Naval Architecture & Ocean Engineering

[2009] [15]

Innovative ship design

- Variational Mehod and Approximation-

June 2009

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$\sum \mathbf{F} = m\mathbf{a}$, $\mathbf{a} = 0$ in equilibrium

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2 weak form : 'weighted average'*

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FEM

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Summary
Variables and Equations

If we are interested in finding the displacement components in a body, we may reduce the system of equations to <u>three equations with three unknown displacement components.</u>

18 Variables
- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Given : Body force	X, Y, Z
Find : Displacement	$u,\ v,\ w$

18 Equations

6 Equations of force equilibrium

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \qquad \sum M_x = \tau_{yz} - \tau_{zy} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \qquad \sum M_y = \tau_{xz} - \tau_{zx} = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \qquad \sum M_z = \tau_{xy} - \tau_{yx} = 0$$

6 Relations btw. *Strain and Displacement*

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations btw. *6 Strain and 6 Stress*

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x \qquad , \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y \qquad , \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_z \qquad , \tau_{zx} = \frac{E}{2(\nu+1)} \gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

3 ← 1, 2

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0$$

$$(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 v + Y = 0$$

$$(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 w + Z = 0$$

3 Variables
3 Equations

X, Y, Z: bodyforce in x,y, and z direction repectively

$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ $\quad \Theta = \sigma_x + \sigma_y + \sigma_z$

μ, λ: Lame Elastic constant

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$

G: Shear Moldulus

ν: Poisson's Ratio

E: Young's Modulus

Summary

18 Variables → 15 Variables

9 Stress $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement u, v, w

18 Equations → 15 Equations

If we are interested in finding only the stress components in a body, we may reduce the system of equations to <u>six equations with six unknown stress components</u>

Given : Body force X, Y, Z

Find : Stress $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

3

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial X}{\partial x}+\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Y}{\partial y}+\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Z}{\partial z}+\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=0$$

$$\left(\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\right)+\nabla^2\tau_{xy}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y}=0$$

$$\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)+\nabla^2\tau_{yz}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z}=0$$

$$\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)+\nabla^2\tau_{zx}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x}=0$$

⇒ **6 Variables**
6 Equations

2

<u>6 Equations of force equilibrium</u>

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

$$\sum F_z=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0$$

$$\sum M_x=\tau_{yz}-\tau_{zy}=0$$

$$\sum M_y=\tau_{xz}-\tau_{zx}=0$$

$$\sum M_z=\tau_{xy}-\tau_{yx}=0$$

<u>6 Relations</u> btw. <u>Strain and Displacement</u>

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

Compatibility equations <u>3 independent Equations</u>

$$\begin{cases}\dfrac{\partial^2\varepsilon_x}{\partial y^2}+\dfrac{\partial^2\varepsilon_y}{\partial x^2}=\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y}\\[2ex]\dfrac{\partial^2\varepsilon_y}{\partial z^2}+\dfrac{\partial^2\varepsilon_z}{\partial y^2}=\dfrac{\partial^2\gamma_{yz}}{\partial y\partial z}\\[2ex]\dfrac{\partial^2\varepsilon_z}{\partial x^2}+\dfrac{\partial^2\varepsilon_x}{\partial z^2}=\dfrac{\partial^2\gamma_{zx}}{\partial z\partial x}\end{cases}\quad\text{or}\quad\begin{cases}2\dfrac{\partial^2\varepsilon_x}{\partial y\partial z}=\dfrac{\partial}{\partial x}\left(-\dfrac{\partial\gamma_{yz}}{\partial x}+\dfrac{\partial\gamma_{zx}}{\partial y}+\dfrac{\partial\gamma_{xy}}{\partial z}\right)\\[2ex]2\dfrac{\partial^2\varepsilon_y}{\partial z\partial x}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial\gamma_{yz}}{\partial x}-\dfrac{\partial\gamma_{zx}}{\partial y}+\dfrac{\partial\gamma_{xy}}{\partial z}\right)\\[2ex]2\dfrac{\partial^2\varepsilon_z}{\partial x\partial y}=\dfrac{\partial}{\partial z}\left(\dfrac{\partial\gamma_{yz}}{\partial x}+\dfrac{\partial\gamma_{zx}}{\partial y}-\dfrac{\partial\gamma_{xy}}{\partial z}\right)\end{cases}$$

1

<u>6 Relations</u> btw. 6 Strain and 6 Stress

$$\sigma_x=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_x,\quad \tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_y,\quad \tau_{yz}=\frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_z,\quad \tau_{zx}=\frac{E}{2(\nu+1)}\gamma_{zx}$$

$$,\ e=\varepsilon_x+\varepsilon_y+\varepsilon_z$$

X, Y, Z: bodyforce in x,y, and z direction repectively

$e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ $\quad\Theta=\sigma_x+\sigma_y+\sigma_z$

μ, λ: Lame Elastic constant

$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}$

G: Shear Moldulus

ν: Poisson's Ratio

E: Young's Modulus

Classification

$$\int_0^1 (-u''v)dx = \left[-u'v\right]_0^1 + \int_0^1 (u'v')dx$$

$$\sum_{i=1}^{N} a_i \left(\sum_{j=1}^{N} c_j \underbrace{\left\{ \int_0^1 \left[\phi_i'(x)\phi_j'(x) + \phi_i(x)\phi_j(x) \right] dx \right\}}_{k_{ij}} \right) = \sum_{i=1}^{N} a_i \underbrace{\int_0^1 x\phi_i(x)\,dx}_{F_i}$$

whenever a smooth 'classical(strong)' solution to a (D.E.) problem exists, it is also the solution of the weak problem[3]

Weak Form[2]

$$\int_0^1 (-u'' + u - x)v\,dx = 0$$

$$u(0)=0,\ u(1)=0$$

multiply v and integration

integration by part and demand the test functions vanish at the endpoints

$$\int_0^1 (-u'v' + uv - xv)\,dx = 0$$

Approximate Method[4]

