Lecture 8:

Comb Resonator Design (4)
- Intro. to Fluidic Dynamics (Damping) -

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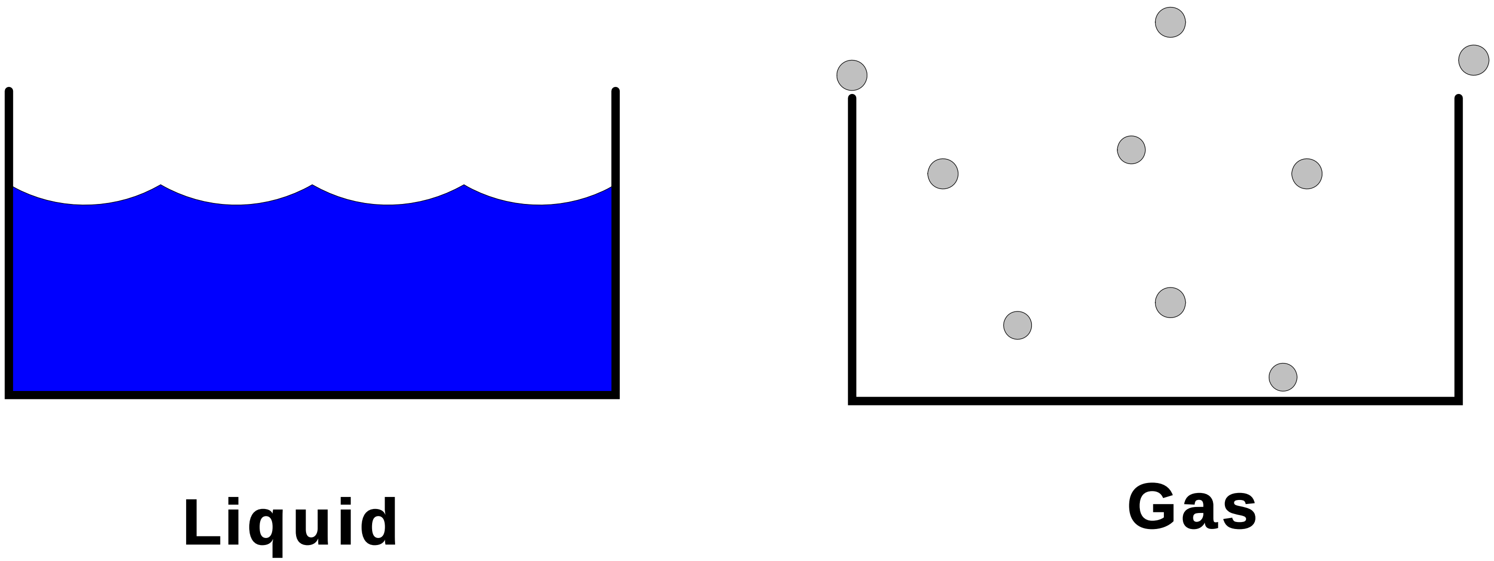
Two Types of Fluids

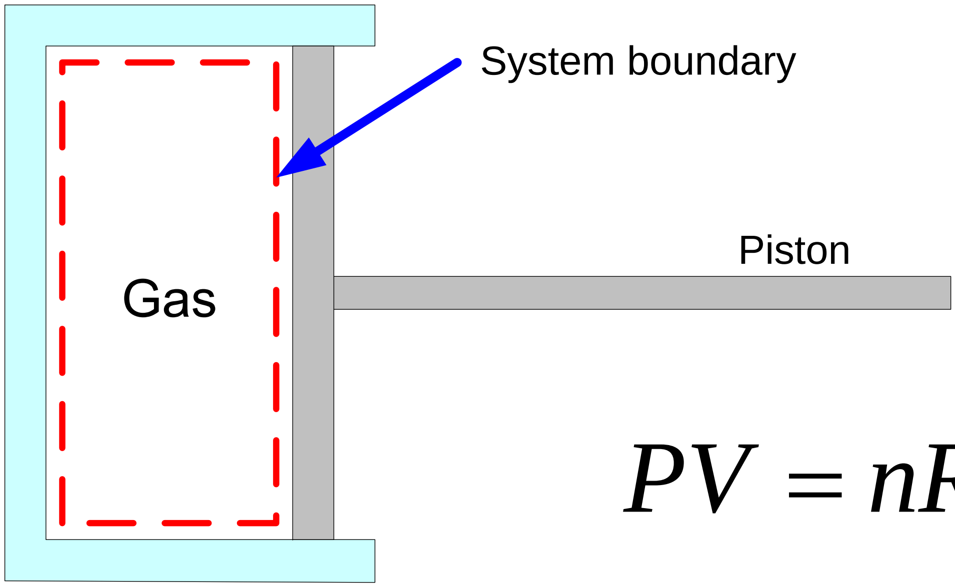
- **Liquid** : molecules are free to change position but are bound by cohesive forces. (fixed volume)

- **Gas** : molecules are unrestricted. (no shape or volume)



Methods of Analysis

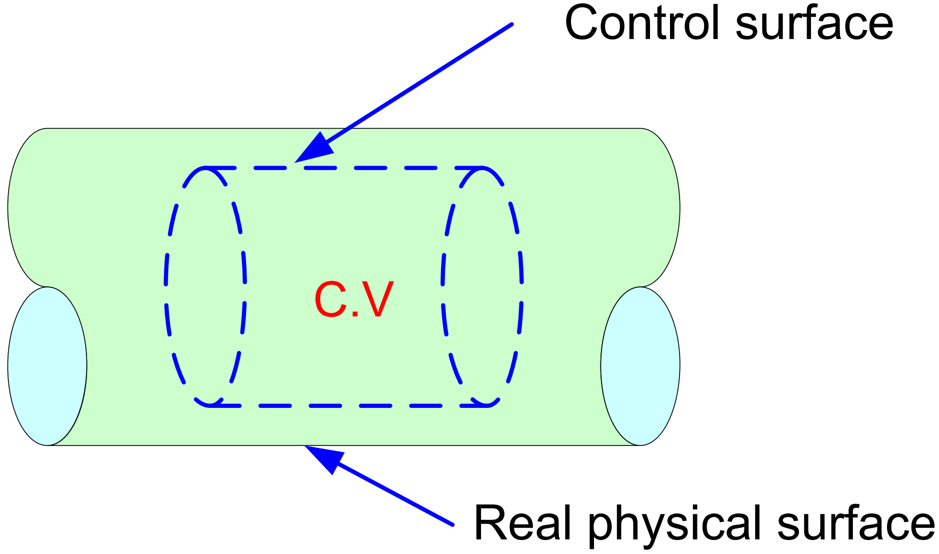
- **System**
  - It is defined as a fixed, identifiable quantities of mass.
  - No mass crosses the system boundary.
  - Heat and work may cross the boundary of the system.



$$PV = nRT$$

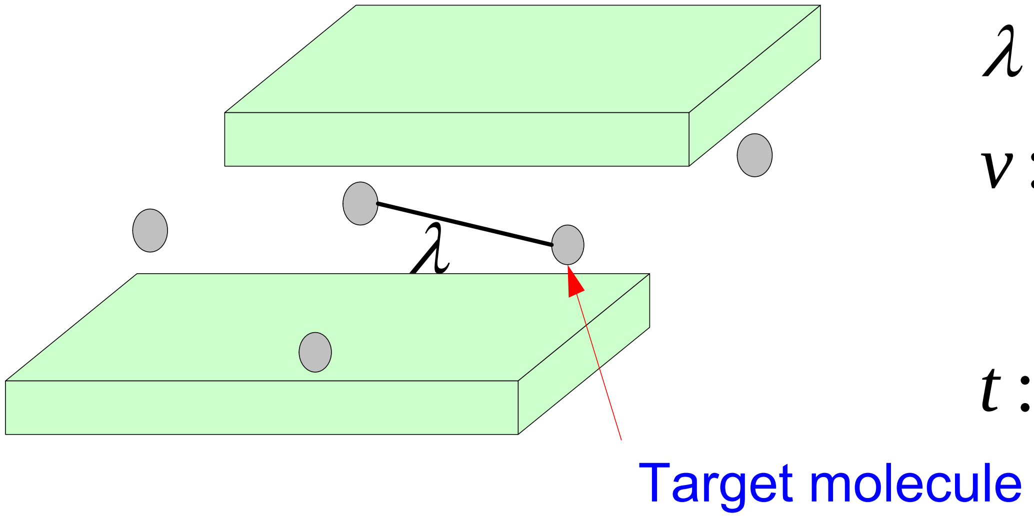
Methods of Analysis (cont'd)

- **Control volume**
  – An arbitrary volume in space through which fluid flows.
- **Control surface**
  – The geometric boundary of the control volume.



