2. Electrostatics in Material Media

A. Electrical Properties of Materials

1) Constitutive parameters of a material medium

- Electric permittivity ϵ (F/m): Dielectrics (Insulators), Plasmas, ...
- Magnetic permeability μ (H/m): Magnetic materials (Paramagnetic, Diamagnetic, Ferromagnetic)
- Electric conductivity σ (S/m): Conductors (Metals, Plasmas, ...)

Homogeneous material
if its constitutive parameter is independent of position (constant).

Linear material
if its constitutive parameter is independent of the external field.

Isotropic material
if its constitutive parameter is independent of direction.

2) Constitutive relations

For a simple (linear, homogeneous, isotropic) medium,

$$\begin{cases} \boldsymbol{D} = \varepsilon \boldsymbol{E} & \text{(Chapter 3)} \\ \boldsymbol{B} = \mu \boldsymbol{H} & \text{(Chapter 5)} \\ \boldsymbol{J} = \sigma \boldsymbol{E} & \text{(Chapter 4)} \end{cases}$$

In general,

$$\boldsymbol{D} = \overleftrightarrow{\varepsilon} \cdot \boldsymbol{E}, \quad \boldsymbol{B} = \overleftrightarrow{\mu} \cdot \boldsymbol{H}, \quad J = \overleftrightarrow{\sigma} \cdot E$$

Notes)

$\epsilon = \epsilon_o$ in free space; scalar in dielectrics;

$\epsilon(E)$ in ferroelectric materials (nonlinear)

$\overleftrightarrow{\epsilon}$ in magnetized plasmas, some crystals, TiO_2, quartz, ...

$\mu = \mu_o$ in free space and plasmas; scalar in magnetic media;

$\mu(H)$ in ferromagnetic materials (nonlinear)

$\sigma = 0$ in free space & insulators; scalar in conductors;

$\overleftrightarrow{\sigma}$ in magnetized plasmas

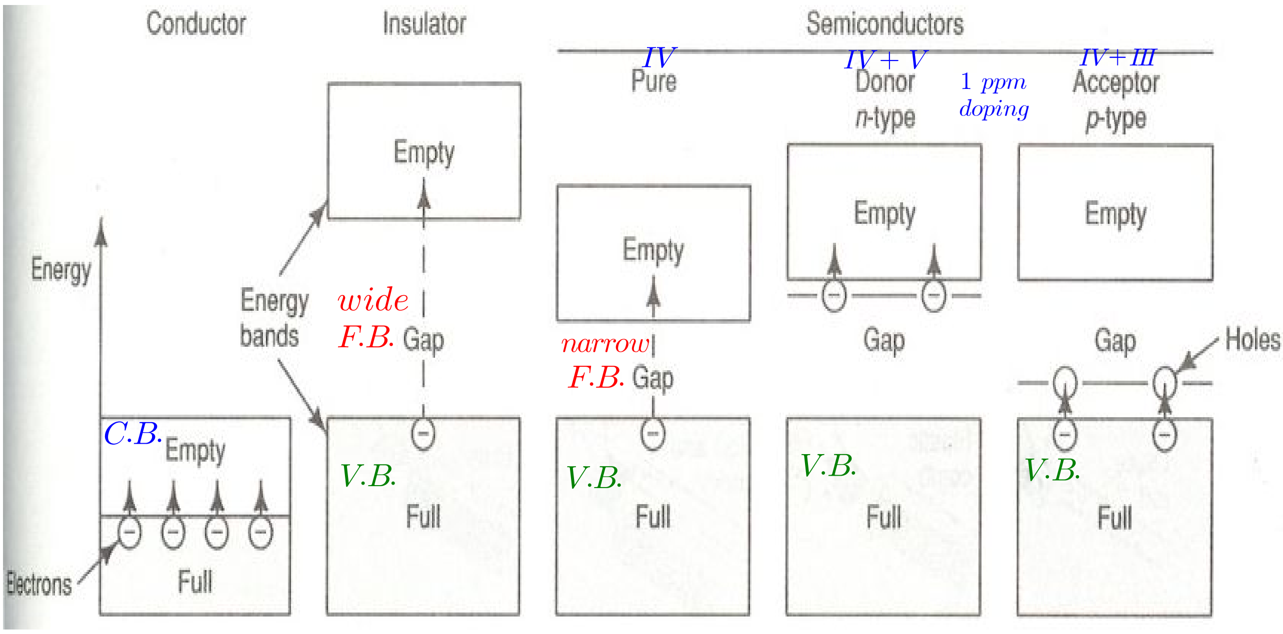
3) Classification of materials according to their electrical properties

