$, $h_1 = 1,\ h_2 = R,\ h_3 = R\sin\theta$ (28);

$$\nabla = \hat{R} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{R} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{R\sin\theta} \frac{\partial}{\partial \phi}$$ (2-57)$_{sph.}$

In toroidal coordinates $(u_1, u_2, u_3) = (r, \phi, \theta)$, $h_1 = 1,\ h_2 = R = R_o + r\cos\theta,\ h_3 = r$ (47);

$$\nabla \equiv \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta}$$ (2-57)$_{tor.}$

Notes) Properties of ∇ operator:

i) $\nabla (f + g) = \nabla f + \nabla g$

ii) $\nabla (fg) = f \nabla g + g \nabla f$

iii) $\nabla f^n = n f^{n-1} \nabla f$

B. Divergence of Vector Field

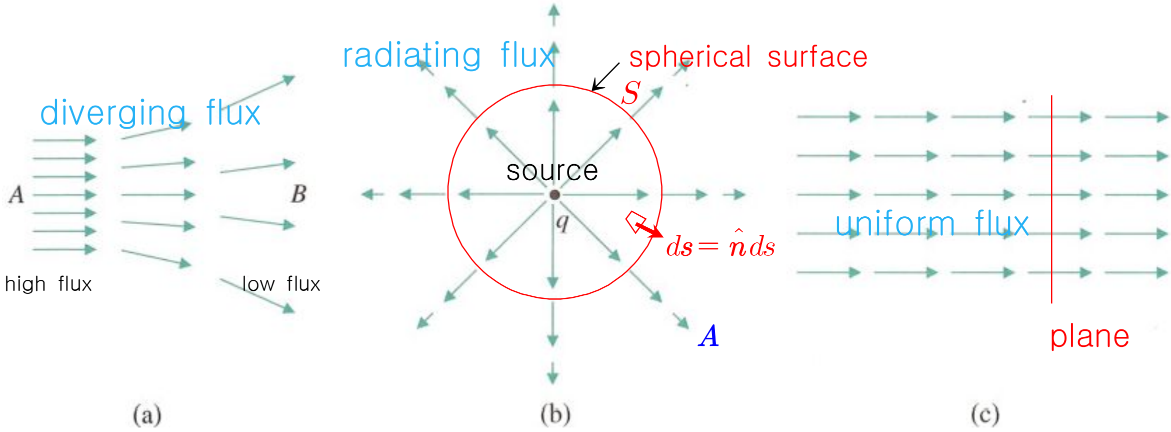


FIGURE 2−17 Flux lines = Directed field lines

1) Definition of divergence

Physical definition:

Net outward flux of the vector quantity per unit volume

Mathematical definition:

$$div\, \mathbf{A} \equiv \nabla \cdot \mathbf{A} \triangleq \lim_{\triangle v \to 0} \left(\frac{\oint_S \mathbf{A} \cdot d\mathbf{s}}{\triangle v} \right) \quad (2\text{-}58)$$

2) Calculation of divergence in orthogonal curvilinear coordinates

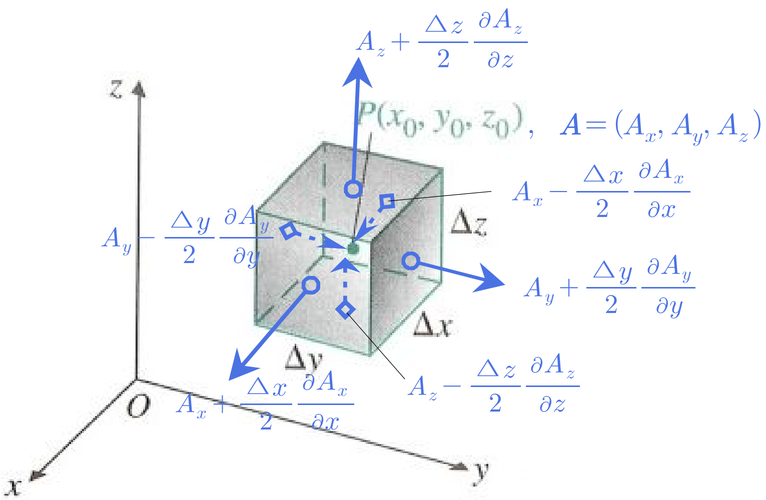


FIGURE 2−18 A differential volume in Cartesian coord.

$$\oint_S \mathbf{A} \cdot d\mathbf{s} = \left[\int_{front} + \int_{back} + \int_{right} + \int_{left} + \int_{top} + \int_{bottom} \right] \mathbf{A} \cdot d\mathbf{s} \quad (2\text{-}59)$$

$$\int_{front} \boldsymbol{A} \cdot d\boldsymbol{s} = \boldsymbol{A}_f \cdot \triangle \boldsymbol{s}_f = \boldsymbol{A}_f \cdot \hat{x}(\triangle y \triangle z)$$

$$= A_x(x_o + \triangle x/2,\, y_o,\, z_o)\triangle y\, \triangle z \quad \text{(2-60)}$$

Taylor expansion at (x_o, y_o, z_o) ↘ ; H.O.T. → 0 for $\triangle x \ll 0$

$$= \left[A_x|_o + \frac{\triangle x}{2} \frac{\partial A_x}{\partial x}\Big|_o + H.O.T. \right] \triangle y \triangle z \quad \text{(2-61)}$$