Mean Free Path

- **Mean free path (λ)**
  - A distance that the molecule travels before it reacts with another molecules.



$$\lambda = \bar{v} \cdot t$$

v : *average velocity of the molecule*

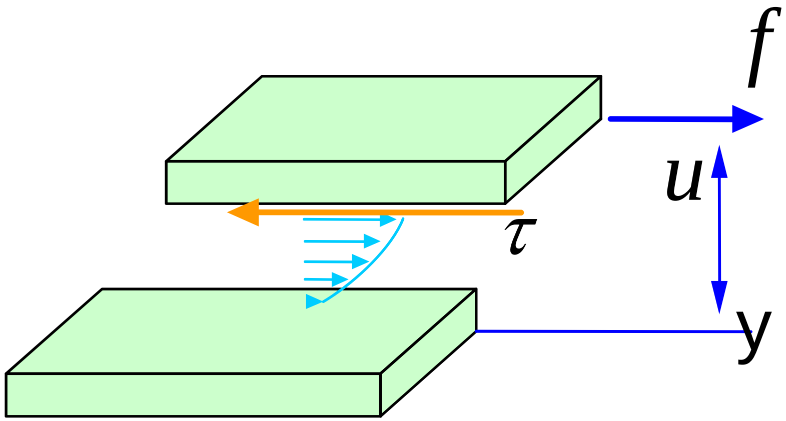
t : *mean free time*

$$\text{mean free path}\ : \lambda\ [m] = \frac{5 \times 10^{-5}}{P\ [Torr]}\ \ (\text{ in air, at room temperature })$$

$$ex)\ P = 1\ atm \rightarrow \lambda = 60\ nm$$

Viscosity

- **Absolute viscosity (μ)**
  - The shearing stress (τ) and rate of shearing strain (du/dy) can be related with a relationship of the form.



$$\tau = \mu \frac{du}{dy}$$

μ : viscosity (absolute)

$$[\mu] = (N/m^2)\cdot(s)$$

f =force , u=velocity, A= area of the plate,
d =distance between the plates, τ= shear stress

Viscosity (cont'd)

- **Kinetic viscosity (ν)**

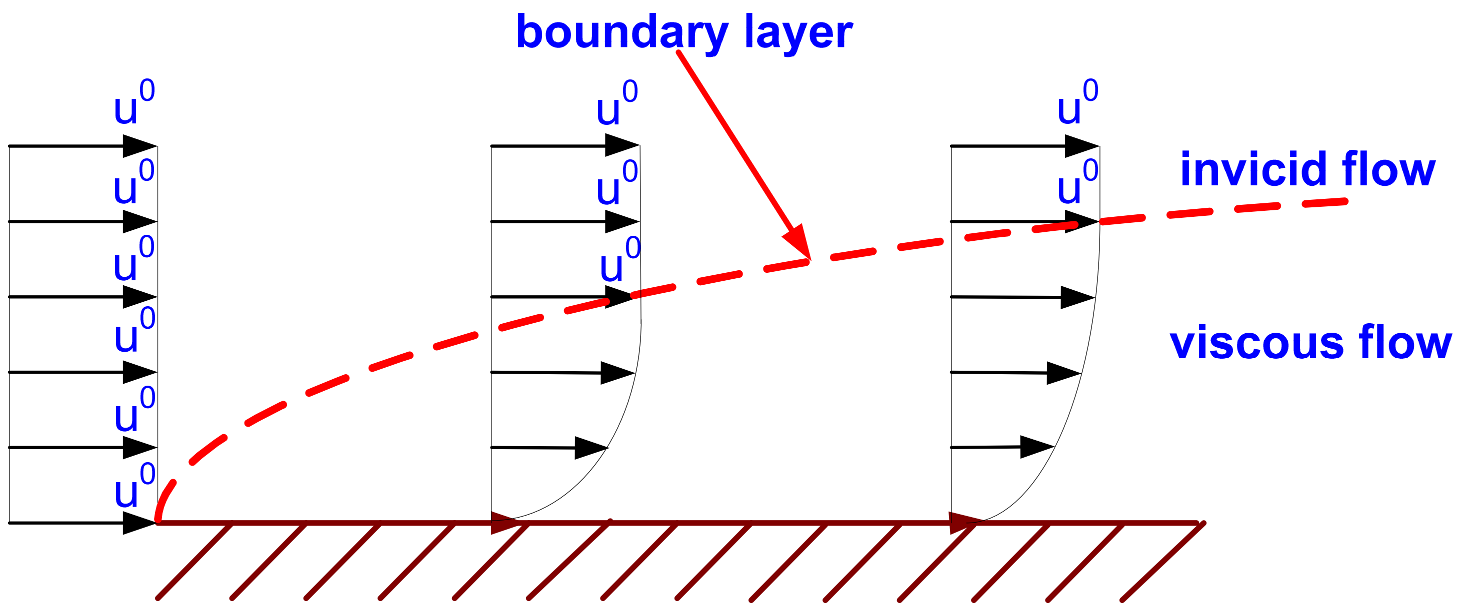
$$\nu = \frac{\mu}{\rho} \; : [\nu] = m^2 / s$$

$$\mu : \text{absolute viscosity, } \rho : \text{density}$$

  - For gases, kinetic viscosity increases with temperature, whereas for liquids, kinetic viscosity decreases with increasing temperature.

Fluid Motions

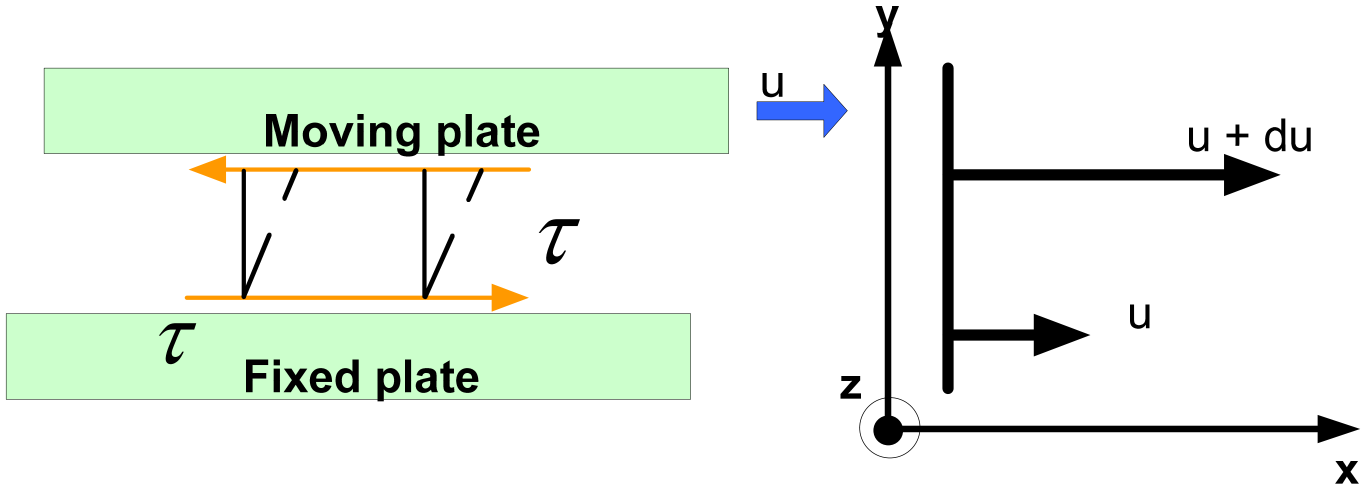
- **Viscous flow**
  – The shear stress gives rise to change the velocity of the fluid at the interface between the solid and fluid.
- **Inviscid flow**
  – The effect of shear stress is so small as to be negligible.



