# Heat and Mass Transfer

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# 20

# NUMERICAL, GRAPHICAL, AND ANALOG METHODS IN THE ANALYSIS OF HEAT CONDUCTION

# Introduction

**The differential equations,**
**based on the differential energy balance,**
**can be written readily enough.**

**However,**
**the mathematical boundary conditions are difficult to apply**
**when the physical boundaries of the object are complex and**
**when the temperatures at these boundaries are not uniform.**

**In general, few analytical solutions exist for heat-transfer systems which do not possess some elements of symmetry in both shape and temperature**

# Numerical, Graphical, and Analog Methods

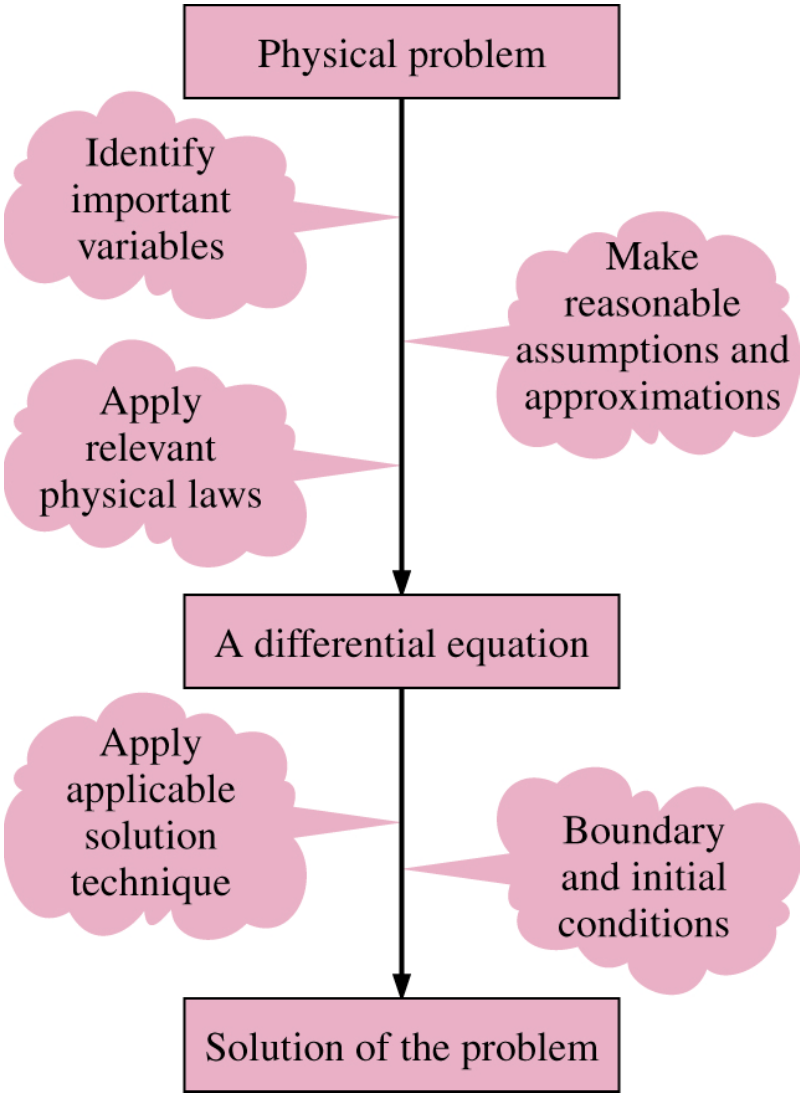
No Analytical solution

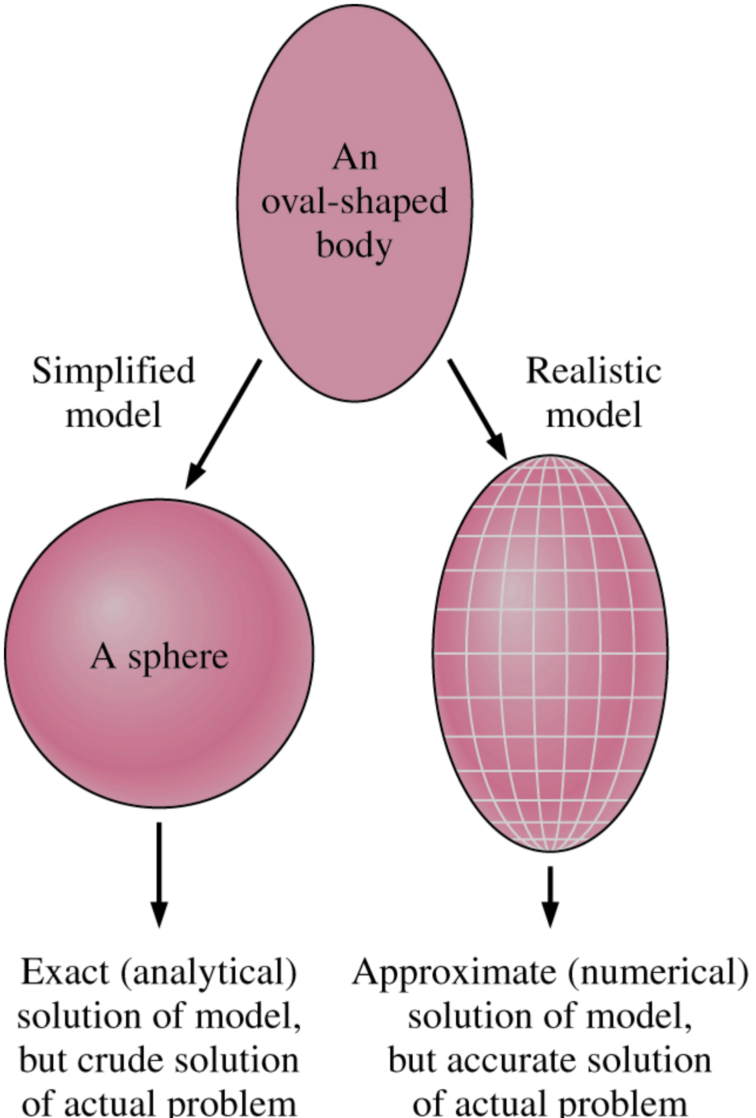
↓

**Numerical, Graphical, and Analog Methods**

Accuracy?
Performing time?
Converge?

# Analytical solution vs. Numerical solution



# Analytical Solution vs. Numerical Solution

**Numerical solutions are also accurate.**

It should not be be assumed that the numerical and graphical solutions are necessarily less accurate than the analytical solutions. In fact, many of them can be made as accurate as desired by mere repetition of routine steps which, if performed an infinite number of times, would give an exact solution.

# Analytical Solution vs. Numerical Solution

**An analytical solution must be carried out to the end before any answer whatsoever is obtained.**

This characteristic of numerical method is often an advantage, because the accuracy required of a solution is usually known beforehand, and the solution can be stopped when this is reached.

