3. Inductance, Magnetic Energy and Forces

A. Inductors and Inductance

1) Inductors and Magnetic flux linkage

Inductor = A device for storing magnetic energy arranged in an appropriate shape of conductor (Magnetic analogue of capacitor)

(e.g.) coils, loops, solenoids, ...

Magnetic flux linkage Λ

= Total magnetic flux linking a given conducting structure or circuit of the inductor

$$\Lambda \triangleq \begin{cases} N\Phi \ \textit{for coils of N turns} \\ \Phi \ \textit{for other inductors} \end{cases} \quad \text{(Wb)} \qquad (5\text{-}75,\ 5\text{-}78)$$

where $\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$: magnetic flux (5-23)

2) Inductance

= Magnetic flux linkage / Current flowing thru the inductor

$$L \equiv \frac{d\Lambda}{dI} = \frac{\Lambda}{I} \quad \text{(H = Wb/A)} \qquad (5\text{-}77,\ 5\text{-}79)$$

for a linear medium

Notes) i) $L = \dfrac{\Lambda}{I} = \dfrac{N\int_S \mathbf{B}\cdot d\mathbf{s}}{(1/\mu_o)\oint_C \mathbf{B}\cdot d\mathbf{l}} = f(\mu, \textit{geometry})$

: Depends only on material & geometry ; Independent of B and I

($\therefore B^{\uparrow}$ *as* $I^{\uparrow}$ by Ampere's or Biot-Savart law)

ii) Two magnetically coupled loops

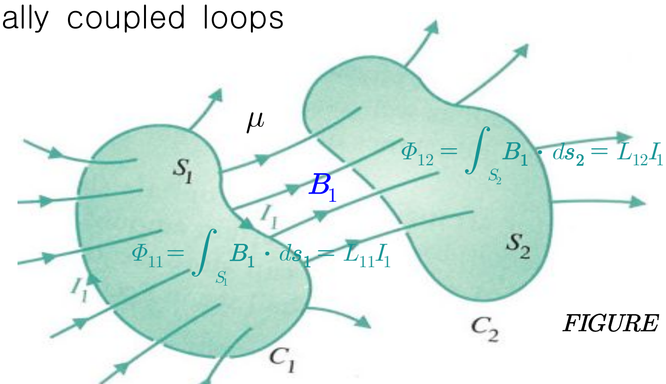


FIGURE 5-14

Self-inductance of loop C_1:

$$L_{11} = \frac{\Lambda_{11}}{I_1} = \frac{N_1}{I_1}\int_{S_1} \mathbf{B}_1 \cdot d\mathbf{s}_1 \qquad (5\text{-}77)$$

Mutual inductance between two loops $C_1 \& C_2$:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2}{I_1}\int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \quad (= L_{21}) \qquad (5\text{-}79)$$

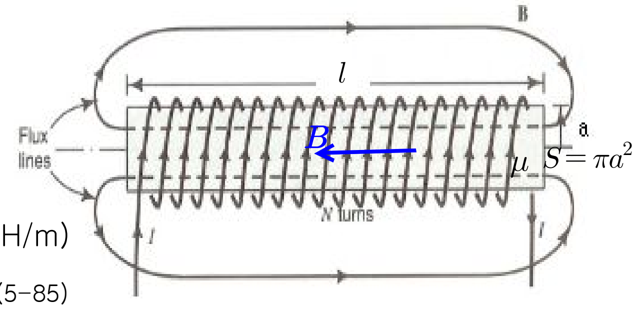
3) Calculations of inductances

a) A very long **solenoid** having n turns per length ($n = N/l$)

Magnetic flux inside the solenoid:

$$\Phi = BS = \mu n I S$$

$(5-12)^{**}$



Inductance per unit length :

$$L' = \frac{L}{l} = \frac{\Lambda / l}{I} = \frac{n\Phi}{I} = \mu n^2 S \quad \text{(H/m)}$$

(5-85)

b) A **toroidal coil** of a rectangular cross-section

Toroidal magnetic field:

$$\text{(5-13)} \Rightarrow \quad B_\phi = \frac{\mu_o N I}{2\pi r} \qquad \text{(5-80)}$$