Dissipation in Fluid Flow

- **Dissipated energy** (D)
  - Work is required to move the plate, but this work is not stored as potential or kinetic energy in the plate or fluid.
  - The differential volume is deformed by the shear force.
  - The work done on the differential volume.

$$D = \tau \cdot du_x dxdz$$



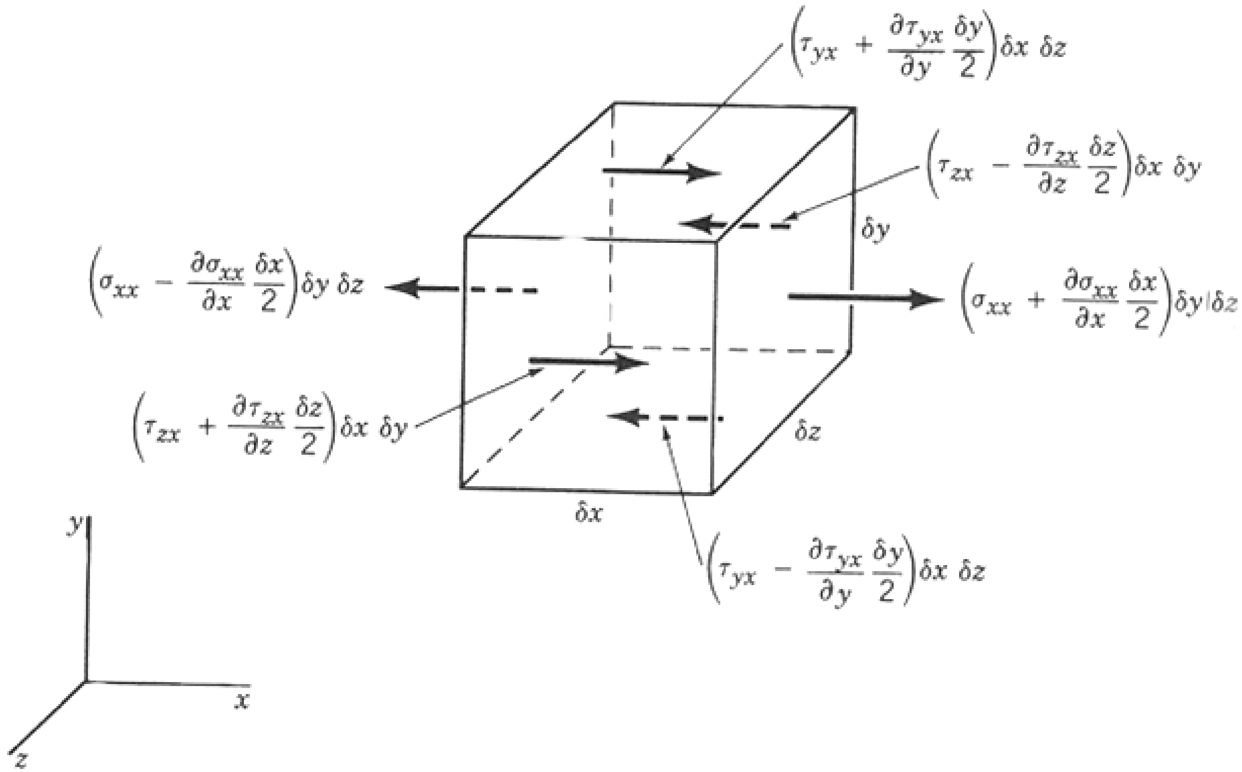
Stress Field

- **Surface force**
  - It acts on the boundaries of a medium through direct contact. (pressure, stress etc)

- **Body force**
  - Forces developed without physical contact, and distributed over the volume of fluid. (gravitational force etc)

Momentum Equation

- **Forces acting on a fluid particle**



<Surface forces in the x direction on a fluid element>

Momentum Equation (cont'd)

- **Surface force (F_s) for x-axis**

$$dF_{sx} = (\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \cdot \frac{dx}{2})dydz - (\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \cdot \frac{dx}{2})dydz$$

$$+ \ (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{dy}{2})dxdz - (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{dy}{2})dxdz$$

$$+ (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{dz}{2})dxdy - (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{dz}{2})dxdy$$

$$= \ (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{yx}}{\partial y})dxdydz \quad \cdots\cdots\cdots\cdots \ (1)$$

Momentum Equation (cont'd)

- **Body force (F_B) for x-axis : gravity**

$$dF_{Bx} = \rho g_x dxdydz \quad \cdots\cdots\cdots\cdots\cdots (2)$$

Then, the total force in x direction $((1)+(2))$

$$dF_x = dF_{sx} + dF_{Bx}$$

$$= (\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}) dxdydz \cdots\cdots\cdots (3)$$

$$similarily, dF_y = (\rho g_y + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z}) dxdydz$$

$$dF_z = (\rho g_z + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial_{yz}}{\partial y}) dxdydz$$

Differential Momentum Equation

From the Newton's 2nd law of motion

$$d\overline{F} = dm\frac{d\overline{v}}{dt}$$

$$(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z})dxdydz$$

$$= \rho dxdydz(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$

$$\therefore \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$

→ Similarly, we can have y and z differential equation of motion.

Navier-Stokes Equation

- **Newtonian fluid**
  - Fluids for which the shearing stress is linearly related to the rate of the shearing strain.

$$\tau_{xy} = \tau_{yz} = \mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}), \qquad \sigma_{xx} = -\rho - \frac{2}{3}\mu\nabla \cdot \bar{v} + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yz} = \tau_{zy} = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}), \qquad \sigma_{yy} = -\rho - \frac{2}{3}\mu\nabla \cdot \bar{v} + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zx} = \tau_{xz} = \mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}), \qquad \sigma_{zz} = -\rho - \frac{2}{3}\mu\nabla \cdot \bar{v} + 2\mu\frac{\partial w}{\partial z}$$

Navier - stokes Equation

x - momentum Equation

$$\rho\frac{\sigma u}{\sigma t} = \rho g x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{\mu(2\frac{du}{dx} - \frac{2}{3}\nabla \cdot \bar{v})\right\}$$

$$+ \frac{\partial}{\partial y}\left\{\mu(\frac{du}{dy} - \frac{\partial v}{\partial x})\right\} + \frac{\partial}{\partial z}\left\{\mu(\frac{dw}{dx} - \frac{\partial u}{\partial z})\right\}$$

Navier-Stokes Equation (cont'd)

- **Navier-Stokes Equation**
  - Equation of motion governing fluid behavior

$$\rho \frac{\partial u}{\partial t} + (\nabla \cdot u)u = \rho g - \nabla(p) + \mu \nabla^2 u$$