Likewise, $\int_{back} \boldsymbol{A} \cdot d\boldsymbol{s} = -A_x(x_o - \triangle x/2,\, y_o,\, z_o)\triangle y\, \triangle z$ (2-62)

H.O.T. → 0 for $\triangle x \ll 0$

$$= -\left[A_x|_o - \frac{\triangle x}{2} \frac{\partial A_x}{\partial x}\Big|_o + H.O.T. \right] \triangle y \triangle z \quad \text{(2-63)}$$

Then, (2-61) + (2-63) gives

$$\left[\int_{front} + \int_{back} \right] \boldsymbol{A} \cdot d\boldsymbol{s} \approx \frac{\partial A_x}{\partial x}\Big|_o \triangle x \triangle y \triangle z = \frac{\partial A_x}{\partial x}\Big|_o \triangle v \quad \text{(2-64)}$$

Similarly, $\left[\int_{right} + \int_{left} \right] \boldsymbol{A} \cdot d\boldsymbol{s} \approx \frac{\partial A_y}{\partial y}\Big|_o \triangle v$ (2-65)

$$\left[\int_{top} + \int_{bottom} \right] \boldsymbol{A} \cdot d\boldsymbol{s} \approx \frac{\partial A_z}{\partial z}\Big|_o \triangle v \quad \text{(2-66)}$$

Finally, (2-64)+(2-65)+(2-66) in (2-59) results in

$$\oint_S \boldsymbol{A} \cdot d\boldsymbol{s} \approx \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)\Big|_o \triangle v \quad \text{(2-67)}$$

$\lim_{\triangle v \to 0} \frac{(2-67)}{\triangle v}$ in (2-58) yields in Cartesian coordinates,

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{(2-68)}$$

Generalization in orthogonal curvilinear coordinates (u_1, u_2, u_3):

$$\nabla \cdot \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3) \right] \quad \text{(2-70)}$$

$$= \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u_i}\left(\frac{h_1 h_2 h_3}{h_i} A_i \right) \left[\equiv \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i}\left(\frac{h_1 h_2 h_3}{h_i} A_i \right) \text{ by summation convention} \right] \quad \text{(52)}$$

In cylindrical coordinates $(u_1, u_2, u_3) = (r, \phi, z)$, $h_1 = 1,\ h_2 = r,\ h_3 = 1$ (22) ;

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad \text{(2-70)}_{\text{cyl.}}$$

In spherical coordinates $(u_1, u_2, u_3) = (R, \theta, \phi)$, $h_1 = 1,\ h_2 = R,\ h_3 = R\sin\theta$ (28);

$$\nabla \cdot \boldsymbol{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi} \quad \text{(2-70)}_{\text{sph.}}$$

In toroidal coordinates $(u_1, u_2, u_3) = (r, \phi, \theta)$, $h_1 = 1,\ h_2 = R = R_o + r\cos\theta,\ h_3 = r$ (47);

$$\nabla \cdot \boldsymbol{A} = \frac{1}{Rr}\left[\frac{\partial}{\partial r}(R r A_r) + \frac{\partial}{\partial \phi}(r A_\phi) + \frac{\partial}{\partial \Theta}(R A_\Theta) \right] \quad \text{(2-70)}_{\text{tor.}}$$

Notes)

$\nabla \cdot \boldsymbol{A} = 0$

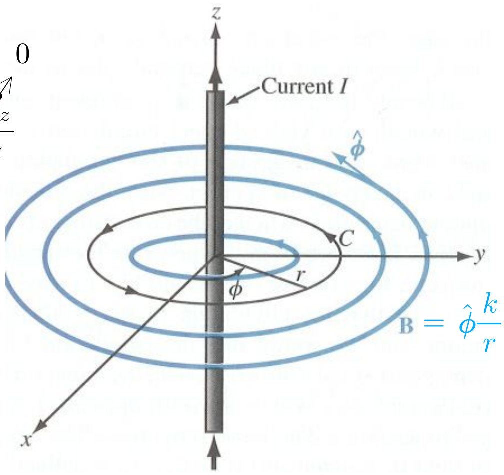
$\Longrightarrow$ $\boldsymbol{A}$: Solenoidal field = Divergenceless field = Divergence-free field

i) Azimuthal magnetic field produced by straight wire current

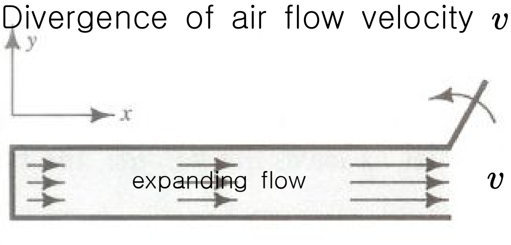
$\boldsymbol{B} = \hat{\phi}\dfrac{k}{r}$ in $(2\text{-}70)_{\text{cyl.}}$:

$$\nabla \cdot \boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r}(r\underbrace{B_r}_{0}) + \frac{1}{r}\frac{\partial \overbrace{B_\phi}}{\partial \phi}_{\to 0} + \frac{\partial \overbrace{B_z}^{0}}{\partial z}$$

$$= 0$$



ii) Divergence of air flow velocity $\boldsymbol{v}$

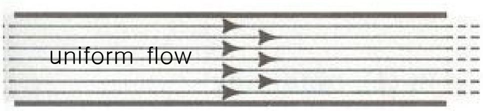


$\boldsymbol{v} = \hat{x}\, kx \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{v} = \dfrac{\partial v_x}{\partial x} = k$