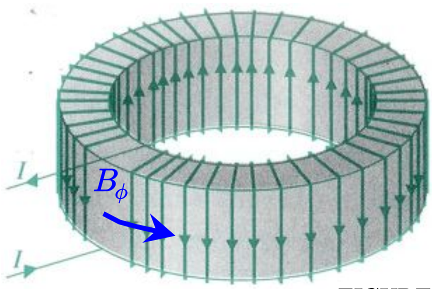
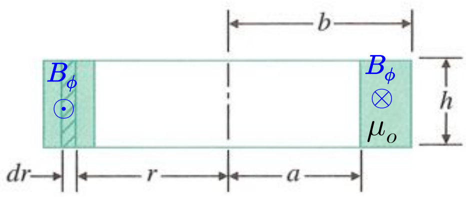


FIGURE 5−15



Toroidal magnetic flux:

$$\Phi = \int B_\phi \cdot ds_\phi$$

$$= \int_a^b \left(\hat{\phi} \frac{\mu_o N I}{2\pi r} \right) \cdot (\hat{\phi} h\, dr)$$

$$= \frac{\mu_o N I h}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_o N I h}{2\pi} ln \frac{b}{a}$$

Flux llinkage: $\Lambda = N\Phi = \dfrac{\mu_o N^2 I h}{2\pi} ln \dfrac{b}{a}$

$$\therefore \quad L = \frac{\Lambda}{I} = \frac{\mu_o N^2 h}{2\pi} ln \frac{b}{a} \quad \text{(H)} \qquad \text{(5-81)}$$

c) A long **coaxial transmission line**

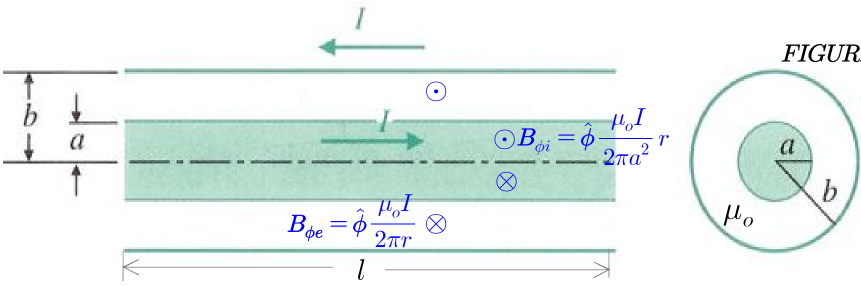


FIGURE 5−16

External flux linkage : $\Lambda_e = \Phi_e = \displaystyle\int_{S_\phi} B_{\phi e} \cdot ds_\phi = \int_a^b \left(\hat{\phi} \frac{\mu_o I}{2\pi r} \right) \cdot (\hat{\phi}\, l\, dr) = \frac{l \mu_o I}{2\pi} \ln \frac{b}{a}$

External inductance per unit length: $L_e' \equiv \dfrac{L_e}{l} = \dfrac{\Lambda_e / l}{I} = \dfrac{\mu_o}{2\pi} \ln \dfrac{b}{a}$

a fraction of total current I

Internal flux linkage : $\Lambda_i = \displaystyle\int_{S_\phi} \left(\frac{\pi r^2}{\pi a^2} \right) B_{\phi i} \cdot ds_\phi = \int_0^a \left(\frac{\pi r^2}{\pi a^2} \right) \left(\hat{\phi} \frac{\mu_o I r}{2\pi a^2} \right) \cdot (\hat{\phi}\, l\, dr) = \frac{l \mu_o I}{8\pi}$

Internal inductance per unit length: $L_i' \equiv \dfrac{L_i}{l} = \dfrac{\Lambda_i / l}{I} = \dfrac{\mu_o}{8\pi}$

Total inductance per unit length: $L' = L'_i + L'_e = \dfrac{\mu_o}{8\pi} + \dfrac{\mu_o}{2\pi} \ln \dfrac{b}{a}$ (5-90)

d) A long **two-wire transmission line** of radius a separated by d

(5-90) ⟹ Internal self-inductance:

$$L'_i = 2\times\frac{\mu_o}{8\pi} = \frac{\mu_o}{4\pi}$$

External self-inductance:

$$L'_e = \frac{\Phi'}{I}$$

$$= \frac{1}{I}\int_a^{d-a}(B_{y1}+B_{y2})\,dx$$

$$= \frac{\mu_o}{2\pi}\int_a^{d-a}\left(\frac{1}{x}+\frac{1}{d-x}\right)dx$$

$$= \frac{\mu_o}{\pi}ln\left(\frac{d-a}{a}\right) \approx \frac{\mu_o}{\pi}\ln\frac{d}{a}$$