In terms of the atomic model,

- Conductors: Free electrons in the outermost shell of the atom
- Insulators (Dielectrics): Confined electrons to their orbits in the atom
- Semiconductors: Small number of freely movable charges

In terms of the band theory,

- Conductors: Partially filled electrons in the conduction band
- Insulators (Dielectrics): Completely filled electrons in the valence band with a wide forbidden band gap
- Semiconductors: Completely filled electrons in the valence band with a narrow forbidden band gap



$F.B.$: Forbidden Band
$C.B.$: Conduction Band
$V.B.$: Valence Band

III족	IV족	V족
B		
Al	Si	P
Ga	Ge	As
In		Sb

(cf) Conductivity σ (S/m) $\Rightarrow$ Appendix B-4

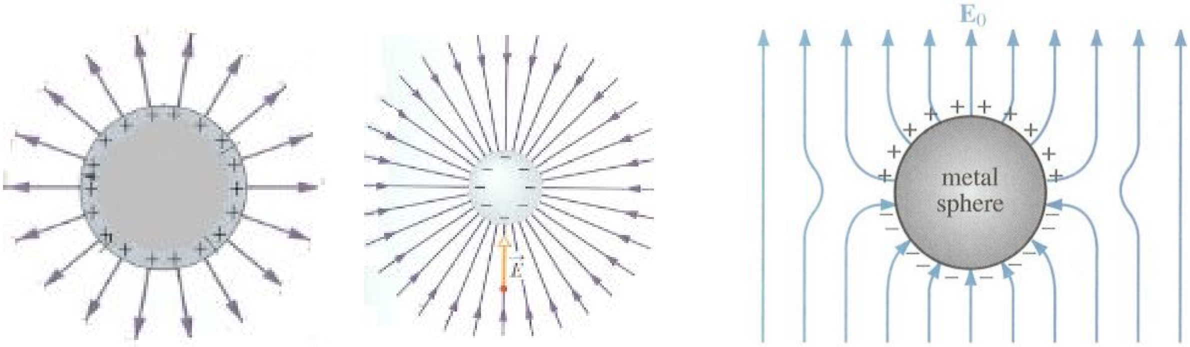
Metals (Fe, Al, Au, Cu, Ag, ...), Fusion plasmas : $10^7 \sim 10^8$

Semiconductors (Si, Ge) : 4.4×10^{-4}, 2.2

Dielectrics (Glass, Porcelain, Rubber, Mica, Quartz) : $10^{-12} \sim 10^{-17}$

B. Conductors in Static Electric Field

After introducing (a) positive or (b) negative charges inside a conducting sphere, or (c) placing it in an external electric field,



Inside a conductor under static conditions,

no free charges $\Rightarrow$ $\rho_v = 0$ (3-43)

Gauss's law: $\oint_S \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{1}{\epsilon_o} \int_V \rho_v dv = 0 \quad \Rightarrow \quad \boldsymbol{E} = 0$ (3-44)

$\boldsymbol{E} = -\nabla V = 0 \quad \Rightarrow \quad V = constant\ everywhere$

On a conductor surface under static conditions,

$\boldsymbol{E} \perp$ Conductor surface everywhere

i.e., $V_{cond.\ surf.} =$ constant (Equipotential surface)

Boundary conditions (Field relation at the interface between two media):

At a conductor-free space interface,

i) $\nabla \times \boldsymbol{E} = 0 \quad \Rightarrow \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$

$$\Rightarrow \oint_{abcda} \boldsymbol{E} \cdot d\boldsymbol{l} \overset{\Delta h \to 0}{=} E_t \Delta w = 0$$

$\Rightarrow$ $E_t = 0$ or $\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0$ (3-45)

