

*Lecture 7:*

# Comb Resonator Design (3)

## - Intro. to Dynamics -

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Dong-il "Dan" Cho

School of Electrical Engineering and Computer Science, Seoul  
National University

Nano/Micro Systems & Controls Laboratory

Email: [dicho@snu.ac.kr](mailto:dicho@snu.ac.kr)

URL: <http://nml.snu.ac.kr>

# Kinematics & Dynamics of Particles

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1) *Kinematics* : a branch of dynamics that deals with the geometry of motion (i.e. acceleration, velocities, and displacement) apart from consideration of mass and forces.

$$\left. \begin{aligned} \vec{v} &= \frac{d\vec{s}}{dt} \\ \vec{a} &= \frac{d\vec{v}}{dt} \end{aligned} \right\} dt = \frac{d\vec{s}}{\vec{v}} = \frac{d\vec{v}}{\vec{a}}$$

$$v \cdot d\vec{v} = a \cdot d\vec{s}$$

$$\int_{v_1}^{v_2} v \cdot d\vec{v} = \int_{s_1}^{s_2} a \cdot d\vec{s}$$



# Kinematics & Dynamics of Particles (cont'd)

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2) *Force momentum principle :*

$$\vec{f} = \frac{d\vec{p}}{dt} \quad \vec{p} = m\vec{v} \text{ (momentum)}$$

3) *Constitutive relations :*

$$\vec{p} = m\vec{v} \text{ (momentum)}$$

$$f_s = k \cdot s$$

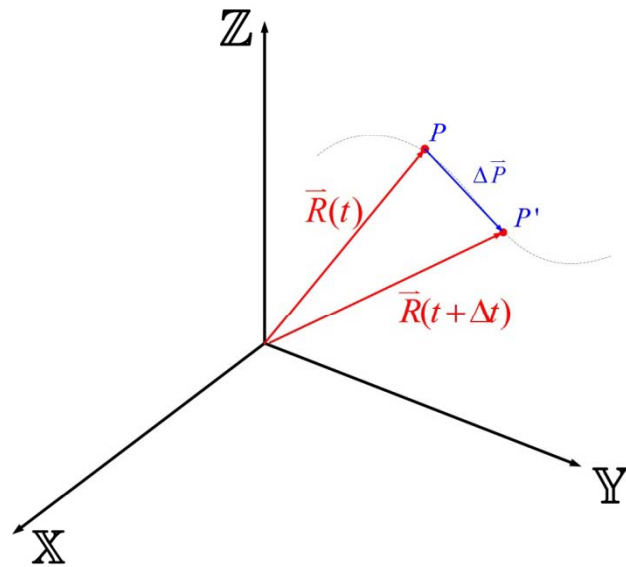
$$f_g = mg$$

$$f_f = \mu N$$



# Kinematics of Particles

- *Particle in moving space*



$\Delta \vec{P}$  = displacement vector

$\vec{R}(t)$  = position vector

$\vec{v}(t)$  = velocity vector =  $\frac{d\vec{R}(t)}{dt}$

$\vec{a}(t)$  = acceleration vector =  $\frac{d\vec{v}(t)}{dt}$



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# Kinematics of Particles (cont'd)

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- ***Inertial Reference Frame (IRF)***

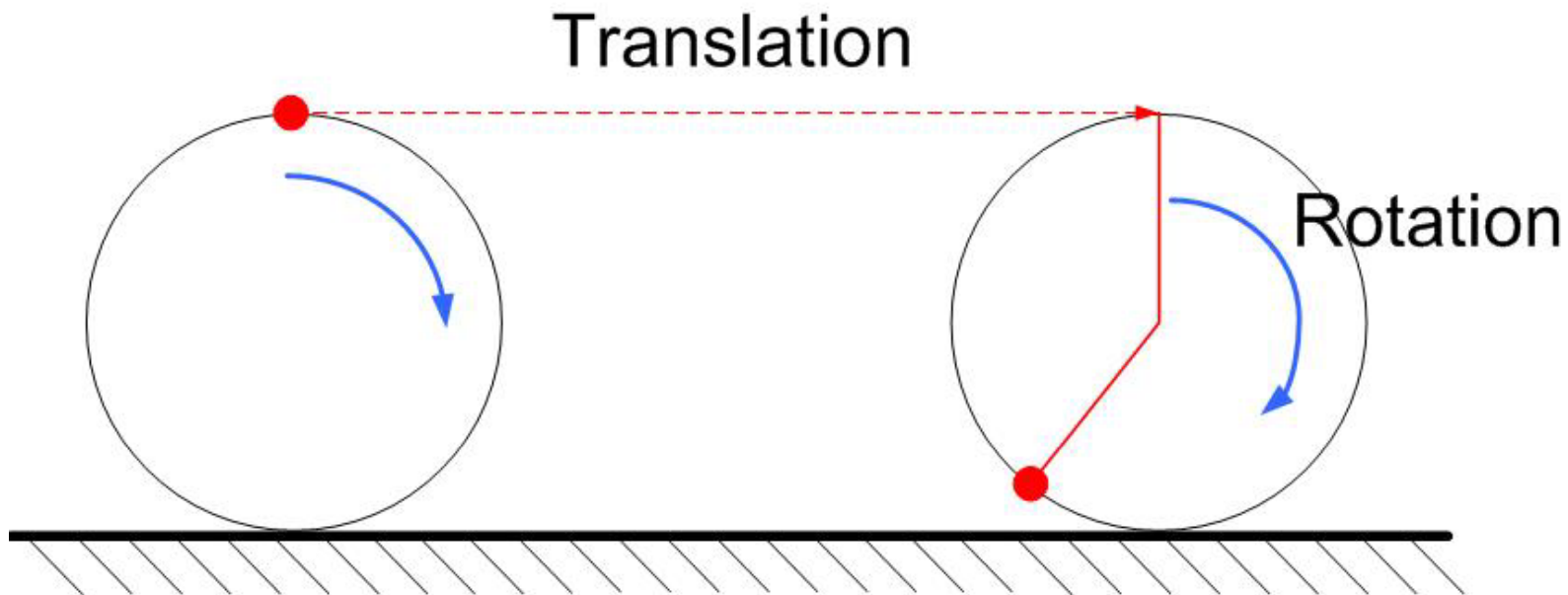
- $\mathcal{OXYZ} (\underline{U}_x, \underline{U}_y, \underline{U}_z)$
- Non-accelerating and non-rotating
- An isolated particle maintains constant velocity

$$\Leftrightarrow \vec{f} = \frac{d\vec{p}}{dt} \quad (\vec{p} = \text{momentum})$$