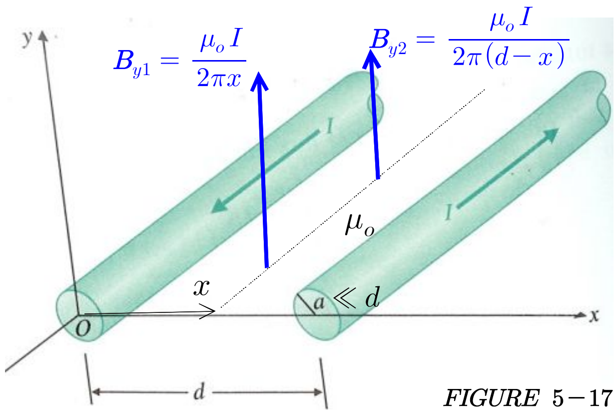


FIGURE 5-17

Total self-inductance per unit length: $L' = L'_i + L'_e = \frac{\mu_o}{\pi}\left(\frac{1}{4}+\ln\frac{d}{a}\right)$ (5-95)

e) A long straight wire & a rectangular loop

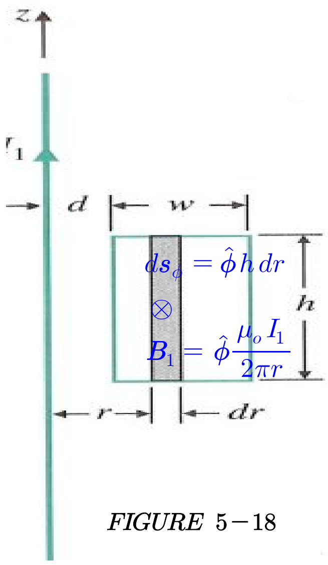


FIGURE 5-18

Mutual inductance:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{1}{I_1}\int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2$$

$$= \int_d^{d+w}\left(\frac{\mu_o}{2\pi r}\right)h\,dr$$

$$= \frac{\mu_o h}{2\pi}\int_d^{d+w}\frac{dr}{r}$$

$$= \frac{\mu_o h}{2\pi}ln\left(1+\frac{w}{d}\right) \quad \text{(H)} \qquad \text{(5-95)}$$

B. Magnetic Energy

1) Magnetic energy stored in an inductor

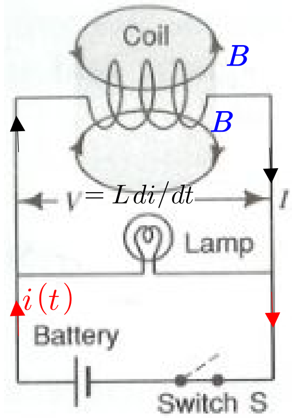
= Total energy in building up the current I in the inductor

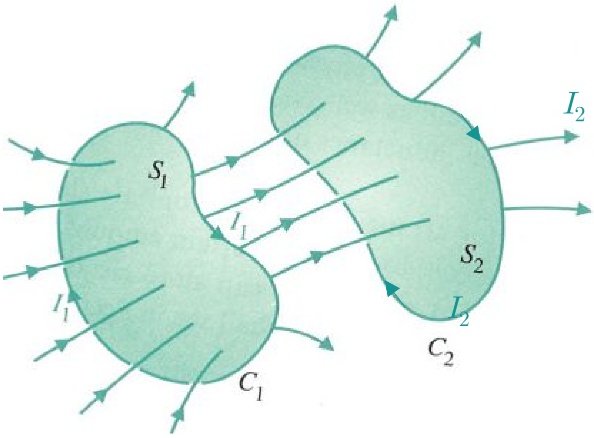
$$v = L\,di/dt$$

$$W_m = \int P\,dt = \int v\,i\,dt = L\int_0^I i\,di$$

$$= \frac{1}{2}LI^2 = \frac{1}{2}\Lambda I = \frac{1}{2}\frac{\Lambda^2}{L} \quad \text{(J)} \qquad (5\text{-}104)$$

$\leftrightarrow$(3-110)





