Chap. 5. Descrete-Time Fourier Transform

5.1 REPRESENTATION OF APERIODIC SIGNALS: THE DISCRETE-TIME FOURIER TRANSFORM

5.1.1 Development of the Discrete-Time Fourier Transform

In Section 4.1 [eq. (4.2) and Figure 4.2], we saw that the Fourier series coefficients for a continuous-time periodic square wave can be viewed as samples of an envelope function and that, as the period of the square wave increases, these samples become more and more finely spaced. This property suggested representing an aperiodic signal $x(t)$ by first constructing a periodic signal $\tilde{x}(t)$ that equaled $x(t)$ over one period. Then, as this period approached infinity, $\tilde{x}(t)$ was equal to $x(t)$ over larger and larger intervals of time, and the Fourier series representation for $\tilde{x}(t)$ converged to the Fourier transform representation for $x(t)$. In this section, we apply an analogous procedure to discrete-time signals in order to develop the Fourier transform representation for discrete-time aperiodic sequences.

Consider a general sequence $x[n]$ that is of finite duration. That is, for some integers N_1 and N_2, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$. A signal of this type is illustrated in Figure 5.1(a). From this aperiodic signal, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period, as illustrated in Figure 5.1(b). As we choose the period N to be larger, $\tilde{x}[n]$ is identical to $x[n]$ over a longer interval, and as $N \to \infty$, $\tilde{x}[n] = x[n]$ for any finite value of n.

Let us now examine the Fourier series representation of $\tilde{x}[n]$. Specifically, from eqs. (3.94) and (3.95), we have

Take $N_1 = N_2 = \frac{1}{3}N$

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \tag{5.1}$$

$\Rightarrow x[n] = \lim_{N\to\infty} x_N[n]$

$$x_N[n] \triangleq \begin{cases} x[n], & -N_1 \leq n \leq N_2 \\ 0, & \text{otherwise} \end{cases}$$

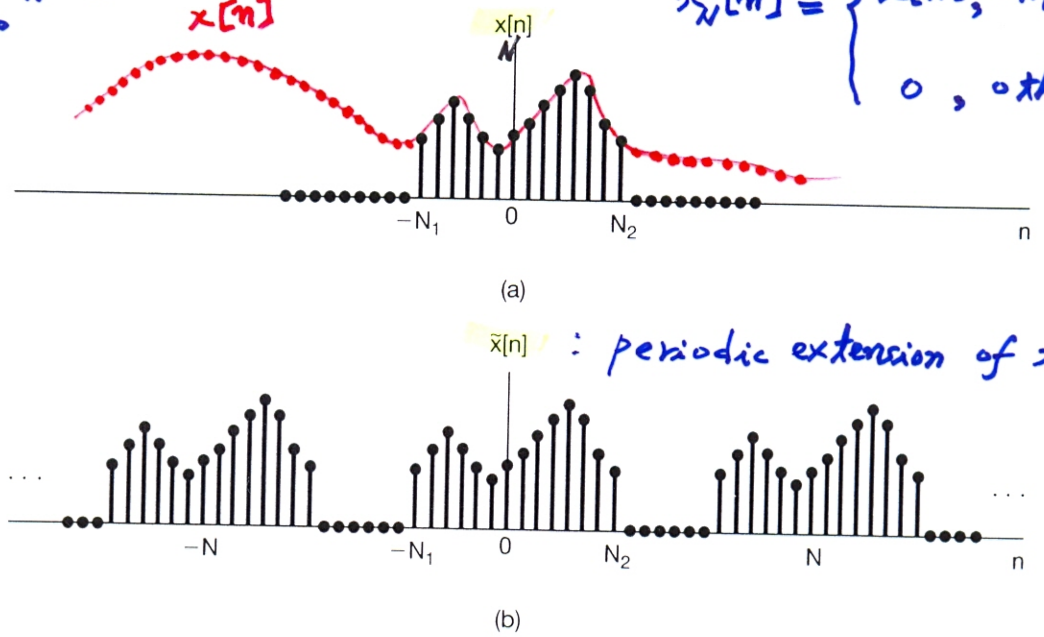

Figure 5.1 (a) Finite-duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}. \tag{5.2}$$

(*) Since $x_N[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \leq n \leq N_2$, it is convenient to choose the interval of summation in eq. (5.2) to include this interval, so that $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation. Therefore,

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x_N[n] e^{-jk(2\pi/N)n}, \tag{5.3}$$

where in the second equality in eq. (5.3) we have used the fact that $x[n]$ is zero outside the interval $-N_1 \leq n \leq N_2$. Defining the function

$$X_N(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} x_N[n] e^{-j\omega n}, \quad X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = \lim_{N\to\infty} X_N(e^{j\omega}) \tag{5.4}$$

we see that the coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N} X_N(e^{jk\omega_0}), \tag{5.5}$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining eqs. (5.1) and (5.5) yields

$$x_N[n] := \tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X_N(e^{jk\omega_0}) e^{jk\omega_0 n}. \tag{5.6}$$

Since $\omega_0 = 2\pi/N$, or equivalently, $1/N = \omega_0/2\pi$, eq. (5.6) can be rewritten as

$$x_N[n] = \tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X_N(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0. \tag{5.7}$$

As with eq. (4.7), as N increases ω_0 decreases, and as $N \to \infty$ eq. (5.7) passes to an integral. To see this more clearly, consider $X(e^{j\omega})e^{j\omega n}$ as sketched in Figure 5.2. From

$$(5.7) \Rightarrow x[n] = \lim_{N\to\infty} x_N[n] = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{k\in\langle N\rangle} X_N(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

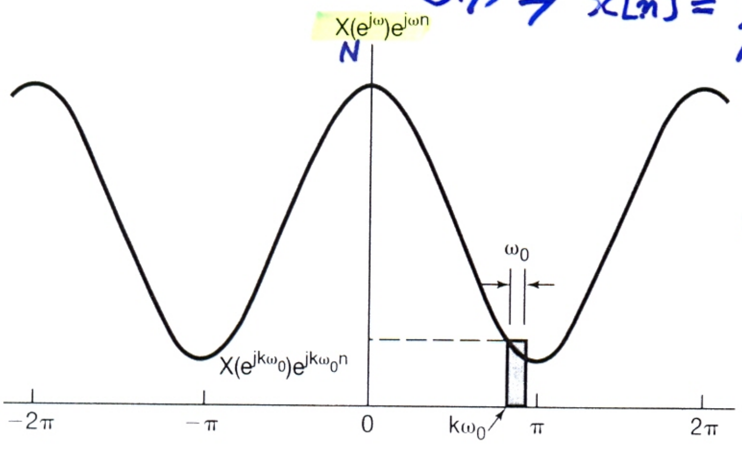

Figure 5.2 Graphical interpretation of eq. (5.7).

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eq. (5.4), $X(e^{j\omega})$ is seen to be periodic in ω with period 2π, and so is $e^{j\omega n}$. Thus, the product $X(e^{j\omega})e^{j\omega n}$ will also be periodic. As depicted in the figure, each term in the summation in eq. (5.7) represents the area of a rectangle of height $X(e^{jk\omega_0})e^{jk\omega_0 n}$ and width ω_0. As $\omega_0 \to 0$, the summation becomes an integral. Furthermore, since the summation is carried out over N consecutive intervals of width $\omega_0 = 2\pi/N$, the total interval of integration will always have a width of 2π. Therefore, as $N \to \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$\tilde{X}(e^{j\omega}) \to X(e^{j\omega})$,

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

where, since $X(e^{j\omega})e^{j\omega n}$ is periodic with period 2π, the interval of integration can be taken as *any* interval of length 2π. Thus, we have the following pair of equations:

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, && (5.8)\\ X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. && (5.9)\end{aligned}}$$

Equations (5.8) and (5.9) are the discrete-time counterparts of eqs. (4.8) and (4.9). The function $X(e^{j\omega})$ is referred to as the *discrete-time Fourier transform* and the pair of equations as the *discrete-time Fourier transform pair.* Equation (5.8) is the *synthesis equation,* eq. (5.9) the *analysis equation.* Our derivation of these equations indicates how an aperiodic sequence can be thought of as a linear combination of complex exponentials. In particular, the synthesis equation is in effect a representation of $x[n]$ as a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $X(e^{j\omega})(d\omega/2\pi)$. For this reason, as in continuous time, the Fourier transform $X(e^{j\omega})$ will often be referred to as the *spectrum* of $x[n]$, because it provides us with the information on how $x[n]$ is composed of complex exponentials at different frequencies.