(a)

$\nabla \cdot \boldsymbol{v} \neq 0$

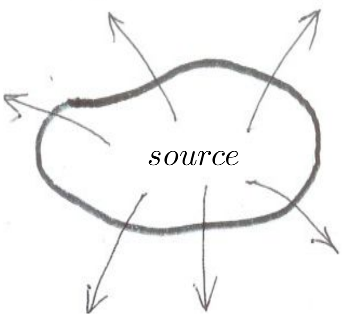
(b)

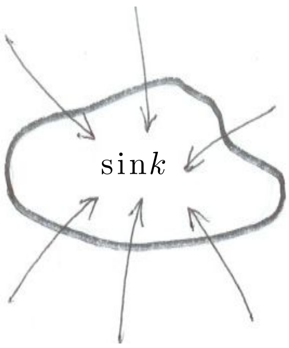


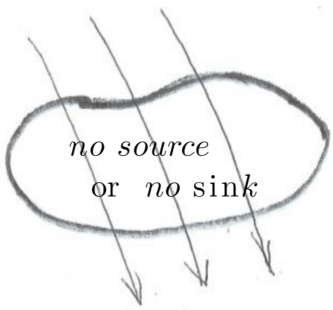
$\boldsymbol{v} = \hat{x}\, k \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{v} = \dfrac{\partial v_x}{\partial x} = 0$

(c)

iii) $\nabla \cdot \boldsymbol{A}$ is a measure of the strength of the flow source or sink.







$\nabla \cdot \boldsymbol{A} \gtrless 0$

(outflux $\gtrless$ influx)

$\nabla \cdot \boldsymbol{A} = 0$

(outflux = influx)

3) Divergence (or Gauss's) theorem

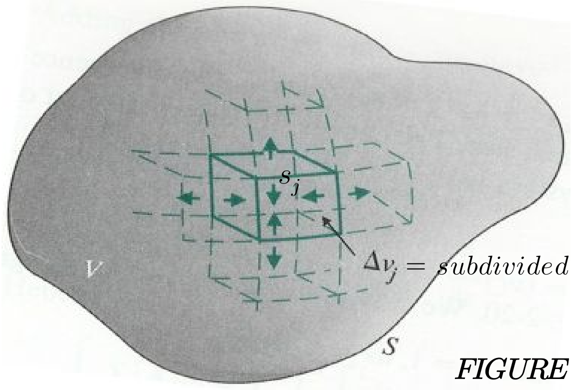


FIGURE 2−19

$$\int_V \nabla \cdot \boldsymbol{A}\, dv = \oint_S \boldsymbol{A} \cdot d\boldsymbol{s} \qquad (2\text{-}75)$$

volume integral of the divergence

= total outflux thru surface S bounding volume V

(Proof) From (2-58),

$$\lim_{\Delta v_j \to 0} \sum_{j=1}^{N \gg 1} (\nabla \cdot \boldsymbol{A})_j \Delta v_j = \lim_{\Delta v_j \to 0} \sum_{j=1}^{N \gg 1} \oint_{S_j} \boldsymbol{A} \cdot d\boldsymbol{s}$$

by definition of volume integral ⟶ ⇓ ⇓ ⟵ by canceling contributions from internal surfaces S_j

$$\int_V \nabla \cdot \boldsymbol{A}\, dv = \oint_S \boldsymbol{A} \cdot d\boldsymbol{s} \quad \Rightarrow \quad (2\text{-}75)$$

(e.g. 2-13)

Spherical shell volume enclosed by a multiply connected surface

$$\nabla \cdot \boldsymbol{F} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 F_R) \qquad F_R = kR$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(kR^3) = 3k$$

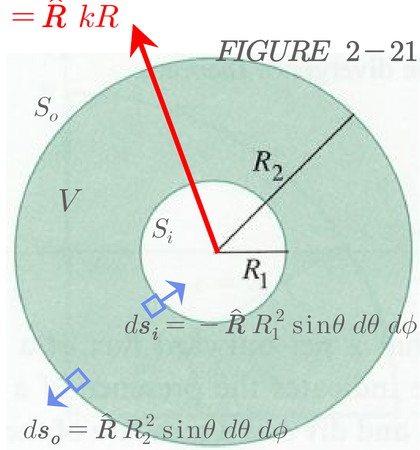


FIGURE 2−21

$$\int_V \nabla \cdot \boldsymbol{F}\, dv = \int_V 3k\, dv$$

$$= 3k \int_V dv = 3k \frac{4\pi}{3}(R_2^3 - R_1^3)$$

$$= 4\pi k (R_2^3 - R_1^3) \qquad (2\text{-}82)$$

$$\oint_S \boldsymbol{A} \cdot d\boldsymbol{s} = \left[\oint_{outer} + \oint_{inner}\right] \boldsymbol{F} \cdot d\boldsymbol{s}$$