# Kinematics of Particles (cont'd)

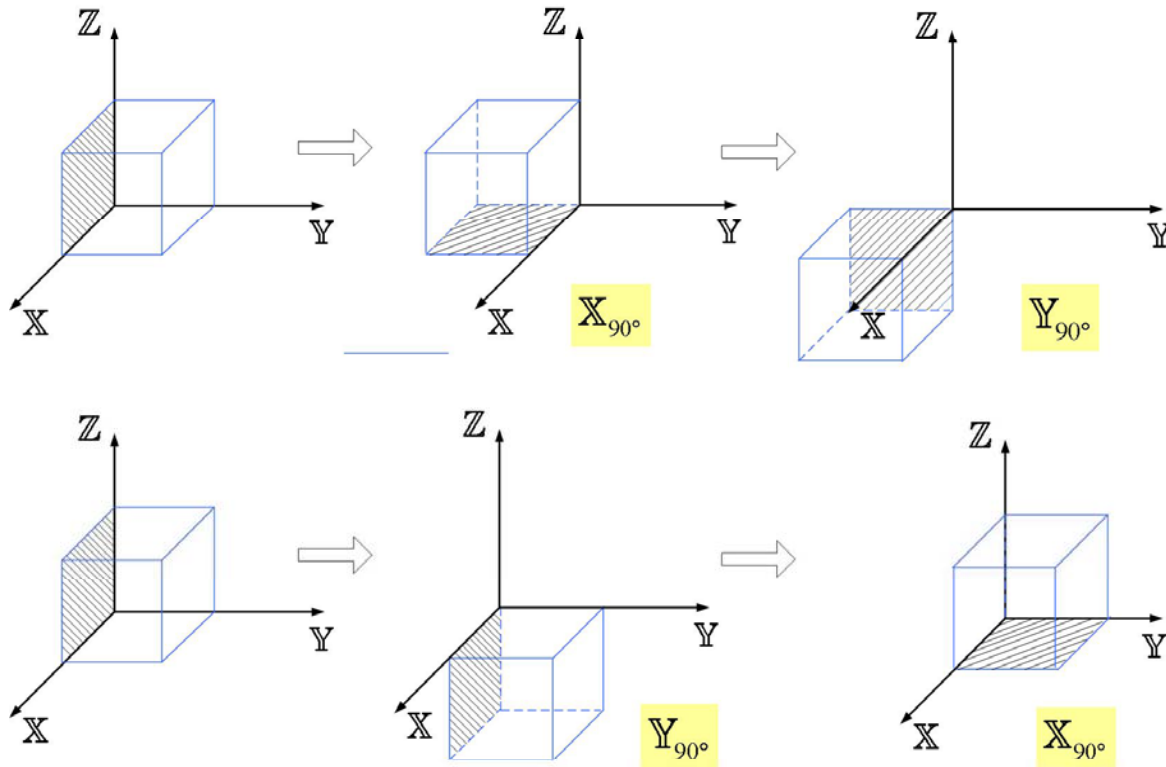
- *Any motion can be described by the sum of a single translation plus a single rotation.*



# Kinematics of Particles (cont'd)

- *Finite rotations are not vectors.*  
: Sequence is important.

$$\{X\}_{90^\circ} + \{Y\}_{90^\circ} \neq \{Y\}_{90^\circ} + \{X\}_{90^\circ}$$



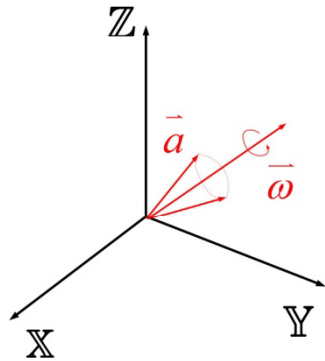
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# Kinematics of Particles (cont'd)

- *Rate of change of rotating constant vector*



$$\frac{d\vec{a}}{dt} = \vec{\omega} \times \vec{a}$$

[Ref] S. H. Crandall et al., "Dynamics of Mechanical and Electromechanical Systems", McGraw-Hill, pp. 51 – 53, 1970.

- *Rate of change of rotating and changing vector*  
rotating frame, **A**

$$\frac{d\vec{A}}{dt} = \dot{A}_x \vec{u}_x + \dot{A}_y \vec{u}_y + \dot{A}_z \vec{u}_z + A_x \vec{\omega} \times \vec{u}_x + A_y \vec{\omega} \times \vec{u}_y + A_z \vec{\omega} \times \vec{u}_z$$

$$\left. \frac{dA}{dt} = \frac{\partial \vec{A}}{\partial t} \right)_{rel} + \vec{\omega} \times \vec{A}$$



$$\left. \frac{d}{dt} = \frac{\partial}{\partial t} \right)_{rel} + \vec{\omega} \times$$



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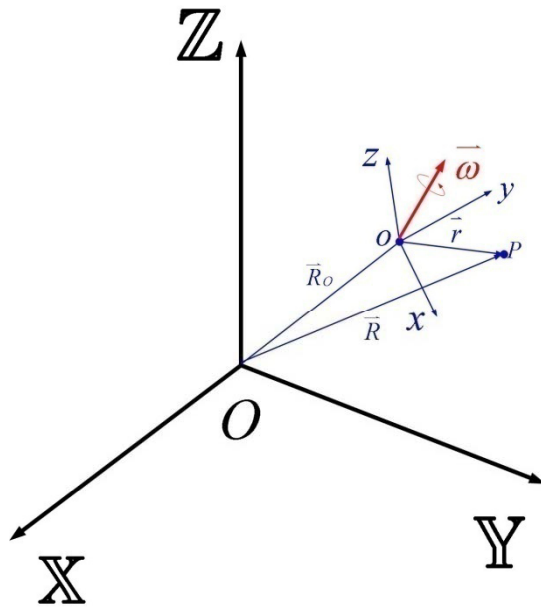
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# Kinematics of Particles (cont'd)

- Kinematics using intermediate reference frame (irf)*



$$IRF = \mathcal{OXYZ}$$

$$irf = oxyz$$



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# Kinematics of Particles (cont'd)

- **Motion of P defined in the irf (oxyz):**

$$\vec{r}(t) = \text{position vector} = x\vec{u}_x + y\vec{u}_y + z\vec{u}_z$$

Let  $\left. \frac{\partial}{\partial t} \right)_{rel}$  be the time differentiation in oxyz.