Note also that, as in continuous time, our derivation of the discrete-time Fourier transform provides us with an important relationship between discrete-time Fourier series and transforms. In particular, the Fourier coefficients a_k of a periodic signal $\tilde{x}[n]$ can be expressed in terms of equally spaced *samples* of the Fourier transform of a finite-duration, aperiodic signal $x[n]$ that is equal to $\tilde{x}[n]$ over one period and is zero otherwise. This fact is of considerable importance in practical signal processing and Fourier analysis, and we

Remark 1:

<proof>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\omega n}$$

$$\Rightarrow a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-jk\omega_0 n}$$

$$= \frac{1}{N}X(e^{jk\omega_0}) \quad \text{where} \quad \omega_0 \triangleq \frac{2\pi}{N}. \quad \square$$

Remark 2:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

$e^{j\omega n} \Rightarrow$ The low (high) frequencies in discrete time are the values of ω near even (odd) multiples of π. (ref. Fig. 5.3)

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eq. (5.4), $X(e^{j\omega})$ is seen to be periodic in ω with period 2π, and so is $e^{j\omega n}$. Thus, the product $X(e^{j\omega})e^{j\omega n}$ will also be periodic. As depicted in the figure, each term in the summation in eq. (5.7) represents the area of a rectangle of height $X(e^{jk\omega_0})e^{j\omega_0 n}$ and width ω_0. As $\omega_0 \rightarrow 0$, the summation becomes an integral. Furthermore, since the summation is carried out over N consecutive intervals of width $\omega_0 = 2\pi/N$, the total interval of integration will always have a width of 2π. Therefore, as $N \rightarrow \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$\tilde{X}(e^{j\omega}) \rightarrow X(e^{j\omega})$,

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$$= \frac{1}{N}X(e^{jk\omega_0}) \quad \text{where } \omega_0 \triangleq \frac{2\pi}{N}. \qquad \square$$

Remark 2:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

$e^{j\omega n} \Rightarrow$ The low (high) frequencies in discrete time are the values of ω near even (odd) multiples of π. (ref. Fig. 5.3) riodicity of $e^{j\omega n}$ as a function of ω: $\omega = 0$ and $\omega = 2\pi$ yield the same signal. Signals at frequencies near these values or any other even multiple of π are slowly varying and therefore are all appropriately thought of as low-frequency signals. Similarly, the high frequencies in discrete time are the values of ω near odd multiples of π. Thus, the signal $x_1[n]$ shown in Figure 5.3(a) with Fourier transform depicted in Figure 5.3(b) varies more slowly than the signal $x_2[n]$ in Figure 5.3(c) whose transform is shown in Figure 5.3(d).

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega}$$

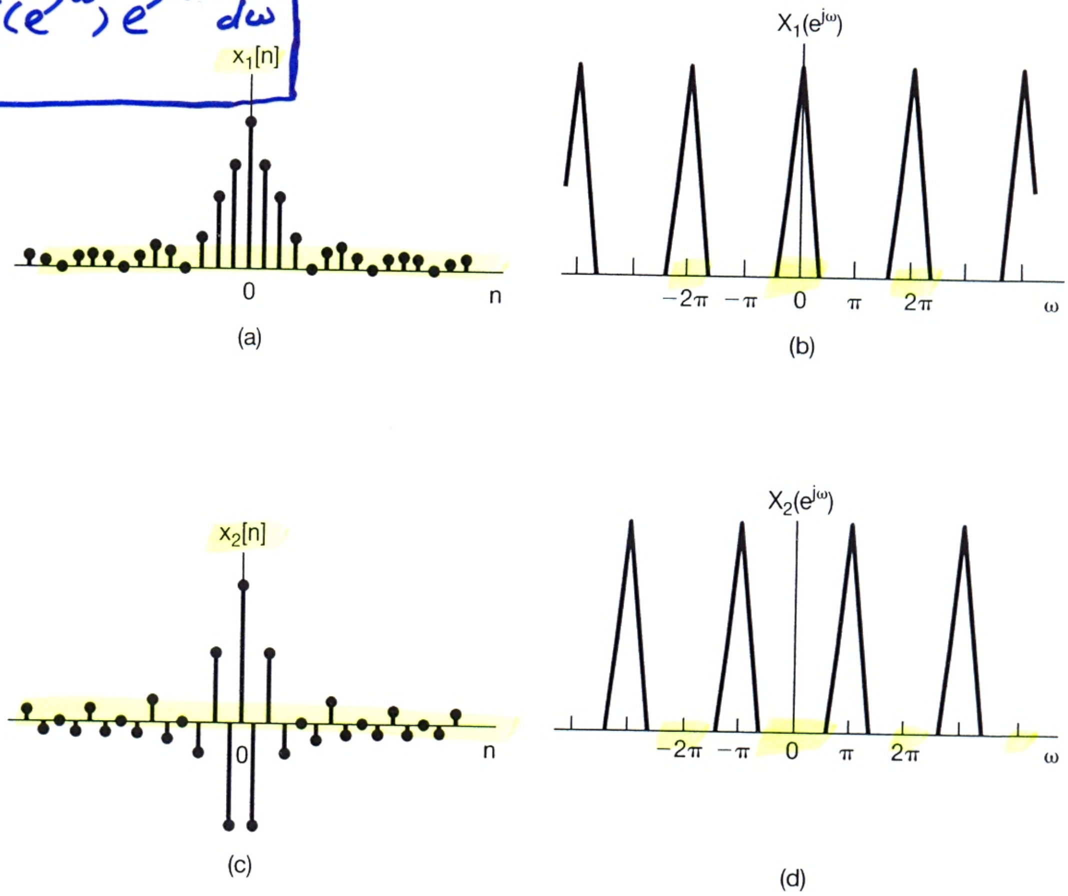


Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

5.1.2 Examples of Discrete-Time Fourier Transforms

To illustrate the discrete-time Fourier transform, let us consider several examples.

Example 5.1

Consider the signal

$$x[n] = a^n u[n], \qquad |a| < 1.$$

§5.1.2 Examples of Discrete-Time Fourier Transforms

Example 5.1: $x[n] = a^n u[n]$, $|a| < 1$

In this case,

$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} \quad \left(= \sum x[n] e^{-j\omega n}\right)$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

The magnitude and phase of $X(e^{j\omega})$ are shown in Figure 5.4(a) for $a > 0$ and in Figure 5.4(b) for $a < 0$. Note that all of these functions are periodic in ω with period 2π.

$1 > a > 0$

$-1 < a < 0$

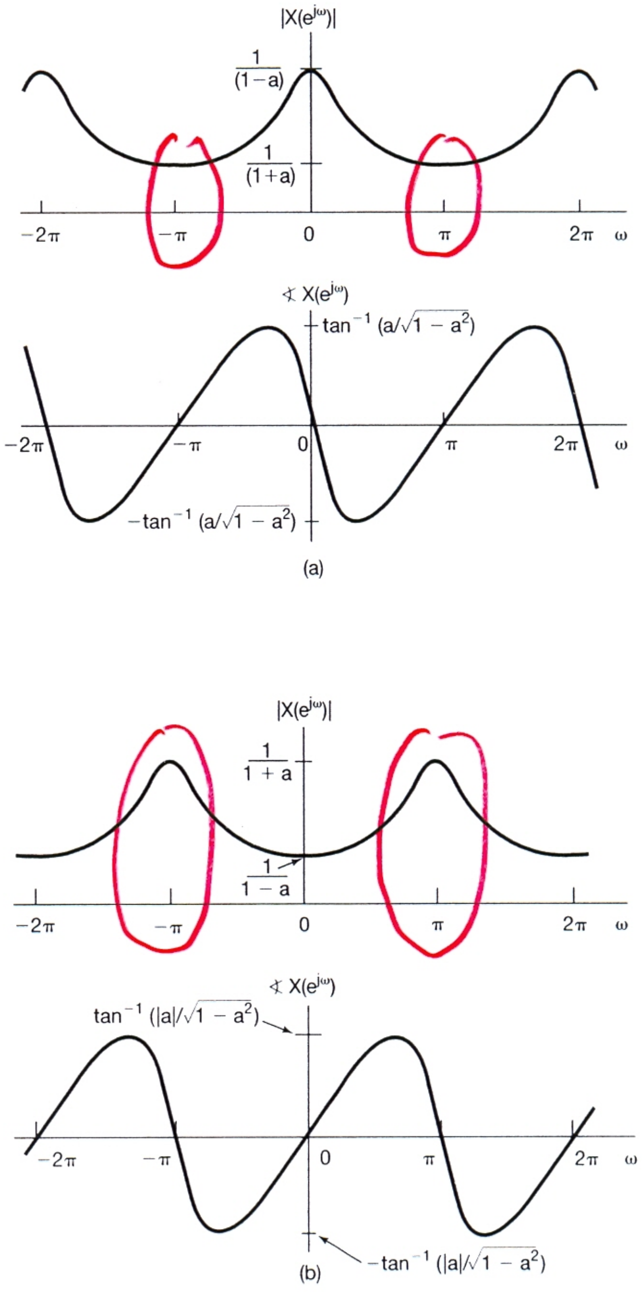


Figure 5.4 Magnitude and phase of the Fourier transform of Example 5.1 for (a) $a > 0$ and (b) $a < 0$.