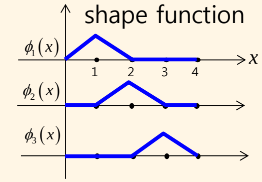
- **Collocation**
- **Least Square**
- **Galerkin** → **FEM**

$$\sum_{i=1}^{n} c_j k_{ij} = F_i$$

$$u(x) \approx \sum_{j=1}^{n} c_j \phi_j(x)$$

$$v(x) \approx \sum_{i=1}^{n} a_i \phi_i(x)$$

shape function



Differential Equation (ODE/PDE)

Ex.) $-u'' + u = x,\ 0 < x < 1,$

$u(0)=0,\ u(1)=0$

Ex.) $\dfrac{d}{dx}\left(T\dfrac{dy}{dx}\right) + \rho\omega^2 y + p = 0$

multiply δy and integration

$$\int_0^l \left(\frac{d}{dx}\left(T\frac{dy}{dx}\right) + \rho\omega^2 y + p \right)\delta y\,dx$$

Work and Energy Principle

Variational formulation

$$\delta \int_0^l \left[\frac{1}{2}\rho\omega^2 y^2 + py - \frac{T}{2}\left(\frac{dy}{dx}\right)^2 \right] dx = 0$$

integration by part and B/C

Approximate Method

Rayleigh-Ritz

assume:

$$y(x) \approx \phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x)$$

- **Variation and integration**
- **Integration and variation**

Leibnitz formula[1]

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \frac{d}{dx}\int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

problem of a "hereditary' nature[5]

Integral Equations

Volterra

$$\alpha(x)y(x) = F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi$$

Fredholm

$$\alpha(x)y(x) = F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi$$

Approximate Method

$$\sum_{k=1}^{n} c_k s_k(x) \approx F(x)\ ,\ s_k(x) = \phi_k(x) - \lambda\int_a^b K(x,\xi)y(\xi)d\xi$$

Collocation $\quad \sum_{k=1}^{n} c_k s_k(x_i) = F(x_i)$

Galerkin $\quad \sum_{k=1}^{n} c_k \int_a^b \psi_i(x)s_k(x)dx = \int_a^b \psi_i(x)F(x)dx$

$$,\psi(x) = \sum_{k=1}^{n} a_k\phi_i(x)$$

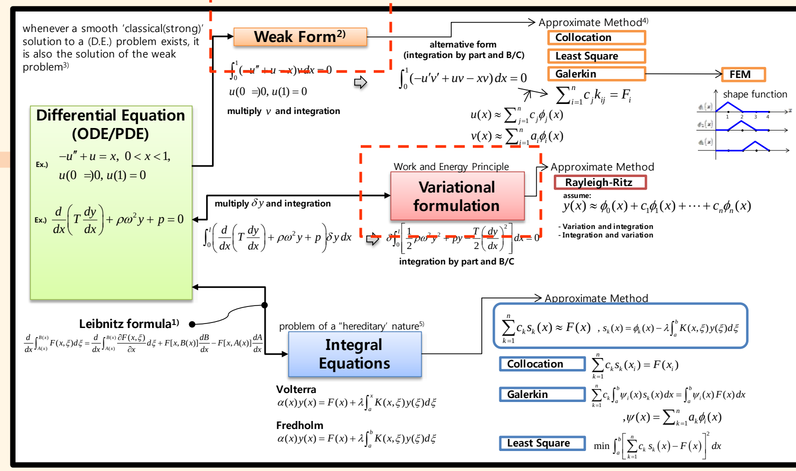
Least Square $\quad \min \int_a^b \left[\sum_{k=1}^{n} c_k s_k(x) - F(x) \right]^2 dx$

?!

what is the relationship between 'week form' and 'Variational formulation'?

1) Jerry, A.j., Introduction to Integral Equations with Applications, Marcel Dekker Inc., 1985, p19~25
2) 'variational statement of the problem' -Becker, E.B., et al, Finite Elements An Introduction, Volume 1, Prentice-Hall, 1981, p4
3) Becker, E.B., et al, Finite Elements An Introduction, Volume 1, Prentice-Hall, 1981, p2 . See also Betounes, Partial Differential Equations for Computational Science, Springer, 1988, p408 "…the weak solution is actually a strong (or classical) solution…"
4) some books refer as 'Method of Weighted Residue' from the Finite Element Equation point of view and they have different type depending on how to choose the weight functions. See also Fletcher,C.A.J., "Computational Galerkin Methods", Springer, 1984
5) Jerry, A.j., Introduction to Integral Equations with Applications, Marcel Dekker Inc., 1985, p1 "Problems of a 'hereditary' nature fall under the first category, since the state of the system u(t) at any time t depends by the definition on all the previous states u(t-τ) at the previous time t-τ ,which means that we must sum over them, hence involve them under the integral sign in an integral equation.

Classification



?

what is the relationship between 'week form' and 'Variational formulation'?

