3. Capacitance, Electrostatic Energy and Forces

A. Capacitors and Capacitances

1) Capacitors (or Condenser) with dielectric media

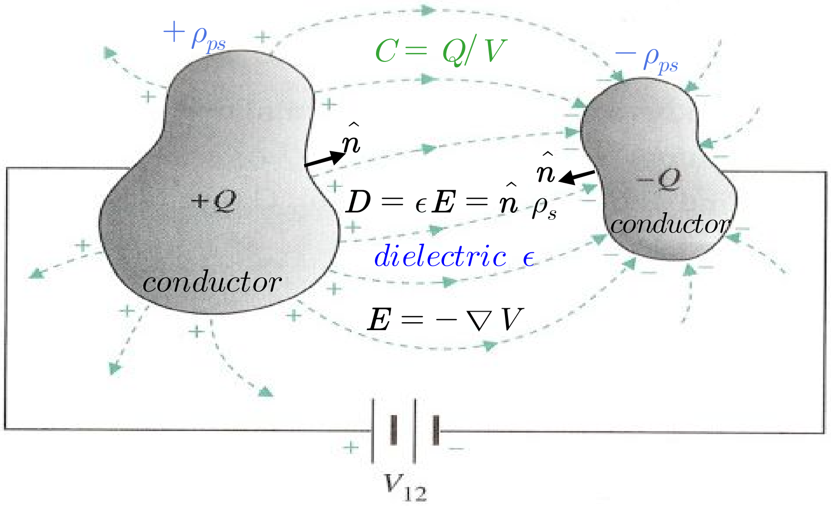


FIGURE 3−19

Capacitor = a device for storing electric charge (i.e., electric energy) consisting of two conductors separated by a dielectric

2) Capacitance

= Charge on one conductor / applied voltage (potential difference)

for an isolated conductor / *for the two conductors*

$$C = \frac{Q}{V} \quad \text{or} \quad C = \frac{Q}{V_{12}} \qquad \text{(C/V = F)} \qquad \text{(3-85, 3-86)}$$

where

$$Q = \oint_S \rho_s \, ds = \oint_S \mathbf{D} \cdot \hat{n} \, ds = \oint_S \epsilon \mathbf{E} \cdot d\mathbf{s} \qquad (1)$$

$$V = V_{12} = -\int_{P_2 \text{ or } \infty}^{P_1} \mathbf{E} \cdot d\mathbf{l} \qquad (2)$$

Note) $$C = \frac{Q}{V_{12}} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\mathbf{l}} = f(\epsilon, S, L) \qquad (3)$$

: Depends only on material & geometry;