**velocity in oxyz :**

$$\vec{v}_{rel} = \left. \frac{\partial \vec{r}}{\partial t} \right)_{rel} = \dot{x}\vec{u}_x + \dot{y}\vec{u}_y + \dot{z}\vec{u}_z$$

**acceleration in oxyz :**

$$\vec{a}_{rel} = \left. \frac{\partial^2 \vec{r}}{\partial t^2} \right)_{rel} = \ddot{x}\vec{u}_x + \ddot{y}\vec{u}_y + \ddot{z}\vec{u}_z$$



# Kinematics of Particles (cont'd)

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## Position of P in OXYZ

$$\vec{R}(t) = \vec{R}_O(t) + \vec{r}(t)$$

## Velocity of P in OXYZ

$$\vec{v}(t) = \frac{d\vec{R}}{dt} = \frac{d\vec{R}_O}{dt} + \frac{d\vec{r}}{dt} = \dot{\vec{R}}_O + \left[ \frac{\partial}{\partial t} \right]_{rel} + \vec{\omega} \times \vec{r} = \dot{\vec{R}}_O + \vec{v}_{rel} + \vec{\omega} \times \vec{r}$$



# Kinematics of Particles (cont'd)

## Acceleration of P in $\mathcal{OXYZ}$

$$\begin{aligned}
 \bar{a}(t) &= \frac{d\bar{v}(t)}{dt} = \frac{d}{dt}(\dot{\bar{R}}_O) + \frac{d}{dt}(\bar{v}_{rel}) + \frac{d}{dt}(\bar{\omega} \times \bar{r}) \\
 &= \ddot{\bar{R}}_O + \left[ \frac{\partial}{\partial t} \right]_{rel} + \bar{\omega} \times \bar{v}_{rel} + \frac{d}{dt}(\bar{\omega}) \times \bar{r} + \bar{\omega} \times \frac{d}{dt}(\bar{r}) \\
 &= \ddot{\bar{R}}_O + \left( \frac{\partial \bar{v}_{rel}}{\partial t} \right)_{rel} + \bar{\omega} \times \bar{v}_{rel} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \left[ \frac{\partial}{\partial t} \right]_{rel} + \bar{\omega} \times \bar{r} \\
 &= \ddot{\bar{R}}_O + \bar{a}_{rel} + \bar{\omega} \times \bar{v}_{rel} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \left( \frac{\partial \bar{r}}{\partial t} \right)_{rel} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) \\
 &= \ddot{\bar{R}}_O + \bar{a}_{rel} + 2 \cdot (\bar{\omega} \times \bar{v}_{rel}) + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})
 \end{aligned}$$

Frame

Particle  
irf

Coriolis' accel.

Frame  
angular accel.

Centripital  
accel.



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# Dynamics of Particles

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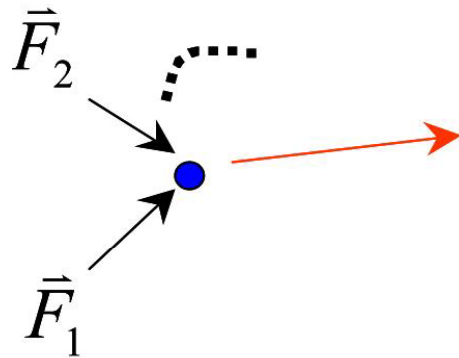
- **Dynamics** is based on kinematics and

Newton's second law :  $\vec{f} = \frac{d\vec{p}}{dt}$

- A reference system in which Newton's second law is valid is called an inertial system
- All systems moving with constant and linear velocities are inertial systems.



# Dynamics of Particles (cont'd)



- When a particle is subjected to forces :

$$\vec{F}_1, \vec{F}_2, \dots$$

$$\vec{F}_R = \sum_i \vec{F}_i = m\vec{a}$$

- We can separate :

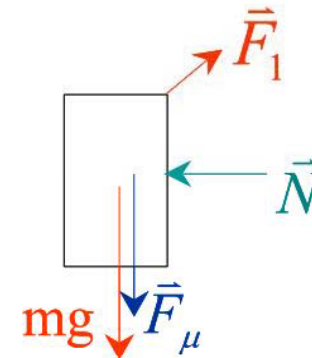
$$F_{Rx} = ma_x$$

$$F_{Ry} = ma_y$$

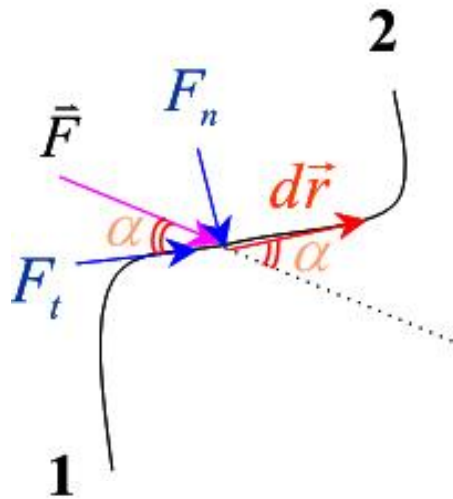
$$F_{Rz} = ma_z$$

[Equation of motion]

- \* Free body diagram :