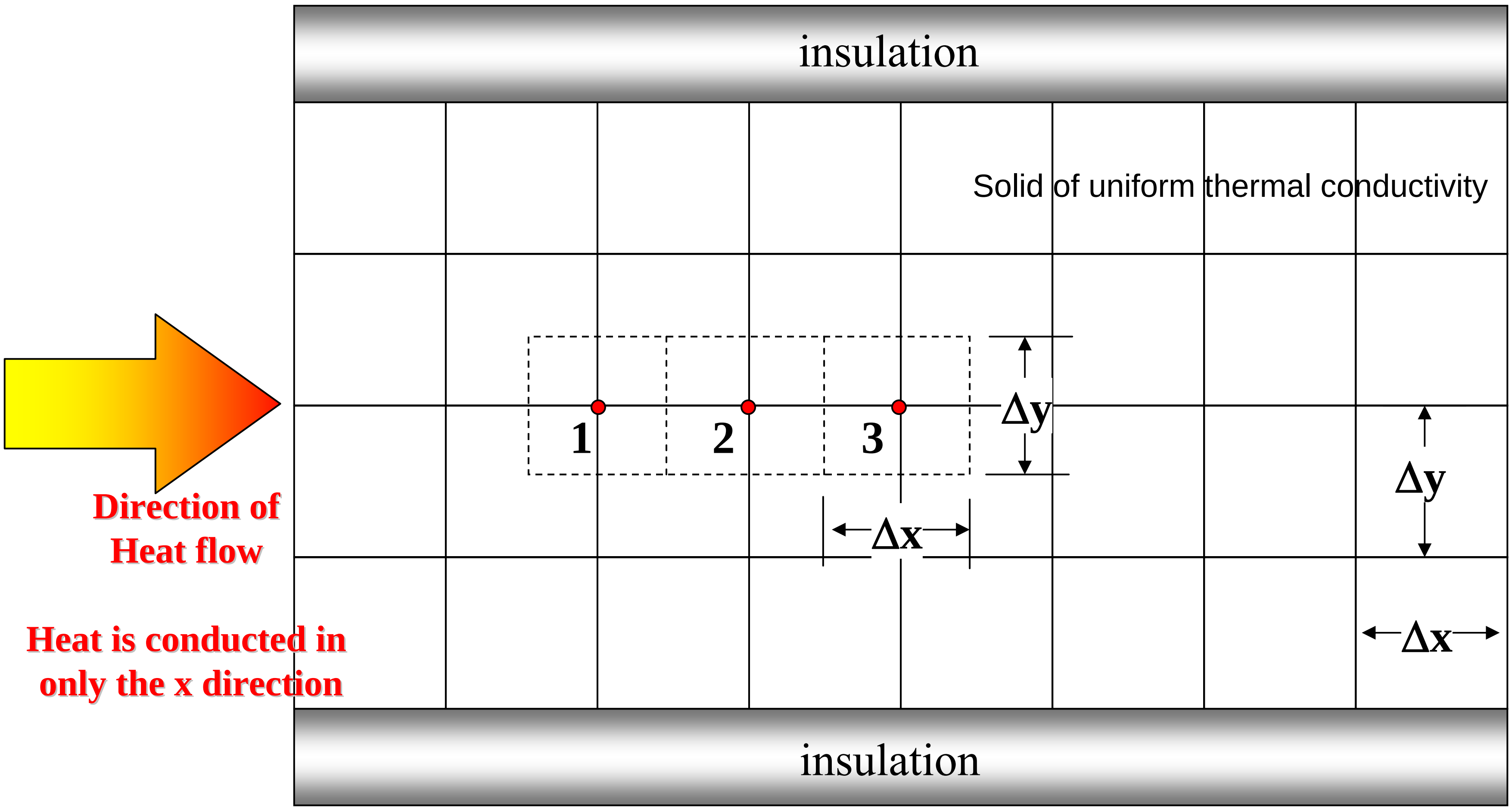
# Analytical Solution vs. Numerical Solution

**Sometimes numerical procedures do not readily converge to the correct result.**

The principal disadvantage of numerical methods is that they are ***tedious to perform***, but with the use of high-speed computer, this disadvantage is overcome. Sometimes numerical procedures, however, do not readily converge to the correct result, no matter how many iterations are performed.

# Steady State Conduction

## Relaxation technique applied to 1-D conduction




$\Delta x = \Delta y$

# Steady State Conduction

As the heat in the element surrounding point 1 flows into the block surrounding point 2, the rate of heat flow will be

$$q_{12} = -kA\frac{dt}{dx} = k\Delta z\Delta y\frac{t_1 - t_2}{\Delta x} \tag{20-1}$$

The rate of heat flow from the block solid around point 2 to the block around point 3 will be

$$q_{23} = -kA\frac{dt}{dx} = k\Delta z\Delta y\frac{t_2 - t_3}{\Delta x} \tag{20-2}$$

The rate at which heat accumulates at point 2 is

$$q_{12} - q_{23} = k\Delta z(t_1 + t_3 - 2t_2)\cancelto{1}{\frac{\Delta y}{\Delta x}} \tag{20-3}$$

# Steady State Conduction

Eq. (20-3) can also be written as

$q'_2$: residuals

$$\frac{q_{12}-q_{23}}{k\Delta z}=t_1+t_3-2t_2 \qquad (20\text{-}4)$$

If the system is at steady state, $q_{12}$ equals $q_{23}$, and Eq. (20-4) can be rearranged to give

$$t_2=\frac{t_1+t_3}{2} \qquad (20\text{-}5)$$

If an arbitrary temperature distribution is chosen for all the points in the system, it can be tested for correctness using Eq. (20-5). If the temperature $t_2$ is in error, it may be revised by adding or subtracting (dimensionless temperature)

$$\frac{1}{2}\frac{q_{12}-q_{23}}{k\Delta z}$$

to the temperature $t_2$.