Independent of Q and V $\Rightarrow$ $Q \uparrow$ *as* $V \uparrow$

(e.g. 3-15) Parallel-plate capacitor

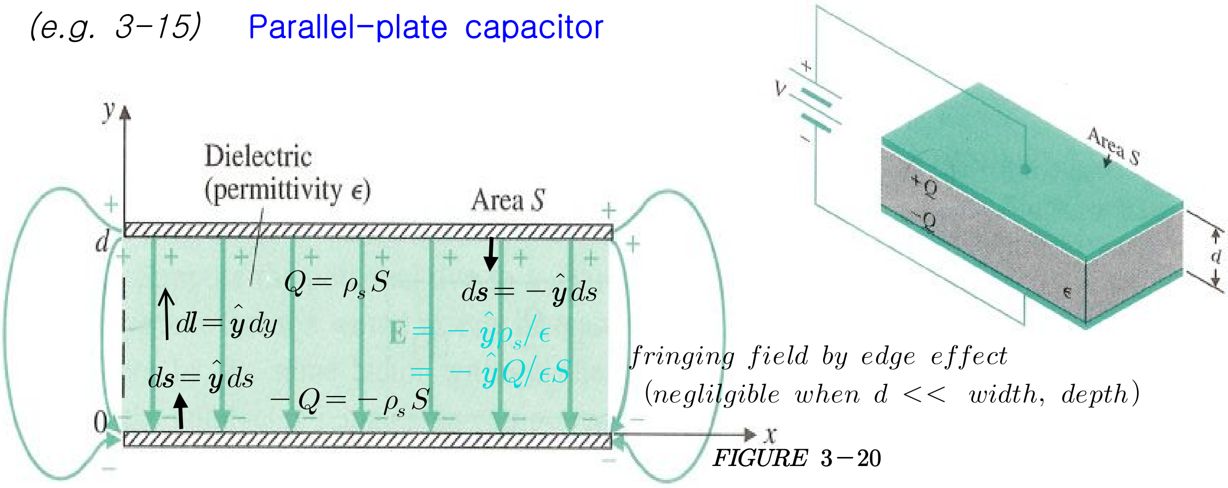


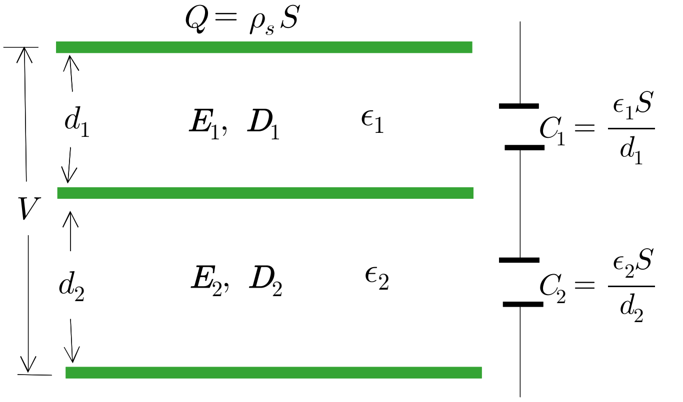
FIGURE 3−20

$$C = \frac{Q}{V_{12}} = \frac{Q}{Ed} = \frac{Q}{(Q/\epsilon S)\,d} = \epsilon\frac{S}{d} = \epsilon_o\epsilon_r\frac{S}{d} = 8.85\,\epsilon_r\frac{S}{d} \quad \text{(pF)} \qquad (3\text{-}87)$$

or

$$= \frac{\oint_S \epsilon \mathbf{E}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\mathbf{l}} = \frac{\oint_S (-\hat{y}\rho_s)\cdot(-\hat{y}\,ds)}{-\int_0^d (-\hat{y}\rho_s/\epsilon)\cdot(\hat{y}\,dy)} = \frac{\rho_s S}{(\rho_s/\epsilon)d} = \epsilon\frac{S}{d}$$

(cf) Series connection



$$C = \frac{Q}{V} = \frac{D_1 S}{E_1 d_1 + E_2 d_2}$$

$D_{1n} = D_{2n}$

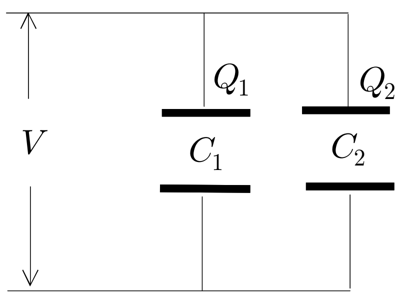
$$= \frac{\epsilon_1 E_1 S}{E_1 d_1 + (\epsilon_1 E_1/\epsilon_2) d_2}$$

$$= \frac{S}{d_1/\epsilon_1 + d_2/\epsilon_2}$$

$$= \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$\Rightarrow \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \qquad (4)$$

Parallel connection



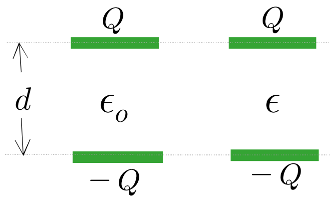
$$C = \frac{Q}{V} = \frac{Q_1 + Q_2}{V}$$

$$= \frac{C_1 V + C_2 V}{V}$$

$$= C_1 + C_2 \qquad (5)$$

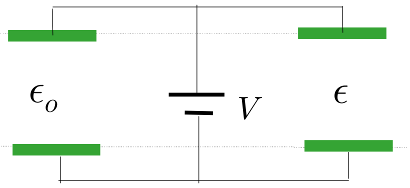
Notes)