: no tangential comp. $\boldsymbol{E}$

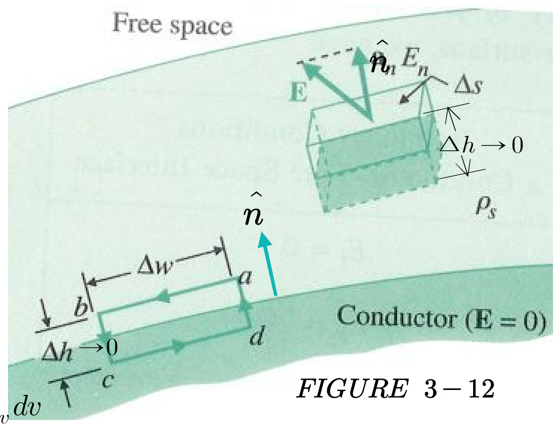


FIGURE 3−12

ii) $\nabla \cdot \boldsymbol{E} = \rho_v / \epsilon_o \Rightarrow \oint_S \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{1}{\epsilon_o} \int_V \rho_v dv$

$$\Rightarrow \oint_{pillbox} \boldsymbol{E} \cdot d\boldsymbol{s} \overset{\Delta h \to 0}{=} E_n \Delta S = \frac{\rho_s \Delta S}{\epsilon_o}$$

$\Rightarrow$ $E_n = \frac{\rho_s}{\epsilon_o}$ or $\hat{\boldsymbol{n}} \cdot \boldsymbol{E} = \frac{\rho_s}{\epsilon_o}$ (3-46)

: normal comp. $\boldsymbol{E} \propto$ surface charge density *(e.g. 3-9)* Spherical conducting shell ($R_i < R < R_o$)

with a point charge Q at the center

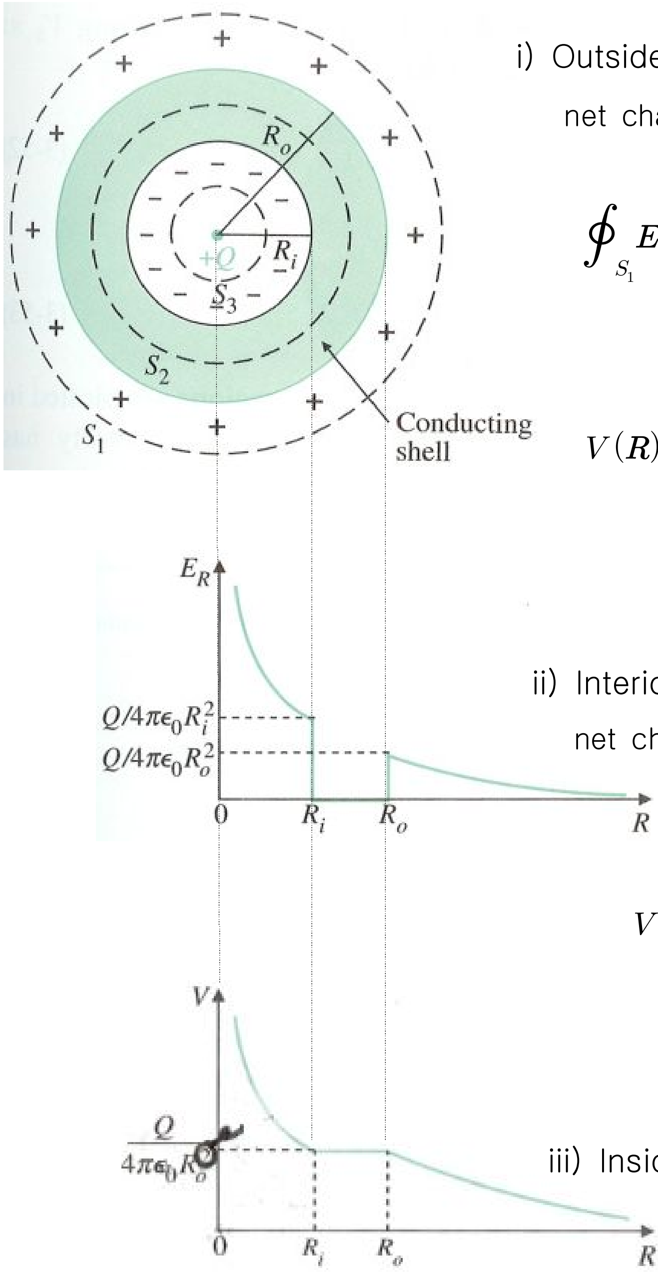


FIGURE 3 − 13

i) Outside the shell ($R > R_o$)