(e.g.) *FIGURE* 5-14

Magnetic energy stored in two loops

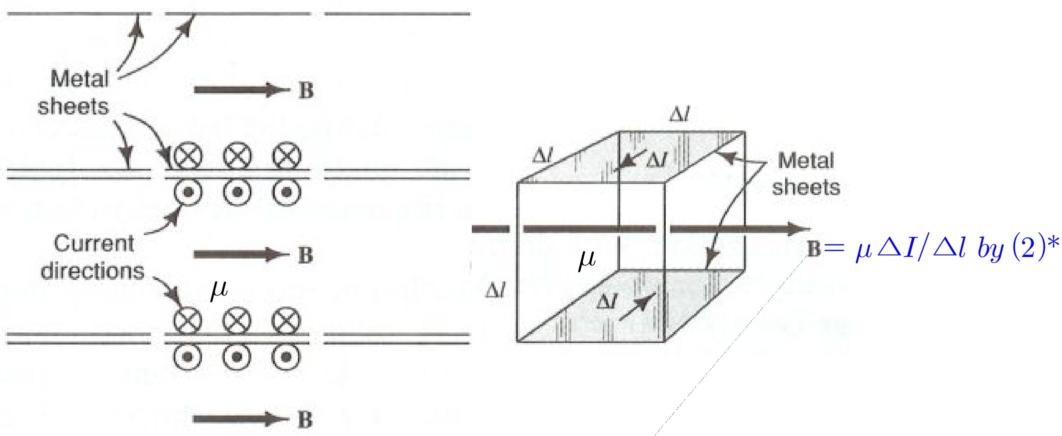
= Total work done in raising currents (I_1, I_2)

$$W_m = W_{11} + W_{21} + W_{22}$$

$$= \frac{1}{2}L_{11}I_1^2 + L_{21}I_1I_2 + \frac{1}{2}L_{22}I_2^2 \quad (5\text{-}104)$$

2) Magnetic energy in terms of fields

Consider a magnetic field space which is assumed to be filled with small unit cubes of a volume $\triangle v = \triangle l^3$ with up- and down-current sheets $\triangle I/\triangle l$.



Inductance of each cell : $\triangle L = \frac{\Delta\Phi}{\Delta I} = \frac{B\,\triangle l^2}{\Delta I} = \frac{\mu\Delta I\Delta l}{\Delta I} = \mu\,\Delta l$ (2)**

Magnetic energy stored in each cell : $\triangle W_m = \frac{1}{2}\Delta L(\Delta I)^2 \overset{(2)*,(2)**}{=} \frac{B^2}{2\mu}\Delta v$

Magnetic energy density:

$$w_m = \lim_{\Delta v \to 0}\triangle W_m = \frac{B^2}{2\mu} \overset{(5-65)}{=} \frac{\mu H^2}{2} = \frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B} = \frac{\mu_o H^2}{2} + \frac{\mu_o MH}{2} \quad \text{(J/m}^3\text{)}$$

(3-108) $\updownarrow$ (5-104)

∴ Magnetic energy stored in the field: $W_m = \int_V w_m \, dv$ (5-107)

$$W_m = \frac{1}{2}\int_V \boldsymbol{H}\cdot\boldsymbol{B}\, dv = \frac{1}{2}\int_V \frac{B^2}{\mu}\, dv = \frac{1}{2}\int_V \mu H^2 dv \quad \text{(J)}$$ (5-105, 106) ↕ (3−105,106)

for a simple medium ($\boldsymbol{B} = \mu \boldsymbol{H}$)

Consequently, the self-inductance of the inductor can be obtained from the stored magnetic energy density by (5-104):

$$L = \frac{2W_m}{I^2}$$ (5-109)

(e.g.1) A very long **solenoid**

$$w_m = \frac{W_m}{V} = \frac{1}{2V} L I^2 = \frac{1}{2V}(\mu n^2 S l) I^2 = \frac{1}{2}\mu n^2 I^2 = \frac{B^2}{2\mu}$$

(5−85) $B = \mu n I$ (5−12)**

(ex. 5.9) A **toroidal coil** of a rectangular cross-section *FIGURE* 5−15

(5-104) $\Rightarrow \quad W_m = \frac{1}{2} L I^2 = \frac{1}{2}\Lambda I = \frac{1}{2} N \Phi I = \frac{\mu_o N^2 I^2 h}{4\pi} ln\frac{b}{a}$

(5-109) $\Rightarrow \quad L = \frac{2W_m}{I^2} = \frac{\mu_o N^2 h}{2\pi} ln\frac{b}{a}$ = (5-81)