$$= \oint_{S_o} \boldsymbol{F} \cdot d\boldsymbol{s}_o + \oint_{S_i} \boldsymbol{F} \cdot d\boldsymbol{s}_i$$

$$= \int_0^{2\pi} \left\{ \int_0^{\pi} \left[(kR_2)R_2^2 - (kR_1)R_1^2 \right] \sin\theta\, d\theta \right\} d\phi$$

$$= 4\pi k (R_2^3 - R_1^3) \qquad (2\text{-}83)$$

C. Curl of Vector Field

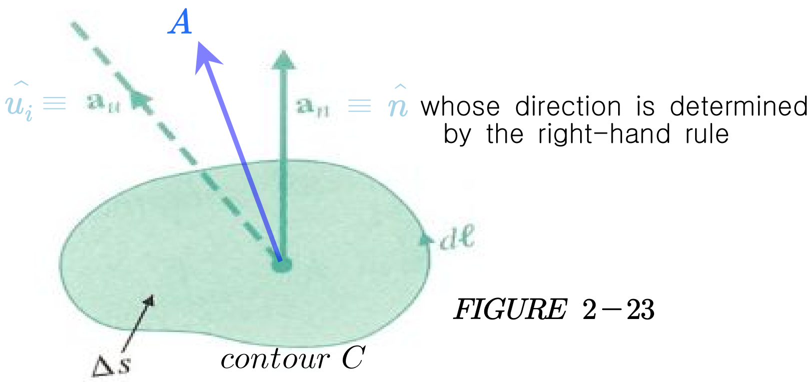


FIGURE 2−23

1) Definition of curl

Physical definition:

Vector that has the maximum circulation of $\boldsymbol{A}$ per unit area

Mathematical definition:

$$curl\,\boldsymbol{A}\,(\text{or}\ \ rot\,\boldsymbol{A}) \equiv \nabla\times\boldsymbol{A} \triangleq \lim_{\Delta s\to 0}\left(\frac{\hat{n}\oint_C \boldsymbol{A}\cdot d\boldsymbol{l}}{\Delta s}\right)_{\max} \qquad (2\text{-}85)$$

Notes)

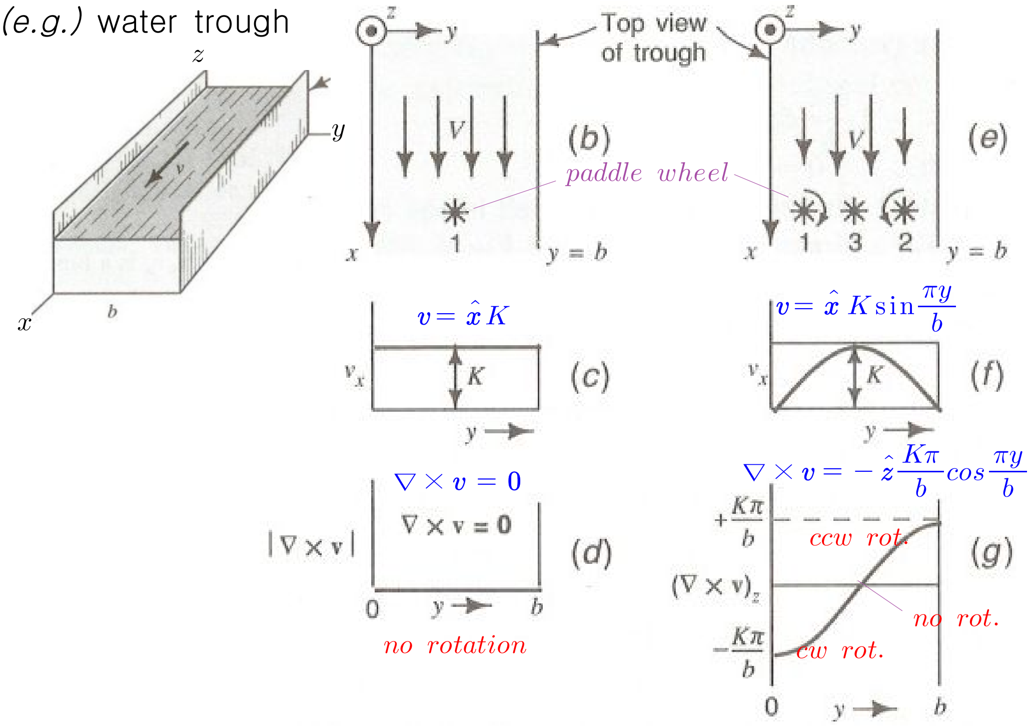
Circulation of $\boldsymbol{A}$ around contour $C \triangleq \oint_C \boldsymbol{A}\cdot d\boldsymbol{l}$ (2−84)

i) If $\boldsymbol{A}=\boldsymbol{F}$ = force, then $\oint_C \boldsymbol{F}\cdot d\boldsymbol{l}$ = work done by the force

ii) If $\boldsymbol{A}=\boldsymbol{E}$ = electric field, $\oint_C \boldsymbol{E}\cdot d\boldsymbol{l}$ = e.m.f.(electromotive force)

iii) If $\boldsymbol{A}=\boldsymbol{v}$ = flow velocity, $\oint_C \boldsymbol{v}\cdot d\boldsymbol{l}$ = circulation of fluid

(e.g.) water trough



∴ $\nabla\times\boldsymbol{A}$ is a measure of the strength of the vortex source or sink.