ρ : *mass density* μ : *absolute viscosity*

p : *pressure* u : *velocity* g : *gravity*

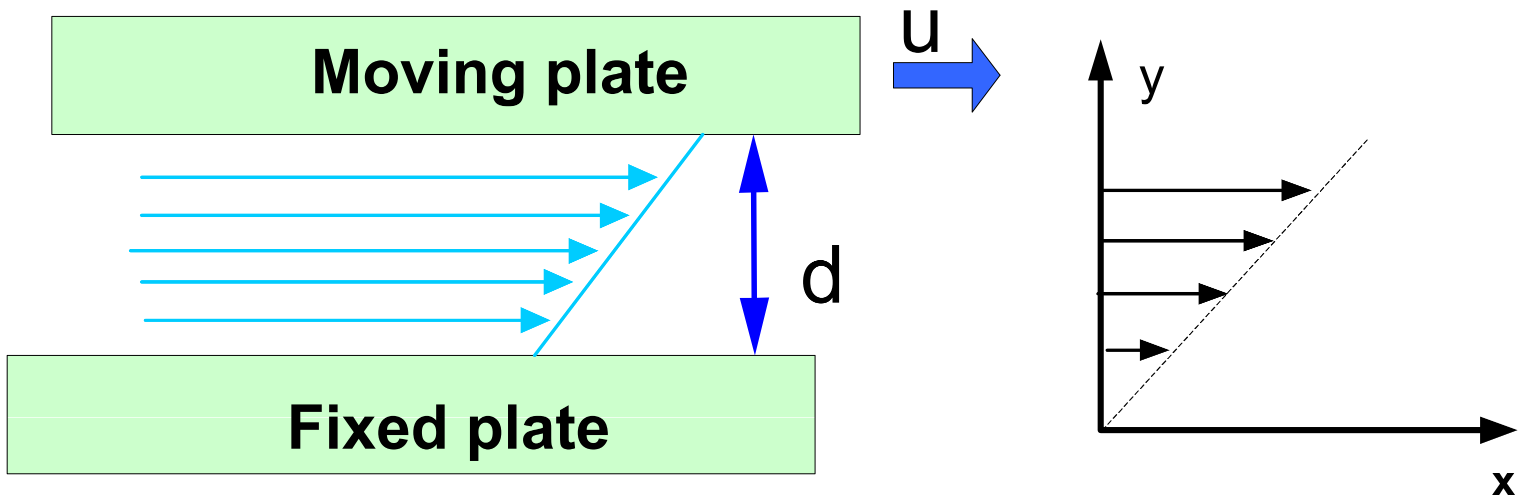
[Ref.] Bruce R. Munson "Fundamentals of Fluid Mechanics", 2^{nd} ed.

Navier-Stokes Equation (cont'd)

- Apply the Navier-Stokes equation to some special cases
  - **Couette flow**
  - **Stokes flow**

Couette Flow

- **Couette flow**
  - We obtain a linear velocity profile for the fluid film underneath the plate, as shown in Figure.



[Ref.] Y.-H. Cho, A. P. Pisano, and R. T. Howe, "Viscous damping model for laterally oscillating microstructures," IEEE JMEMS, Vol. 3, No. 2, pp. 81-87, 1994.

X. Zhang and W. C. Tang, "Viscous air damping in laterally driven microresonators," MEMS 1994, pp. 199-204, 1994

Couette Flow (cont'd)

- **Navier- Stokes Equation** : $\rho \frac{\partial u}{\partial t} + (\nabla \bullet u) u = \rho g - \nabla(p) + \mu \nabla^2 u$
  - Neglecting the time dependent term, **du/dt=0**. And we assume no pressure gradient, **dp/dx=0**. Then we have

$$\nabla^2 u = 0 \xrightarrow{1-D} \frac{\partial^2 u_x(y)}{\partial y^2} = 0$$

  - No-slip boundary conditions (**zero velocity** at the surface of the stationary plate), than we have

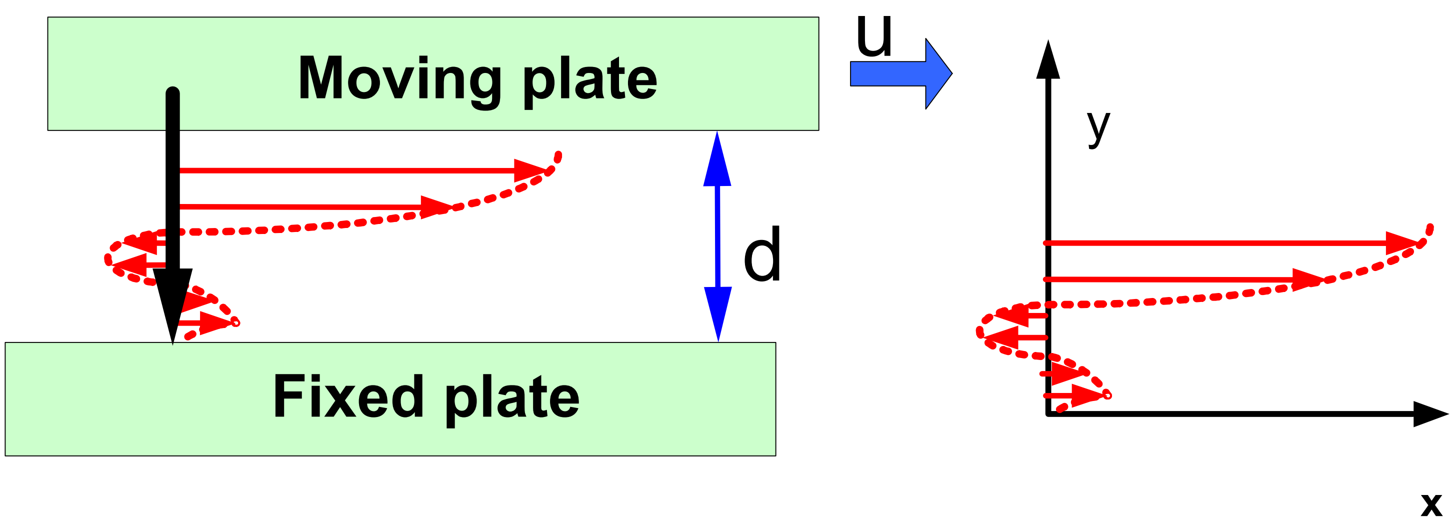
$$u_x = \frac{y}{d} u_0$$

  - **Dissipated energy :**

$$D = \frac{\pi}{w} u_0^{\,2} (\frac{\mu}{d})$$

Stokes Flow

- **Stokes flow**
  - The steady state velocity profile, u(y,t), in the fluid is governed by time term of the Navier-Stokes Equation.



[Ref.] Y.-H. Cho, A. P. Pisano, and R. T. Howe, "Viscous damping model for laterally oscillating microstructures," IEEE JMEMS, Vol. 3, No. 2, pp. 81-87, 1994.

X. Zhang and W. C. Tang, "Viscous air damping in laterally driven microresonators,"MEMS 1994, pp. 199-204, 1994

Stokes Flow (cont'd)

- **Navier-Stokes Equation** : $\rho \frac{\partial u}{\partial t} + (\nabla \bullet u)u = \rho g - \nabla(p) + \mu \nabla^2 u$
  - And we assume **no pressure gradient**. Then we have

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial y^2} \quad v : kinetic\ vis\cos ity$$