(e.g. 5-13) A long **coaxial transmission line**

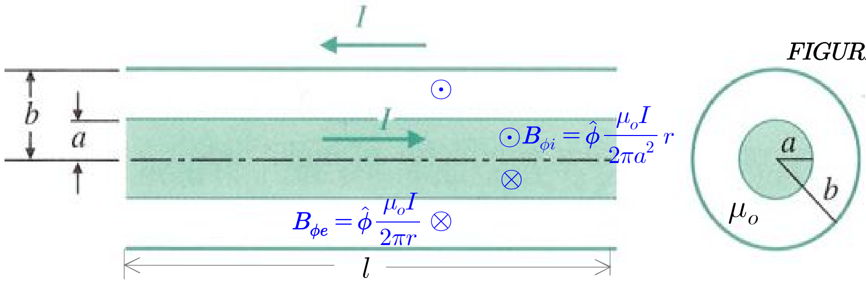


FIGURE 5−16

External magnetic energy per unit length:

$$W'_{me} = \frac{1}{l}\int_V w_{me}\, dv = \frac{1}{2\mu_o l}\int_{S_{\phi e}} B_{\phi e}^2\, dv = \frac{1}{2\mu_o}\int_a^b \left(\frac{\mu_o I}{2\pi r}\right)^2 2\pi r dr = \frac{\mu_o I^2}{4\pi} ln\frac{b}{a}$$ (5-111)

Internal magnetic energy per unit length:

$$W'_{mi} = \frac{1}{l}\int_V w_{mi}\, dv = \frac{1}{2\mu_o l}\int_{S_{\phi i}} B_{\phi i}^2\, dv = \frac{1}{2\mu_o}\int_0^a \left(\frac{\mu_o I r}{2\pi a^2}\right)^2 2\pi r dr = \frac{\mu_o I^2}{16\pi}$$ (5-110)

Total magnetic energy per unit length:

$$W'_m = W'_{mi} + W'_{me} = \frac{\mu_o I^2}{16\pi} + \frac{\mu_o I^2}{4\pi} ln\frac{b}{a}$$

Self-inductance per unit length:

(5-109) $\Rightarrow \quad L' = \frac{2W'_m}{I^2} = \frac{\mu_o}{16\pi} + \frac{\mu_o}{4\pi} ln\frac{b}{a}$ = (5-90)

C. *Magnetic Forces and Torques*

1) Magnetic forces

a) Magnetic force on a moving charge q

$$F_m = q\boldsymbol{u} \times \boldsymbol{B} \quad \text{(N)} \qquad (5\text{-}113)$$

Notes) i) Total electromagnetic force on a charge q

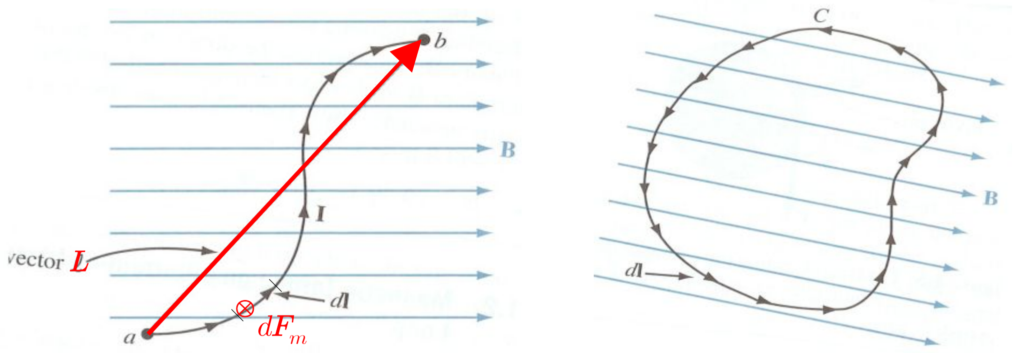
$$\boldsymbol{F} = \boldsymbol{F}_e + \boldsymbol{F}_m = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \quad \text{(N)} : \text{Lorentz's force} \qquad (5\text{-}5)$$

ii) Electromagnetic body force (= e.m. force per unit volume)

in charged particle systems such as plasmas

$$\boldsymbol{f} = \boldsymbol{f}_e + \boldsymbol{f}_m = nq(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = \rho_v \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \qquad (5\text{-}5)^*$$

b) Magnetic force on a current-carrying conducting wire



Current due to free electrons of charge density $\rho_v = ne$ moving with a velocity $\boldsymbol{u}$ along the wire : $I = JS = \rho_v u S = -neuS$

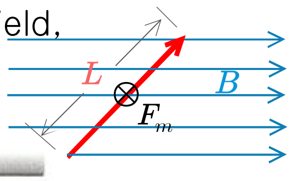
Magnetic force on a current-carrying $d\boldsymbol{l}$ with a cross-sectional area S :

$$d\boldsymbol{F}_m = Q\boldsymbol{u} \times \boldsymbol{B} = -(neSdl)\boldsymbol{u} \times \boldsymbol{B} = I\,d\boldsymbol{l} \times \boldsymbol{B} \quad \text{(N)} \qquad (5\text{-}115)$$