# Steady State Conduction

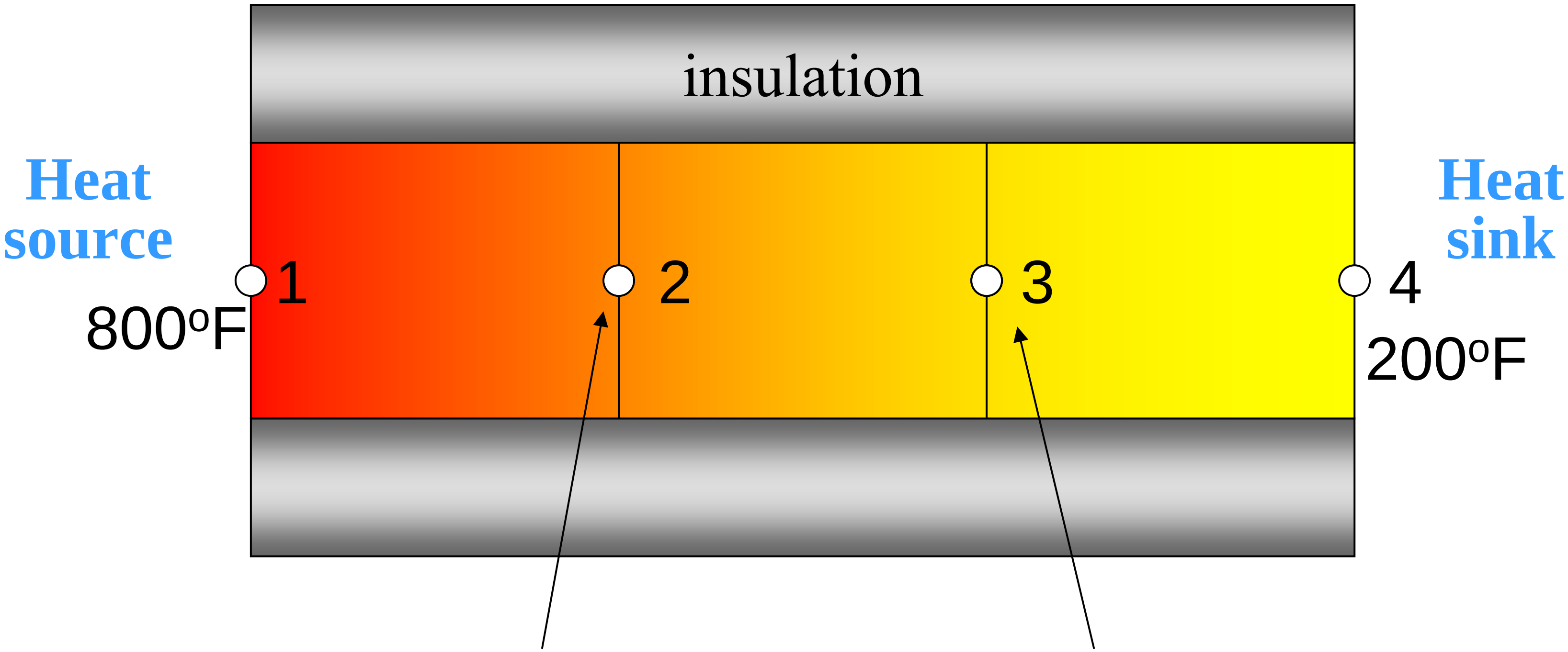
$$\frac{1}{2}\frac{q_{12}-q_{23}}{k\Delta z}$$ Dimensionless temperature

If all but one of the temperature at the nodes of the grid are correct, then alternations at this point will destroy the consistency among the temperatures at the remaining points. However, the magnitude of the correction diminishes by halves with succeeding points and if not too great initially, soon vanishes as a significant quantity.

# Relaxation techniques for one-dimensional conduction

**Ex 20-1**

**A solid rod conducts heat from a heat source at 800°F to a heat sink at 200°F. The side of the rod are insulated as shown in Fig.20-2. Find the temperature distribution in the rod.**




**Residuals:**

$$q_2' = \frac{q_{12} - q_{23}}{k\Delta z} \qquad q_3' = \frac{q_{23} - q_{34}}{k\Delta z}$$

# Relaxation techniques for one-dimensional conduction

## Ex 20-1

**Step 0**
Guessing a value for each of the unknown t

$$q'_2=t_1+t_3-2t_2$$
$$q'_3=t_2+t_4-2t_3$$

**Step 1**
Half of each of $q'_2$ & $q'_3$ is then added or subtracted, as the sign indicates, to the respective $t_2$ *and* $t_3$.

**Step 2**
Using the new temperature distribution, new $q'_2$ & $q'_3$ are calculated and new $t_2$ *and* $t_3$ once more computed

| Step | Point | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | | 3 | | 4 |
| | $t_1$ | $t_2$ | $q'_2$ | $t_3$ | $q'_3$ | $t_4$ |
| 0 | 800 | 700 | -300 | 300 | 300 | 200 |
| 1 | 800 | 550 | 150 | 450 | -150 | 200 |
| 2 | 800 | 625 | -75 | 375 | 75 | 200 |
| 3 | 800 | 587 | 39 | 413 | -39 | 200 |
| 4 | 800 | 607 | -21 | 393 | 21 | 200 |
| 5 | 800 | 596 | 12 | 404 | -12 | 200 |
| 6 | 800 | 602 | -6 | 398 | 6 | 200 |
| 7 | 800 | 599 | 3 | 401 | -3 | 200 |
| 8 | 800 | 600.5 | … | 399.5 | … | 200 |