# Work



- The work done by a force on a particle in a displacement of  $d\vec{r}$

$$dU = \vec{F} \cdot d\vec{r} = F \cdot ds \cos \alpha, \quad ds = |d\vec{r}|$$

if  $\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$

$$d\vec{r} = dx \vec{i} + dy \vec{j} + dz \vec{k}$$

Then  $\vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$

$\therefore$  The total work done :  $U = \int \vec{F} \cdot d\vec{r}$

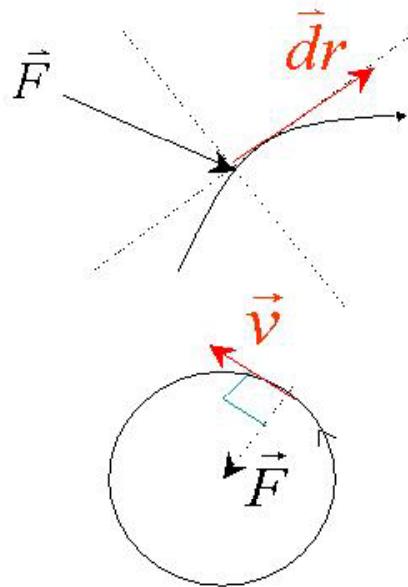
Now  $dU = F_t \cdot ds \quad (F \cos \alpha = F_t)$   
 $= ma_t \cdot ds$



# Kinetic Energy

- Total work done by external force

$$\begin{aligned}
 W_{1-2} &= \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 F dr_t = \int_1^2 m a_t dr_t = \int_1^2 m \frac{dv}{dt} dr_t \\
 &= \int_1^2 m \frac{dv}{dt} dr_t = \int_1^2 m \frac{dv}{dt} v dt = \int_1^2 m v dv \\
 &= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = T_2 - T_1 = \Delta T
 \end{aligned}$$



**Total work done by external = change in K.E.**

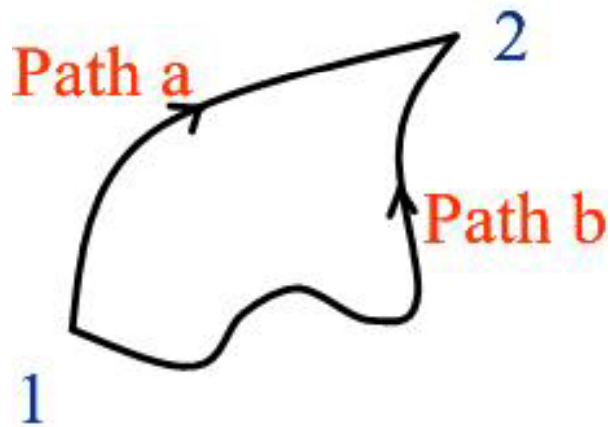
$$\text{Power : } P = \frac{dU}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$





# Conservative Force and Potential Energy

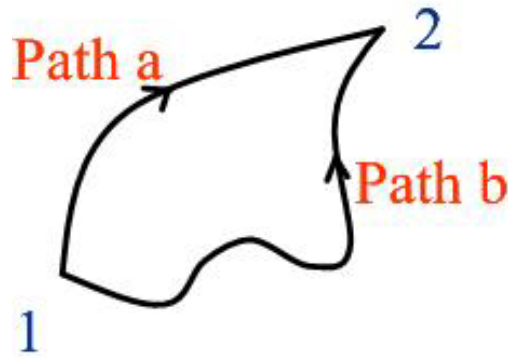
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- A conservative force is a force having the characteristic that the work done by the force on the particle depends on the net change in position and not on the actual path followed by the particle.
- e.g.
  - gravitational force
  - elastic force
  - electrostatic force



# Conservative Force and Potential Energy (cont'd)



- Work done to move P from 1 to 2 is independent of path

$$\int_{1(\text{path } a)}^2 \vec{F} \cdot d\vec{r} = \int_{1(\text{path } b)}^2 \vec{F} \cdot d\vec{r} = V - V_{ref}$$

where  $V$  is the potential depending on position  $\vec{r}$ ,  $\vec{F}$  is a conservative force

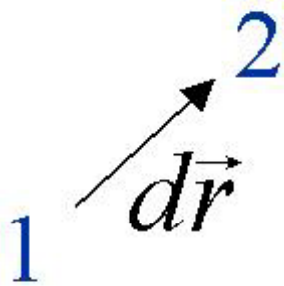
- The potential is written as

$$V(\vec{r}) = V_{ref} - \int_{S_0}^S \vec{F} \cdot d\vec{r}$$

$$\text{at } S_0, V = V_{ref}$$



# Conservative Force and Potential Energy (cont'd)



when  $d\vec{r}$  is very small,

$$dV = -\int_{S_0}^{S_2} \vec{F} \cdot d\vec{r} - \left(-\int_{S_0}^{S_1} \vec{F} \cdot d\vec{r}\right)$$

$$= -\int_{S_1}^{S_2} \vec{F} \cdot d\vec{r}$$

$$= -\vec{F} \cdot d\vec{r}$$

$$= -F_x dx - F_y dy - F_z dz$$

Since  $V$  is a function of  $\vec{r}$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \quad \text{(exact differential)}$$

$$F_x = -\frac{\partial V}{\partial x} ; F_y = -\frac{\partial V}{\partial y} ; F_z = -\frac{\partial V}{\partial z}$$



# Work-Energy Equation

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$$\begin{aligned}\Delta K.E. &= \int \sum \vec{F} \cdot d\vec{r} \\ &= \int \left( \sum \vec{F}^{(C)} + \sum \vec{F}^{(0)} \right) \cdot d\vec{r}\end{aligned}$$

- $\sum \vec{F}^{(C)}$  - all conservative forces
- $\sum \vec{F}^{(0)}$  - non-conservative forces (frictional force), depend on paths

$$\int \sum \vec{F}^{(C)} \cdot d\vec{r} = -\Delta V$$

- If we specify

$$\int \sum \vec{F}^{(0)} \cdot d\vec{r} = W, \text{ dissipative work}$$

$$\therefore \Delta K.E. = -\Delta V + W$$



# Work-Energy Equation (cont'd)

$$\therefore \Delta T + \Delta V = W$$

(workdone)

increase

decrease

: work-energy eq.