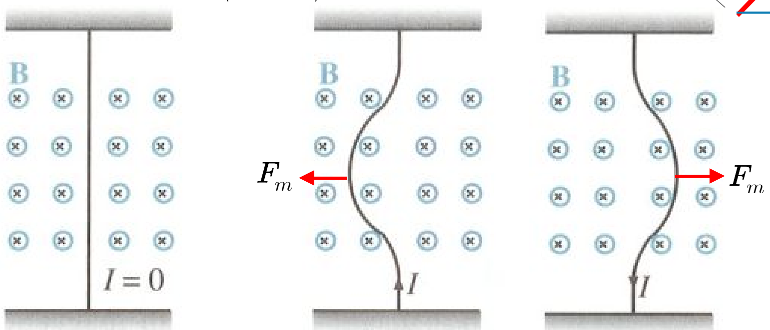
Magnetic force on the whole current-carrying wire of contour C :

$$\boldsymbol{F}_m = I \int_C d\boldsymbol{l} \times \boldsymbol{B} \quad \text{(N)} \qquad (5\text{-}116)$$

Notes) i) For a curved wire in a uniform $\boldsymbol{B}$ field,



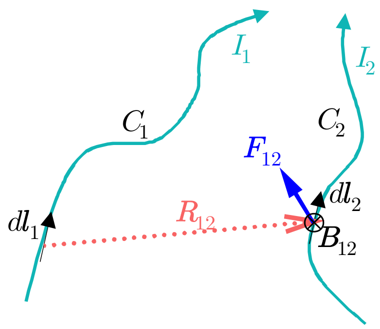
$$\boldsymbol{F}_m = I\left(\int_a^b d\boldsymbol{l}\right) \times \boldsymbol{B} = I\,\boldsymbol{L} \times \boldsymbol{B}$$



ii) For a closed circuit in a uniform $\boldsymbol{B}$ field,

$$\boldsymbol{F}_m = I\left(\oint_C d\boldsymbol{l}\right) \times \boldsymbol{B} = 0$$

c) Ampere's law of force between two current-carrying circuits



Magnetic field caused by I_1:

$$B_{12} = \frac{\mu_o I_1}{4\pi} \int_{C_1} \frac{dl_1 \times \widehat{R_{12}}}{R_{12}^2} \qquad \text{(5-117b)}$$

Force on circuit C_2 caused by I_1:

$$F_{12} = I_2 \int_{C_2} dl_2 \times B_{12} \qquad \text{(5-117a)}$$

(5-117b) in (5-117a) $\Rightarrow$ Ampere's law of force:

$$F_{12} = \frac{\mu_o}{4\pi} I_2 I_1 \int_{C_2}\int_{C_1} \frac{dl_2 \times (dl_1 \times \widehat{R_{12}})}{R_{12}^2} = -F_{21} \ \text{(N)} \qquad \text{(5-118)} \leftrightarrow \text{(3-13)}$$

Coulomb's law of force

(e.g. 5-14) Two parallel current-carrying wires ($I_1 \upuparrows I_2$)

(5-12) $\Rightarrow$ $B_{12}(d) = -\hat{x}\frac{\mu_o I_1}{2\pi d}$ (5-120)

Force per unit length on wire 2:

(5-117a) $\Rightarrow$ $F_{12}{}' = I_2(\hat{z} \times B_{12})$

(5-120) $\Rightarrow$ $F_{12}{}' = -\hat{y}\frac{\mu_o I_1 I_2}{2\pi d}$

: attractive force

Note) If $I_1 \updownarrows I_2$,

$$F_{12}{}' = +\hat{y}\frac{\mu_o I_1 I_2}{2\pi d}: \text{ repulsive force}$$

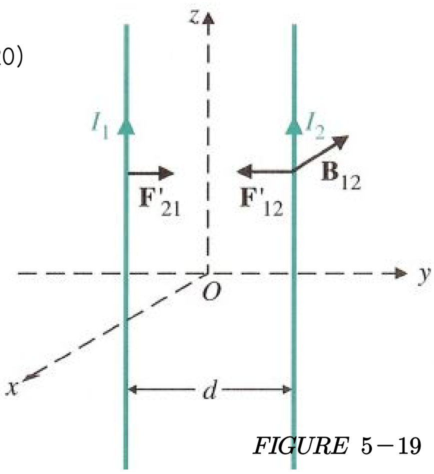


FIGURE 5−19

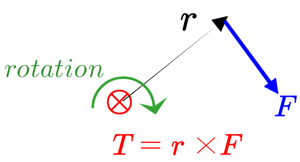
2) Magnetic torque on a current-carrying loop