Making the substitution of variables $m = -n$ in the second summation, we obtain

$$X(e^{j\omega}) = \sum_{n=0}^{\infty}(ae^{-j\omega})^n + \sum_{m=1}^{\infty}(ae^{j\omega})^m.$$

Both of these summations are infinite geometric series that we can evaluate in closed form, yielding

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\ &= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}. \end{aligned}$$

In this case, $X(e^{j\omega})$ is real and is illustrated in Figure 5.5(b), again for $0 < a < 1$.

Example 5.3

Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}, \tag{5.10}$$

which is illustrated in Figure 5.6(a) for $N_1 = 2$. In this case,

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}. \tag{5.11}$$

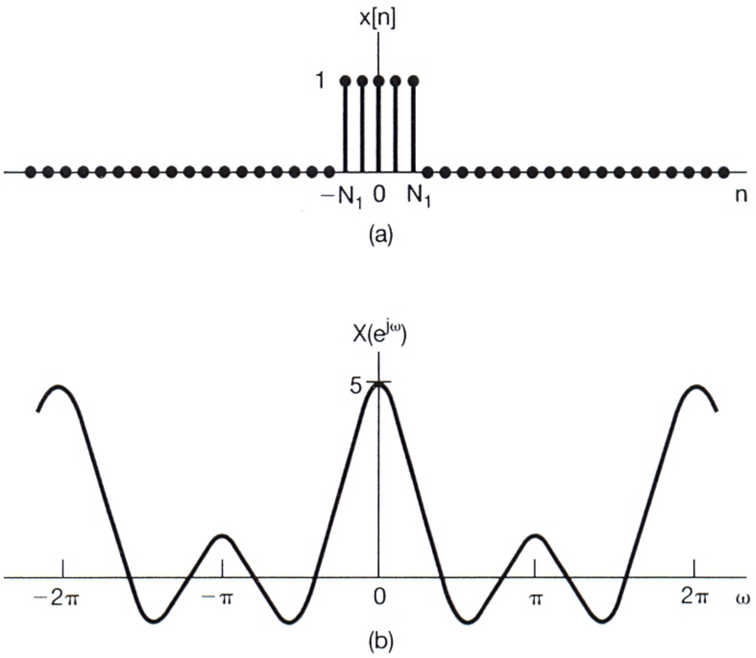


Figure 5.6 (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

Using calculations similar to those employed in obtaining eq. (3.104) in Example 3.12, we can write

$X(j\omega) = \frac{\sin \omega T_1}{\omega/2}$ (4.17)

$$X(e^{j\omega}) = \frac{\sin \omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}. \tag{5.12}$$

This Fourier transform is sketched in Figure 5.6(b) for $N_1 = 2$. The function in eq. (5.12) is the discrete-time counterpart of the sinc function, which appears in the Fourier transform of the continuous-time rectangular pulse (see Example 4.4). An important difference between these two functions is that the function in eq. (5.12) is periodic with period 2π, whereas the sinc function is aperiodic.

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

Although the argument we used to derive the discrete-time Fourier transform in Section 5.1.1 was constructed assuming that $x[n]$ was of arbitrary but finite duration, eqs. (5.8) and (5.9) remain valid for an extremely broad class of signals with infinite duration (such as the signals in Examples 5.1 and 5.2). In this case, however, we again must consider the question of convergence of the infinite summation in the analysis equation (5.9). The conditions on $x[n]$ that guarantee the convergence of this sum are direct counterparts of the convergence conditions for the continuous-time Fourier transform.[1] Specifically, eq. (5.9) will converge either if $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty, \quad \text{i.e. } x \in l_1(-\infty,\infty) \tag{5.13}$$

or if the sequence has finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty. \quad \text{i.e. } x \in l_2(-\infty,\infty) \tag{5.14}$$

In contrast to the situation for the analysis equation (5.9), there are generally no convergence issues associated with the synthesis equation (5.8), since the integral in this equation is over a finite interval of integration. This is very much the same situation as for the discrete-time Fourier series synthesis equation (3.94), which involves a finite sum and consequently has no issues of convergence associated with it either. In particular, if we approximate an aperiodic signal $x[n]$ by an integral of complex exponentials with frequencies taken over the interval $|\omega| \leq W$, i.e.,

$\exists$ Gibbs Phenomenon

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{5.15}$$

Then, $\hat{x}[n] = x[n]$ for $W = \pi$

[1]For discussions of the convergence issues associated with the discrete-time Fourier transform, see A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1989), and L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975).

Recall: $X(e^{j\omega}) = \lim_{\substack{N_1 \to \infty \\ N_2 \to \infty}} \sum_{n=-N_1}^{N_2} x[n]\, e^{-j\omega n}$ (5.9)' then $\hat{x}[n] = x[n]$ for $W = \pi$. Thus, much as in Figure 3.18, we would expect not to see any behavior like the Gibbs phenomenon in evaluating the discrete-time Fourier transform synthesis equation. This is illustrated in the following example.

Example 5.4

Let $x[n]$ be the unit impulse; that is,

$$x[n] = \delta[n].$$

In this case the analysis equation (5.9) is easily evaluated, yielding

$$X(e^{j\omega}) = 1.$$

In other words, just as in continuous time, the unit impulse has a Fourier transform representation consisting of equal contributions at all frequencies. If we then apply eq. (5.15) to this example, we obtain

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\, d\omega = \frac{\sin Wn}{\pi n}. \tag{5.16}$$

(cf.) Fig. 4.11, p.296

This is plotted in Figure 5.7 for several values of W. As can be seen, the frequency of the oscillations in the approximation increases as W is increased, which is similar to what we observed in the continuous-time case. On the other hand, in contrast to the continuous-time case, the amplitude of these oscillations decreases relative to the magnitude of $\hat{x}[0]$ as W is increased, and the oscillations disappear entirely for $W = \pi$.

5.2 THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

Recall:
$e^{j\omega_0 t} \notin L_1(-\infty,\infty)$
$\Rightarrow \nexists$ a formal Fourier Transform
$X(j\omega) = 2\pi\delta(\omega-\omega_0)$

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain. To derive the form of this representation, consider the signal

$$x[n] = e^{j\omega_0 n}. \quad \notin \ell_1(-\infty,\infty) \tag{5.17}$$

$\Rightarrow \nexists$ a formal Fourier Transform

In continuous time, we saw that the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. Therefore, we might expect the same type of transform to result for the discrete-time signal of eq. (5.17). However, the discrete-time Fourier transform must be periodic in ω with period 2π. This then suggests that the Fourier transform of $x[n]$ in eq. (5.17) should have impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. In fact, the Fourier transform of $x[n]$ is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l), \tag{5.18}$$

<Proof> $x[n] = \frac{1}{2\pi}\int_{\omega_1}^{\omega_1+2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$

$= \frac{2\pi}{2\pi}\sum_{l=-\infty}^{+\infty}\int_{\omega_1}^{\omega_1+2\pi} \delta(\omega-\omega_0-2\pi l)\, e^{j\omega n}\, d\omega = e^{j(\omega_0+2\pi l_1)n}$