Ex: Rotating String) **differential equation**

$$\frac{d}{dx}\left(T\frac{dy}{dx}\right) + \rho\omega^2 y = -p$$

multiply δy and integration

virtual displacement (it has a physical meaning)

$$\int_0^l \left(\frac{d}{dx}\left(T\frac{dy}{dx}\right) + \rho\omega^2 y + p\right)\delta y\, dx = 0$$

integration by part and B/C

$$\delta\int_0^l \left[\frac{1}{2}\rho\omega^2 y^2 + py - \frac{T}{2}\left(\frac{dy}{dx}\right)^2\right] dx + \left[T\frac{dy}{dx}\delta y\right]_0^l = 0$$

Variational formulation:

$$\delta\int_0^l \left[\frac{1}{2}\rho\omega^2 y^2 + py - \frac{T}{2}\left(\frac{dy}{dx}\right)^2\right] dx = 0$$

self-adjoint form
Natural B/C

physical meaning : "Minimize the difference of kinetic energy and potential energy"

Ritz method

$$y(x) \approx \sum_{j=1}^{n} c_j\phi_j(x)$$

a similarity

multiply v and 'week form'

$$\int_0^l \left(\frac{d}{dx}\left(T\frac{dy}{dx}\right) + \rho\omega^2 y\right) v\, dx = \int_0^l -p v\, dx$$

Weighted Residual

$$\int_0^l \left(\frac{d}{dx}\left(T\frac{dy}{dx}\right) + \rho\omega^2 y + p\right) v\, dx = 0$$

if is has a meaning of minimizing the weighted residual, we may rewrite is as

$$\delta\int_0^l \left(\frac{d}{dx}\left(T\frac{dy}{dx}\right) + \rho\omega^2 y + p\right) v\, dx = 0$$

in case of using Galerkin method, weight function can be regarded as a kind of y since they have same basis functions

$$y(x) \approx \sum_{j=1}^{n} c_j\phi_j(x)$$
$$v(x) \approx \sum_{i=1}^{n} a_i\phi_i(x)$$

$$\delta\int_0^l \left(\frac{d}{dx}\left(T\frac{dy}{dx}\right) + \rho\omega^2 y + p\right) y\, dx = 0$$

Classification

what is the relationship between 'week form' and 'Variational formulation'?

Our reference to certain weak forms of boundary-value problems as "variational" statements arises from the fact that, whenever the operators involved possess <u>a certain symmetry</u>, a weak form of the problem can be obtained which is precisely that arising in standard problems in the calculus of variations.

1) Becker E.B., Finite Elements An Introduction, Volume1, Prentice-Hall, 1981

Classification

$$(Au,u)=\int_a^b Au'^2dx$$

what is the relationship between 'week form' and 'Variational formulation'?

Remark 14.1[1] In the case of positive definite operators, the Galerkin method brings nothing new in comparison with the Ritz method ; the two methods lead to the solution of identical systems of linear equations and to identical sequences of approximate solution. However, the possibility of the application of the Galerkin method is much broader that that of the Ritz method.

The Galerkin method, which is characterized by the condition $(Au_n - f, \varphi_k)=0, \quad k=1,\dots,n$ does not impose beforehand any essential restrictive conditions on operator A

It is in no way necessary that the operator A be positive definite, it need not even be symmetric, above all it need not ne linear. Formally speaking, the Galerkin method can thus applied even in the case of very general operators

Remark 14.2· Although both the Ritz and the Galerkin methods lead to the <u>same results in the case of linear positive operators</u>, the basic ideas of these methods are entirely different.

ex: deflection of beam

$\left[EIu''\right]''=q(x)$ with the B/C $u(0)=u'(0)=0,\ u(l)=u'(l)=0,$

or

minimize the function of energy

$$\frac{1}{2}\int_0^l EI(u'')^2dx-\int_0^l q\,u\,dx$$

Definition 8.15[2] An operator A is called positive in its domain D_A if it is symmetric and if for all $u \in D_A$, the relations

$(Au,u)\geq 0$ and $(Au,u)=0 \Rightarrow u=0$ in D_A hold.

If, moreover, there exists a constant $C>0$ such that for all $u \in D_A$ the relation $(Au,u)\geq C^2\|u\|^2$ holds,

then the operator A is called *positive definite* in D_A

1) Rectorys, K. Variational Methods in Mathematics, Science and Engineering, Second edition, D.Reidel Publishing, 1980, p163
2) Rectorys, K. Variational Methods in Mathematics, Science and Engineering, Second edition, D.Reidel Publishing, 1980, p106

Classification

what is the relationship between 'week form' and 'Variational formulation'?

Remark 14.2. Although both the Ritz and the Galerkin methods lead to the same results in the case of linear positive operators, the basic ideas of these methods are entirely different.

ex: deflection of beam

$\left[EIu''\right]'' = q(x)$ with the B/C $u(0) = u'(0) = 0,\ u(l) = u'(l) = 0,$

or

minimize the function of energy

$$\frac{1}{2}\int_0^l EI(u'')^2 dx - \int_0^l q\,u\,dx$$

approximate solution $u_n = \sum_{k=1}^{n} a_i\varphi_i,$ where φ_i satisfy the B/C

Ritz method

Galerkin method : multiply φ and integration

$$a_1\int_0^l \left(EI\varphi_1''\right)''\varphi_1\,dx + a_2\int_0^l \left(EI\varphi_2''\right)''\varphi_1\,dx + \cdots + a_n\int_0^l \left(EI\varphi_n''\right)''\varphi_1\,dx = \int_0^l q\,\varphi_1\,dx$$

$$a_1\int_0^l \left(EI\varphi_1''\right)''\varphi_2\,dx + a_2\int_0^l \left(EI\varphi_2''\right)''\varphi_2\,dx + \cdots + a_n\int_0^l \left(EI\varphi_n''\right)''\varphi_2\,dx = \int_0^l q\,\varphi_2\,dx$$

$$\vdots$$

$$a_1\int_0^l \left(EI\varphi_1''\right)''\varphi_n\,dx + a_2\int_0^l \left(EI\varphi_2''\right)''\varphi_n\,dx + \cdots + a_n\int_0^l \left(EI\varphi_n''\right)''\varphi_n\,dx = \int_0^l q\,\varphi_n\,dx$$

different?