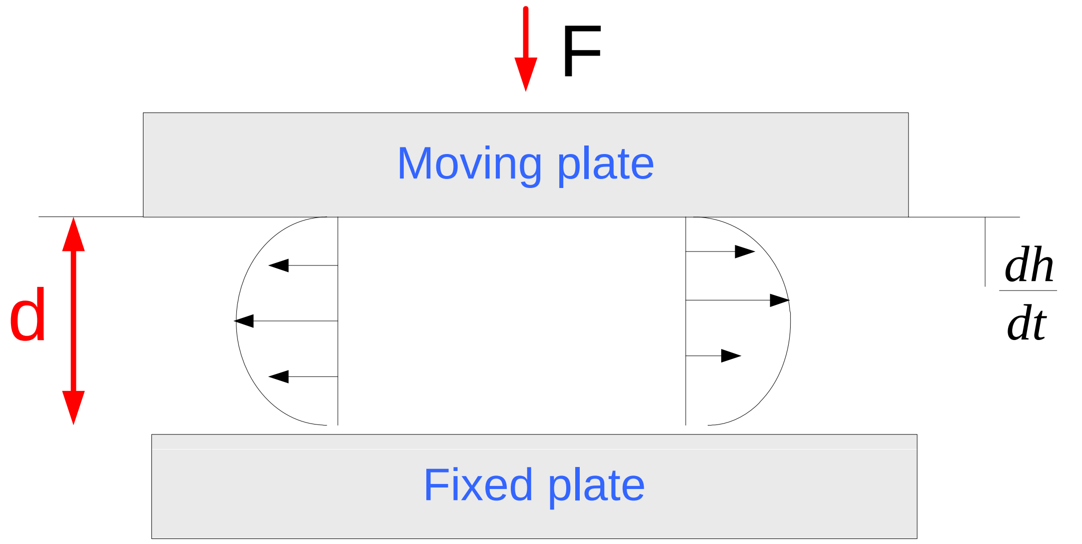
  - No-slip boundary conditions (i.e., **zero velocity** at the surface of the stationary plate, Than we have

$$u_x = u_0 e^{j(\beta y - wt)} \quad where\ \beta = \sqrt{w / 2v}$$

  - **Dissipated energy :** $D = \frac{\pi}{w} u_0^{\ 2} (\mu \beta \frac{\sinh 2\beta d + \sin 2\beta d}{\cosh 2\beta d - \cos 2\beta d})$

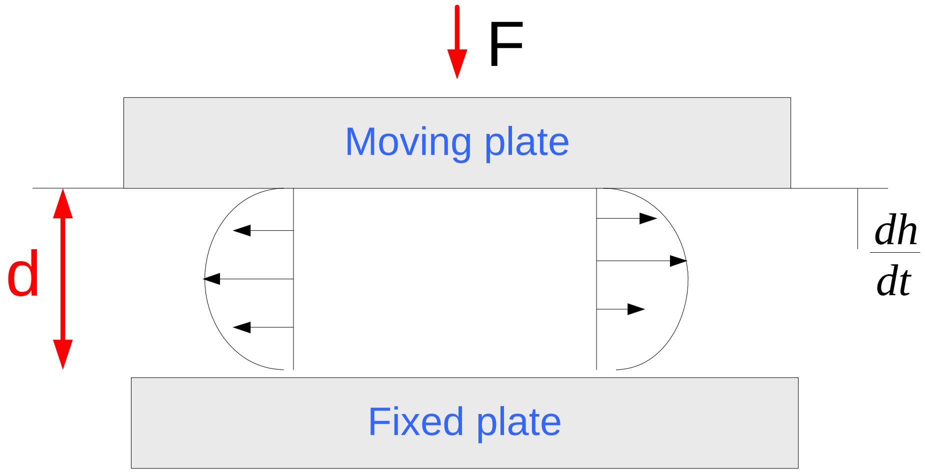
Squeeze Film

- **Squeeze-film damping**
  - Dissipated force
  - Vertical Motion of upper plate relative to fixed bottom plate with viscous fluid between plates
  - Viscous drag during flow creates dissipative force on plate opposing motion



Squeeze Film (cont'd)

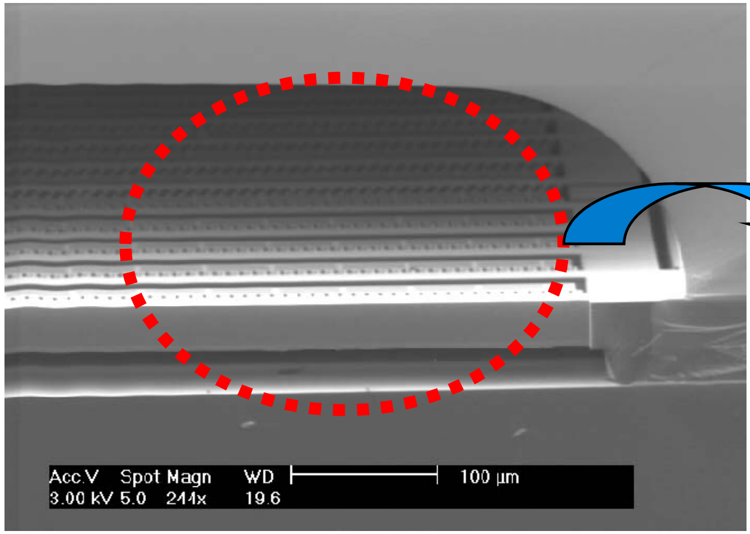
- Assumption
  - **No pressure gradient** transverse to the plate
  - Gap, h, is much smaller than the lateral dimensions of plate.



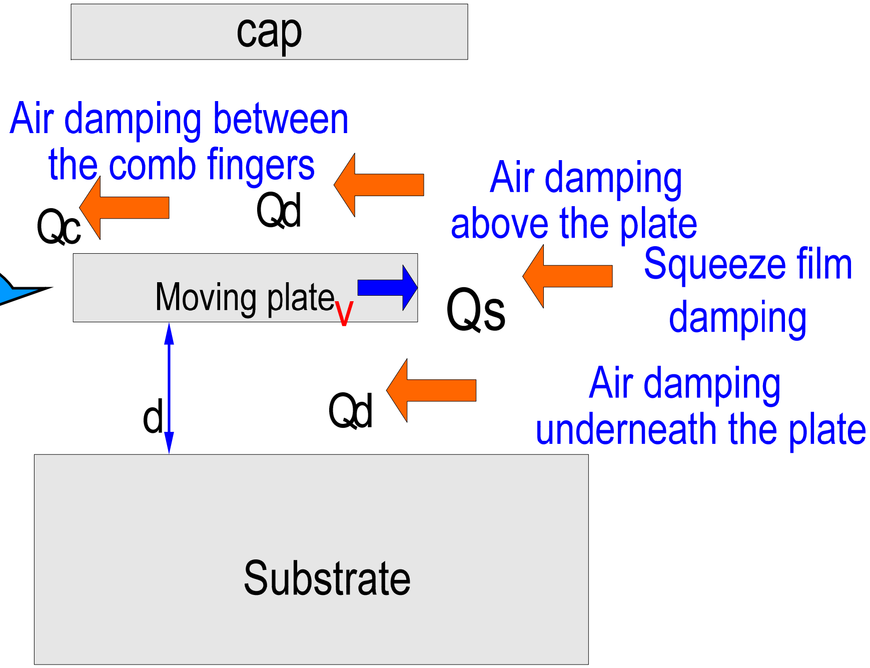
$$F = \int_{L/2}^{L/2} pwdx = \frac{\mu w L^3}{d^3}\frac{dh}{dt} = bh'$$

$$damping\ \ coefficient : b = \frac{\mu w L^3}{d^3}$$

Quality Factor Analysis



Laterally driven microresonator



- Stokes flow model → Qd, Qc
- Squeeze film model → Qs

Quality Factor Analysis (cont'd)

- **Damping Coefficient & Quality factor**
  - **System equation**

$$F_{ext} = Mx'' + bx' + Kx$$

$$x'' + \frac{b}{M}x' + \frac{K}{M}x = x'' + 2\zeta\omega_n + \omega_n^{\ 2}$$

  - **Damping ratio** : $\zeta = \dfrac{1}{2Q}$, **Natural frequency** : $\omega_n = \sqrt{\dfrac{K}{M}}$