net charge = point charge (Q)
+ inner induced surface charge (−Q)
+ outer induced surface charge (+Q)

$$\oint_{S_1} \boldsymbol{E} \cdot d\boldsymbol{s} = E_R 4\pi R^2 = Q/\epsilon_o$$

$$\Rightarrow \quad \boldsymbol{E}(R) = \hat{\boldsymbol{R}} E_R = \hat{\boldsymbol{R}} \frac{Q}{4\pi\epsilon_o R^2}$$

$$V(\boldsymbol{R}) = -\int_{\infty}^{R} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{\infty}^{R} \frac{Q}{4\pi\epsilon_o R^2} dR$$

$$\Rightarrow \quad V(R) = \frac{Q}{4\pi\epsilon_o R}$$

ii) Interior of conducting shell ($R_i < R < R_o$)

net charge = point charge (Q)
+ inner induced surface charge (−Q)
= 0

$$\boldsymbol{E}(R) = \boldsymbol{0}$$

$$V(\boldsymbol{R}) = -\int_{\infty}^{R_o} \frac{Q}{4\pi\epsilon_o R^2} dR - \int_{R_o}^{R} 0\, dR$$

$$\Rightarrow \quad V(R) = \frac{Q}{4\pi\epsilon_o R_o} \quad = \text{constant}$$

iii) Inside the shell ($R < R_i$)

net charge = point charge (Q)

$$\boldsymbol{E}(R) = \hat{\boldsymbol{R}} \frac{Q}{4\pi\epsilon_o R^2}$$

$$V(\boldsymbol{R}) = -\int_{\infty}^{R_o} \frac{Q}{4\pi\epsilon_o R^2} dR - \int_{R_o}^{R_i} 0\, dR - \int_{R_i}^{R} \frac{Q}{4\pi\epsilon_o R^2} dR$$

$$= \frac{Q}{4\pi\epsilon_o R_o} + 0 + \left(\frac{Q}{4\pi\epsilon_o R} - \frac{Q}{4\pi\epsilon_o R_i} \right)$$

$$\Rightarrow \quad V(R) = \frac{Q}{4\pi\epsilon_o} \left(\frac{1}{R} + \frac{1}{R_o} - \frac{1}{R_i} \right)$$

C. Dielectrics in Static Electric Field

1) Polarization

Displacement of the entire negative charge (electrons) relative to positive charge (nucleus) in dielectrics