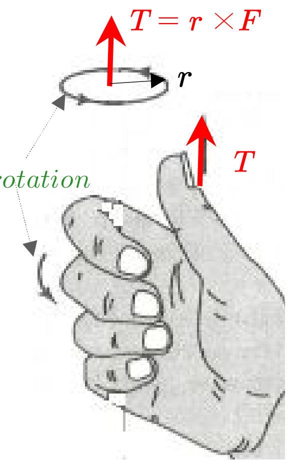
Torque = Cross product of lever arm about the axis and force

$\Rightarrow$ Moment of inertia: Mechanical moment

$$T = r \times F \quad \text{(N·m)}$$

(Right-hand rule for directions)





Consider a circular loop in a uniform magnetic field $\boldsymbol{B} = \boldsymbol{B}_{\perp} + \boldsymbol{B}_{\parallel}$

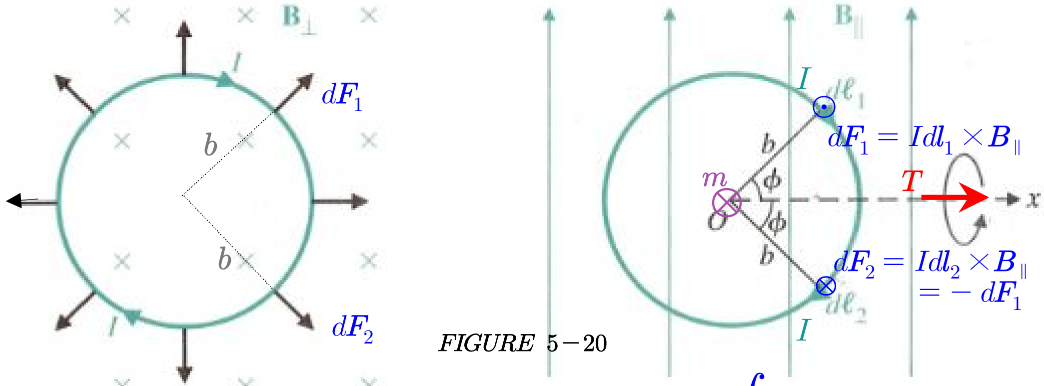


FIGURE 5−20

$\boldsymbol{F}_m = I \oint d\boldsymbol{l} \times \boldsymbol{B}_{\perp} = 0$

$d\boldsymbol{T} = \boldsymbol{b} \times (d\boldsymbol{F}_1 + d\boldsymbol{F}_2) = 0$

$\boldsymbol{F}_m = I \oint d\boldsymbol{l} \times \boldsymbol{B}_{\parallel} = 0$

$d\boldsymbol{T} = \boldsymbol{b} \times (d\boldsymbol{F}_1 + d\boldsymbol{F}_2)$

Differential torque produced by $d\boldsymbol{F}_1$ and $d\boldsymbol{F}_2$:

$$d\boldsymbol{T} = \boldsymbol{b} \times (d\boldsymbol{F}_1 + d\boldsymbol{F}_2) = \hat{\boldsymbol{x}}(dF)\,2b\sin\phi = \hat{\boldsymbol{x}}(Idl\,B_{\parallel}\sin\phi)\,2b\sin\phi$$

$$= \hat{\boldsymbol{x}}\,2Ib^2 B_{\parallel} \sin^2\phi \, d\phi \qquad (5\text{-}122)$$

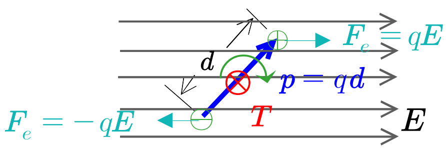
Magnetic torque on a current-carrying loop in a constant B :

$$\boldsymbol{T} = \int d\boldsymbol{T} = \hat{\boldsymbol{x}}\,2Ib^2 B_{\parallel} \int_0^{\pi} \sin^2\phi \, d\phi = \hat{\boldsymbol{x}}\, I(\pi b^2) B_{\parallel} = \hat{\boldsymbol{x}}\, m B_{\parallel} \qquad (5\text{-}123)$$

$$\Rightarrow \quad \boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B} \quad (\text{N·m}) \qquad (5\text{-}124)$$

$\Rightarrow$ $\boldsymbol{T}$ tends to turn the current loop until $\boldsymbol{m}$ and $\boldsymbol{B}$ are in the same direction