2) Calculation of curl in orthogonal curvilinear coordinates

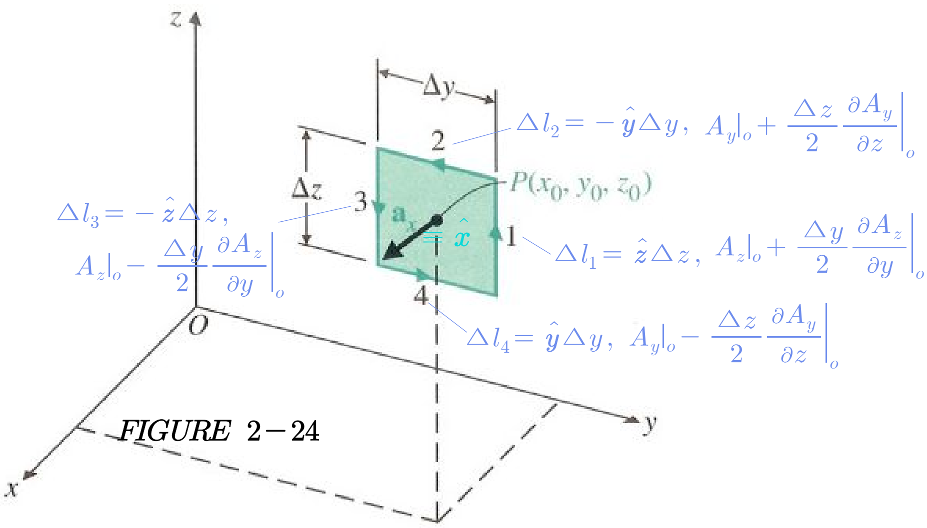


FIGURE 2−24

u_i component of $\nabla \times \boldsymbol{A}$ in orthogonal curvilinear coordinates(u_1, u_2, u_3):

$$(\nabla \times \boldsymbol{A})_{u_i} = \hat{u_i} \cdot (\nabla \times \boldsymbol{A}) \overset{(2-85)}{=} \lim_{\triangle s_{u_i} \to 0}\left(\frac{1}{\triangle s_{u_i}} \oint_{C_{u_i}} \boldsymbol{A} \cdot d\boldsymbol{l}\right) \tag{2-86}$$

In Cartesian coordinates $(u_1, u_2, u_3)=(x, y, z)$,

$$(\nabla \times \boldsymbol{A})_x = \hat{x} \cdot (\nabla \times \boldsymbol{A}) = \lim_{\triangle y \triangle z \to 0}\left(\frac{1}{\triangle y \triangle z} \oint_{\square_{1,2,3,4}} \boldsymbol{A} \cdot d\boldsymbol{l}\right) \tag{2-87}$$

$$\oint_{\square_{1,2,3,4}} \boldsymbol{A} \cdot d\boldsymbol{l} = \left[\int_1 + \int_2 + \int_3 + \int_4\right] \boldsymbol{A} \cdot d\boldsymbol{l} \tag{2-87)*}$$

$$\int_1 \boldsymbol{A} \cdot d\boldsymbol{l} = \boldsymbol{A_1} \cdot \triangle \boldsymbol{l_1} = \boldsymbol{A_1} \cdot \hat{z} \triangle z$$

$$= A_z(x_o, y_o + \triangle y/2, z_o) \triangle z$$

Taylor expansion at (x_o, y_o, z_o) → 0 for $\triangle y \ll 1$

$$= \left[A_z|_o + \frac{\triangle y}{2}\frac{\partial A_z}{\partial y}\Big|_o + \text{H.O.T.}\right] \triangle z \tag{2-89}$$

Likewise, $\int_3 \boldsymbol{A} \cdot d\boldsymbol{l} = \boldsymbol{A_3} \cdot \triangle \boldsymbol{l_3} = -\boldsymbol{A_3} \cdot \hat{z} \triangle z$

0 for $\triangle y \ll 1$

$$= -\left[A_z|_o - \frac{\triangle y}{2}\frac{\partial A_z}{\partial y}\Big|_o + \text{H.O.T.}\right] \triangle z \tag{2-91}$$

Then, (2-89) + (2-91) gives

$$\left[\int_1 + \int_3\right] \boldsymbol{A} \cdot d\boldsymbol{l} \approx \frac{\partial A_z}{\partial y}\Big|_o \triangle y \triangle z \tag{2-92}$$

Similarly, $\left[\int_2 + \int_4\right] \boldsymbol{A} \cdot d\boldsymbol{l} \approx -\frac{\partial A_y}{\partial z}\Big|_o \triangle y \triangle z$ (2-93)

Finally, (2-92)+(2-93) in (2-87) results in

$$(\nabla \times \boldsymbol{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{2-94}$$