$= e^{j\omega_0 n}$ □

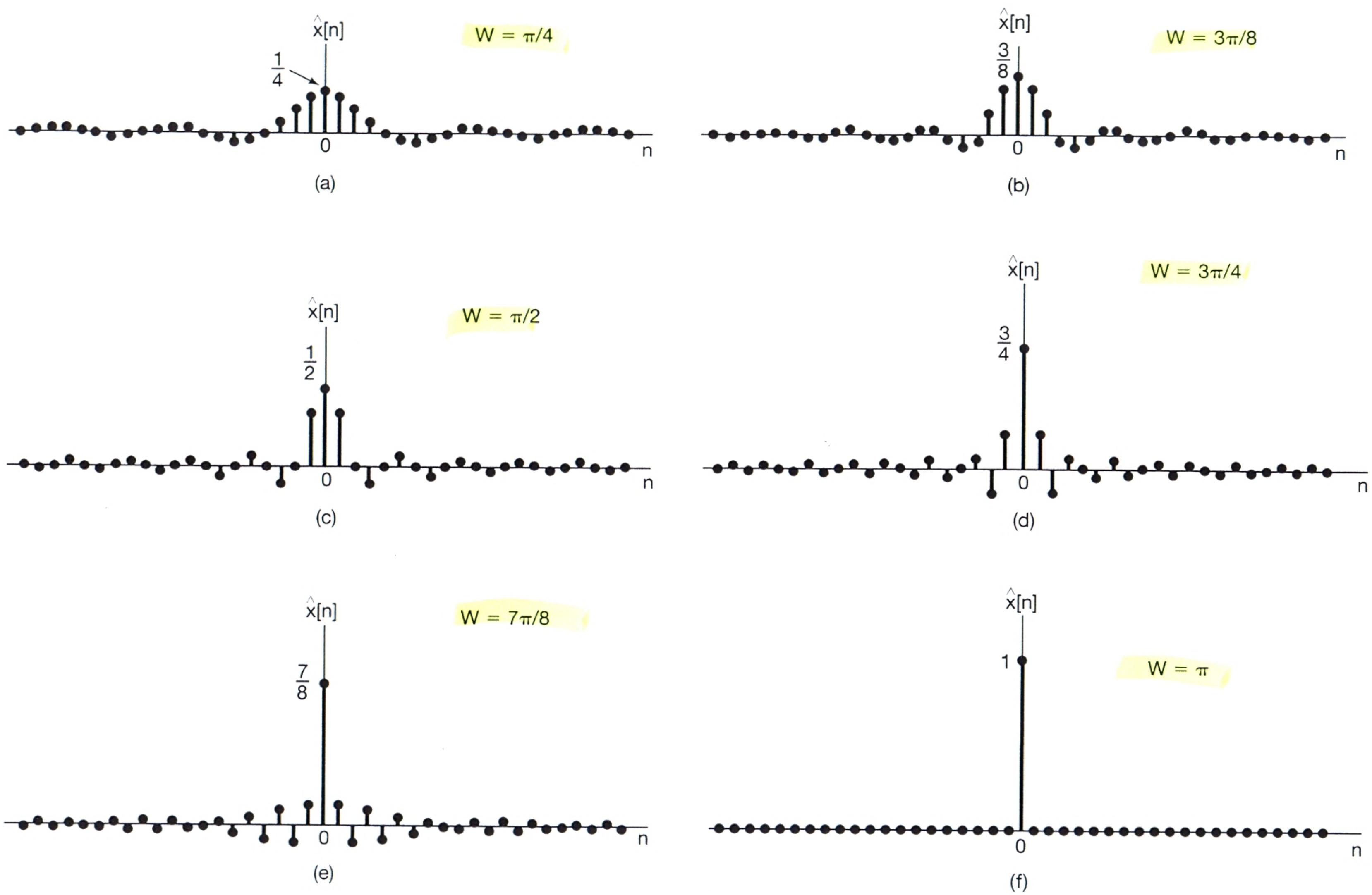


Figure 5.7 Approximation to the unit sample obtained as in eq. (5.16) using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} \qquad (5.19)$$

$$a_k = a_{k+N}$$

$$\Rightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\underbrace{\omega_0}_{=\frac{2\pi}{N}}) \qquad (5.20)$$

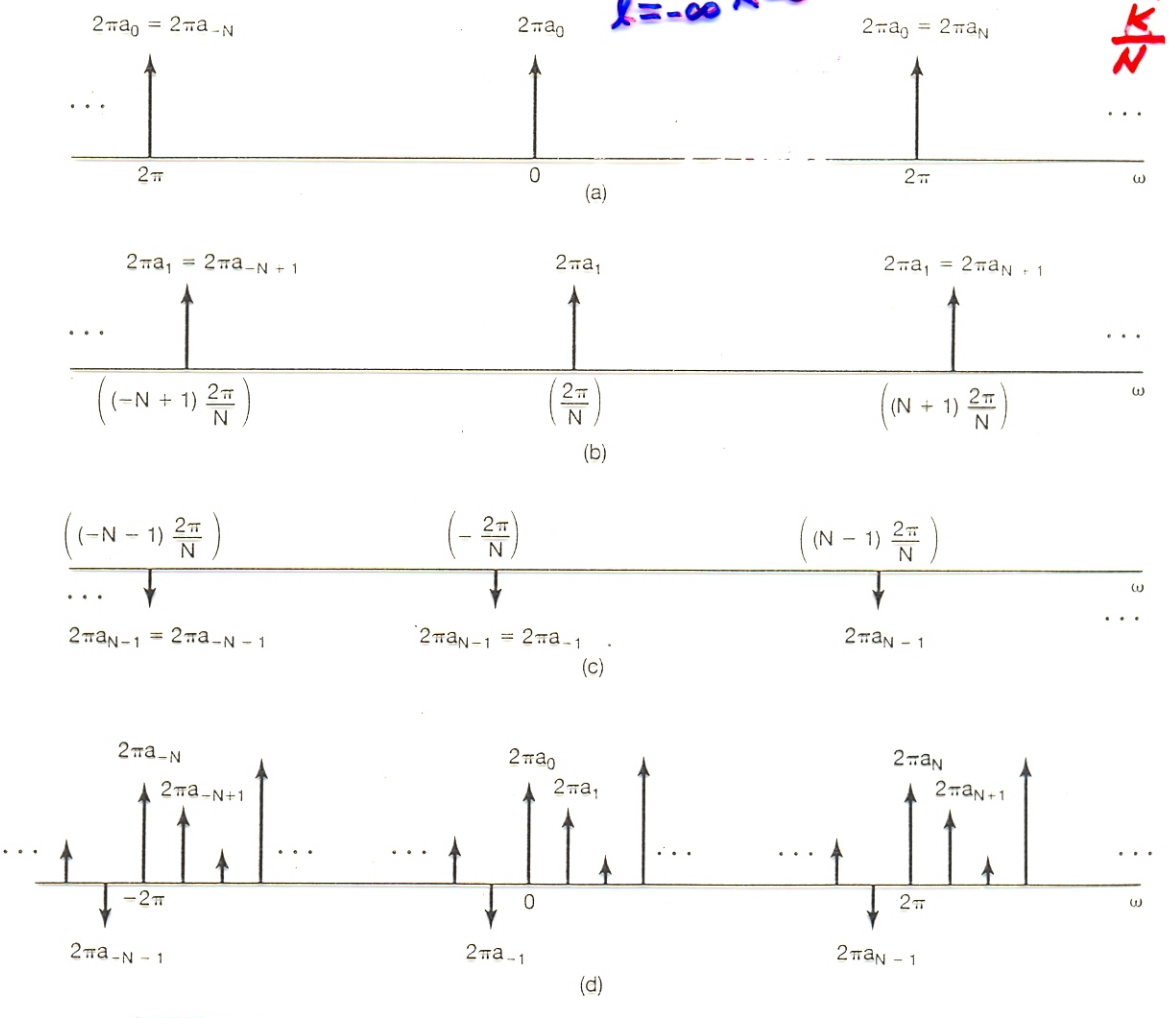
$$(\text{cf.})\ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \qquad (4.22)$$

<Proof>

$$(\text{Method I})\ X(e^{j\omega}) = \sum_{\bar{k}=0}^{N-1} a_{\bar{k}} \mathcal{F}\{e^{j\bar{k}\omega_0 n}\}$$

$$= \sum_{\bar{k}=0}^{N-1} a_{\bar{k}} \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \bar{k}\omega_0 - 2\pi l)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{\bar{k}=0}^{N-1} 2\pi a_{\bar{k}} \delta\left(\omega - 2\pi \frac{\bar{k}+lN}{N}\right), \qquad \frac{\bar{k}+lN}{N} = \frac{k}{N}$$



$\boxed{\text{Figure 5.9}}$ Fourier transform of a discrete-time periodic signal: (a) Fourier transform of the first term on the right-hand side of eq. (5.21); (b) Fourier transform of the second term in eq. (5.21); (c) Fourier transform of the last term in eq. (5.21); (d) Fourier transform of $x[n]$ in eq (5.21).

$$(\text{Method II})\ \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{\bar{\omega}}^{\bar{\omega}+2\pi} \left\{ \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \right\} e^{j\omega n} d\omega$$

$$= \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} = x[n]$$

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} \quad (5.19)$$

$$\Rightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \quad (5.20)$$

$$\omega_0 = \frac{2\pi}{N} \qquad a_k = a_{k+N}$$

$$(\text{cf.})\ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \quad (4.22)$$

<Proof>

(Method I)

$$X(e^{j\omega}) = \sum_{\bar{k}=0}^{N-1} a_{\bar{k}} \mathcal{F}\{e^{j\bar{k}\omega_0 n}\}$$

$$= \sum_{\bar{k}=0}^{N-1} a_{\bar{k}} \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \bar{k}\omega_0 - 2\pi l)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{\bar{k}=0}^{N-1} 2\pi a_{\bar{k}} \delta\left(\omega - 2\pi \frac{\bar{k}+lN}{N}\right), \qquad \frac{\bar{k}+lN}{N} = \frac{k}{N}$$