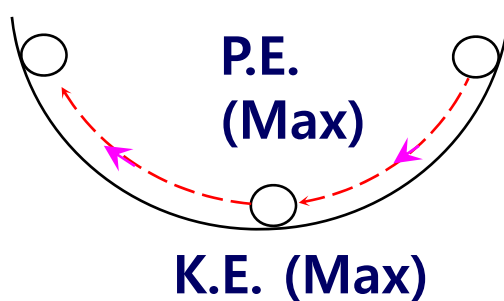
If  $W = 0$ , then

$$\Delta T + \Delta V = 0$$

The conservation of mechanical energy.

$$\therefore (T_2 - T_1) + (V_2 - V_1) = 0$$

$$\therefore T_2 + V_2 = T_1 + V_1$$



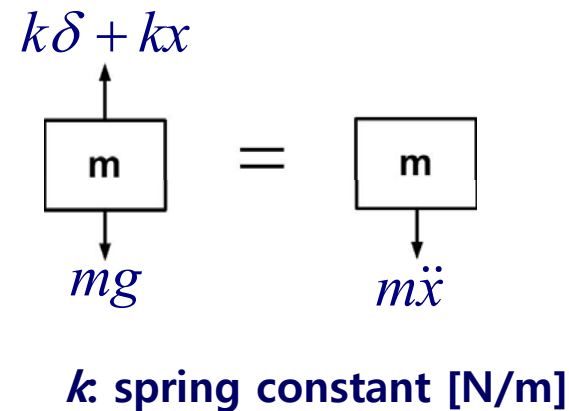
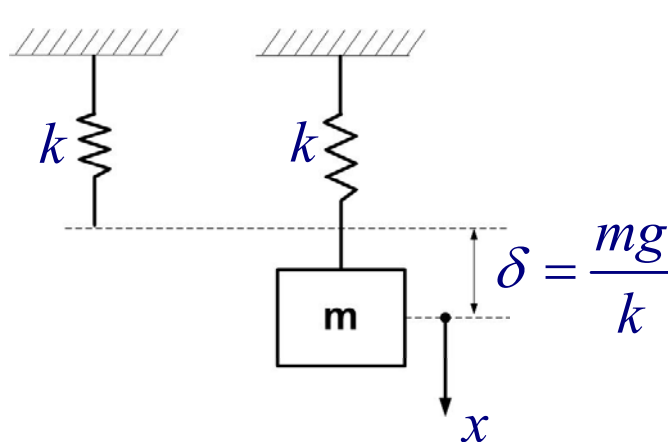
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# Free Vibration of Particle

- **Free-vibration:** absence of any imposed external forces
  - Undamped free vibration:
    - (1) Translational motion



The equation of motion:  $-k\delta - kx + mg = m\ddot{x} = F$

$$\Rightarrow \boxed{m\ddot{x} + kx = 0}$$

(simple harmonic motion)



## Free Vibration of Particle (cont'd)

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or  $\ddot{x} + \omega_n^2 x = 0$

where  $\omega_n = \sqrt{\frac{k}{m}}$  [rad / s]  
= natural frequency of the vibration

The solution:  $x = A \cos(\omega_n t) + B \sin(\omega_n t)$   
 $= C \sin(\omega_n t + \psi)$

where  $C$  : amplitude  
 $\psi$  : phase angle

**Unknown factors are determined by initial conditions**

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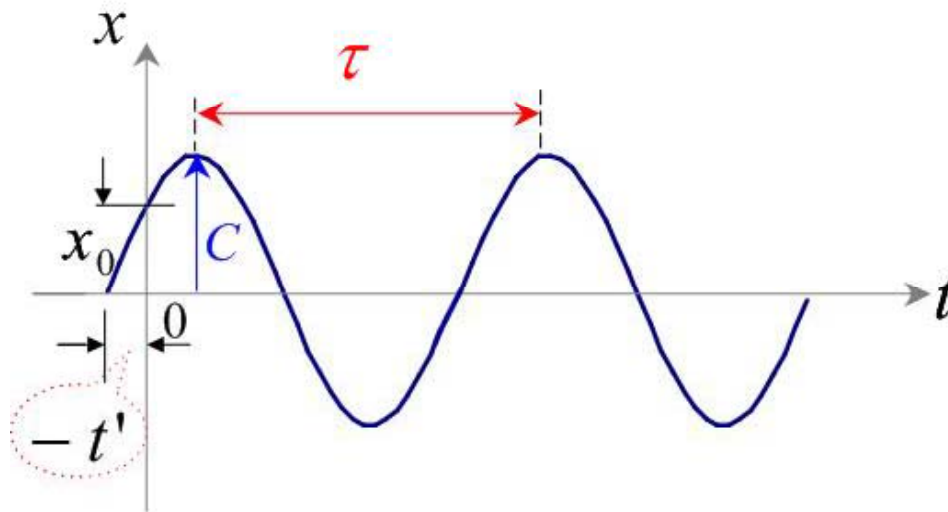


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## Free Vibration of Particle (cont'd)



$$\left. \begin{aligned} x_0 &= C \sin \psi \\ 0 &= C \sin(-\omega_n t' + \psi) \end{aligned} \right\} \Rightarrow -\omega_n t' + \psi = 0$$