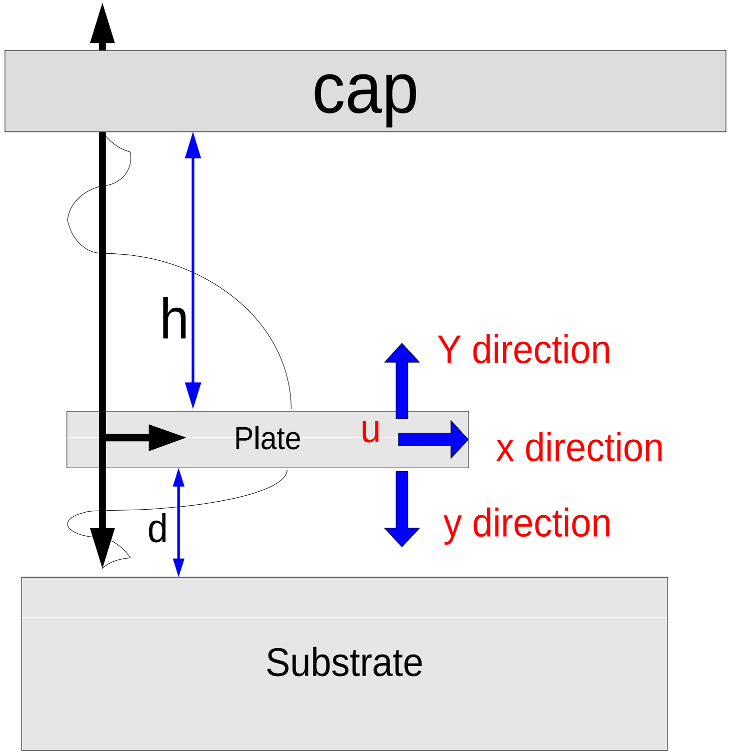
  - **Damping coefficient** : $b = 2M\zeta\omega_n = \dfrac{\sqrt{MK}}{Q}$

  - **Quality factor** : $Q = \dfrac{\sqrt{MK}}{b}$

Quality Factor Analysis (cont'd)

- Parameter (1 atm, at room temperature)
  - Natural frequency : f = 5.071 kHz
  - Absolute viscosity : $\mu = 1.8 \times 10^{-5}$ kg/m·sec
  - Kinetic viscosity : $v = 1.5 \times 10^{-5}$ m^2/sec
  - Density of silicon : 2330 kg/m^3
  - Distance of inter-plate (sacrificial gap) d=20 μm
  - Distance of inter-plate (between structure and cap) h=150 μm
  - Distance of inter-combs : dc=2 μm
  - Mass: 42 μg
  - Spring stiffness: 137.0 N/m

Stokes Flow Model (Qd)



$$put\ u = u_0 e^{j(\beta y - wt)} \quad \text{where} \quad \beta = \sqrt{\frac{w}{v}}\ ,$$

$$D = \frac{1}{w}\int_0^{2\pi} \tau_0 u d(wt) \quad , \tau_0 = -\mu \frac{du}{dy} (frictional\ shear)$$

$$D = \frac{\pi}{w} u_0^{\ 2} \mu\beta (\frac{\sinh 2\beta y + \sin 2\beta y}{\cosh 2\beta y - \cos 2\beta y})$$

$(a)\ 0 < y < d_1$

$$D = \frac{\pi}{w} u_0^{\ 2} \mu\beta (\frac{\sinh 2\beta d + \sin 2\beta d}{\cosh 2\beta d - \cos 2\beta d})$$

$$Qd_1 = \frac{2\pi W}{AD} = \frac{1}{\mu A\beta}\sqrt{MK} (\frac{\cosh 2\beta d - \cos 2\beta d}{\sinh 2\beta d + \sin 2\beta d}) \quad \cdots\cdots(1)$$

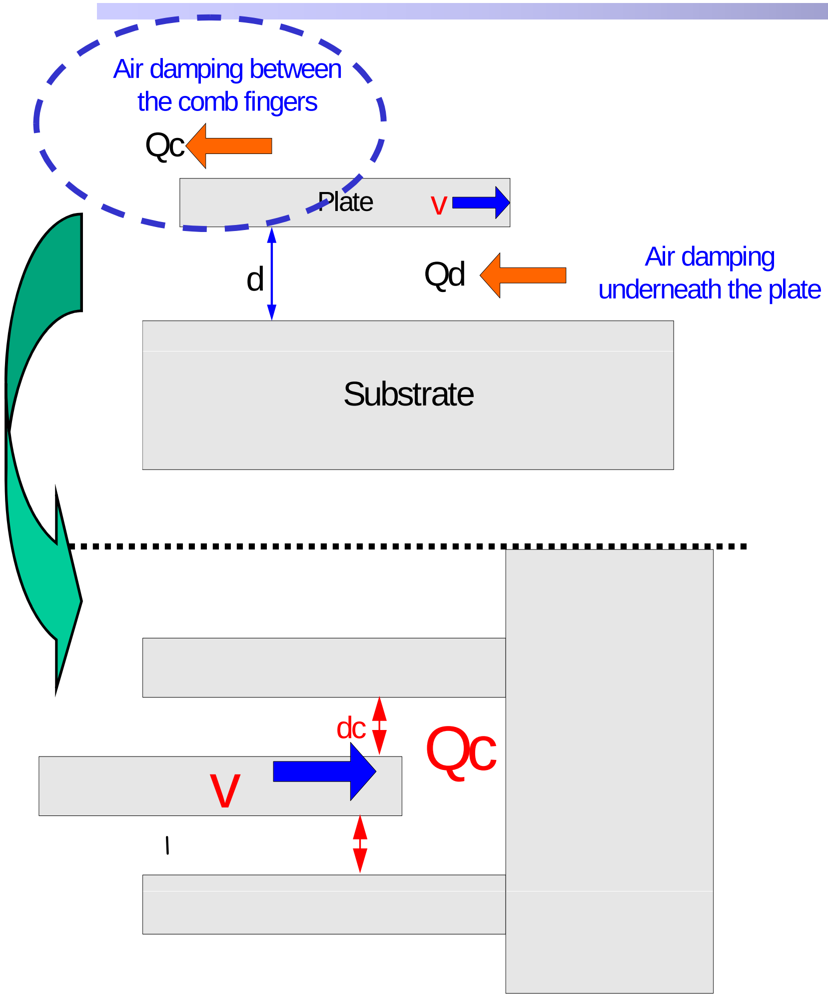
$W:\ strain\ energy,\ A:\ plate\ area,\ \ D:\ dissipate\ energy$

$(b)\ d_2 < y < d_3\ \ h = d_3 - d_2$

$$Qd_2 = \frac{2\pi W}{AD} = \frac{1}{\mu A\beta}\sqrt{MK} (\frac{\cosh 2\beta h - \cos 2\beta h}{\sinh 2\beta h + \sin 2\beta h}) \quad \cdots\cdots(2)$$

[Ref.] T. Y. Song, S. J. Park, Y. H. Park, D. H. Kwak, H. H. Ko, and D. I. Cho, “Quality Factor in Micro-gyroscopes,“ APCOT MNT 2004, pp. 916-920, 2004

Stokes Flow Model (Qc)



$$put\ u = u_0 e^{j(\beta y - wt)} \quad \text{where} \quad \beta = \sqrt{\frac{w}{v}}\ ,$$

$$D = \frac{1}{w}\int_0^{2\pi} \tau_0 u d(wt) \quad , \tau_0 = -\mu \frac{du}{dy} (frictional\ shear)$$

$$D = \frac{\pi}{w} u_0^{\ 2} \mu\beta \left(\frac{\sinh 2\beta y + \sin 2\beta y}{\cosh 2\beta y - \cos 2\beta y}\right)$$