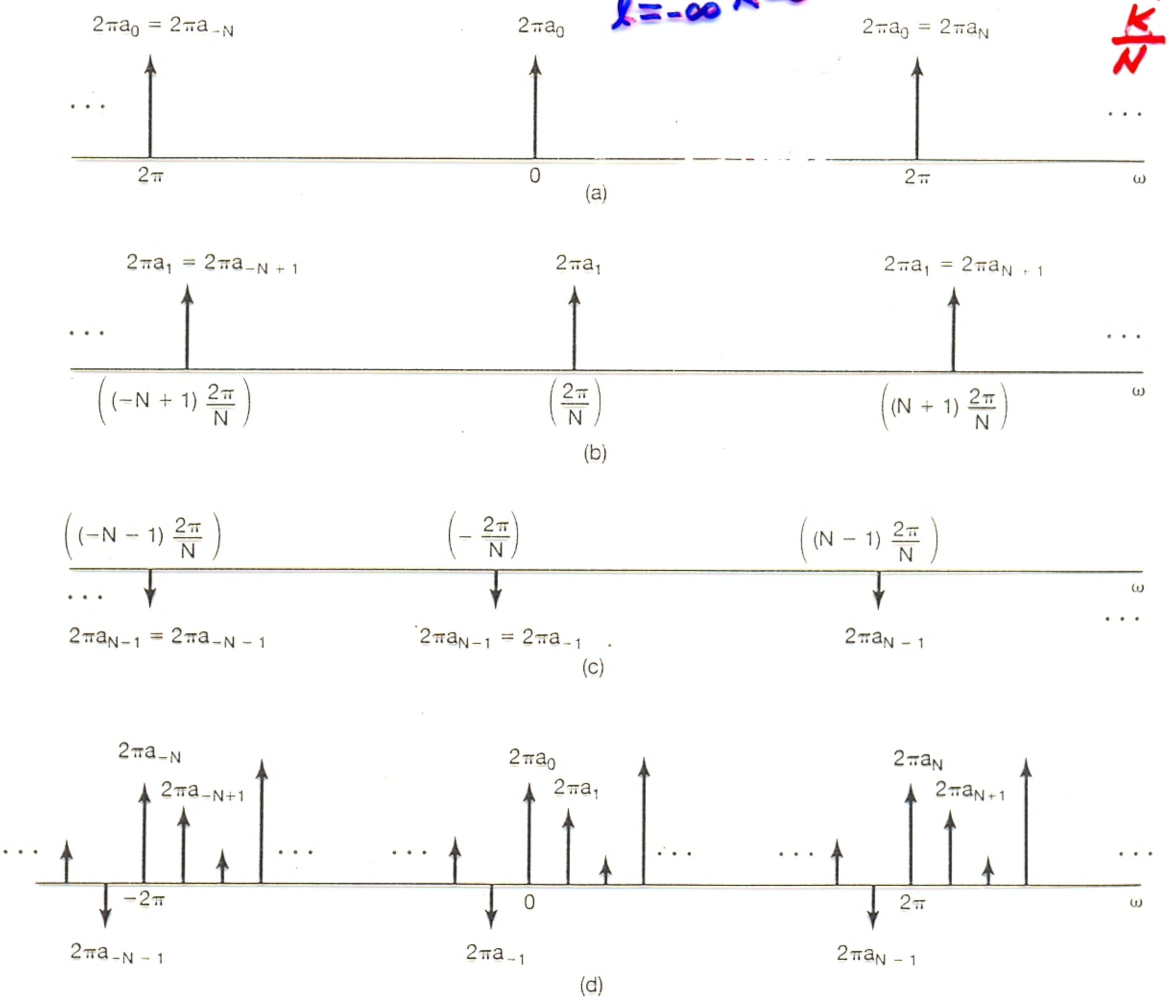


Figure 5.9 Fourier transform of a discrete-time periodic signal: (a) Fourier transform of the first term on the right-hand side of eq. (5.21); (b) Fourier transform of the second term in eq. (5.21); (c) Fourier transform of the last term in eq. (5.21); (d) Fourier transform of $x[n]$ in eq (5.21).

(Method II)

$$\mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{\bar{\omega}}^{\bar{\omega}+2\pi} \left\{\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)\right\} e^{j\omega n} d\omega$$

$$= \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} = x[n]$$

<Example 5.6> 5-9

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \Rightarrow a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Rightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

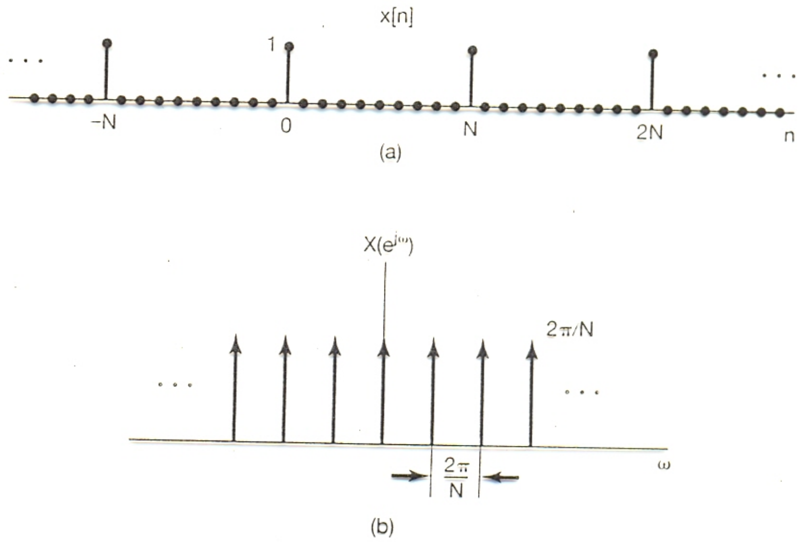
Choosing the interval of summation as $0 \le n \le N-1$, we have

$$a_k = \frac{1}{N}. \tag{5.26}$$

Using eqs. (5.26) and (5.20), we can then represent the Fourier transform of the signal as

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right). \tag{5.27}$$

which is illustrated in Figure 5.11(b).



$\boxed{\text{Figure 5.11}}$ (a) Discrete-time periodic impulse train; (b) its Fourier transform.

5.3 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega}) \quad (5.28)$$

$$\mathcal{F}\{a x_1[n] + b x_2[n]\} = a X_1(e^{j\omega}) + b X_2(e^{j\omega}) \quad (5.29)$$

$$\mathcal{F}\{x[n-n_0]\} = e^{-j\omega n_0} X(e^{j\omega}) \quad (5.30)$$

$$\mathcal{F}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)}) \quad (5.31)$$

$$\left(\begin{array}{l} \mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega) \\ \mathcal{F}\{e^{j\omega_0 t} x(t)\} = X(j(\omega-\omega_0)) \end{array} \right)$$

5-9

<Example 5.6>

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \Rightarrow a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Rightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

Choosing the interval of summation as $0 \le n \le N-1$, we have

$$a_k = \frac{1}{N}. \tag{5.26}$$

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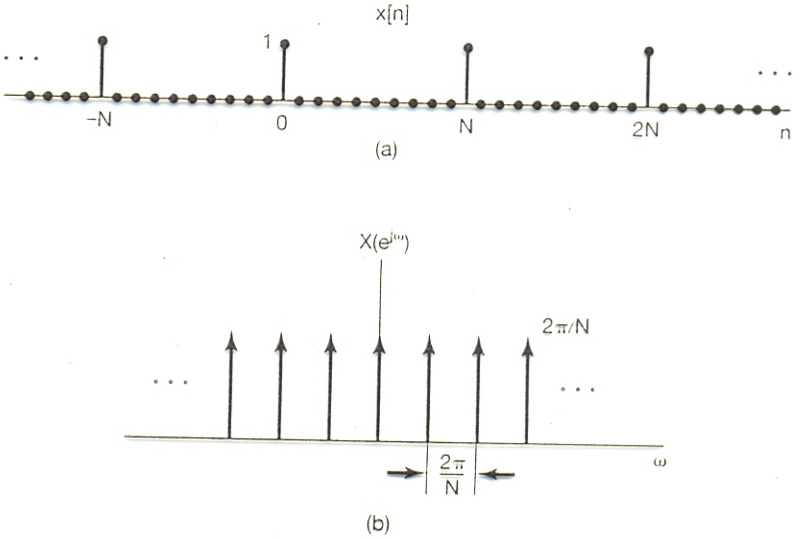

Figure 5.11 (a) Discrete-time periodic impulse train; (b) its Fourier transform.

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and

$$\boxed{e^{j\omega_0 n}x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).} \tag{5.31}$$

Equation (5.30) can be obtained by direct substitution of $x[n-n_0]$ into the analysis equation (5.9), while eq. (5.31) is derived by substituting $X(e^{j(\omega-\omega_0)})$ into the synthesis equation (5.8).

As a consequence of the periodicity and frequency-shifting properties of the discrete-time Fourier transform, there exists a special relationship between ideal lowpass and ideal highpass discrete-time filters. This is illustrated in the next example.