i) For the same charging Q



$$Q_1 = Q_2 \quad (= Q)$$
$$\rho_{s1} = \rho_{s2} \quad (= Q/S)$$
$$D_1 = D_2 \quad (= Q/S)$$
$$E_1 = D/\epsilon_o > E_2 = D/\epsilon$$
$$V_1 = E_1 d > V_2 = E_2 d$$
$$\frac{E_2}{E_1} = \frac{V_2}{V_1} = \frac{C_1}{C_2} = \frac{W_2}{W_1} = \frac{1}{\epsilon_r}$$

ii) For the same applied voltage V



$$V_1 = V_2 \quad (= V)$$
$$E_1 = E_2 \quad (= V/d)$$
$$D_1 = \epsilon_o E_1 < D_2 = \epsilon E_2$$
$$\rho_{s1} < \rho_{s2}$$
$$\frac{D_2}{D_1} = \frac{\rho_{s2}}{\rho_{s1}} = \frac{Q_2}{Q_1} = \frac{C_2}{C_1} = \frac{W_2}{W_1} = \epsilon_r$$

(e.g. 3-16) Cylindrical capacitor of coaxial line with $b - a \ll L$

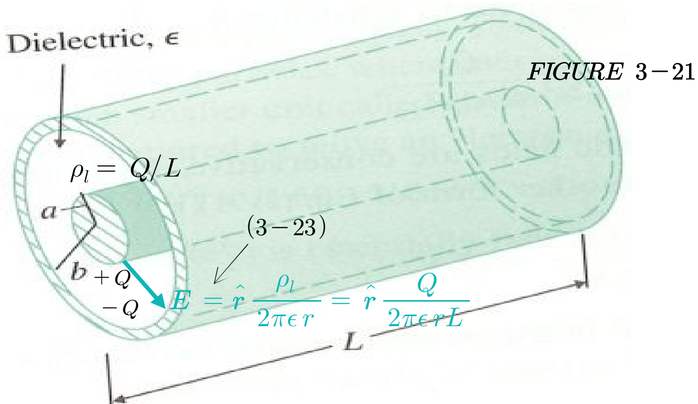
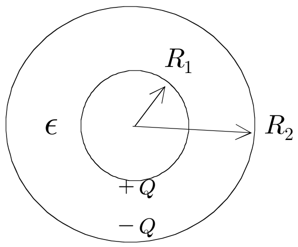


FIGURE 3−21

$$V_{ab} = -\int_a^b \mathbf{E} \cdot dl = -\int_a^b \hat{r}\frac{\rho_l}{2\pi\epsilon r} \cdot (\hat{r}dr) = -\frac{\rho_l}{2\pi\epsilon}\int_a^b \frac{dr}{r} = -\frac{Q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)$$

$$\therefore\ C = \frac{Q}{V_{ab}} = \frac{2\pi\epsilon L}{\ln(b/a)} \quad \text{or} \quad \frac{C}{L} = \frac{2\pi\epsilon}{\ln(b/a)} = \frac{24.2\epsilon_r}{\log(b/a)} \ (\text{pF/m}) \qquad (3\text{-}90)$$

(e.g.) Spherical capacitor of two concentric spheres

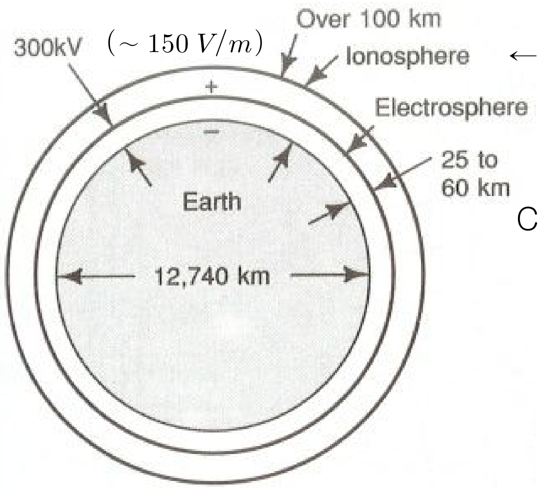


$$C = \frac{Q}{V_{R_1R_2}} = \frac{Q}{\frac{Q}{4\pi\epsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$
$$= \frac{4\pi\epsilon\, R_1 R_3}{(R_2 - R_1)} \qquad (6)$$