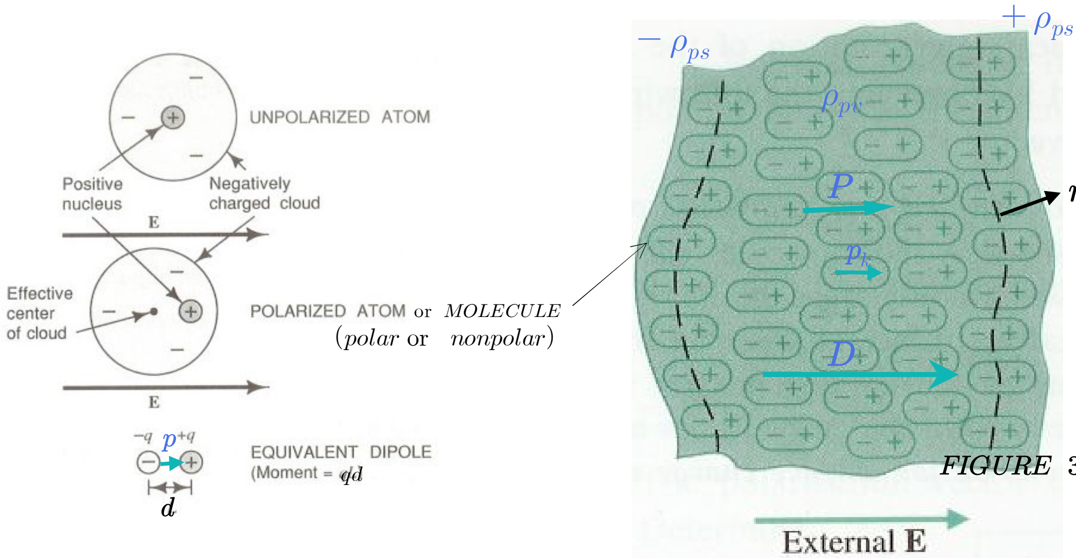


FIGURE 3−14

In dielectrics, ∃ bound polarization charges

not free to move very far and be extracted from dielectrics

→ fictitious isolated charges → theoretical electric field

(cf) In conductors, ∃ free electrons

can be detached and migrate

→ true movable charges → measurable electric field

a) Polarization field $\boldsymbol{P}$

= Electric dipole moments per unit volume

$$\boldsymbol{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{p}_k}{\Delta v} \qquad (\text{C/m}^2) \qquad (3\text{-}53)$$

Electrostatic potential due to $\boldsymbol{P}$:

$$(3\text{-}36) \Rightarrow \quad dV = \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{R}}}{4\pi\epsilon_o R^2} dv' \quad \Rightarrow \quad V = \frac{1}{4\pi\epsilon_o} \int_{V'} \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{R}}}{R^2} dv' \qquad (3\text{-}56)$$

b) Polarization surface charge density ρ_{ps}

$$(3\text{-}53) \quad \Rightarrow \quad \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = \frac{n\Delta v}{\Delta v} q d = \frac{(\Delta Q_s)\, d}{\Delta v} = \frac{(\Delta Q_s) d}{(\Delta s) d} = \rho_{ps}$$

$$\Rightarrow \quad \rho_{ps} = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} \qquad (\text{C/m}^2) \qquad (3\text{-}57)$$

c) Polarization volume charge density ρ_{ps}

Net charge remaining within V bounded by the polarized surface:

$$Q_v = \underline{\int_V \rho_{pv}\, dv} = -\oint_S \rho_{ps}\, ds = -\oint_S \boldsymbol{P} \cdot \hat{\boldsymbol{n}}\, ds = \underline{\int_V (-\nabla \cdot \boldsymbol{P})\, dv}$$

$$\Rightarrow \quad \rho_{pv} = -\nabla \cdot \boldsymbol{P} \quad (\text{C/m}^3)$$: bound charges (3−59)

Notes)

Potential due to the surface and volume charge distributions in a polarized dielectric: (3-39), (3-38) $\Rightarrow$

$$V(\boldsymbol{R}) = \frac{1}{4\pi\epsilon_o}\left(\oint_{S'} \frac{\rho_{ps}(\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|}ds' + \int_{V'} \frac{\rho_{pv}(\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|}dv'\right) \quad (3\text{-}60)$$

Consequently, electric field intensity can be found from $\boldsymbol{E} = -\nabla V$.

2) Electric flux density and generalized Gauss's law

In a dielectric, from (3-3) and (3-59)

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_v}{\epsilon_o} + \frac{\rho_{pv}}{\epsilon_o}$$ (free charges + bound charges = total charges)

$$\Rightarrow \quad \nabla \cdot (\epsilon_o \boldsymbol{E} + \boldsymbol{P}) = \rho_v \quad (3\text{-}61)$$

The electric flux density (or displacement field) is defined as

$$\boldsymbol{D} = \epsilon_o \boldsymbol{E} + \boldsymbol{P} \quad (\text{C/m}^2) \quad (3\text{-}62)$$

(3-62) in (3-61) $\Rightarrow$

Differential form of Generalized Gauss's law for any medium :

$$\nabla \cdot \boldsymbol{D} = \rho_v \quad (\text{C/m}^3) \quad (3\text{-}63)$$: free charges

(3-63) using the divergence theorem $\Rightarrow$

Integral form of Generalized Gauss's law for any medium:

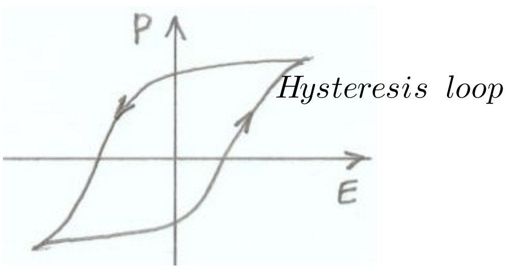
$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = \int_V \rho_v dv = Q \quad (\text{C}) \quad (3\text{-}65)$$: total free charge