Example 5.7

In Figure 5.12(a) we have depicted the frequency response $H_{\text{lp}}(e^{j\omega})$ of a lowpass filter with cutoff frequency ω_c, while in Figure 5.12(b) we have displayed $H_{\text{lp}}(e^{j(\omega-\pi)})$—that is, the frequency response $H_{\text{lp}}(e^{j\omega})$ shifted by one-half period, i.e., by π. Since high frequencies in discrete time are concentrated near π (and other odd multiples of π), the filter in Figure 5.12(b) is an ideal highpass filter with cutoff frequency $\pi-\omega_c$. That is,

$$H_{\text{hp}}(e^{j\omega}) \hat{=} H_{\text{lp}}(e^{j(\omega-\pi)}). \tag{5.32}$$

As we can see from eq. (3.122), and as we will discuss again in Section 5.4, the frequency response of an LTI system is the Fourier transform of the impulse response of the system. Thus, if $h_{\text{lp}}[n]$ and $h_{\text{hp}}[n]$ respectively denote the impulse responses of

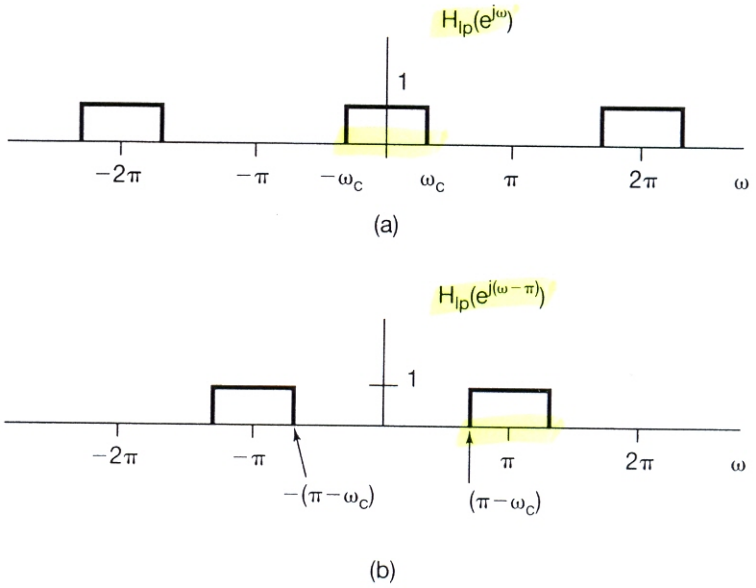


Figure 5.12 (a) Frequency response of a lowpass filter; (b) frequency response of a highpass filter obtained by shifting the frequency response in (a) by $\omega = \pi$ corresponding to one-half period.

Note : $h_{hp}[n] = \mathcal{F}^{-1}\{H_{hp}(e^{j\omega})\} = \mathcal{F}^{-1}\{H_{lp}(e^{j(\omega-\pi)})\}$
$= e^{j\pi n}\,\mathcal{F}^{-1}\{H_{lp}(e^{j\omega})\} = (-1)^n h_{lp}[n]$
(5.34)

5.3.4 Conjugation and Conjugate Symmetry

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega}).} \tag{5.35}$$

Also, if $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is,

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{real}].} \tag{5.36}$$

From this, it follows that $\mathcal{R}e\{X(e^{j\omega})\}$ is an even function of ω and $\mathcal{I}m\{X(e^{j\omega})\}$ is an odd function of ω. Similarly, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function. Furthermore,

$$\mathcal{E}v\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\}$$

and

$$\mathcal{O}d\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m\{X(e^{j\omega})\},$$

where $\mathcal{E}v$ and $\mathcal{O}d$ denote the even and odd parts, respectively, of $x[n]$. For example, if $x[n]$ is real and even, its Fourier transform is also real and even. Example 5.2 illustrates this symmetry for $x[n] = a^{|n|}$.

5.3.5 Differencing and Accumulation

In this subsection, we consider the discrete-time counterpart of integration—that is, accumulation—and its inverse, first differencing. Let $x[n]$ be a signal with Fourier transform $X(e^{j\omega})$. Then, from the linearity and time-shifting properties, the Fourier transform pair for the first-difference signal $x[n] - x[n-1]$ is given by

$$\boxed{x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega}).} \tag{5.37}$$

$$\boxed{\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k).} \tag{5.39}$$

Remarks

Note : if $y[n] \in l_1(-\infty, \infty)$ or $y[n] \in l_2(-\infty, \infty)$, then $\exists\, Y(e^{j\omega})$ formally.

Furthermore, $\lim_{n \to \infty} y[n] = 0 \quad (*)$.

Consider $y[n] \triangleq \sum_{m=-\infty}^{n} x[m] = y[n-1] + x[n]$

$$\Rightarrow Y(e^{j\omega}) = e^{-j\omega} Y(e^{j\omega}) + X(e^{j\omega}) \Rightarrow \boxed{Y(e^{j\omega}) = \frac{X(e^{j\omega})}{1 - e^{-j\omega}}}$$

($(*) \Rightarrow \sum_{m=-\infty}^{+\infty} x[m] = X(e^{j0}) = 0$. This with (5.39) $\Rightarrow$)

< Proof of (5.39) >

$$u[n] = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(n) \quad \text{where } \mathrm{sgn}(n) \triangleq \begin{cases} -1, & n<0 \\ 1, & n\geq 0 \end{cases} \qquad (*1)$$

$$\mathcal{F}\{1\} = \sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - 2\pi k) \quad \text{from (5.18), p.367} \qquad (*2)$$

$$e^{-a|n|}\mathrm{sgn}(n) \xrightarrow{\mathcal{F}} \frac{1}{1-e^{-(a+j\omega)}} + \frac{1}{1-e^{(a-j\omega)}}$$

as $a \to 0$

$$\Longrightarrow \mathcal{F}\{\mathrm{sgn}(n)\} = \frac{2}{1-e^{-j\omega}} \qquad (*3)$$

By (*1), (*2), and (*3),

$$\mathcal{F}\{u[n]\} = \frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k) \qquad (*4)$$

(The result of Example 5.8)

$$\Rightarrow \mathcal{F}\left\{\sum_{m=-\infty}^{n} x[m]\right\} = \mathcal{F}\{u[n] * x[n]\}$$

$$= \mathcal{F}\{u[n]\}\, X(e^{j\omega})$$

$$= \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{+\infty} X(e^{j\omega})\,\delta(\omega-2\pi k)$$

$$= \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{+\infty} X(e^{j2\pi k})\,\delta(\omega-2\pi k)$$

(by (1.69))

$$= (5.39) \quad (\text{by } 5.28)) \qquad \square$$

<u>Remark</u> : < Example 5.8 > X

That is,

$$\boxed{x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega}).} \tag{5.42}$$

5.3.7 Time Expansion

Because of the discrete nature of the time index for discrete-time signals, the relation between time and frequency scaling in discrete time takes on a somewhat different form from its continuous-time counterpart. Specifically, in Section 4.3.5 we derived the continuous-time property

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right). \tag{5.43}$$

However, if we try to define the signal $x[an]$, we run into difficulties if a is not an integer. Therefore, we cannot slow down the signal by choosing $a < 1$. On the other hand, if we let a be an integer other than ± 1—for example, if we consider $x[2n]$—we do not merely speed up the original signal. That is, since n can take on only integer values, the signal $x[2n]$ consists of the even samples of $x[n]$ alone.

There is a result that does closely parallel eq. (5.43), however. Let k be a positive integer, and define the signal

$$x_{(k)}[n] \triangleq \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases} \tag{5.44}$$

As illustrated in Figure 5.13 for $k = 3$, $x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal. Intuitively, we can think of $x_{(k)}[n]$ as a slowed-down version of $x[n]$. Since $x_{(k)}[n]$ equals 0 unless n is a multiple of k, i.e., unless $n = rk$, we see that the Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}.$$

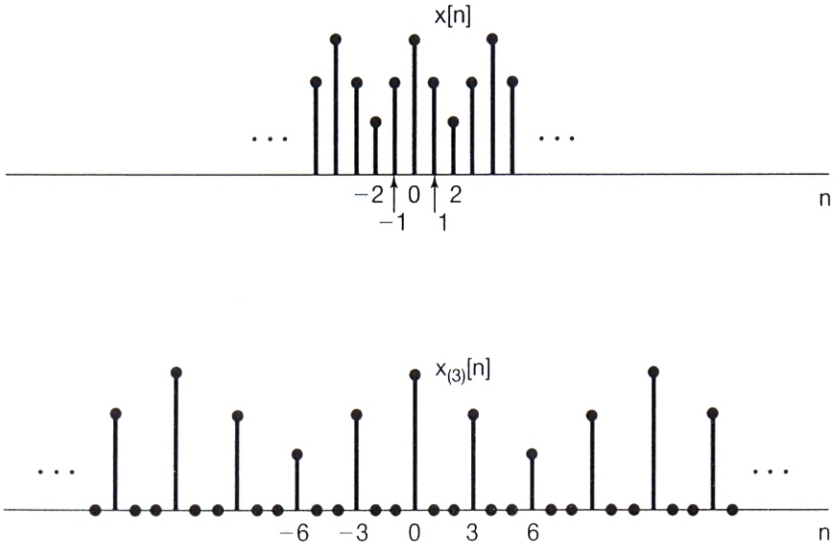


Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

Furthermore, since $x_{(k)}[rk] = x[r]$, we find that

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is, $\left(\text{cf.}:\ x_{(m)}[n] \xleftrightarrow{\mathcal{F}} a_k/m\right)$

$$\boxed{x_{(k)}[n] \xleftrightarrow{\mathcal{F}} X(e^{jk\omega}).} \tag{5.45}$$

Note that as the signal is spread out and slowed down in time by taking $k > 1$, its Fourier transform is compressed. For example, since $X(e^{j\omega})$ is periodic with period 2π, $X(e^{jk\omega})$ is periodic with period $2\pi/k$. This property is illustrated in Figure 5.14 for a rectangular pulse.