For $R_2 \rightarrow \infty$ (Capacitor of a single isolated sphere),

$$C = 4\pi\epsilon R_1 \qquad (6)^*$$

(cf) Earth capacitor



Ionosphere $\leftarrow$ *plasma* $(e+i)$ *by solar radiation* (UV)

Electrosphere $\leftarrow$ *ionized molecules by cosmic ray*

Capacitance of Earth-Electrosphere:

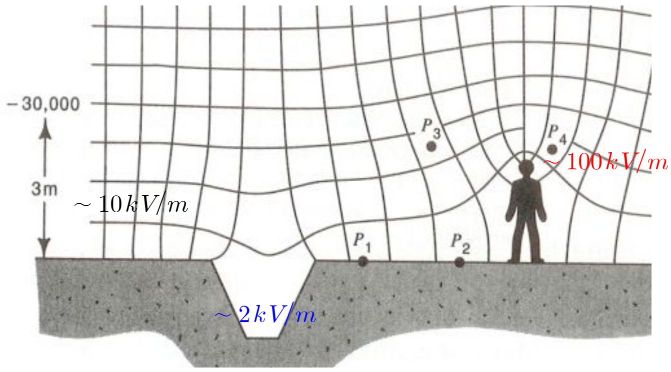
(6)* $\Rightarrow \quad C \approx 0.18$ (F)

$$Q = \rho_s S \approx 1.65 \times 10^6 \quad \text{(C)}$$

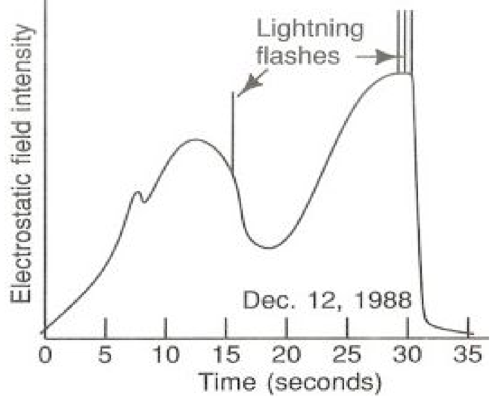
$$W_e = \frac{1}{2}\frac{Q^2}{C} \approx 7 \times 10^{12} \quad \text{(J)}$$

($12\,V$-$100\,A \cdot hr$ battery $\times 10^6$)