$$\therefore \psi = \omega_n t'$$

$$\tau = 2\pi / \omega_n = \text{period} = 1/f_n$$

$$\text{where } f_n : \text{natural frequency [Hz]} = \omega_n / 2\pi$$



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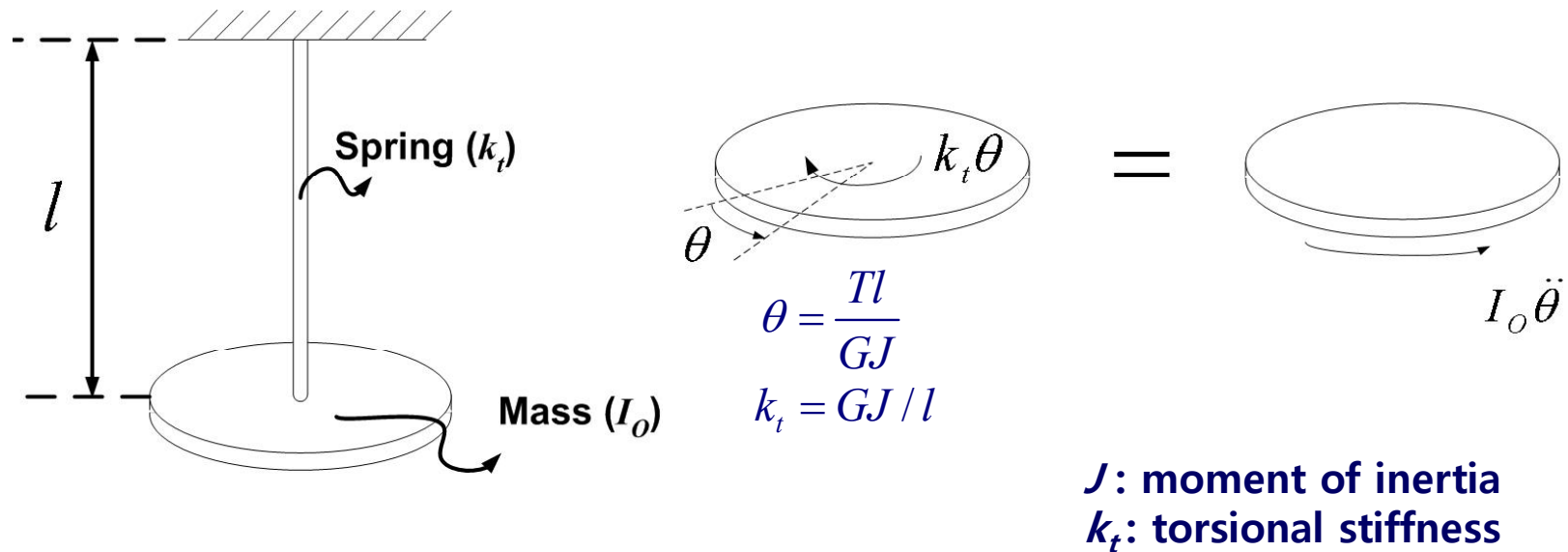
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# Free Vibration of Particle (cont'd)

## (2) Torsional motion



The equation of motion:  $I_o \ddot{\theta} + k_t \theta = 0$

$$\Rightarrow \omega_n = \sqrt{\frac{k_t}{I_o}} = \sqrt{\frac{GJ}{I_o l}}$$



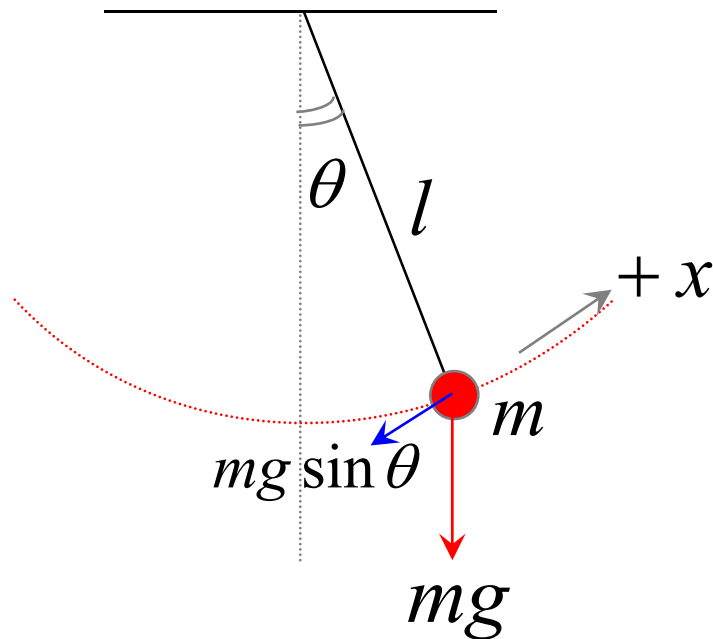
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# Free Vibration of Particle (cont'd)

(3) Simple pendulum: (small oscillation)



$$\sin \theta \approx \theta$$

$$\begin{aligned} F &= -mg \sin \theta \approx -mg\theta \\ &= -mgx / l = m\ddot{x} \end{aligned}$$

$$\therefore \ddot{x} + \frac{g}{l} x = 0$$

$$\therefore \omega_n = \sqrt{\frac{g}{l}}$$

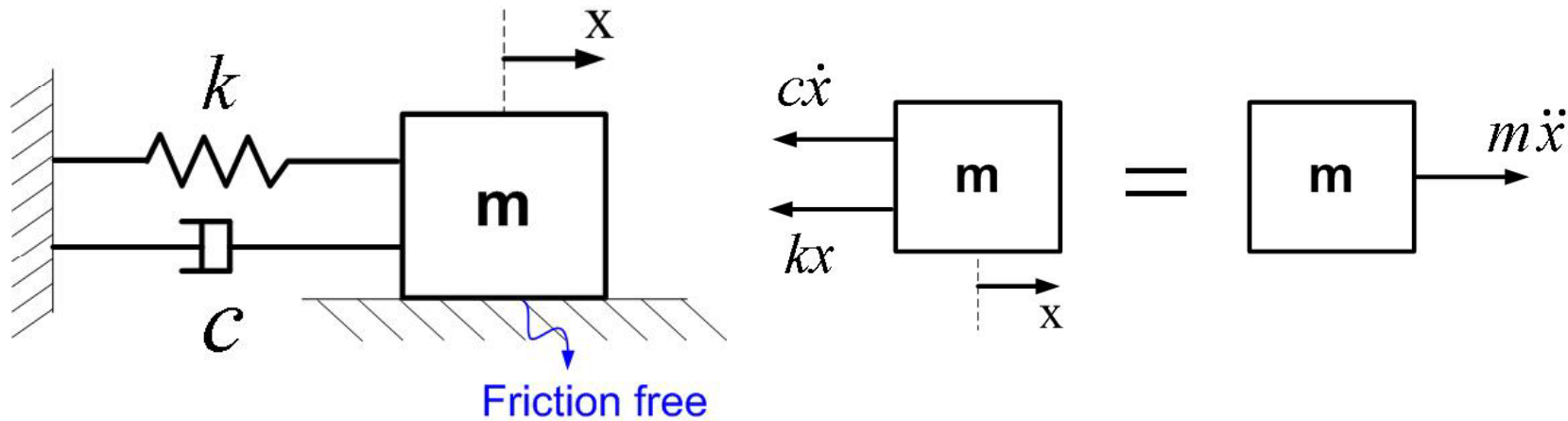


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# Damped Vibration of Particle



The equation of motion :  $\sum F = -kx - c\dot{x} = m\ddot{x}$

where  $c$  = viscous damping constant [Ns/m]  
(viscous damping coefficient)

$$\text{Then } m\ddot{x} + c\dot{x} + kx = 0$$



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# Damped Vibration of Particle (cont'd)

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Let the solution  $x = ce^{\lambda t}$