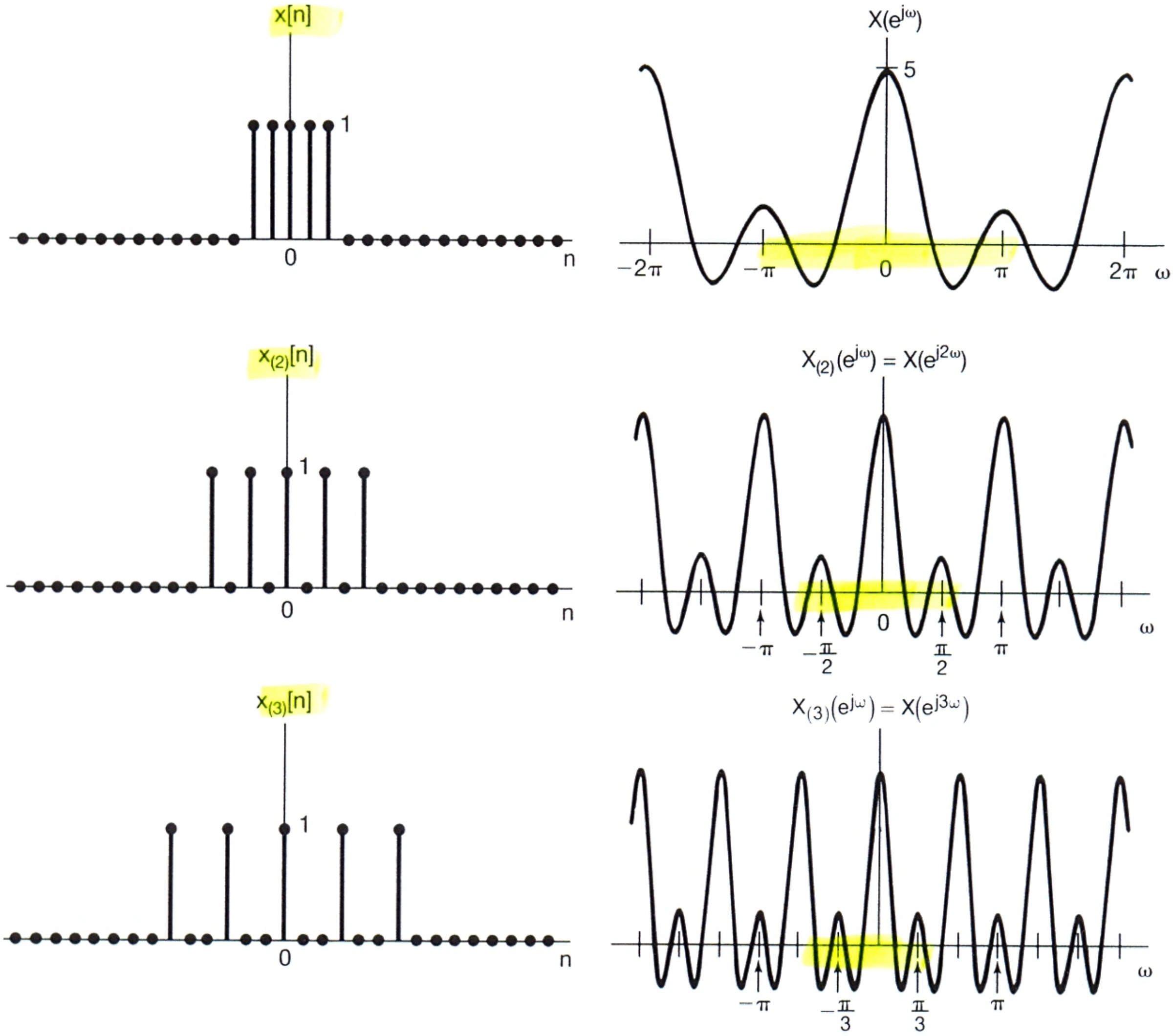


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

§5.3.8 Differentiation in Freq.

$$\mathcal{F}\{n\,x[n]\} = j\,\frac{dX(e^{j\omega})}{d\omega} \qquad (5.46)$$

since

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jn\,x[n]\,e^{-j\omega n} \quad \text{if } \sum_{n=-\infty}^{\infty} |n\,x[n]| < \infty$$

§5.3.9 Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]\,x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\,x[n] \int_{2\pi} X^*(e^{j\omega})\,e^{-j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega}) \left[\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\,d\omega \qquad (5.49)$$

energy-density spectrum

§5.4 The Convolution Property

$$\mathcal{F}\{x[n] * h[n]\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} h[n-k]\,x[k]\right\}$$

$$= \sum_{n}\left\{\sum_{k} h[n-k]\,x[k]\right\} e^{-j\omega n}$$

$$= \sum_{\bar{n}}\sum_{k} h[\bar{n}]\,x[k]\,e^{-j\omega(\bar{n}+k)}$$

$$= \sum_{\bar{n}} h[\bar{n}]\,e^{-j\omega\bar{n}} \sum_{k} x[k]\,e^{-j\omega k}$$

$$= H(e^{j\omega})\,X(e^{j\omega}) \qquad (5.48)$$

if $h \in l_1(-\infty, \infty)$ and $x \in l_1(-\infty, \infty)$

<Example 5.12>

response and frequency response of an LTI system are a Fourier transform pair, we can determine the impulse response of the ideal lowpass filter from the frequency response using the Fourier transform synthesis equation (5.8). In particular, using $-\pi \leq \omega \leq \pi$ as the interval of integration in that equation, we see from Figure 5.17(a) that

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{\sin \omega_c n}{\pi n}, \end{aligned} \tag{5.50}$$

which is shown in Figure 5.17(b).

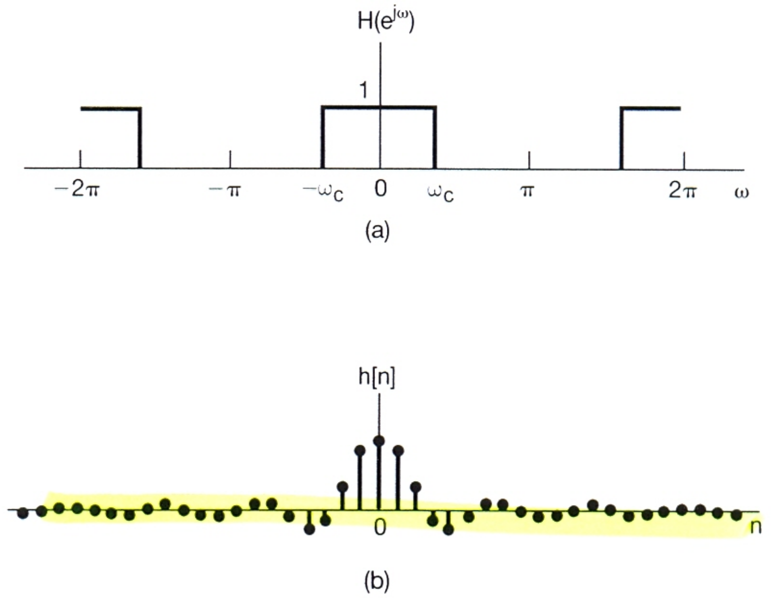


Figure 5.17 (a) Frequency response of a discrete-time ideal lowpass filter; (b) impulse response of the ideal lowpass filter.

In Figure 5.17, we come across many of the same issues that surfaced with the continuous-time ideal lowpass filter in Example 4.18. First, since $h[n]$ is not zero for $n < 0$, the ideal lowpass filter is not causal. Second, even if causality is not an important issue, there are other reasons, including ease of implementation and preferable time domain characteristics, that nonideal filters are generally used to perform frequency-selective filtering. In particular, the impulse response of the ideal lowpass filter in Figure 5.17(b) is oscillatory, a characteristic that is undesirable in some applications. In such cases, a trade-off between frequency-domain objectives such as frequency selectivity and time-domain properties such as nonoscillatory behavior must be made. In Chapter 6, we will discuss these and related ideas in more detail.

As the following example illustrates, the convolution property can also be of value in facilitating the calculation of convolution sums.