(cf) Electric torque acting on the electric dipole



$$\boldsymbol{T} = \boldsymbol{p} \times \boldsymbol{E} \qquad (5\text{-}124)^*$$

3) Application of the magnetic torque to d-c motors

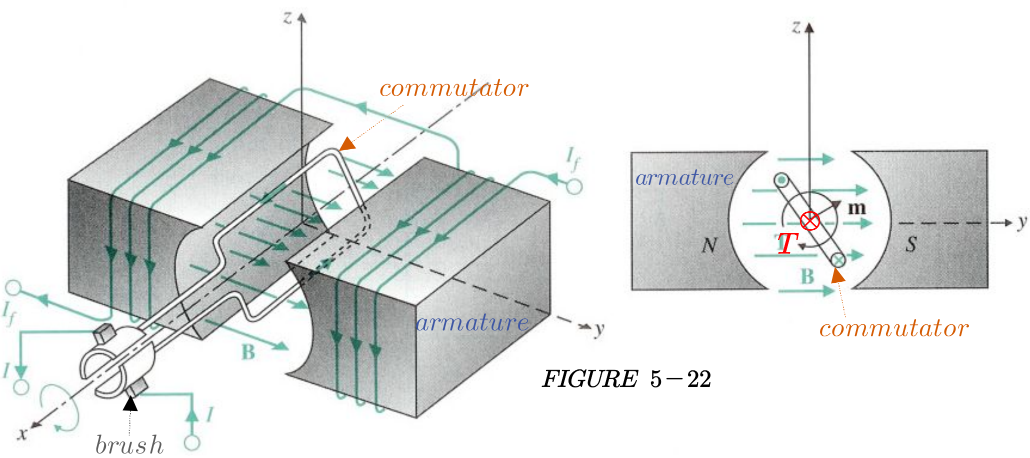


FIGURE 5−22

D. *Forces and Torques in Terms of Magnetic Energy*

1) **Magnetic force** based on the principle of virtual displacement

Mechanical work dW done by the system for virtual displacement dl due to magnetic force $\boldsymbol{F}_\Phi$ on a current-carrying circuit under the constant-flux condition

= A decrease of the stored magnetic energy dW_e

$$\Rightarrow \quad dW = \underline{\boldsymbol{F}_\Phi \cdot d\boldsymbol{l}} = -dW_m = \underline{-(\nabla W_m)\cdot d\boldsymbol{l}}$$

$$\Rightarrow \quad \boldsymbol{F}_\Phi = -\nabla W_m \quad \text{(N)} \qquad (5\text{-}130)\leftrightarrow(3\text{-}115)$$

In Cartesian coordinates,

$$(\boldsymbol{F}_\Phi)_x = -\frac{\partial W_m}{\partial x} \qquad (5\text{-}131)$$

2) **Magnetic torque** based on the principle of virtual displacement

Magnetic torque about a given axis (z-axis) on a current-carrying circuit under the condition of constant flux linkage:

$$(\boldsymbol{T}_\Phi)_z = -\frac{\partial W_m}{\partial \phi} \quad \text{(N·m)} \qquad (5\text{-}132)$$

(e.g. 5-16) An electromagnet of an N-turn coil

Magnetic energy stored in two air gaps:

$$dW_m = dW_{m,gap} = 2\left(\frac{B^2}{2\mu_o}Sdy\right) = \frac{\Phi^2}{\mu_o S}dy \qquad (5\text{-}133)$$

Force in y-direction by gap change dy:

$$\boldsymbol{F}_\Phi = \hat{y}(F_\Phi)_y$$

$$\overset{(5-131)}{=} -\hat{y}\frac{\partial W_m}{\partial y} \overset{(5-133)}{=} -\hat{y}\frac{\Phi^2}{\mu_o S}$$

: attractive force

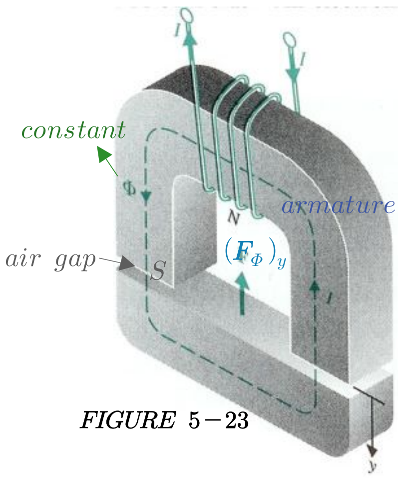


FIGURE 5-23

Homework Set 7

1) P.5-13
2) P.5-15
3) P.5-17
4) P.5-19
5) P.5-21