607−21/2
=596.5

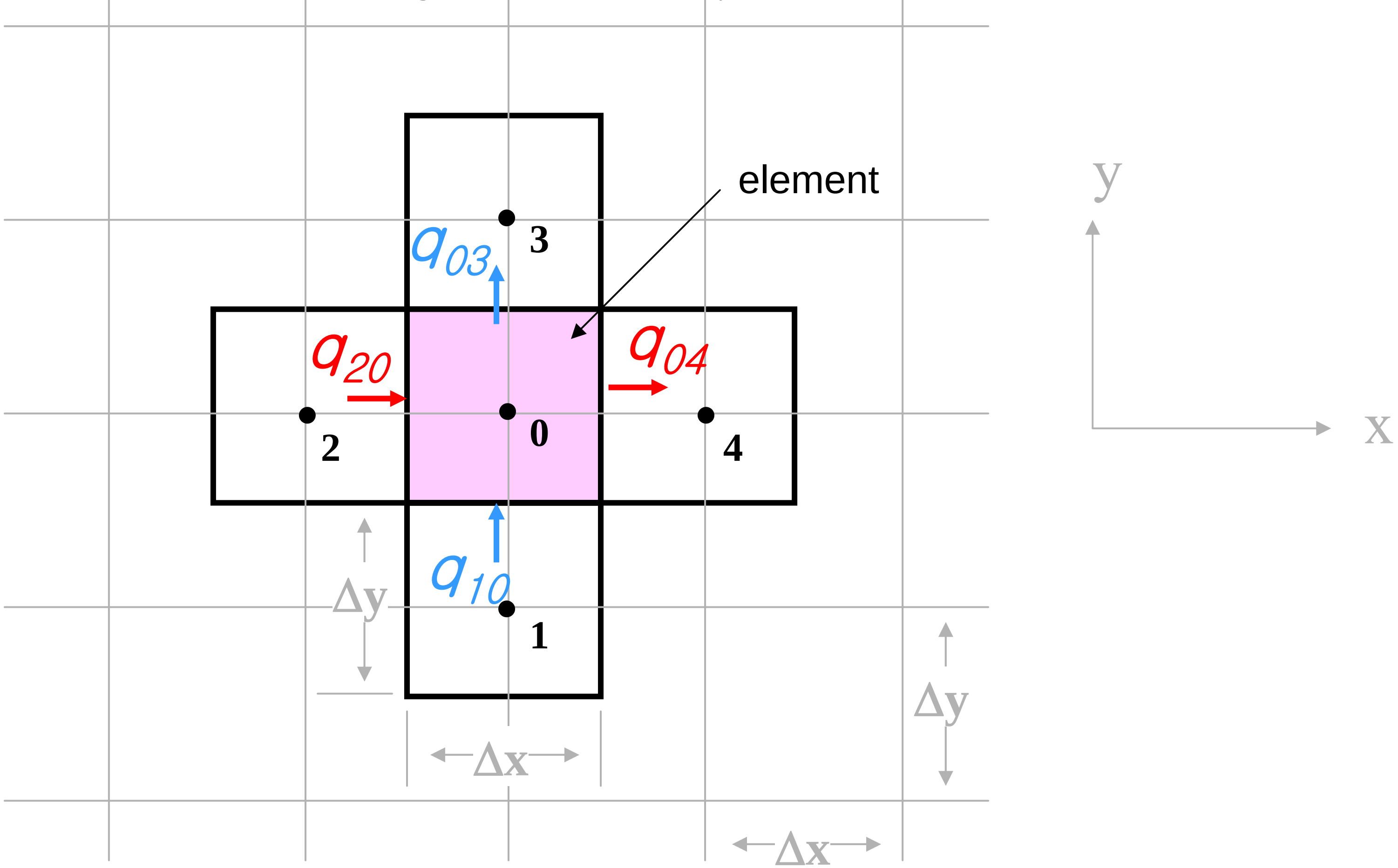
800+404−(2)(596)
=12

Relax this residual to zero

The repetition of this simple process gives figures nearly equal to the correct temperature after 8 steps

# Two Dimension Relaxation

Consider the solid to be conducting heat in x and y directions but not in the z direction



# Two Dimension Relaxation

If the solid has a thickness Δz in the z direction, we can write

$$q_{10} = k\Delta z\Delta x\frac{t_1 - t_0}{\Delta y} \qquad (20\text{-}6)$$

$$q_{20} = k\Delta z\Delta y\frac{t_2 - t_0}{\Delta x} \qquad (20\text{-}7)$$

$$q_{03} = k\Delta z\Delta x\frac{t_0 - t_3}{\Delta y} \qquad (20\text{-}8)$$

$$q_{04} = k\Delta z\Delta y\frac{t_0 - t_4}{\Delta x} \qquad (20\text{-}9)$$

# Energy Balance on an element

An equation for the net heat flux into the element

$$q_{10} + q_{20} - q_{03} - q_{04} = k\Delta z(t_1 + t_2 + t_3 + t_4 - 4t_0)$$

$$q_0' = \frac{q_{10} + q_{20} - q_{03} - q_{04}}{k\Delta z} = t_1 + t_2 + t_3 + t_4 - 4t_0$$

If the heat is being transferred at steady state, $q'_0$=0

$$t_0 = \frac{t_1 + t_2 + t_3 + t_4}{4}$$

Arithmetic average of the temperature at the four surrounding points.

# An approach to this problem

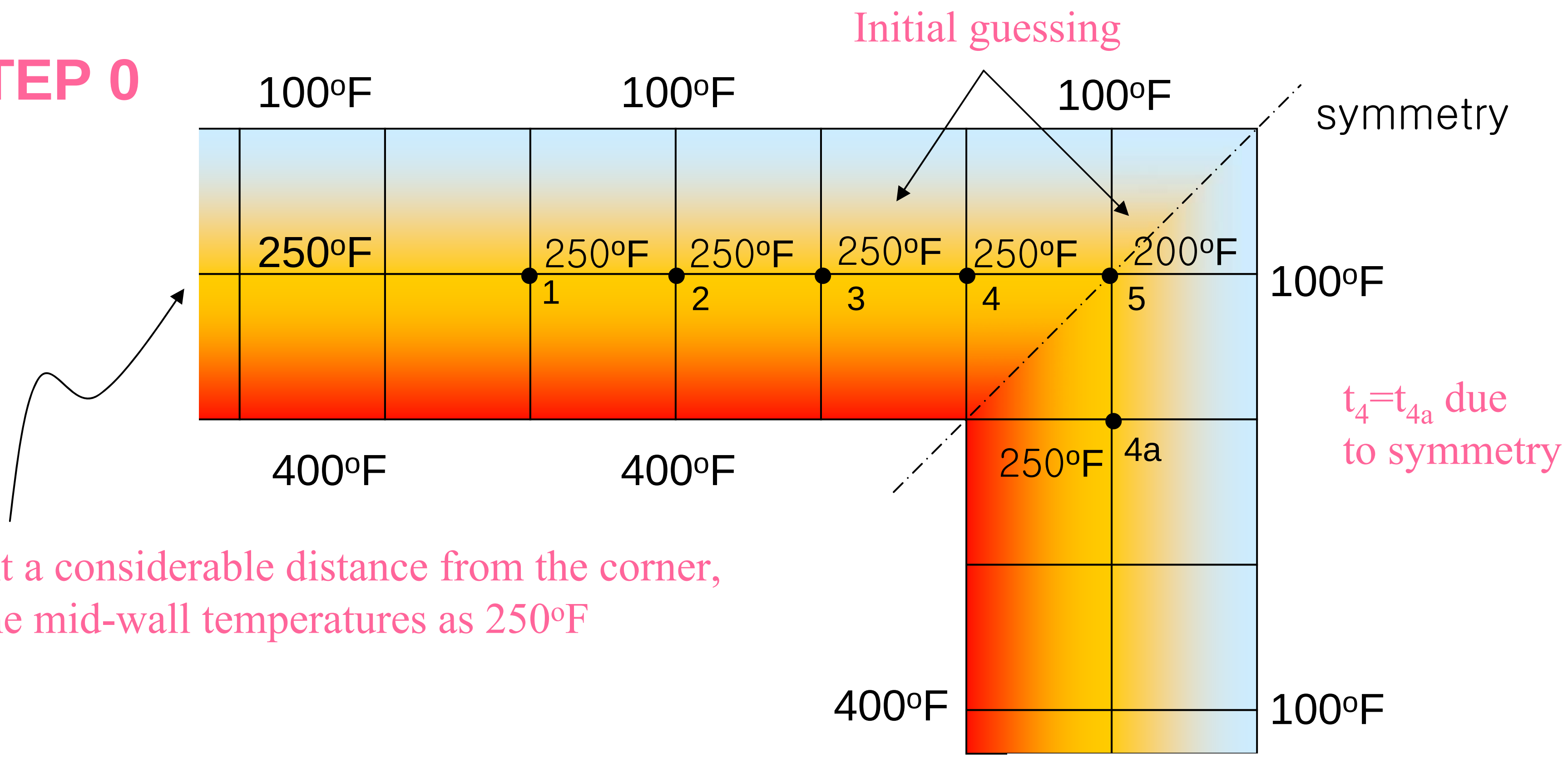
… is to set
the residual $q'_0$ quantities
at all nodal points
equal to zero.