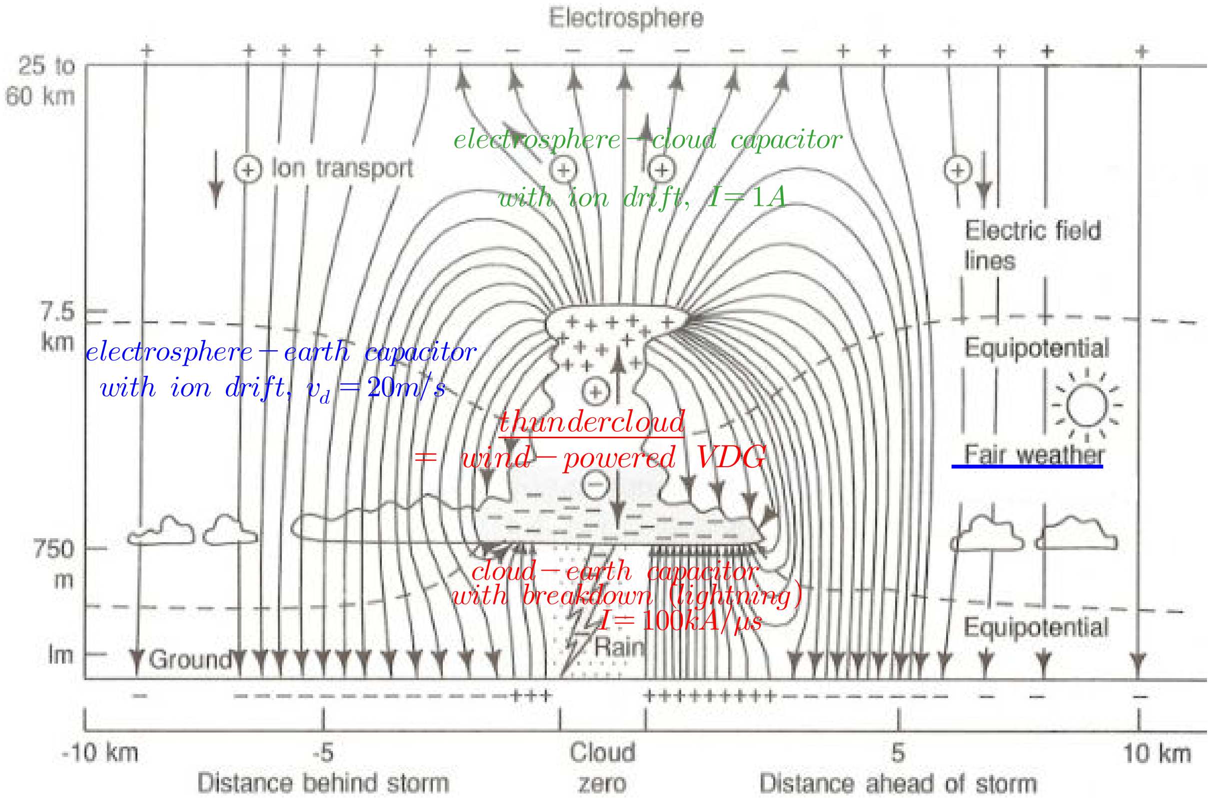
Equipotentials under a thundercloud



$\boldsymbol{E}$ during passage of a thundercloud



Maynard Hill's model of $\boldsymbol{E}$ of the atmospheric(earth-electrosphere) capacitor



B. Electrostatic Energy and Forces

1) Electric energy stored in a charge distribution

Chap.3-1D.2): Eq.(12) $dW = q\,dV$ with Eq. (13)

$\Rightarrow$ Work required to bring a charge q from infinity against $\boldsymbol{E}$

= Electrostatic potential energy stored in $\boldsymbol{E}$

$$W_e = q\,V \tag{7}$$

Energy stored in the assembly of two point charges :

$$\underline{W_2 = Q_2 V_2} = Q_2 \frac{Q_1}{4\pi\epsilon_o R_{12}} = Q_1 \frac{Q_2}{4\pi\epsilon_o R_{12}} = \underline{Q_1 V_1}$$

$$\Rightarrow \quad W_2 = \frac{1}{2}(Q_1 V_1 + Q_2 V_2) = \frac{1}{2}\sum_{k=1}^{2} Q_k V_k \tag{3-93}$$

Energy stored in the assembly of three point charges :

$$W_3 = W_2 + Q_3 V_3 = Q_2 V_2 + Q_3\left(\frac{Q_1}{4\pi\epsilon_o R_{13}} + \frac{Q_2}{4\pi\epsilon_o R_{23}}\right)$$

$$= \frac{1}{4\pi\epsilon_o}\left(\frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_3}{R_{23}}\right)$$

$$= \frac{1}{4\pi\epsilon_o}\frac{1}{2}\left[Q_1\left(\frac{Q_2}{R_{12}} + \frac{Q_3}{R_{13}}\right) + Q_2\left(\frac{Q_1}{R_{12}} + \frac{Q_3}{R_{23}}\right) + Q_3\left(\frac{Q_1}{R_{13}} + \frac{Q_2}{R_{23}}\right)\right]$$

$$= \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3) = \frac{1}{2}\sum_{k=1}^{3} Q_k V_k \tag{3-96}$$

....

Energy stored in a system of discrete point charges :

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \quad \text{(J or eV)} \tag{3-97}$$

Energy stored in a continuous distribution of charge :

$$W_e = \frac{1}{2}\int_{V'} V\,dQ' = \frac{1}{2}\int_{V'} \rho_v V\,dv' \quad \text{(J or eV)} \tag{3-101}$$

(e.g. 3-17)

Assembling a uniform of sphere of charge

Work or energy in bringing up dQ_R:

$$dW_e = V_R\,dQ_R$$

Total work or energy required to assemble:

$$W_e = \int dW_e = \int V_R\,dQ_R$$

$$= \frac{4\pi}{3\epsilon_o}\rho_v^2 \int_0^b R^4\,dR = \frac{4\pi\rho_v^2 b^5}{15\epsilon_o} = \frac{3\,Q^2}{20\pi\epsilon_o b}$$