$$(c)\ 0 < y < dc$$

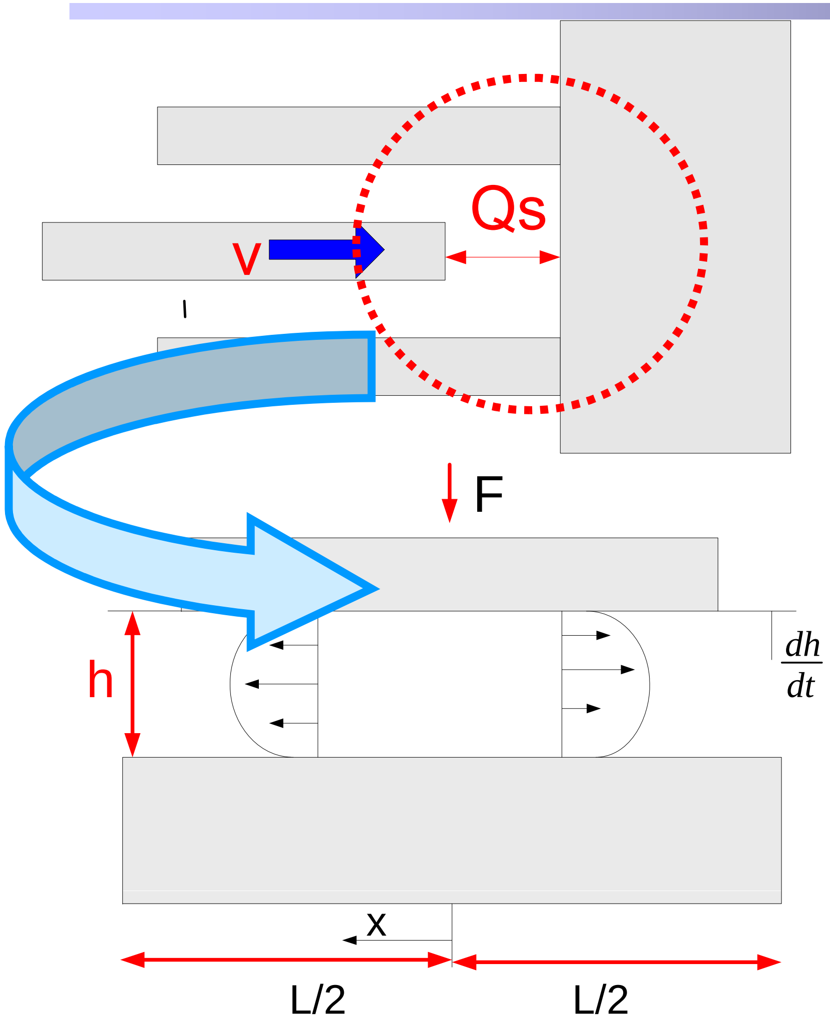
$$D = \frac{\pi}{w} u_0^{\ 2} \mu\beta \left(\frac{\sinh 2\beta dc + \sin 2\beta dc}{\cosh 2\beta dc - \cos 2\beta dc}\right)$$

$$Qc = \frac{2\pi W}{AD} = \frac{1}{\mu A\beta}\sqrt{MK}\left(\frac{\cosh 2\beta dc - \cos 2\beta dc}{\sinh 2\beta dc + \sin 2\beta dc}\right) \cdots\cdots (3)$$

W: strain energy, A: plate area, D: dissipate energy

Qc between comb fingers

Squeeze Film Model (Qs)



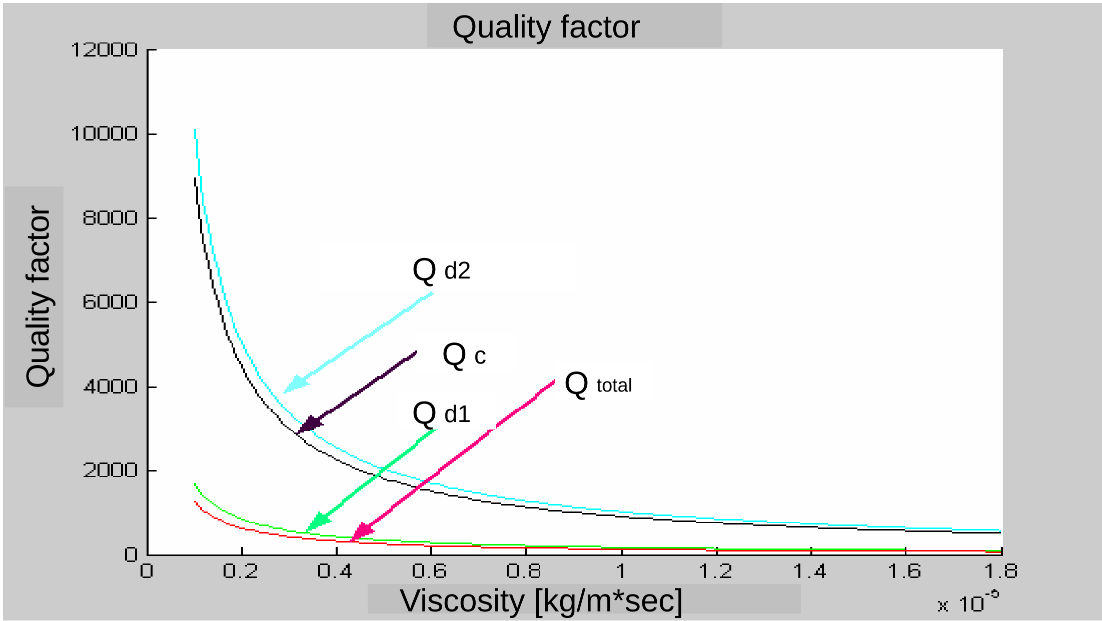
$$F = \int_{-L/2}^{L/2} pwdx = \frac{\mu w L^3}{d^3}\frac{dh}{dt}$$

$$b = \frac{\mu w L^3}{d^3}$$

$$(d)\ Qs = \frac{d^3}{\mu w L^3}\sqrt{MK} \cdots\cdots\cdots(4)$$

Quality Factor Analysis Results

- Quality factor versus viscosity



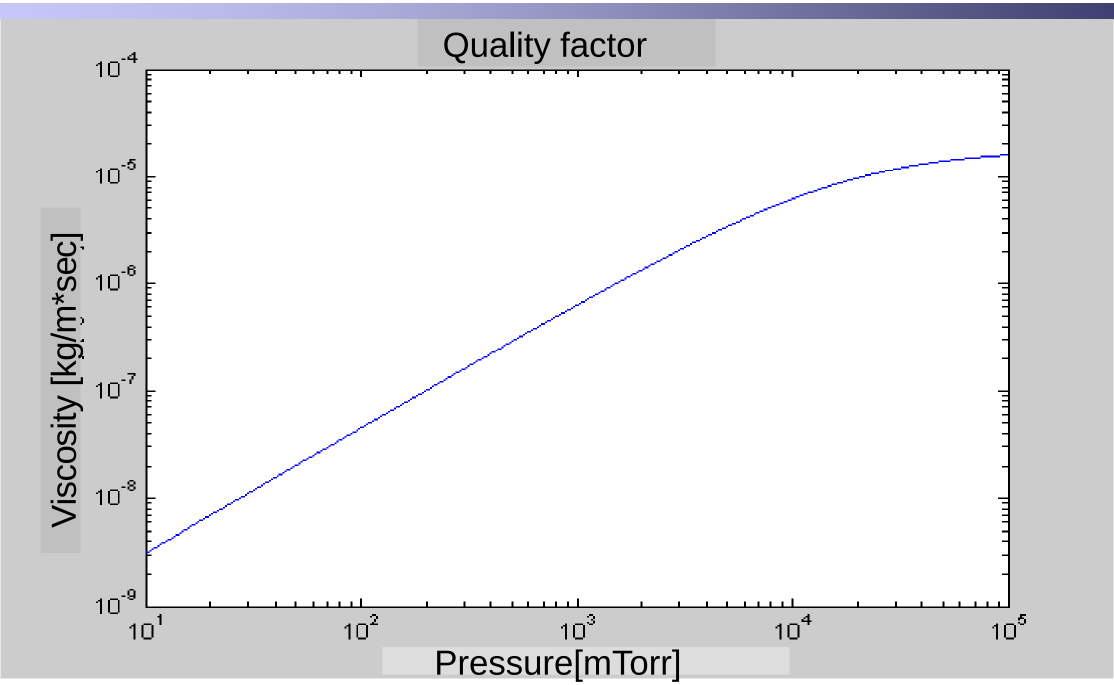
	Quality factor	Damping coefficient
(a)	93	$8.15e^{-4}$
(b)	561	$1.35e^{-4}$
(c)	497	$1.53e^{-4}$
(d)	$5.12e^{10}$	$1.48e^{-12}$
Total	65	$1.2e^{-3}$

$$total\ \ Quality\ \ factor : Q_{total}$$

$$\frac{1}{Q_{total}} = \frac{1}{Qd_1} + \frac{1}{Qd_2} + \frac{1}{Qc} + \frac{1}{Qs}, \quad b = \frac{\sqrt{MK}}{Q}$$