This generates a system of linear equations
for the temperatures at the nodal points
which can be solved
using direct or iterative methods

# Ex 20-2 Two-dimensional Relaxation

The wall of a furnace has an inside temperature of 400ºF and an outside temperature of 100ºF. Find the centerline temperature of the wall.

**STEP 0**




At a considerable distance from the corner, the mid-wall temperatures as 250ºF

# Ex 20-2 Two-dimensional Relaxation

Initial guessing:

Because of the plane of symmetry through the corner, the temperature at point 4 will be the same as at point 4 will be the same as at point 4a.

At a considerable distance from the corner the mid-wall temperature will obviously be 250$^{o}$F.

If we choose all the centerline temperatures as 250$^{o}$F, we shall introduce some error, because the temperature at point 5 would seem to be lower. Therefore we choose $t_5$=200$^{o}$F and $t_1$=$t_2$=$t_3$=$t_4$=250$^{o}$F.

# Ex 20-2 Two-dimensional Relaxation

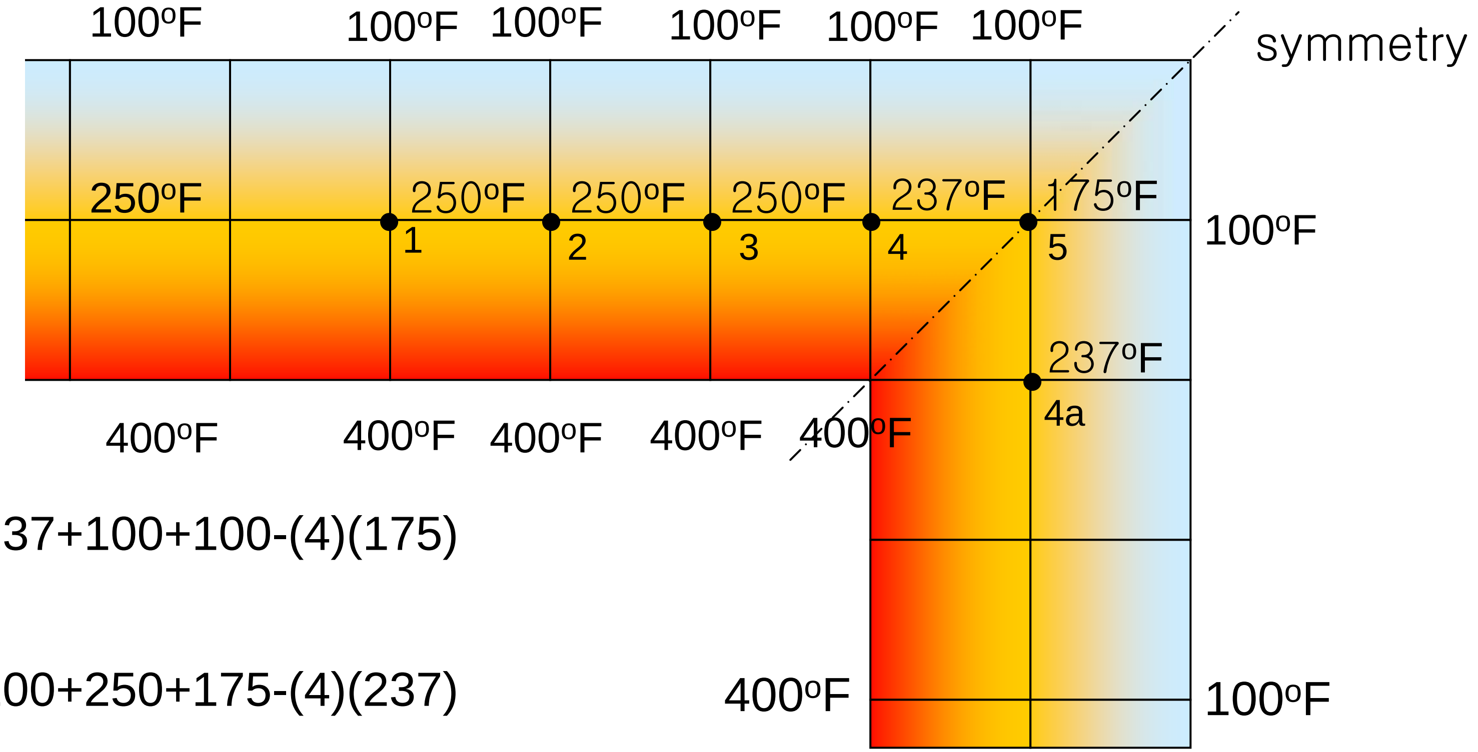
In step 0:

$$q'_1 = 250+100+250+400-(4)(250) = 0$$
$$q'_2 = 250+100+250+400-(4)(250) = 0$$
$$q'_3 = 250+100+250+400-(4)(250) = 0$$
$$q'_4 = 250+100+200+400-(4)(250) = -50$$
$$q'_5 = 250+100+100+250-(4)(200) = -100$$

We then add or subtract one-fourth (1/4) the value of $q'$ to or from each corresponding temperature to obtain the new temperature profile, with which we repeat the calculations until the values of $q'$ are negligible.