3) Constitutive relation and electric material properties

For a simple (homogeneous, linear, isotropic) medium,

$$\boldsymbol{P} = \epsilon_o \chi_e \boldsymbol{E} \quad (3\text{-}66)$$

(cf) For a nonlinear ferroelectric material,



where χ_e is the electric susceptibility. Then, (3-62) becomes

$$\boldsymbol{D} = \epsilon_o \boldsymbol{E} + \boldsymbol{P} = \epsilon_o(1+\chi_e)\boldsymbol{E} = \epsilon_o\epsilon_r \boldsymbol{E} = \epsilon \boldsymbol{E}$$

Dielectric constant (or Relative permittivity):

$$\epsilon_r = 1 + \chi_e = \epsilon/\epsilon_o \quad (3\text{-}68)$$

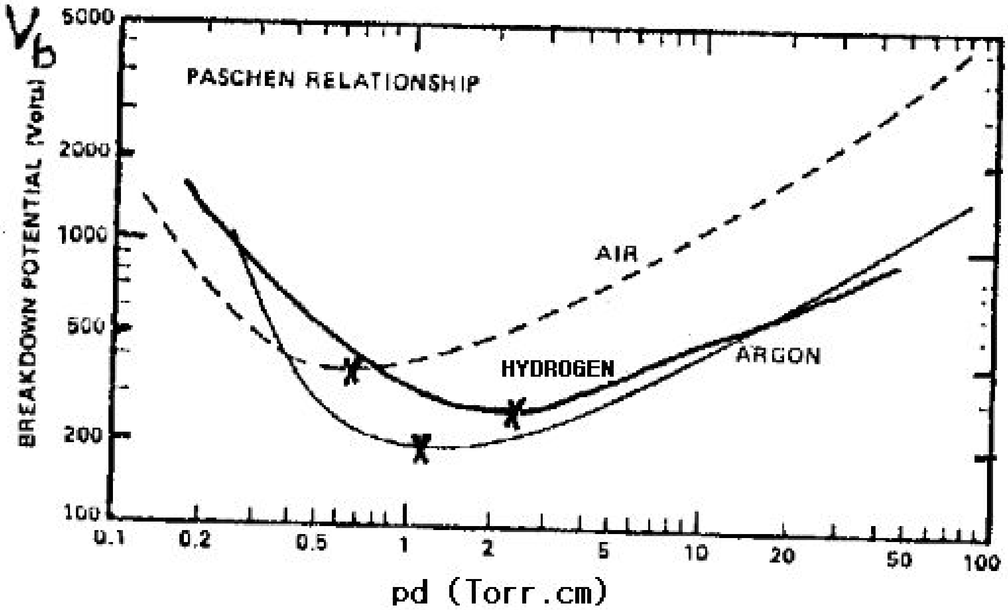
Dielectric strength E_b = Critical E where the dielectric breakdown occurs.

(cf) Table 3-1 & Appendix B-3

(e.g.)

	ϵ_r	E_b (kV/mm)
air	1.0006	3
paper	2-4	15
rubber	2.3-4	25
bakelite	5	20
glass	4-10	30
mica	6	200

Breakdown voltage V_b for gas discharges $\Rightarrow$ Paschen curves



p=gas pressure ; d=distance between the electrodes

For dry atmospheric air in uniform E

$$V_b = 3000d + 1.35 \quad kV$$

$$E_b \approx \frac{dV}{dx} \approx 3 \quad kV/mm$$

For most of pure gases,

$$V_{b,\min} \approx 100 - 500 \quad V$$

$$pd = 0.1 - 10 \quad Torr\ cm$$

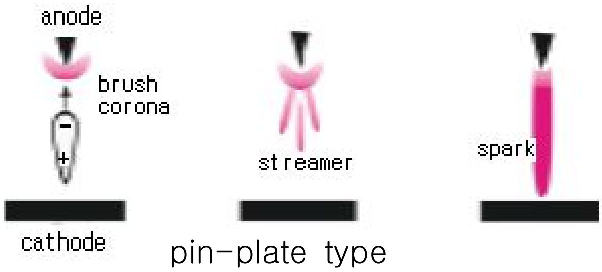
Notes)

$\boldsymbol{E}$ tends to be higher at the conductor surface of a larger curvature.