Also, y- and z-components can be found by a cyclic order in x, y, and z as follows:

$$\nabla \times \boldsymbol{A} = \hat{\boldsymbol{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\boldsymbol{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\boldsymbol{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad (2\text{-}95)$$

$$= \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad (2\text{-}96)$$

Generalization in orthogonal curvilinear coordinates (u_1, u_2, u_3):

$$\nabla \times \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \widehat{u_1} h_1 & \widehat{u_2} h_2 & \widehat{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad (2\text{-}97)$$

$$(\nabla \times \boldsymbol{A})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial}{\partial u_j}(h_k A_k) \quad (53)$$

or $= \varepsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial}{\partial u_j}(h_k A_k)$ by summation convention (53)*

In cylindrical coordinates $(u_1, u_2, u_3) = (r, \phi, z)$, $h_1 = 1$, $h_2 = r$, $h_3 = 1$ (22) ;

$$\nabla \times \boldsymbol{A} = \frac{1}{r} \begin{vmatrix} \hat{r} & \hat{\phi} r & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & r A_\phi & A_z \end{vmatrix} \quad (2\text{-}97)_{cyl.}, (2\text{-}98)$$

In spherical coordinates $(u_1, u_2, u_3) = (R, \theta, \phi)$, $h_1 = 1$, $h_2 = R$, $h_3 = R\sin\theta$ (28);

$$\nabla \times \boldsymbol{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{R} & \hat{\theta} R & \hat{\phi} R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & R A_\theta & (R\sin\theta) A_\phi \end{vmatrix} \quad (2\text{-}70)_{sph.}, (2\text{-}99)$$

In toroidal coordinates $(u_1, u_2, u_3) = (r, \phi, \theta)$, $h_1 = 1$, $h_2 = R = R_o + r\cos\theta$, $h_3 = r$(47);

$$\nabla \times \boldsymbol{A} = \frac{1}{Rr} \begin{vmatrix} \hat{r} & \hat{\Phi} R & \hat{\Theta} r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \Phi} & \frac{\partial}{\partial \Theta} \\ A_r & R A_\Phi & r A_\Theta \end{vmatrix} \quad (2\text{-}70)_{tor.}$$

Note)

$\nabla \times \boldsymbol{A} = 0$

$\Longrightarrow$ $\boldsymbol{A}$: Curl-free field = Irrotational (or lamellar) field due to no rotation

= Conservative field due to $\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = 0$

3) Stokes's theorem

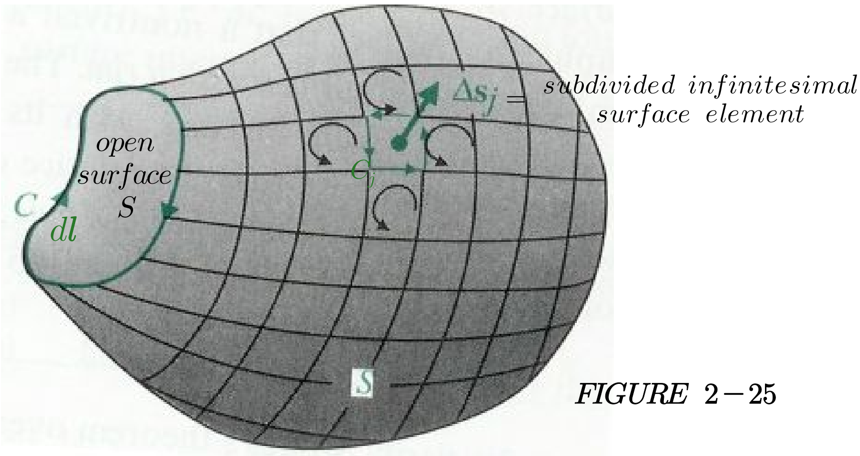


FIGURE 2−25

$$\int_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l} \qquad (2\text{-}103)$$

open surface integral of the curl

= closed line integral along contour C bounding surface S

(Proof) From (2-85),

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N \gg 1} (\nabla \times \boldsymbol{A})_j \cdot \Delta \boldsymbol{s}_j = \lim_{\Delta s_j \to 0} \sum_{j=1}^{N \gg 1} \oint_{C_j} \boldsymbol{A} \cdot d\boldsymbol{l}$$

by definition of surface integral → ⇓   ⇓ ← by canceling contributions from internal contours C_j

$$\int_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l} \quad \Rightarrow \quad (2\text{-}103)$$

Note) For any closed surface S with no open surface with a rim C,

$$\oint_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s} = 0 \qquad (2\text{-}103)^*$$

D. Laplacian Operator

1) Definition of Laplacian

Laplacian = divergence of gradient (of a scalar or a vector)

$$\nabla^2 \triangleq \nabla \cdot \nabla \qquad (54)$$

2) Calculation of Laplacian in orthogonal curvilinear coordinates

$$(52) \Rightarrow \quad \nabla \cdot = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u_i} \left(\frac{h_1 h_2 h_3}{h_i} \right)$$

$$(51) \Rightarrow \quad \nabla_i = \frac{1}{h_i} \frac{\partial}{\partial u_i}$$

(52), (51) in (54):

$$\nabla^2 \triangleq \nabla \cdot \nabla = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u_i} \left(\frac{h_1 h_2 h_3}{h_i} \frac{1}{h_i} \frac{\partial}{\partial u_i} \right) \qquad (55)$$