Then  $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$c_{cr} = 2\sqrt{mk} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n : \text{damping coeff. [N} \cdot \text{s / m]}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} : \text{damping ratio [dimensionless]}$$

so  $\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\frac{c^2 - 4mk}{4m^2}} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$



# Damped Vibration of Particle (cont'd)

The general solution:  $x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

There are three cases:

(1)  $\zeta > 1$  or  $c^2 - 4mk > 0$  (overdamping):

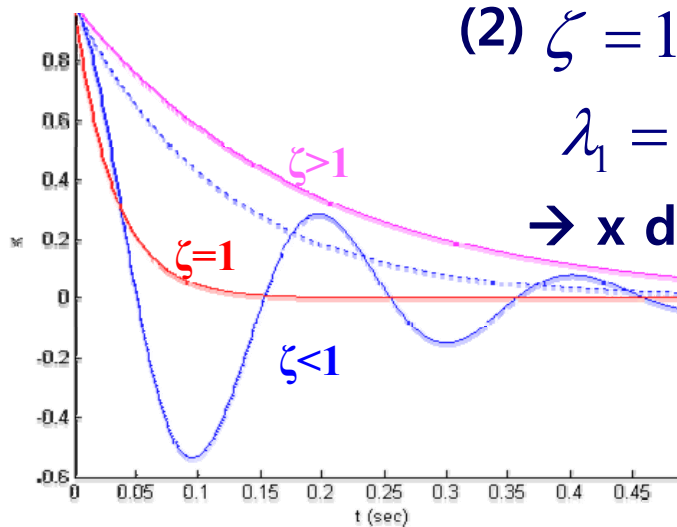
$\lambda_1, \lambda_2$  are distinct real numbers

→  $x$  decays to zero. / No oscillation.

(2)  $\zeta = 1$  or  $c^2 - 4mk = 0$  (critical damping):

$\lambda_1 = \lambda_2 = -\omega_n$ : one real number

→  $x$  decays to zero very fast. / No oscillation.



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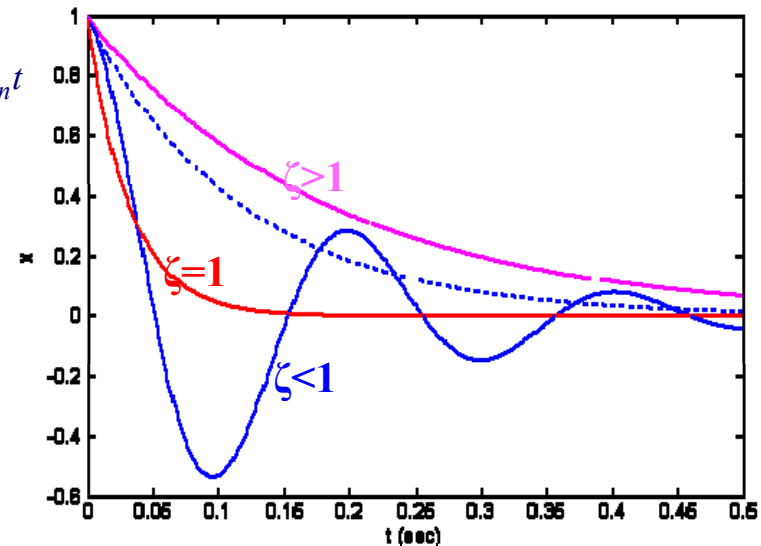
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## Damped Vibration of Particle (cont'd)

(3)  $\zeta < 1$  or  $c^2 - 4mk < 0$  (underdamping):

$$\begin{aligned}x &= \left[ c_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + c_2 e^{-i\sqrt{1-\zeta^2}\omega_n t} \right] \times e^{-\zeta\omega_n t} \\&= \left[ c_1 e^{i\omega_d t} + c_2 e^{-i\omega_d t} \right] e^{-\zeta\omega_n t} \\&= C \cdot \sin(\omega_d t + \psi) e^{-\zeta\omega_n t} \\&= \left[ C \cdot e^{-\zeta\omega_n t} \right] \sin(\omega_d t + \psi)\end{aligned}$$



Where  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$  = damped natural frequency

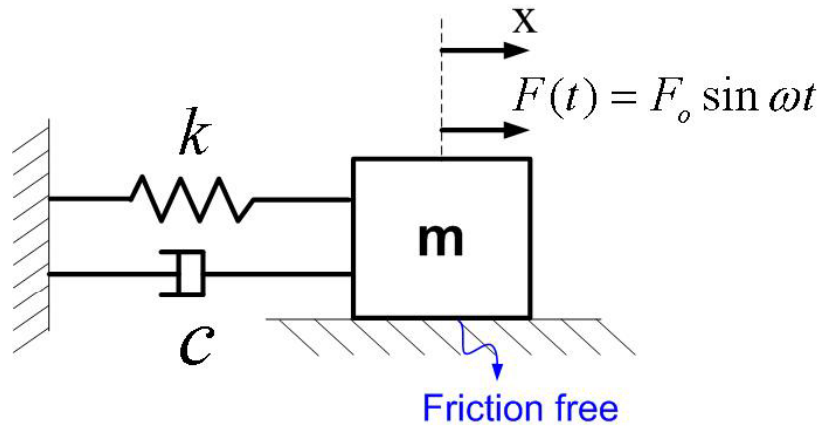


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# Forced Vibration of Particles



The equation of motion:

$$\sum F = -kx - c\dot{x} + F_0 \sin \omega t = m\ddot{x}$$

$$\therefore m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$$

$$\text{i.e. } \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \omega t$$



# Forced Vibration of Particles (cont'd)

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- **Solution of damped forced vibration:**

The equation of motion: 
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{F_o}{m}\sin\omega t$$