# Example 20-2 (Relaxation model)

## Step 1




$q_5{}' = 237+237+100+100-(4)(175)$
$= -26$

$q_4{}' = 400+100+250+175-(4)(237)$
$= -23$

$q_3{}' = 400+100+250+237-(4)(250)$
$= -13$

$q_2{}' = 400+100+250+250-(4)(250)$
$= 0$

# Ex 20-2 Two-dimensional Relaxation

237+237+100+100−(4)(175)
=−26

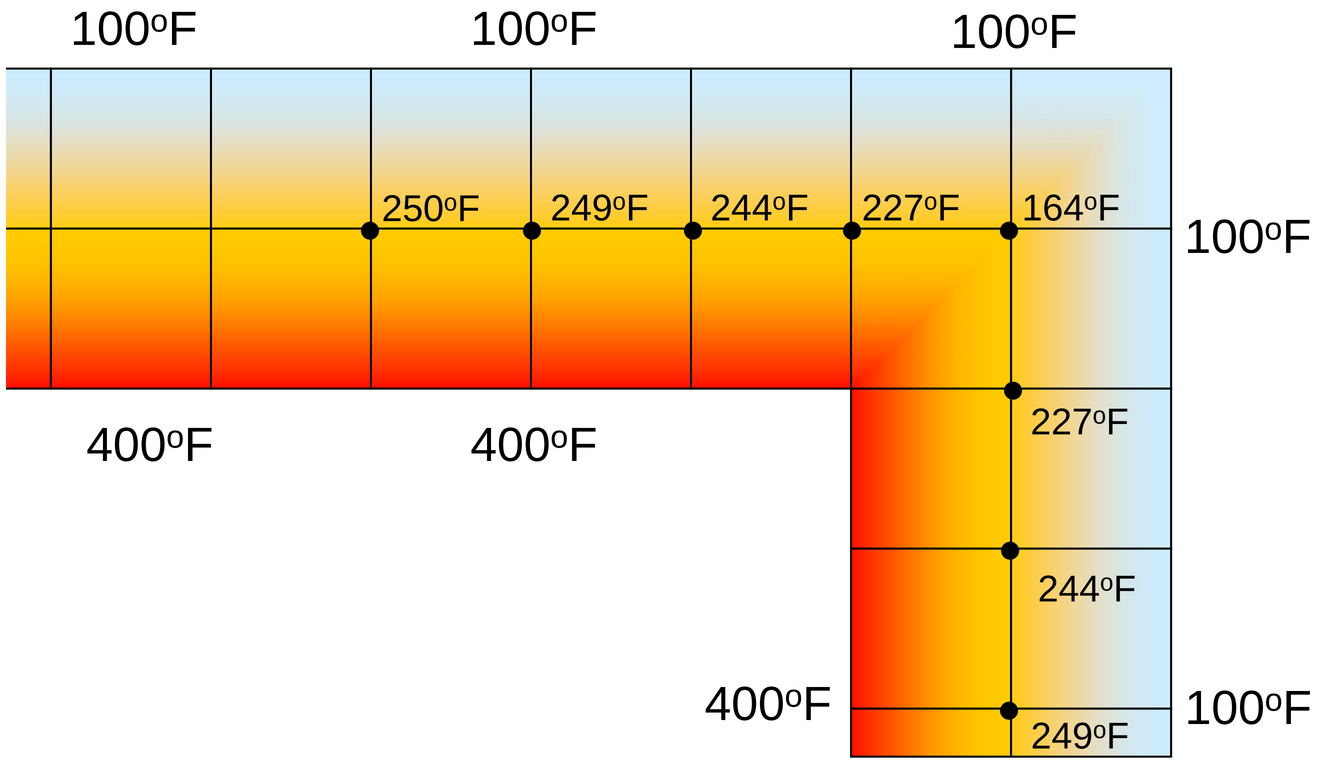
| step | point | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | | 2 | | 3 | | 4 | | 5 | |
| | $t_1$ | $q'_1$ | $t_2$ | $q'_2$ | $t_3$ | $q'_3$ | $t_4$ | $q'_4$ | $t_5$ | $q'_5$ |
| **Initial guess** → 0 | 250 | 0 | 250 | 0 | 250 | 0 | 250 | −50 | 200 | −100 |
| 1 | 250 | 0 | 250 | 0 | 250 | −13 | 237 | −23 | 175 | −26 |
| 2 | 250 | 0 | 250 | −3 | 247 | −7 | 231 | −9 | 168 | −10 |
| 3 | 250 | −1 | 249 | −1 | 245 | −2 | 229 | −6 | 165 | −2 |
| 4 | 250 | −1 | 249 | −2 | 244 | 0 | 227 | 0 | 164 | −2 |

200−(100)/(4)
=25

200−25=175

# Example 20-2 (Relaxation model)

## Step 4



# Relaxation Technique

If the boundary of the system had been curved, → **Smaller Grid**

⇓

**Higher accuracy of the temperature**

# Initial Guessing for Relaxation Technique

**Large Grid**

**Rough Temperature Distribution**

**Good basis for guessing the initial temperature distribution**

**Small Grid**

**Accurate Temperature Distribution**

# Other factor affecting heat transfer

**Heat Generation in the solid, $q_r$**

$$q_0' = t_1 + t_2 + t_3 + t_4 - 4t_0 + \frac{q_r (\Delta x)^2}{k} \qquad \textbf{(20-12)}$$

$$t_0 = \frac{t_1 + t_2 + t_3 + t_4 + \dfrac{q_r (\Delta x)^2}{k}}{4}$$

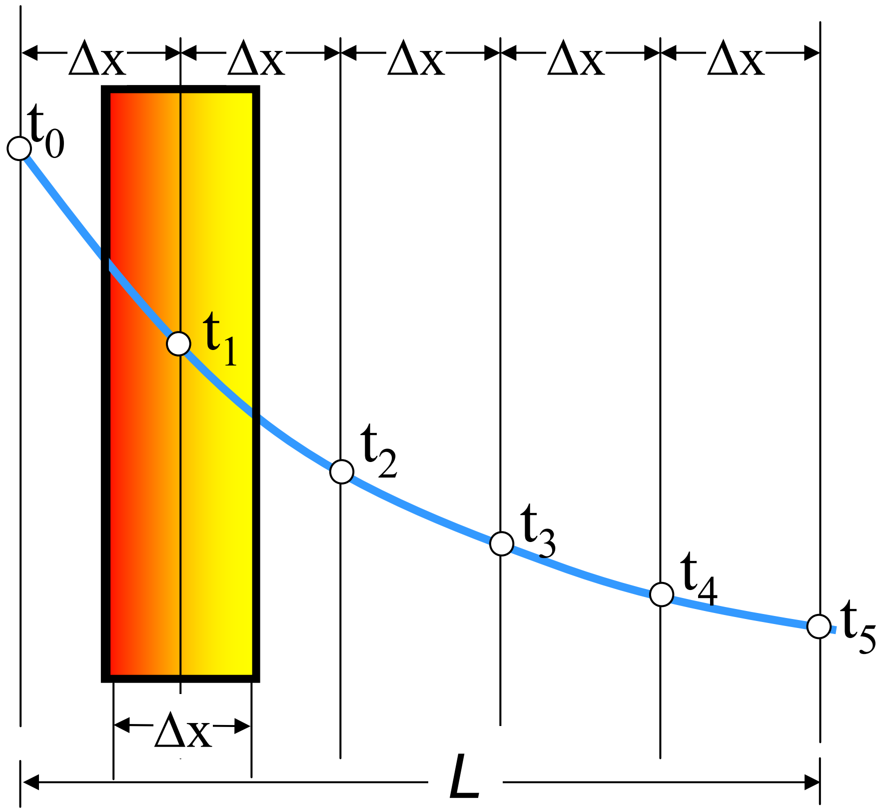
**Relaxation equation for three-dimensional conduction**

$$q_0' = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 - 6t_0 \qquad \textbf{(20-13)}$$

$$t_0 = \frac{t_1 + t_2 + t_3 + t_4 + t_5 + t_6}{6}$$

# Unsteady-State Heat Conduction

At an instant time,




Area: A
Thickness: L

$t'_1$-$t_1$=
Increase in the temperature of the element during the time, $\Delta\theta$.