In Cartesian coordinates $(u_1, u_2, u_3)=(x, y, z)$, $h_1 = h_2 = h_3 = 1$ (10) ;

$$\nabla^2 = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (55)_{Car.}$$

In cylindrical coordinates $(u_1, u_2, u_3) = (r, \phi, z)$, $h_1 = 1$, $h_2 = r$, $h_3 = 1$ (22) ;

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \qquad (55)_{cyl.}$$

In spherical coordinates $(u_1, u_2, u_3) = (R, \theta, \phi)$, $h_1 = 1$, $h_2 = R$, $h_3 = R\sin\theta$ (28);

$$\nabla^2 = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial R}\left(R^2\sin\theta\frac{\partial}{\partial R}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right]$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \qquad (55)_{sph.}$$

In toroidal coordinates $(u_1, u_2, u_3) = (r, \phi, \theta)$, $h_1 = 1$, $h_2 = R = R_o + r\cos\theta$, $h_3 = r$ (47);

$$\nabla^2 = \frac{1}{Rr}\left[\frac{\partial}{\partial r}\left(Rr\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \Phi}\left(\frac{r}{R}\frac{\partial}{\partial \Phi}\right) + \frac{\partial}{\partial \Theta}\left(\frac{R}{r}\frac{\partial}{\partial \Theta}\right)\right]$$

$$= \frac{1}{Rr}\frac{\partial}{\partial r}\left(Rr\frac{\partial}{\partial r}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{Rr^2}\frac{\partial}{\partial \theta}\left(R\frac{\partial}{\partial \theta}\right)\Bigg] \qquad (55)_{tor.}$$

E. Vector Identities

1) Two null identities

a) Identity I

The curl of gradient always results in a null vector.

$$\nabla \times (\nabla V) \equiv \mathbf{0} \qquad (2\text{-}105)$$

(Proof 1) Using Stokes's theorem (2-103), (2-51)

$$\underline{\int_S [\nabla \times (\nabla V)] \cdot d\mathbf{s}} = \oint_C \nabla V \cdot d\mathbf{l} = \oint_C dV \,\underline{= 0} \qquad (2\text{-}106,\ 107)$$

For any surface $d\mathbf{s}$, $\nabla \times (\nabla V) = 0$ $\Rightarrow$ (2-105)

(Proof 2) Using the notation (summation convention) & the symbol ϵ_{ijk},

$$\underline{[\nabla \times (\nabla V)]_i} = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_k}\right)$$

$$= \epsilon_{ikj}\frac{\partial}{\partial x_k}\left(\frac{\partial V}{\partial x_j}\right) \quad \text{by exchanging indices j \& k}$$

$$= -\epsilon_{ijk}\frac{\partial}{\partial x_k}\left(\frac{\partial V}{\partial x_j}\right) \quad \text{by the property of symbol } \epsilon_{ijk}$$

$$= -\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_k}\right) \quad \text{since } \frac{\partial^2}{\partial x_k \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_k}$$

$\underline{= 0}$ because a = − a only for a = 0.

Notes)

$$\nabla \times (\nabla V) = \nabla \times \boldsymbol{A} = 0$$

$\Rightarrow$ $\boldsymbol{A}$: a curl-free (conservative) vector field that can always be expressed as the gradient of a scalar field (∇V).

(e.g.) In electrostatics, $\nabla \times \boldsymbol{E} = \boldsymbol{0}$. Therefore, $\boldsymbol{E}$ can be found from scalar electric potential V such that

$$\boldsymbol{E} = -\nabla V. \tag{2-108}$$

b) Identity II

The divergence of curl always vanishes.

$$\nabla \cdot (\nabla \times \boldsymbol{A}) \equiv 0 \tag{2-109}$$

(Proof 1) Using divergence theorem (2-75) & Stokes's theorem (2-103),

$$\underline{\int_V \nabla \cdot (\nabla \times A)\, dv} = \oint_S (\nabla \times A) \cdot ds \tag{2-110}$$

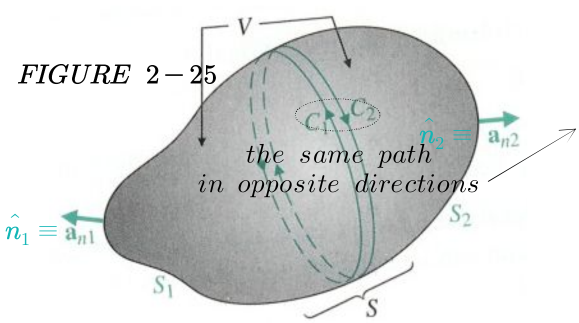
$$= \int_{S_1} (\nabla \times A) \cdot \hat{n}_1 ds + \int_{S_2} (\nabla \times A) \cdot \hat{n}_2 ds$$

$$= \oint_{C_1} A \cdot dl + \oint_{C_2} A \cdot dl$$

$$= \oint_{C_1} A \cdot dl - \oint_{C_1} A \cdot dl$$

$$\underline{= 0}$$

FIGURE 2-25



(Proof 2) Using the notation (summation convention) & the symbol ϵ_{ijk},

$$\underline{\nabla \cdot (\nabla \times A)} = \frac{\partial}{\partial x_i}(\nabla \times A)_i = \frac{\partial}{\partial x_i}\left(\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}\right) = \epsilon_{ijk}\frac{\partial}{\partial x_i}\left(\frac{\partial A_k}{\partial x_j}\right)$$

$$= \epsilon_{jik}\frac{\partial}{\partial x_j}\left(\frac{\partial A_k}{\partial x_i}\right) \text{ by exchanging indices i \& j}$$

$$= -\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial A_k}{\partial x_i}\right) \text{ by the property of symbol } \epsilon_{ijk}$$

$$= -\epsilon_{ijk}\frac{\partial}{\partial x_i}\left(\frac{\partial A_k}{\partial x_j}\right) \text{ since } \frac{\partial^2}{\partial x_j \partial x_i} = \frac{\partial^2}{\partial x_i \partial x_j}$$

$$\underline{= 0} \text{ because a = - a only for a = 0.}$$

Notes)