$$a_1\int_0^l EI\left(\varphi_1''\right)^2 dx + a_2\int_0^l EI\varphi_1''\varphi_2''\,dx + \cdots + a_n\int_0^l EI\varphi_1''\varphi_n''\,dx = \int_0^l q\,\varphi_1\,dx$$

$$a_1\int_0^l EI\varphi_1''\varphi_2''\,dx + a_2\int_0^l EI\left(\varphi_2''\right)^2 dx + \cdots + a_n\int_0^l EI\varphi_2''\varphi_n''\,dx = \int_0^l q\,\varphi_2\,dx$$

$$\vdots$$

$$a_1\int_0^l EI\varphi_1''\varphi_n''\,dx + a_2\int_0^l EI\varphi_2''\varphi_n''\,dx + \cdots + a_n\int_0^l EI\left(\varphi_n''\right)^2 dx = \int_0^l q\,\varphi_n\,dx$$

The Galerkin method starting with the differential equation of the problem and the Ritz method with the respective functional

$$\int_0^l \left(EI\varphi_i''\right)''\varphi_k\,dx = \left[\left(EI\varphi_i''\right)'\varphi_k\right]_0^l - \int_0^l \left(EI\varphi_i''\right)'\varphi_k'\,dx = -\left[\left(EI\varphi_i''\right)\varphi_k'\right] + \int_0^l \left(EI\varphi_i''\right)\varphi_k''\,dx$$

$$= \int_0^l \left(EI\varphi_i''\right)\varphi_k''\,dx$$

identical

1) Rectorys, K. Variational Methods in Mathematics, Science and Engineering, Second edition, D.Reidel Publishing, 1980, p163

Self-adjoint form

differential equation

Ex.)

$$\frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2 y+p=0$$

multiply δy and integrate

$$\int_0^l\left(\frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2 y+p\right)\delta y\,dx$$

$$\int_0^l\frac{d}{dx}\left(T\frac{dy}{dx}\right)\delta y\,dx+\int_0^l\rho\omega^2 y\,\delta y\,dx+\int_0^l p\,\delta y\,dx=0$$

integrating by part
and two end conditions

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2 y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx+\left[T\frac{dy}{dx}\delta y\right]_0^l=0$$

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2 y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx=0$$

variational problem

then the differential equation is obtained by the Euler equation

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$$

$$\text{let } F=\frac{1}{2}\rho\omega^2 y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2$$

Can this procedure be used for any differential equations?

$$\int_0^l (D.E)\delta y\,dx=0 \Longrightarrow \delta I=0$$

while the technique of forming the left directly to the right is a particularly convenient one, and certainly is appropriate in this case, its use in other situation may be less well motivated unless it is verified that the differential equation involved is indeed the Euler equations of some variational problem, $\delta I=0$ whose natural boundary conditions include those which govern the problem at hand*

*Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p184

Self-adjoint form

Can this procedure be used for any differential equations? $\int_0^l (D.E)\delta y\,dx = 0 \Longleftrightarrow \delta I = 0$

while the technique of forming the left directly to the right is a particularly convenient one, and certainly is appropriate in this case, its use in other situation may be less well motivated unless it is verified that the differential equation involved is indeed the Euler equations of some variational problem, $\delta I = 0$ whose natural boundary conditions include those which govern the problem at hand*

Ex.) $(x^2y')' + xy = x \quad ,0 \le x \le l \xrightarrow{\text{equivalent equation}} x^2y'' + 2xy' + xy = x$

$\Rightarrow xy'' + 2y' + y = 1$

multiply δy and integrate

$$\int_0^l \left[(x^2y')' + xy - x\right]\delta y\,dx$$

$$\delta\int_0^l \left(\frac{1}{2}x^2(y')^2 + \frac{1}{2}xy^2 - xy\right)dx + \left[x^2\,y'\,\delta y\right]_0^l = 0$$

if the specified boundary conditions are such that

$$\left[x^2\,y'\,\delta y\right]_0^l = 0$$

the variational problem

$$\delta\int_0^l \left(\frac{1}{2}x^2(y')^2 + \frac{1}{2}xy^2 - xy\right)dx = 0$$

or, after calculating the variation

$$\int_0^l \left[(x^2y')' + xy - x\right]\delta y\,dx$$

multiply δy and integrate

$$\int_0^l \left(xy'' + 2y' + y - 1\right)\delta y\,dx$$

this form cannot be transformed to a proper variational problem, $\delta I = 0$, merely by multiplication by δy and subsequent integration by part

introducing *weighting function x*

In the case of a D.E. of order greater than two, it may happen that no such weighting function exists.

However, it is readily verified that the abbreviated procedure is valid (when appropriate boundary conditions are prescribed) if the governing equation is of the so-called self-adjoint form

$$\mathscr{L}y \equiv (py')' + qy = f \quad , p, q : \text{function of } x \text{ or const.}$$

That is such an equation is the Euler equation of a proper variational problem, $\delta I = 0$ which is equivalent to the condition

$$\int_{x_1}^{x_2} \left(\mathscr{L}y - f\right)\delta y\,dx \ .$$

Self-adjoint form*

nonhomogeneous boundary value problem

$$A_0(x)\frac{d^2y}{dx^2}+A_1(x)\frac{dy}{dx}+A_2(x)y\equiv Ly=f(x)$$

$$\alpha_1 y(a)+\alpha_2 y'(a)=0$$

$$\beta_1 y(b)+\beta_2 y'(b)=0$$

the self-adjoint form, which means $\left(v\,Lu-u\,Lv\right)dx$ **must be exact differential**

$dg=\left(v\,Lu-u\,Lv\right)dx$ **for any two functions** $u(x)$ **and** $v(x)$ **operated on by** L

A very important example in applied mathematics if a self-adjoint differential operator is

$$Lu\equiv\frac{d}{dx}[r(x)\frac{du}{dx}]+[q(x)+\lambda p(x)]u=0$$

which is used with well-known Sturm-Liouville problem

$$Lu\equiv\frac{d}{dx}[r(x)\frac{du}{dx}]+[q(x)+\lambda p(x)]u=0$$

Subject to: $\alpha_1 y(a)+\alpha_2 y'(a)=0$

$\beta_1 y(b)+\beta_2 y'(b)=0$

*Jerry A.J., Introduction to Integral Equations with Applications, Marcel Dekker, 1985, p96-97