[Ref.] T. Y. Song, S. J. Park, Y. H. Park, D. H. Kwak, H. H. Ko, and D. I. Cho, "Quality Factor in Micro-gyroscopes," APCOT MNT 2004, pp. 916-920, 2004

Quality Factor Analysis Result (cont'd)



The viscosity versus the pressure

$$\mu_{effi} = \frac{\mu_0}{1 + 9.658(K_n)^{1.159}}$$

$$\lambda = \frac{5\times 10^{-5}}{P}, Kn = \frac{\lambda}{L}$$

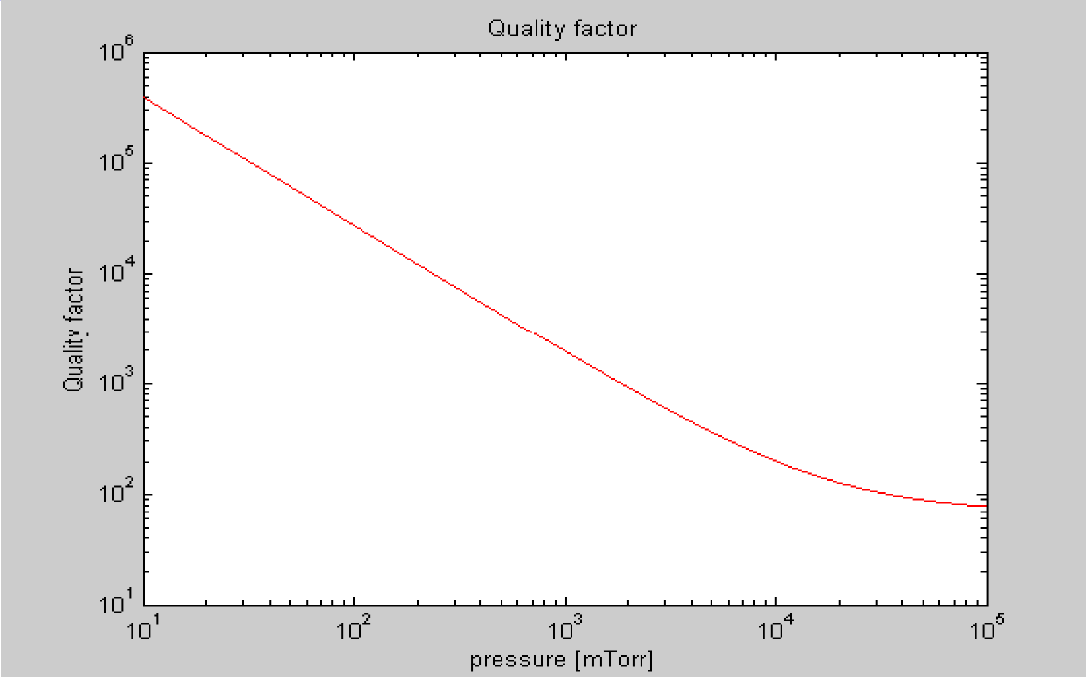
$$\therefore \mu_{effi} = \frac{\mu_0}{1 + 9.658(\frac{5\times 10^{-5}}{LP})^{1.159}}$$

λ: *mean free path*, L: *distance between of the parallel plate*

Kn: *Knudsen number*

[Ref.] M. Bao, H. Yang, H. Yin, and Y. Snu "Energy transfer model for squeeze-film air damping in low vacuum," IOP JMM, Vol. 12, pp 341-346, 2002

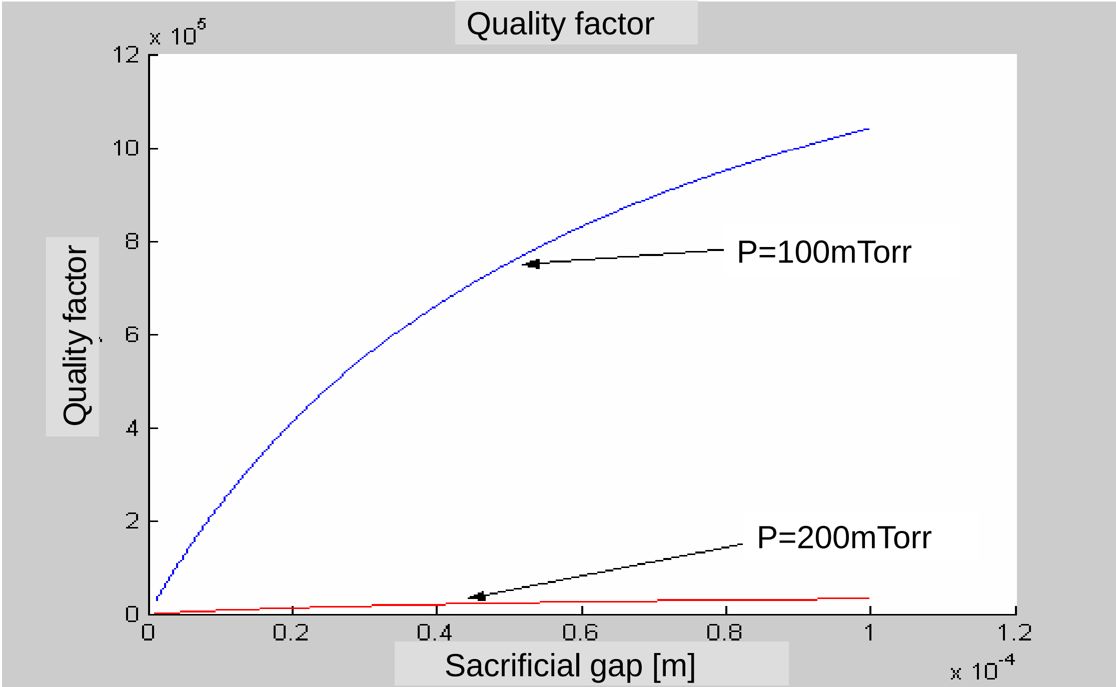
Quality Factor Analysis Result (cont'd)



The Quality factor versus the pressure

	Sacrificial gap	Quality factor	Damping Coefficient (b)
P = 200mTorr	2 um	4.1×10^4	1.8×10^{-6}
P = 200mTorr	20 um	1.2×10^5	6.3×10^{-7}

Quality Factor Analysis Result (cont'd)



The Quality factor vs. the sacrificial gap

	Sacrificial gap	Quality factor	Damping Coefficient (b)
P = 100mTorr	20 um	3.8×10^5	2.0×10^{-7}
P = 200mTorr	20 um	1.2×10^5	6.3×10^{-7}

Reference

- Bruce R. Munson “Fundamentals of Fluid Mechanics”, 2nd ed.
- Y.-H. Cho, A. P. Pisano, and R. T. Howe, "Viscous damping model for laterally oscillating microstructures,” IEEE JMEMS, Vol. 3, No. 2, pp. 81-87, 1994.
- X. Zhang and W. C. Tang, “Viscous air damping in laterally driven microresonators,” MEMS 1994, pp. 199-204, 1994
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- M. Bao, H. Yang, H. Yin, and Y. Snu “Energy transfer model for squeeze-film air damping in low vacuum,” IOP JMM, Vol. 12, pp 341-346, 2002