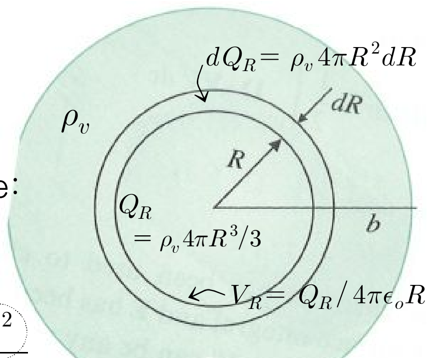


FIGURE 3−22

2) Electrostatic energy in terms of fields

Gauss's law $\rho_v = \nabla \cdot \boldsymbol{D}$ in (3-101) :

$$W_e = \frac{1}{2}\int_V (\nabla \cdot \boldsymbol{D}) V dv = \frac{1}{2}\int_V \nabla \cdot (V\boldsymbol{D})\, dv - \frac{1}{2}\int_V \boldsymbol{D} \cdot \nabla V dv$$

$\nabla \cdot (V\boldsymbol{D}) = V\nabla \cdot \boldsymbol{D} + \boldsymbol{D} \cdot \nabla V$

$$= \frac{1}{2}\oint_S V\boldsymbol{D} \cdot d\boldsymbol{s} + \frac{1}{2}\int_V \boldsymbol{D} \cdot \boldsymbol{E}\, dv \qquad (3\text{-}104)$$

Divergence theorem

0 since $V \propto R^{-1}$, $D \propto R^{-2}$, $S \propto R^2$

$\Rightarrow$ $integral \propto R^{-1} \to 0$ as $R \to \infty$

∴ Electric energy stored in the field :

$$W_e = \frac{1}{2}\int_V \boldsymbol{D} \cdot \boldsymbol{E}\, dv = \frac{1}{2}\int_V \epsilon E^2\, dv \quad \text{(J)} \qquad (3\text{-}105,\ 106)$$

for a simple medium ($\boldsymbol{D} = \epsilon \boldsymbol{E}$)

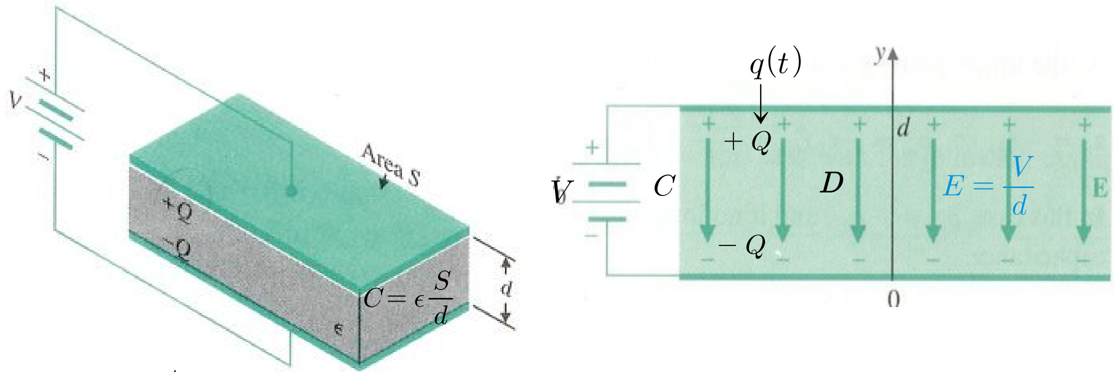
Consequently, electric energy density is

$$w_e = \frac{1}{2}\boldsymbol{D} \cdot \boldsymbol{E} = \frac{1}{2}\epsilon E^2 = \frac{1}{2}\epsilon_o E^2 + \frac{1}{2}PE \quad (\text{J/m}^3) \qquad (3\text{-}108)$$

such that $W_e = \int_V w_e\, dv$ (3-107)