Heat balance for the colored element

$$\frac{kA(t_0 - t_1)}{\Delta x} - \frac{kA(t_1 - t_2)}{\Delta x} = \frac{A\Delta x \rho C_p (t'_1 - t_1)}{\Delta \theta}$$

# Unsteady-State Heat Conduction

$$\frac{kA(t_0 - t_1)}{\Delta x} - \frac{kA(t_1 - t_2)}{\Delta x} = \frac{A \Delta x \rho C_p (t_1' - t_1)}{\Delta \theta}$$

$$t_1' - t_1 = \frac{k}{\rho C_p} \frac{\Delta \theta}{(\Delta x)^2} (t_0 + t_2 - 2t_1) = \frac{1}{M} (t_0 + t_2 - 2t_1) \quad \textbf{(20-14)}$$

$$M = \frac{(\Delta x)^2}{\alpha \Delta \theta} = \frac{1}{Fo} \qquad M = \frac{1}{Fo} = \frac{L^2}{\alpha \theta} = \frac{\dot{Q}_{stored}}{\dot{Q}_{conducted}}$$

$$t_1' = \frac{t_0 + t_1(M - 2) + t_2}{M} \quad \textbf{(20-15)}$$

# Unsteady-State Heat Conduction

If M = 2, Eq (20-15) reduces to $t_1' = \dfrac{t_0 + t_2}{2}$

$$\Delta\theta = \frac{\rho C_p}{k}\frac{(\Delta x)^2}{2} \qquad \textbf{(20-16)}$$

If more accuracy is desired, a smaller value of $\Delta$x, giving finer space mesh with more nodal points, must be used. For a given value of M, this will produce shorter time steps.

# Unsteady-State Heat Conduction

## Choice of values of M

$$M = \frac{1}{Fo} = \frac{L^2}{\alpha\theta} = \frac{\dot{Q}_{stored}}{\dot{Q}_{conducted}}$$

**If M < 2, the solution is unstable.**

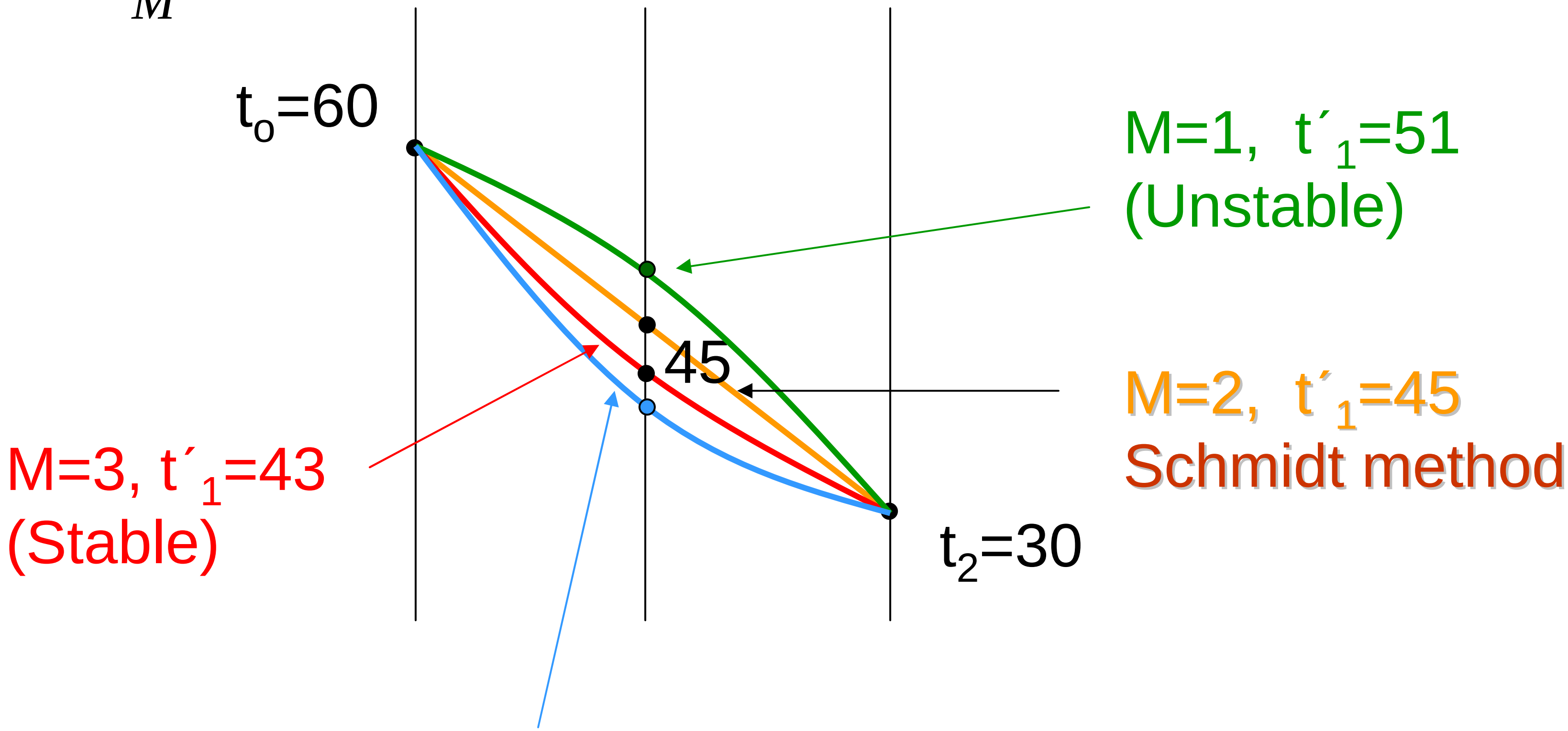
→ Result in the accumulation and amplification of the numerical errors as the solution progresses.