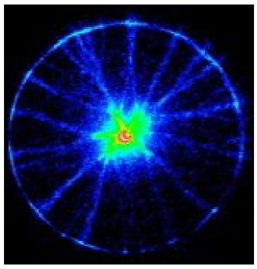
$\Longrightarrow$ Applications:

Lightning arrester

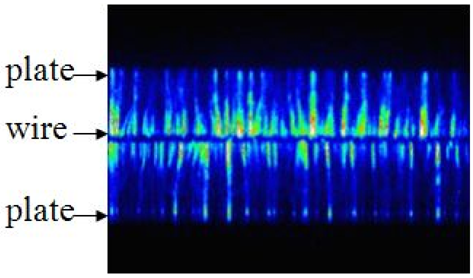
Corona discharges



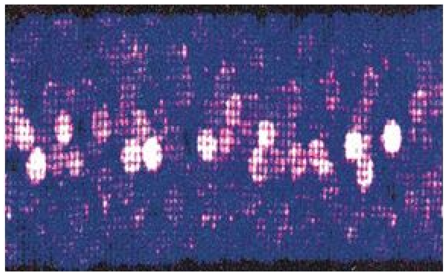
pin-plate type



wire-cylinder type

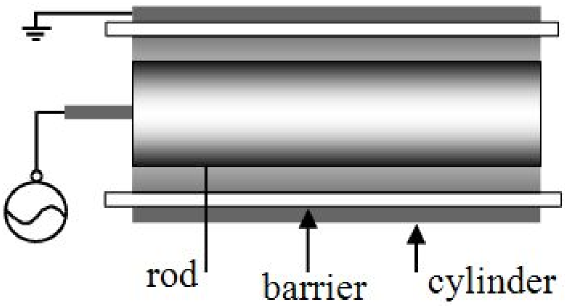


wire-plate type



ferroelectric packed discharge

Dielectric barrier dicharge (DBD)



(e.g. 3-11) Two connected spherical conductors with different curvatures

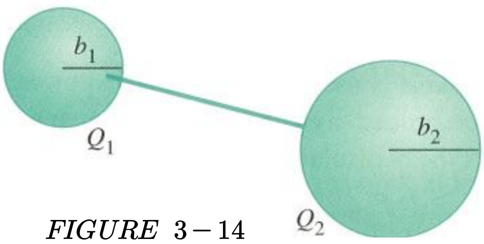


FIGURE 3 – 14

a) Same potential

$$\Longrightarrow \quad \frac{Q_1}{4\pi\epsilon_o b_1} = \frac{Q_2}{4\pi\epsilon_o b_2}$$

$$\Longrightarrow \quad \frac{Q_1}{Q_2} = \frac{b_1}{b_2}$$

: Charging in proportion to radius

b) (3-46) $E_n = \rho_s/\epsilon_o$ and $\rho_s = Q/A = Q/4\pi R^2$

$$\Longrightarrow \quad E_{1n} = Q_1/4\pi\epsilon_o b_1^2, \qquad E_{2n} = Q_2/4\pi\epsilon_o b_2^2$$

$$\Longrightarrow \quad \frac{E_{1n}}{E_{2n}} = \left(\frac{b_2}{b_1}\right)^2 \frac{Q_1}{Q_2} = \frac{b_2}{b_1} > 1$$