3) Electrostatic energy stored in capacitors

(e.g. 3-18) Paralllel-plate capacitor



$E = V/d$ in (3-106):

$$W_e = \frac{1}{2}\int_V \epsilon\left(\frac{V}{d}\right)^2 dv = \frac{1}{2}\epsilon\left(\frac{V}{d}\right)^2 (Sd) = \frac{1}{2}\left(\epsilon\frac{S}{d}\right)V^2 = \frac{1}{2}CV^2 \qquad (3\text{-}109)$$

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C} \qquad (3\text{-}110)$$

($Q = CV$)

Note) Other way of deriving (3-110):

(7) $\Rightarrow$ $dW_e = V dq = \frac{q}{C} dq$ during the charging period

Energy stored in the capacitor = Energy stored in E :

$$W_e = \frac{1}{C}\int_0^Q q\, dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

(e.g. 3-19) Cylindrical capacitor of coaxial line with $b-a \ll L$ (*FIGURE* 3−21)

$$W_e = \frac{1}{2}\int_a^b \epsilon E^2 dv = \frac{1}{2}\int_a^b \epsilon\left(\frac{Q}{2\pi\epsilon r L}\right)^2 (L 2\pi r\, dr)$$

$$= \frac{Q^2}{4\pi\epsilon L}\int_a^b \frac{dr}{r} = \underline{\frac{Q^2}{4\pi\epsilon L} ln\left(\frac{b}{a}\right)} = \underline{\frac{1}{2}\frac{Q^2}{C}}$$

$$\Longrightarrow \quad C = \frac{2\pi\epsilon L}{\ln(b/a)} \quad = (3\text{-}90)$$

4) Electrostatic forces

Consider an isolated system of charged bodies with constant charges on them. The electric force on an object in the system can be determined from W_e based on the Principle of Virtual Displacement.

Mechanical work dW done by the system for virtual displacement dl due to electrostatic force $\boldsymbol{F}_Q$ on charged bodies

= Stored electrostatic energy dW_e

$$\Rightarrow \quad dW = \underline{\boldsymbol{F}_Q \cdot \boldsymbol{dl}} = -\,dW_e \underline{= -(\nabla W_e)\cdot \boldsymbol{dl}}$$

$$\Rightarrow \quad \boldsymbol{F}_Q = -\nabla W_e \quad \text{(N)} \qquad (3\text{-}115)$$

(e.g. 3-19)

Force on the conducting plates of a charged parallel-plate capacitor :

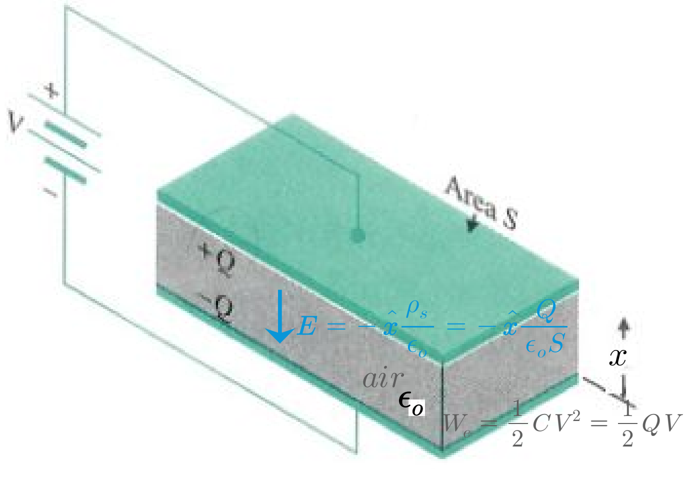
$$V = -\int_L \boldsymbol{E}\cdot \boldsymbol{dl}$$

$$= -\int_0^x \left(-\hat{x}\frac{Q}{\epsilon_o S}\right)\cdot \hat{x}\, dx$$

$$= \frac{Q}{\epsilon_o S} x$$

$$W_e = \frac{1}{2} C V^2 = \frac{1}{2} Q V$$

$$= \frac{Q^2}{2\epsilon_o S} x$$



$$(3\text{-}115) \quad \Rightarrow \quad (\boldsymbol{F}_Q)_x = -\frac{\partial W_e}{\partial x} = -\frac{Q^2}{2\epsilon_o S} \qquad (3\text{-}120)$$