Self-adjoint form*

the self-adjoint form, which means $(v\,Lu - u\,Lv)dx$ must be exact differential

$dg = (v\,Lu - u\,Lv)dx$ for any two functions $u(x)$ and $v(x)$ operated on by L

$$Lu \equiv \frac{d}{dx}[r(x)\frac{du}{dx}] + [q(x) + \lambda p(x)]u = 0 \quad \longleftarrow \quad A_0(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_2(x)y \equiv Ly = f(x)$$

self-adjoint form

$$vLu - uLv = v\frac{d}{dx}[r(x)u'] + v[q(x) + \lambda p(x)]u - u\frac{d}{dx}[r(x)v'] + u[q(x) + \lambda p(x)]v$$

$$= v\frac{d}{dx}[r(x)u'] - u\frac{d}{dx}[r(x)v']$$

$$= vr'u' + vru'' - ur'v' - urv''$$

$$= r(vu'' - uv'') + r'(vu' - uv')$$

$$= r(vu'' - uv'' - v'u' + v'u') + r'(vu' - uv')$$

$$= r(v'u' + vu'' - u'v' - uv'') + r'(vu' - uv')$$

$$= \frac{d}{dx}\left[r(vu' - uv')\right]$$

$$vLu - uLv = \frac{d}{dx}\left[r(vu' - uv')\right]$$

$$\therefore (vLu - uLv)dx = dg, \quad g = \left[r(vu' - uv')\right]$$

Note that the governing differential equation in linear second-order problems can be written compactly in the operator form[1)]

$$Au = f$$

Where A is the differential operator for the problem.

If u and v are arbitrary smooth functions vanishing at *x=0, x=l*, the operator A is said to be formally self-adjoint whenever

$$\int_0^l v\,Au\,dx = \int_0^l u\,Av\,dx$$

it can be shown that an energy functional exists for a given boundary-value problem only when the operator A for the problem is self-adjoint. For self-adjoint problems and, therefore, for all problems derivable from an energy functional and the stiffness matrix resulting from the Rits approximation will always be symmetric.

Clearly, when Ritz method is applicable, it leads the the same system of equations as Galerkin method.

In the general case, the operator was not self-adjoint. For this reason, it is clear that Galerkin method is applicable to a wider class od problems that is Rits method

*Jerry A.J., Introduction to Integral Equations with Applications, Marcel Dekker, 1985, p96-97
1) Becker, E.B., et al, Finite Elements An Introduction, Volume 1, Prentice-Hall, p63 , see also Greenburg, M.D., Application of Green's Functions in Science and Engineering, p6-9 1971, p6-9

Summary : Variational Problem and D.E.

solution of D.E.

solution of Integral Equation

Green Function

Approximation
-Galerkin /Collocation /Least Square

Differential Equation

Ex.)

$$\frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2 y+p=0$$

multiply δy and integrate

$$\int_0^l\left(\frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2 y+p\right)\delta y\,dx$$

$$\int_0^l\frac{d}{dx}\left(T\frac{dy}{dx}\right)\delta y\,dx+\int_0^l\rho\omega^2 y\,\delta y\,dx+\int_0^l p\,\delta y\,dx=0$$

integrating by part
and two end conditions

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx+\left[T\frac{dy}{dx}\delta y\right]_0^l=0$$

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx=0$$

Variational formulation

then the differential equation is obtained by the Euler equation

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$$

let $F=\frac{1}{2}\rho\omega^2y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2$

Approximation
- Rayleigh-Ritz

Variational problems for deformable bodies

Calculus of variation

Calculus of variation are concerned chiefly with the determination of maxima and minima of certain expression involving unknown functions*

Variational problems for deformable bodied

a variational problem can be derived from a differential equation and the associated boundary conditions…such formulations are readily adapted to approximate analysis**

*Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p119
**Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p172

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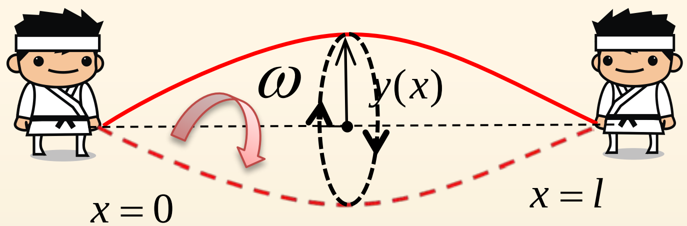
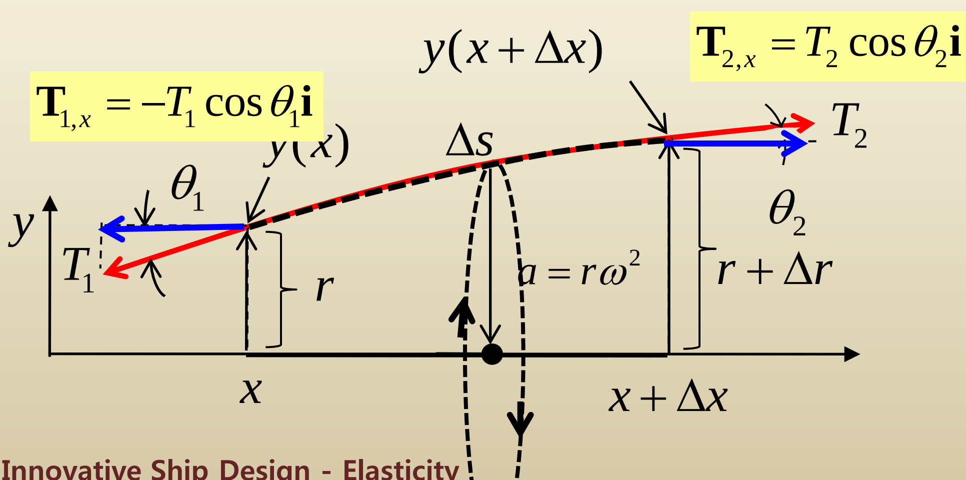
Example : the problem of determining small deflection s of a rotating string