: Higher E at larger curvature

4) Boundary conditions

a) Tangential component of E

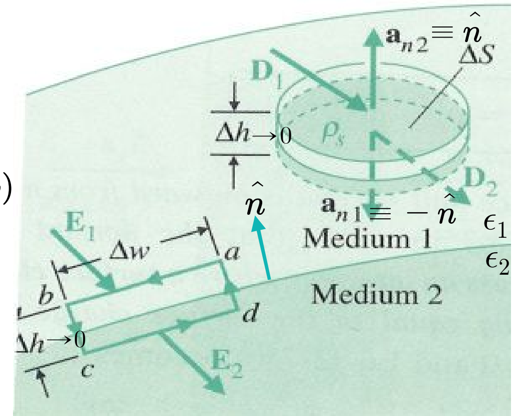
$$\nabla \times \boldsymbol{E} = 0 \quad \Rightarrow \quad \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

$$\Rightarrow \oint_{abcda} \boldsymbol{E} \cdot d\boldsymbol{l} = \boldsymbol{E}_1 \cdot \Delta \boldsymbol{w} + \boldsymbol{E}_2 \cdot (-\Delta \boldsymbol{w})$$

$$\Delta h \to 0 \qquad = (E_{1t} - E_{2t})\,\Delta w = 0$$

$$\Rightarrow \quad E_{1t} = E_{2t} \qquad \text{(V/m)}$$

or $\quad \hat{n} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0 \qquad$ (3-72)



$$\Rightarrow \quad D_{1t}/\epsilon_1 = D_{2t}/\epsilon_2 \quad \text{or} \quad \hat{n} \times (\boldsymbol{D}_1/\epsilon_1 - \boldsymbol{D}_2/\epsilon_2) = 0 \qquad \text{(3-73)}$$

For medium 2 = conductor (dielectric-conductor), $\boldsymbol{E}_2 = 0$.

$$E_{1t} = 0 \quad \text{or} \quad \hat{n} \times \boldsymbol{E}_1 = 0 \quad \Rightarrow \quad \text{(3-45)}$$

b) Normal component of D

$$\nabla \cdot \boldsymbol{D} = \rho_v \quad \Rightarrow \quad \oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = \int_V \rho_v \, dv$$

$$\Rightarrow \oint_{pillbox} \boldsymbol{D} \cdot d\boldsymbol{s} = (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \hat{n}\,\Delta S$$

$$\Delta h \to 0 \qquad = (D_{1n} - D_{2n})\Delta S = \rho_s \Delta S$$

$$\Rightarrow \quad D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2)$$

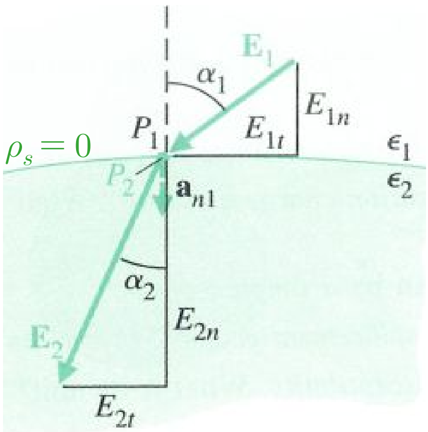
or $\quad \hat{n} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_s \qquad$ (3-75)

For $\rho_s = 0$ (dielectric-dielectric), $D_{1n} = D_{2n}$ & $\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$ (3-77, 78)

For medium 2 = conductor (dielectric-conductor), $\boldsymbol{D}_2 = 0$.

$$D_{1n} = \rho_s \quad \text{or} \quad \hat{n} \cdot \boldsymbol{D} = \rho_s \quad \Rightarrow \quad \text{(3-46)}$$

(e.g. 3-14) At a dielectric-dielectric interface,



(3-72): $E_{1t} = E_{2t}$

(3-78): $\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$

$$\Rightarrow \quad \frac{E_1 \sin\alpha_1}{\epsilon_1 E_1 \cos\alpha_1} = \frac{E_2 \sin\alpha_2}{\epsilon_2 E_2 \cos\alpha_2}$$

$$\Rightarrow \quad \tan\alpha_2 = \frac{\epsilon_2}{\epsilon_1}\tan\alpha_1 = \frac{\epsilon_{r2}}{\epsilon_{r1}}\tan\alpha_1$$

$$E_2 = \sqrt{E_{2t}^2 + E_{2n}^2} = \sqrt{E_{1t}^2 + (\epsilon_1 E_{1n}/\epsilon_2)^2}$$

$$= \sqrt{(E_1 \sin\alpha_1)^2 + (\epsilon_1 E_1 \cos\alpha_1/\epsilon_2)^2}$$

$$= E_1 \sqrt{\sin^2\alpha_1 + (\epsilon_1 \cos\alpha_1/\epsilon_2)^2}$$