**Solution is the sum of a complementary solution and a particular solution**

$$x = x_c + x_p$$

$$x_c = Ce^{-\zeta\omega_n t} \sin(\omega_d t + \psi)$$

**dies down exponentially and is not important**

**Let :**  $x_p = X \sin(\omega t + \phi)$

**then**  $V(t) = X\omega \cos(\omega t + \phi) = V_o \cos(\omega t + \phi)$





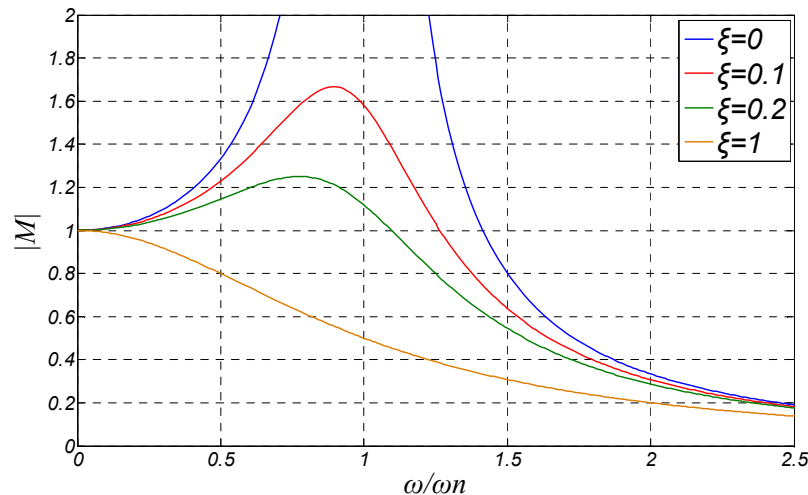
# Forced Vibration of Particles (cont'd)

- The solution is:

$$X = \frac{F_o / k}{\left\{ \left[ 1 - (\omega / \omega_n)^2 \right]^2 + \left[ 2\zeta \omega / \omega_n \right]^2 \right\}^{1/2}}$$

$$\tan \phi = \frac{2\zeta \omega / \omega_n}{1 - (\omega / \omega_n)^2}$$

$$M = \frac{X}{F_o / k} = \frac{1}{\left\{ \left[ 1 - (\omega / \omega_n)^2 \right]^2 + \left[ 2\zeta \omega / \omega_n \right]^2 \right\}^{1/2}}$$



$$\frac{d}{d\omega} \left\{ \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right]^2 + \left( \frac{2\zeta\omega}{\omega_n} \right)^2 \right\}^2 = 0$$

$$\omega_{\text{resonance}} = \omega_n \sqrt{1 - 2\zeta^2}$$

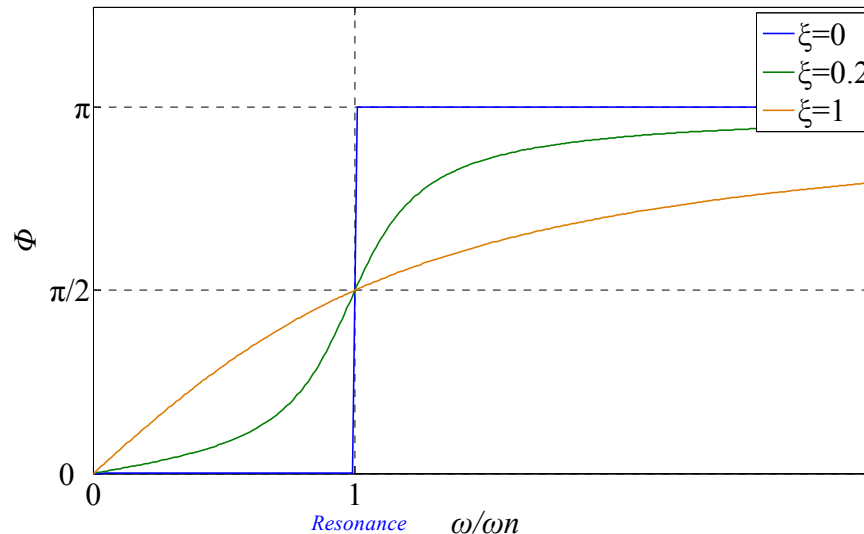


Dong-il "Dan" Cho

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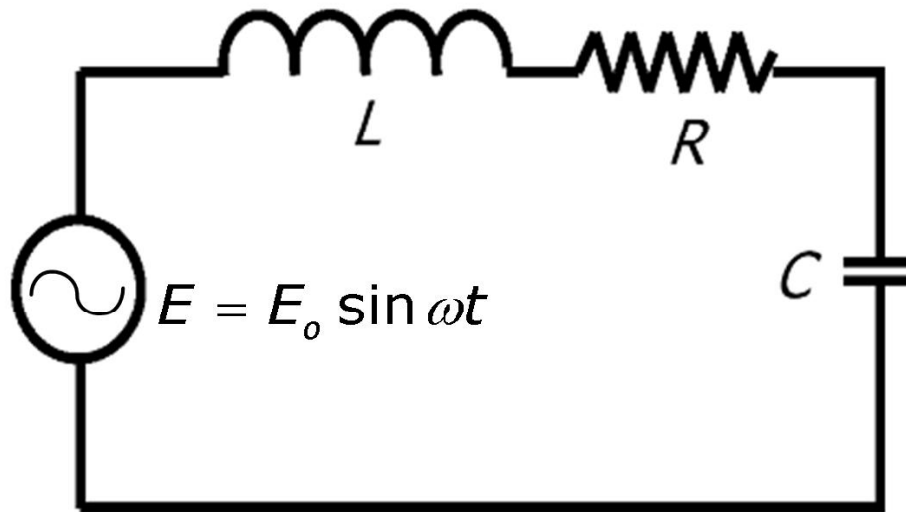
# Forced Vibration of Particles (cont'd)



- (1)  $\omega$  is small,  $\tan\phi > 0$ ,  $\phi \rightarrow 0^+$ ,  $x_p$  in phase with driving force
- (2)  $\omega$  is large,  $\tan\phi < 0$ ,  $\phi \rightarrow 0^-$ ,  $\phi = \pi$ ,  $x_p$  leads the driving force by  $90^\circ$
- (3)  $\omega \rightarrow \omega_n^-$ ,  $\tan\phi \rightarrow +\infty$ ,  $\phi \rightarrow \pi/2^{(-)}$   
 $\omega \rightarrow \omega_n^+$ ,  $\tan\phi \rightarrow -\infty$ ,  $\phi \rightarrow \pi/2^{(+)}$



# Electric Circuit Analogy



q-charge

L-inductance

C-capacitance

R-resistance

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{q(t)}{C} = E$$

$$\dot{q} = i \text{ (current)}$$



# Reference

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