**If M ≥ 2, the solution is stable.**

→ The errors diminish

# Choice of value of M is restricted

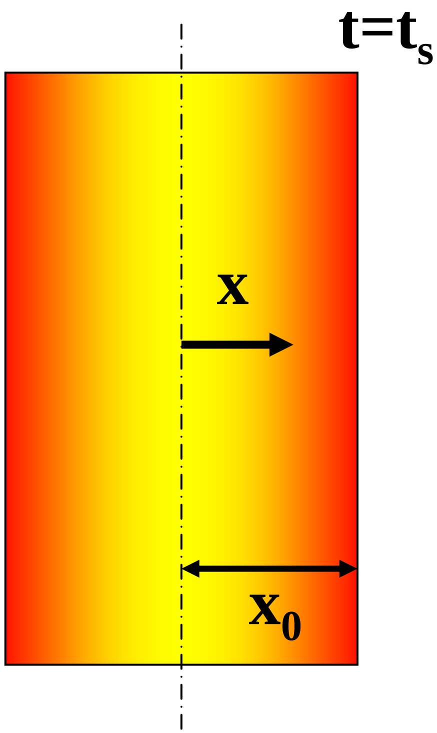
$$t_1' = \frac{t_0 + t_1(M-2) + t_2}{M}$$




Let $t_0$=60, $t_1$=39, $t_2$=30
at some instant in time

# Example 20-3

## Conduction into a plate of finite thickness




$$\frac{\partial t}{\partial \theta} = \alpha \frac{\partial^2 t}{\partial x^2} \qquad (19\text{-}1)$$

$$\text{I.C.} \quad : \quad \text{at } \theta = 0,\ t = t_0 = 70^\circ\text{F}$$

$$\text{B.C. 1}: \quad \text{at } x = x_0 = 1/4\text{"},\ t = t_s = 292^\circ\text{F}$$

$$\text{B.C. 2}: \quad \text{at } x = 0,\ dt/dx = 0$$

$$\theta = ?\ @\ x = 0 \text{ and } t = 290^\circ\text{F}$$

$$\alpha = \frac{k}{\rho C_p} = 0.028\ ft^2/h$$

# Example 20-3

If M=2 is chosen and sheet is considered as six slabs, each of thickness 0.0833 in, the time increment is

$$\Delta\theta = \frac{\rho C_p}{k}\frac{(\Delta x)^2}{2} = \frac{(0.0833/12)^2}{(2)(0.0028)} = 0.00858\ h\ (0.515\ mim)$$

If we carried the solution to the point of finding how long it took for the centerline temperature to reach 290$^o$F, the number of increments would be 18.7 min/0.515 min =36.3.

# Example 20-3:

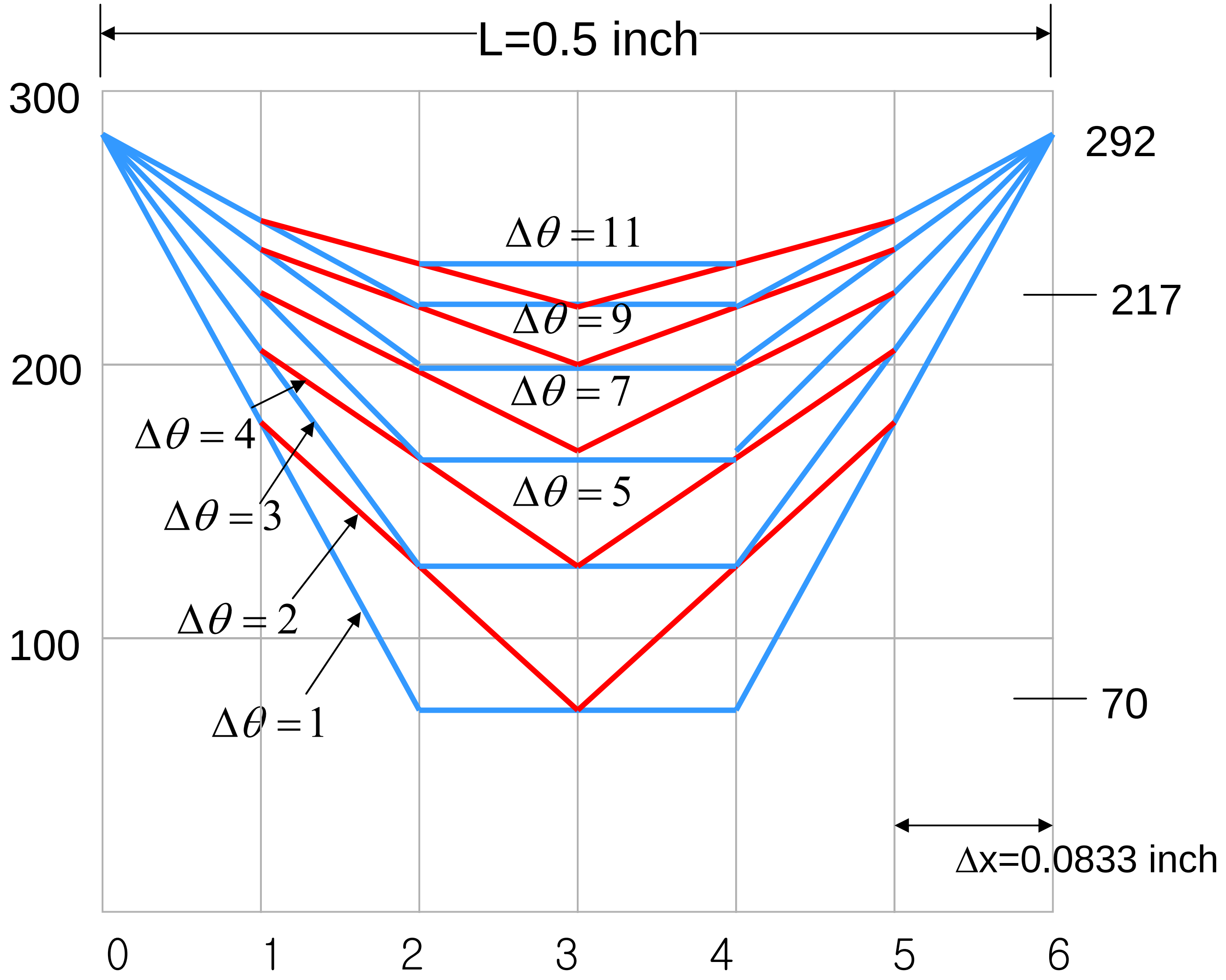
## Numerical Analysis of unsteady state conduction in a rubber sheet

| $\Delta\theta$ | t0 | t1 | t2 | t3 | t4 | t5 | t6 |
|---|---|---|---|---|---|---|---|
| 0 | 292 | 70 | 70 | 70 | 70 | 70 | 292 |
| 1 | 292 | 181 | 70 | 70 | 70 | 181 | 292 |
| 2 | 292 | 181 | 125 | 70 | 125 | 181 | 292 |
| 3 | 292 | 209 | 125 | 125 | 125 | 209 | 292 |
| 4 | 292 | 209 | 167 | 125 | 167 | 209 | 292 |
| 5 | 292 | 230 | 167 | 167 | 167 | 230 | 292 |
| 6 | 292 | 230 | 199 | 167 | 199 | 230 | 292 |
| 7 | 292 | 246 | 199 | 199 | 199 | 246 | 292 |
| 8 | 292 | 246 | 223 | 199 | 223 | 246 | 292 |
| 9 | 292 | 258 | 223 | 223 | 223 | 258 | 292 |
| 10 | 292 | 258 | 240 | 223 | 240 | 258 | 292 |

**9.7($\Delta\theta$) Increment (5 min)**

**$t_3$=217°F**

# Example 20-3: Graphical Solution



# HOMEWORK #3

**PROBLEMS**

**20-2**
**20-4**

**Due on October 24**