Moreover, $\lim\limits_{\substack{N_1\to\infty \\ N_2\to\infty}} \sum_{n=-N_1}^{N_2} |h[n]| = \lim\limits_{\substack{N_1\to\infty \\ N_2\to\infty}} \sum_{n=-N_1}^{N_2} \left|\frac{\sin \omega_c n}{\pi n}\right| \to \infty$

that is, the ideal LPF is not BIBO stable.

Example 5.13 (Use z-transform in chapter 10)

A Practical LPF
See Fig. 5.4 (a)
for $H(e^{j\omega})$
- not oscillatory
- causal

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n],$$

with $|\alpha| < 1$, and suppose that the input to this system is

$$x[n] = \beta^n u[n],$$

with $|\beta| < 1$. Evaluating the Fourier transforms of $h[n]$ and $x[n]$, we have

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \tag{5.51}$$

and

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}, \tag{5.52}$$

so that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}. \tag{5.53}$$

As with Example 4.19 , determining the inverse transform of $Y(e^{j\omega})$ is most easily done by expanding $Y(e^{j\omega})$ by the method of partial fractions. Specifically, $Y(e^{j\omega})$ is a ratio of polynomials in powers of $e^{-j\omega}$, and we would like to express this as a sum of simpler terms of this type so that we can find the inverse transform of each term by inspection (together, perhaps, with the use of the frequency differentiation property of Section 5.3.8). The general algebraic procedure for rational transforms is described in the appendix. For this example, if $\alpha \neq \beta$, the partial fraction expansion of $Y(e^{j\omega})$ is of the form

$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}. \tag{5.54}$$

Equating the right-hand sides of eqs (5.53) and (5.54), we find that

$$A = \frac{\alpha}{\alpha - \beta}, \quad B = -\frac{\beta}{\alpha - \beta}.$$

Therefore, from Example 5.1 and the linearity property, we can obtain the inverse transform of eq. (5.54) by inspection:

$$\begin{aligned} y[n] &= \frac{\alpha}{\alpha - \beta}\alpha^n u[n] - \frac{\beta}{\alpha - \beta}\beta^n u[n] \\ &= \frac{1}{\alpha - \beta}[\alpha^{n+1} u[n] - \beta^{n+1} u[n]]. \end{aligned} \tag{5.55}$$

For $\alpha = \beta$, the partial-fraction expansion in eq. (5.54) is not valid. However, in this case,

$$Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}}\right)^2,$$

which can be expressed as

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right). \tag{5.56}$$

As in Example 4.19, we can use the frequency differentiation property, eq. (5.46), together with the Fourier transform pair

$$\alpha^n u[n] \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\omega}},$$

to conclude that

$$n\alpha^n u[n] \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right).$$

To account for the factor $e^{j\omega}$, we use the time-shifting property to obtain

$$(n+1)\alpha^{n+1} u[n+1] \overset{\mathfrak{F}}{\longleftrightarrow} je^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right),$$

and finally, accounting for the factor $1/\alpha$, in eq. (5.56), we obtain

$$y[n] = (n+1)\alpha^n u[n+1]. \tag{5.57}$$

It is worth noting that, although the right-hand side is multiplied by a step that begins at $n = -1$, the sequence $(n+1)\alpha^n u[n+1]$ is still zero prior to $n = 0$, since the factor $n+1$ is zero at $n = -1$. Thus, we can alternatively express $y[n]$ as

$$y[n] = (n+1)\alpha^n u[n]. \tag{5.58}$$

As illustrated in the next example, the convolution property, along with other Fourier transform properties, is often useful in analyzing system interconnections.

Example 5.14

Consider the system shown in Figure 5.18(a) with input $x[n]$ and output $y[n]$. The LTI systems with frequency response $H_{lp}(e^{j\omega})$ are ideal lowpass filters with cutoff frequency $\pi/4$ and unity gain in the passband.

Let us first consider the top path in Figure 5.18(a). The Fourier transform of the signal $w_1[n]$ can be obtained by noting that $(-1)^n = e^{j\pi n}$ so that $w_1[n] = e^{j\pi n}x[n]$. Using the frequency-shifting property, we then obtain

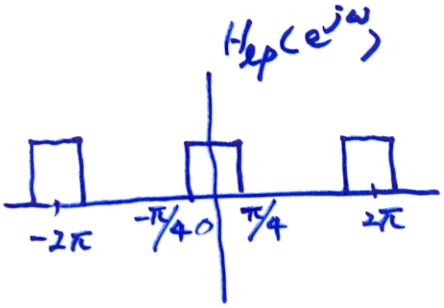
$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)}).$$

The convolution property yields

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)}).$$

Since $w_3[n] = e^{j\pi n}w_2[n]$, we can again apply the frequency-shifting property to obtain

$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}). \end{aligned}$$



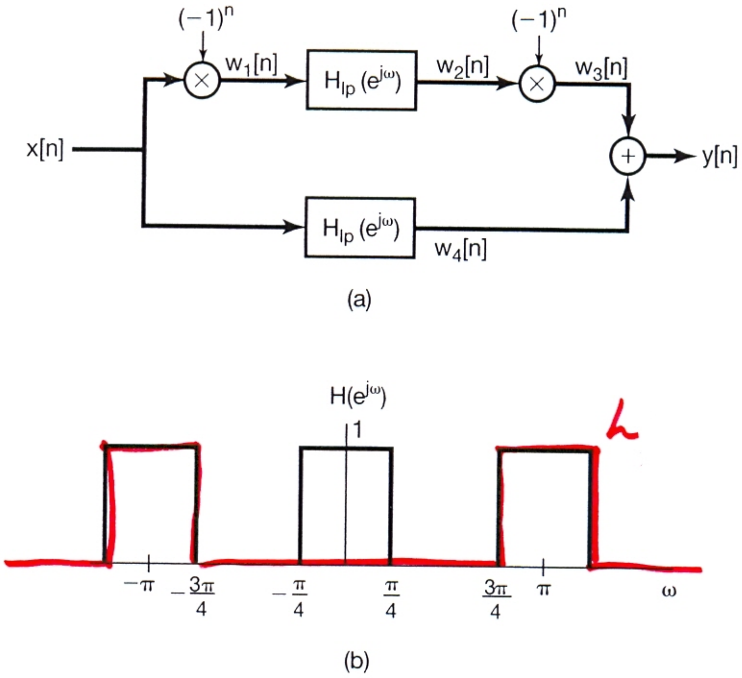


Figure 5.18 (a) System interconnection for Example 5.14; (b) the overall frequency response for this system.

Since discrete-time Fourier transforms are always periodic with period 2π,

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}).$$

Applying the convolution property to the lower path, we get

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega}).$$

From the linearity property of the Fourier transform, we obtain

$$\begin{aligned} Y(e^{j\omega}) &= W_3(e^{j\omega}) + W_4(e^{j\omega}) \\ &= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega}). \end{aligned}$$

Consequently, the overall system in Figure 5.18(a) has the frequency response

$$H(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]$$

which is shown in Figure 5.18(b).

As we saw in Example 5.7, $H_{lp}(e^{j(\omega-\pi)})$ is the frequency response of an ideal highpass filter. Thus, the overall system passes both low and high frequencies and stops frequencies between these two passbands. That is, the filter has what is often referred to as an *ideal bandstop characteristic,* where the stopband is the region $\pi/4 < |\omega| < 3\pi/4$.

It is important to note that, as in continuous time, not every discrete-time LTI system has a frequency response. For example, the LTI system with impulse response $h[n] = 2^n u[n]$ does not have a finite response to sinusoidal inputs, which is reflected in the fact that the Fourier transform analysis equation for $h[n]$ diverges. However, if an LTI system is stable, then, from Section 2.3.7, its impulse response is absolutely summable; that is,

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty. \tag{5.59}$$

Therefore, the frequency response always converges for stable systems. In using Fourier methods, we will be restricting ourselves to systems with impulse responses that have well-defined Fourier transforms. In Chapter 10, we will introduce an extension of the Fourier transform referred to as the z-transform that will allow us to use transform techniques for LTI systems for which the frequency response does not converge.

5.5 THE MULTIPLICATION PROPERTY

In Section 4.5, we introduced the multiplication property for continuous-time signals and indicated some of its applications through several examples. An analogous property exists for discrete-time signals and plays a similar role in applications. In this section, we derive this result directly and give an example of its use. In Chapters 7 and 8, we will use the multiplication property in the context of our discussions of sampling and communications.

Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms. Then

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n},$$

or since

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta, \tag{5.60}$$

it follows that

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\} e^{-j\omega n}. \tag{5.61}$$

Interchanging the order of summation and integration, we obtain

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta. \tag{5.62}$$

The bracketed summation is $X_2(e^{j(\omega-\theta)})$, and consequently, eq. (5.62) becomes

$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.} \tag{5.63}$$