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = \nabla \cdot \boldsymbol{B} = 0$$

$\Rightarrow$ $\boldsymbol{B}$: a divergence-free (solenoidal) vector field that can be expressed as the curl of another vector field ($\nabla \times \boldsymbol{A}$).

(e.g.) For the magnetic flux density $\boldsymbol{B}$, $\nabla \cdot \boldsymbol{B} = \boldsymbol{0}$. Therefore, $\boldsymbol{B}$ can be found from the vector magnetic potential $\boldsymbol{A}$ such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. (2-112)

2) Some other useful vector identities

See the inside of the front cover of the text or 'NRL Plasma Formulary' on the lecture note website.

a) $\nabla(fV) = f\nabla V + V\nabla f$

b) $\nabla \cdot (f\boldsymbol{A}) = f\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f$ (2-114)

c) $\nabla \times (f\boldsymbol{A}) = f\nabla \times \boldsymbol{A} + \nabla f \times \boldsymbol{A}$ (2-115)

d) $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$

e) $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A}(\nabla \cdot \boldsymbol{B})$

f) $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$

g) $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

(Proof d)

$$\begin{aligned}\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) &= \nabla_i (\boldsymbol{A} \times \boldsymbol{B})_i = \nabla_i \epsilon_{ijk} A_j B_k \\ &= \epsilon_{ijk}[(\nabla_i A_j)B_k + A_j(\nabla_i B_k)] \\ &= B_k \epsilon_{ijk} \nabla_i A_j - A_j \epsilon_{jik} \nabla_i B_k \\ &= B_k (\nabla \times \boldsymbol{A})_k - A_j (\nabla \times \boldsymbol{B})_j \\ &= \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})\end{aligned}$$

(Proof g)

$$\begin{aligned}[\nabla \times (\nabla \times \boldsymbol{A})]_i &= \epsilon_{ijk} \nabla_j (\nabla \times \boldsymbol{A})_k \\ &= \epsilon_{ijk} \epsilon_{klm} \nabla_j \nabla_l A_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \nabla_j \nabla_l A_m \\ &= \nabla_i \nabla_j A_j - \nabla_j \nabla_j A_i \\ &= \nabla_i (\nabla \cdot \boldsymbol{A}) - \nabla^2 A_i\end{aligned}$$

$$\Rightarrow \quad \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

F. Field Classification

$\nabla \cdot \boldsymbol{F} = 0$: $\boldsymbol{F}$ = Solenoidal (or Divergenceless or Divergence-free) field

$\nabla \times \boldsymbol{F} = 0$: $\boldsymbol{F}$ = Irrotational (or Conservative or lamellar or Curl-free) field

i) $\nabla \cdot \boldsymbol{F} = 0$ & $\nabla \times \boldsymbol{F} = 0$

(e.g.) In electrostatics in charge free regions, $\nabla \cdot \boldsymbol{E} = 0$, $\nabla \times \boldsymbol{E} = 0$

ii) $\nabla \cdot \boldsymbol{F} = 0$ & $\nabla \times \boldsymbol{F} \neq 0$

(e.g.) In magnetostatics in current carrying medium, $\nabla \cdot \boldsymbol{B} = 0, \nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$

iii) $\nabla \cdot \boldsymbol{F} \neq 0$ & $\nabla \times \boldsymbol{F} = 0$

(e.g.) In electrostatics in charged regions, $\nabla \cdot \boldsymbol{E} = \rho_v/\epsilon$, $\nabla \times \boldsymbol{E} = 0$

iv) $\nabla \cdot \boldsymbol{F} \neq 0$ & $\nabla \times \boldsymbol{F} \neq 0$

(e.g.) In electromagnetics in charged regions with time-varying magnetic fields,

$$\nabla \cdot \boldsymbol{E} = \rho_v/\epsilon, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Helmholtz's Theorem :

Both $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$ are specified everywhere.

$\Rightarrow$ The field vector $\boldsymbol{F}$ is determined.

(The strengths of both the flow and vortex sources are specified.

$\Rightarrow$ The field vector $\boldsymbol{F}$ is determined.)

In the electromagnetic model based on the deductive (axiomatic) approach, $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$ for electromagnetic fields are specified by the fundamental postulates (axioms), which will then develop other theorems and phenomena.

Homework Set 2

1) P.2-18
2) P.2-20
3) P.2-21
4) P.2-23
5) P.2-26
6) P.2-29. In addition, also prove (2-115) by using summation convention and Levi-Civita symbol ϵ_{ijk}.
7) P.2-30