❖ **derivation of the differential equation***



ρ : string density
ω : string angular velocity
T : magnitude of tension



force equilibrium in x-direction

$$\sum \mathbf{F}_x = 0$$

$$\sum \mathbf{F}_x = \mathbf{T}_{1,x} + \mathbf{T}_{2,x}$$

$$= T_1\cos(\pi+\theta_1)\mathbf{i} + T_2\cos\theta_2\mathbf{i}$$

$$= -T_1\cos\theta_1\mathbf{i} + T_2\cos\theta_2\mathbf{i}$$

$$= 0$$

$$\therefore T_1\cos\theta_1 = T_2\cos\theta_2$$

$$\text{let } T_1\cos\theta_1 = T_2\cos\theta_2 = T$$

$$\text{then } T_1 = \frac{T}{\cos\theta_1},\quad T_2 = \frac{T}{\cos\theta_2}$$

*Zill,D.G, Cullen,M.R., Advanced Engineering Mathematics, Third Edition, Jones and Bartlett, 2006, p107

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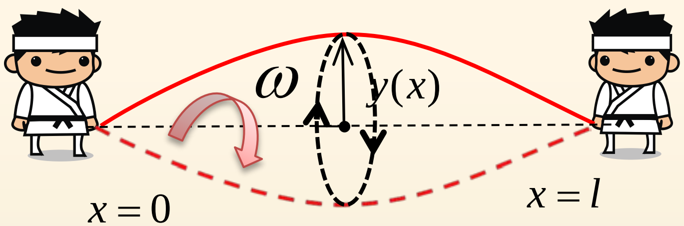

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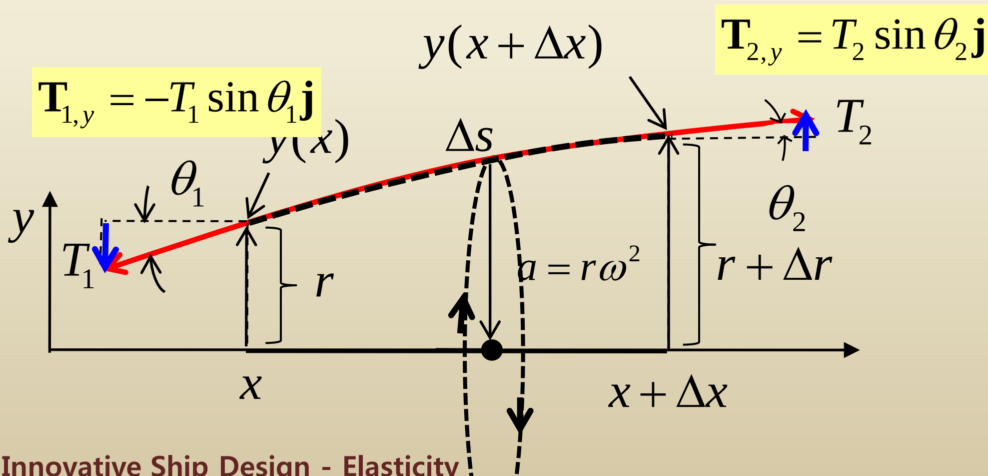

force equilibrium in x-direction

$$T_1=\frac{T}{\cos\theta_1},\quad T_2=\frac{T}{\cos\theta_2}$$

resultant force in y-direction

$$\sum\mathbf{F}_y=\mathbf{T}_{1,y}+\mathbf{T}_{2,y}$$

$$=T_1\sin(\pi+\theta_1)\mathbf{j}+T_2\sin\theta_2\mathbf{j}$$

$$=-T_1\sin\theta_1\mathbf{j}+T_2\sin\theta_2\mathbf{j}$$

$$=-T\frac{\sin\theta_1}{\cos\theta_1}\mathbf{j}+T\frac{\sin\theta_2}{\cos\theta_2}\mathbf{j}$$

$$=-T\tan\theta_1\mathbf{j}+T\tan\theta_2\mathbf{j}$$

$$\because\tan\theta_1=\left.\frac{dy}{dx}\right|_x,\ \tan\theta_2=\left.\frac{dy}{dx}\right|_{x+\Delta x}$$

$$\sum\mathbf{F}_y=T[y'(x+\Delta x)-y'(x)]\mathbf{j}$$

*Zill,D.G, Cullen,M.R., Advanced Engineering Mathematics, Third Edition, Jones and Bartlett, 2006, p107

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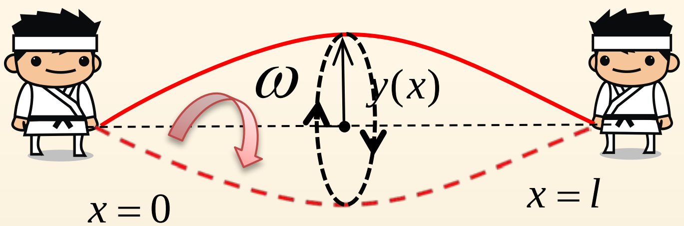

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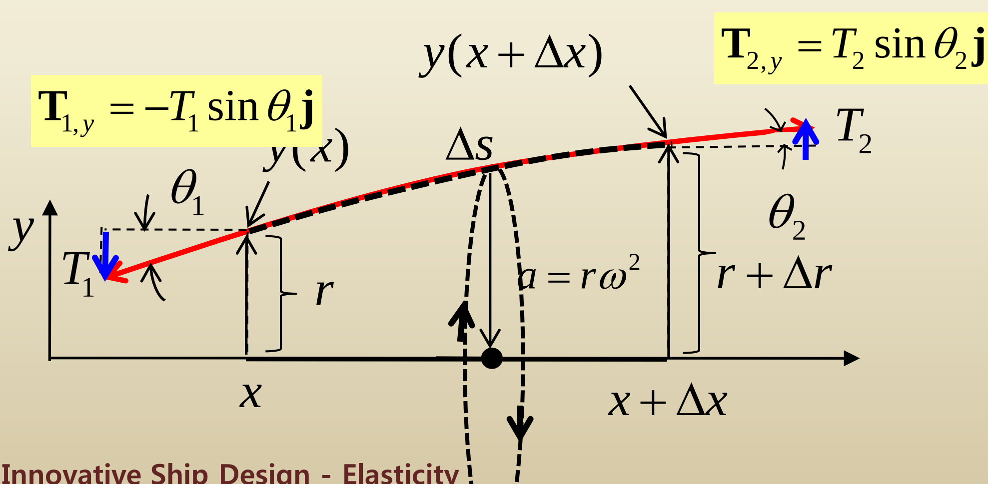

resultant force in y-direction

$$\sum \mathbf{F}_y = T[y'(x+\Delta x)-y'(x)]\mathbf{j}$$

acceleration in y-direction

assum.: $\Delta x \ll 1$

Mass: $m=\rho\Delta s \approx \rho\Delta x$

Centripetal acceleration: $a_y=-r\omega^2$

negative sign : acceleration points in the direction opposite to the positive y direction

When Δx is small,

linearization

$$\boxed{r+\Delta r \approx r} = y$$

$$\therefore a_y = -y\omega^2$$

$$\therefore ma_y \approx -(\rho\Delta x)y\omega^2$$

*Zill,D.G, Cullen,M.R., Advanced Engineering Mathematics, Third Edition, Jones and Bartlett, 2006, p107

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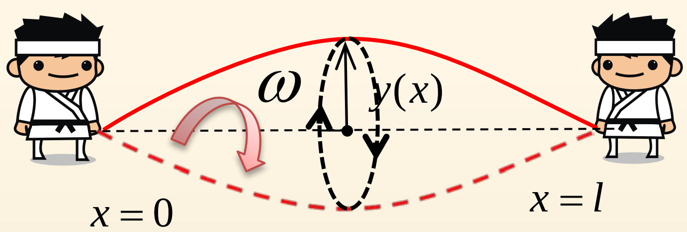

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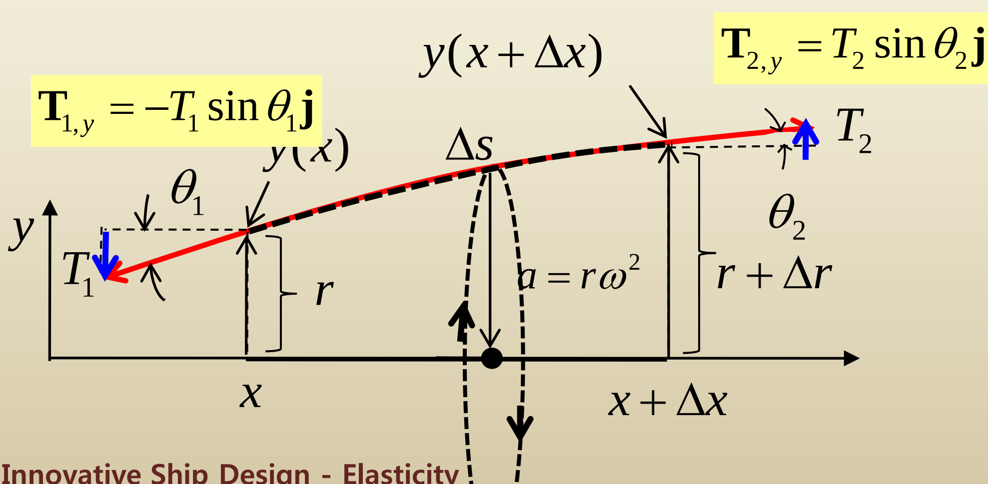


- **resultant force in y-direction**

$$\sum \mathbf{F}_y = T[y'(x+\Delta x) - y'(x)]\mathbf{j}$$

- **acceleration in y-direction**

$$ma_y = -\rho \cdot \Delta x \cdot y \cdot \omega^2$$

- **Newton's 2nd law** $\sum \mathbf{F}_y = ma_y \mathbf{j}$

$$T[y'(x+\Delta x) - y'(x)] = -\rho \cdot \Delta x \cdot y \cdot \omega^2$$

$$T\frac{y'(x+\Delta x) - y'(x)}{\Delta x} + \rho\omega^2 y = 0$$

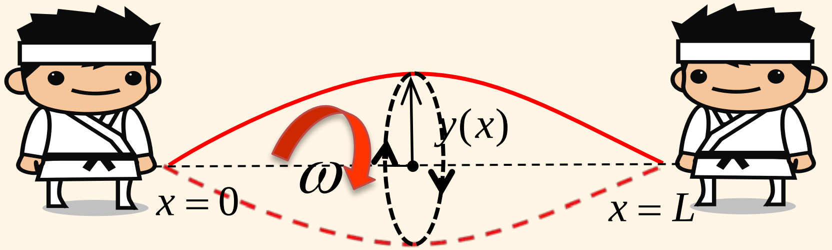
When Δx is small, $\dfrac{y'(x+\Delta x) - y'(x)}{\Delta x} \approx \dfrac{d^2 y}{dx^2}$

$$\therefore T\frac{d^2 y}{dx^2} + \rho\omega^2 y = 0$$

*Zill,D.G, Cullen,M.R., Advanced Engineering Mathematics, Third Edition, Jones and Bartlett, 2006, p107

Physical Meaning of Green Function

Ex) Rotating String



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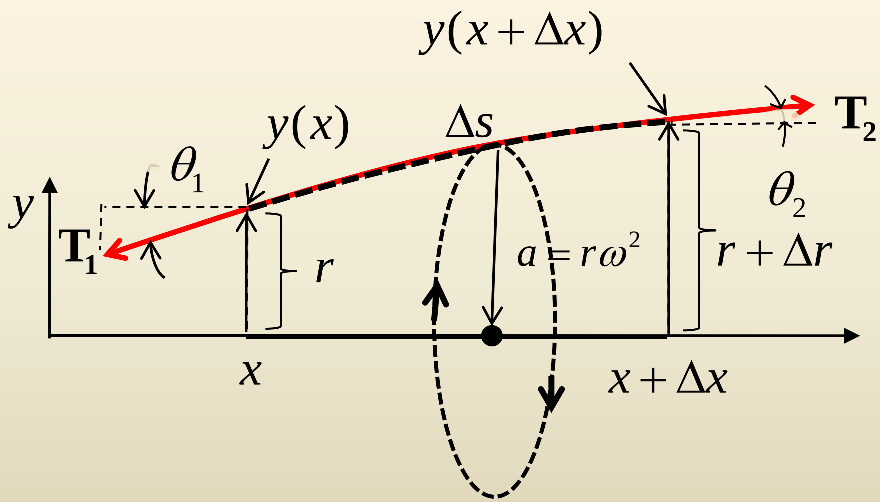

$$\therefore T\frac{d^2y}{dx^2}+\rho\omega^2 y=0$$

$$\boxed{\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)=0,\ y(l)=0}$$

$$\therefore y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi \quad ,K(x,\xi)=\begin{cases}\frac{\xi}{l}(l-x) & when\ \xi<x\\ \frac{x}{l}(l-\xi) & when\ \xi>x\end{cases}$$

Green function

displacement can be occurred with no external force and homogeneous B/C ?

in this example, string's angular velocity are causing the displacement. If tension is zero, this equation is not valid. With non zero tension, displacement is affected by the string's angular velocity and in the equation it is λ .

Even in the case of homogeneous B/C and no external force (actually, it means the nonhomogeneous term in the equation), there could be 'a source' causing 'motion' of the system in the equation *

* this statement is just writer's note, sea also Morse, P.M, Feshbach, H, Methods of theoretical Physics, McGraw-Hill, 1953, p791~p793

Variational problems for deformable bodies

a variational problem can be derived from a differential equation and the associated boundary conditions…such formulations are readily adapted to approximate analysis
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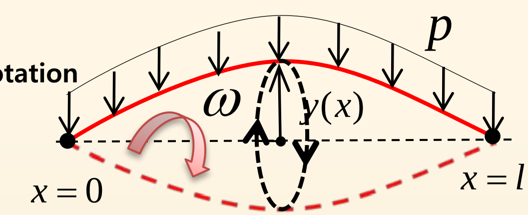
Example : the problem of determining small deflection s of a rotating string

❖The governing differential equation

$$\therefore \frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2 y+p=0$$

y : the displacement of a point from the axis of rotation
ρ : linear mass density
ω : string angular velocity
T : magnitude of tension
p : intensity of a distributed radial load



Recall, *

▪Regular Sturm-Liouville Problem B.V.P

Solve : $$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$$

Subject to: $$A_1 y(a)+B_1 y'(a)=0$$
$$A_2 y(b)+B_2 y'(b)=0$$

p, q, r, r' real-valued functions continuous on an interval $[a,b]$

$r(x)>0,\ p(x)>0$

for every x in the interval $[a,b]$

A_1, B_1 are not both zero

A_2, B_2 are not both zero

*Zill, D.G., Cullen, M.R., Advanced Engineering Mathematics, Third Edition, Jones and Bartlet, 2006, ch.12.5

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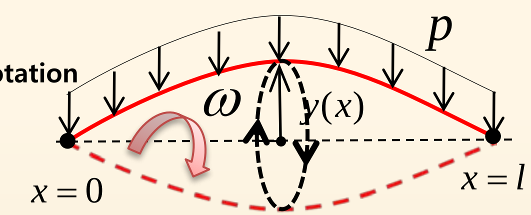

❖**in order to formulate a corresponding variational problem**

multiply δy **and integrate over** $(0,l)$

$$\int_0^l \frac{d}{dx}\left(T\frac{dy}{dx}\right)\delta y\,dx+\int_0^l \rho\omega^2 y\,\delta y\,dx+\int_0^l p\,\delta y\,dx=0$$

$$-\int_0^l \delta\left(\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right)dx+\int_0^l \delta\left(\frac{1}{2}\rho\omega^2 y^2\right)dx+\int_0^l \delta(py)\,dx=0$$

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2 y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx+\left[T\frac{dy}{dx}\delta y\right]_0^l=0$$

- $p\delta y=\delta(py)$
- $\rho\omega^2 y\delta y=\delta\left(\frac{1}{2}\rho\omega^2 y^2\right)$
- $\int_0^l \frac{d}{dx}\left(T\frac{dy}{dx}\right)\delta y\,dx=\left[T\frac{dy}{dx}\delta y\right]_0^l-\int_0^l T\frac{dy}{dx}\frac{d(\delta y)}{dx}dx$

$$=\left[T\frac{dy}{dx}\delta y\right]_0^l-\int_0^l T\frac{dy}{dx}\delta\frac{dy}{dx}dx$$

$$\because \frac{d}{dx}\delta y=\delta\frac{d}{dx}y$$

$$\int_0^l \frac{d}{dx}\left(T\frac{dy}{dx}\right)\delta y\,dx=\left[T\frac{dy}{dx}\delta y\right]_0^l-\int_0^l \delta\left(\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right)dx$$

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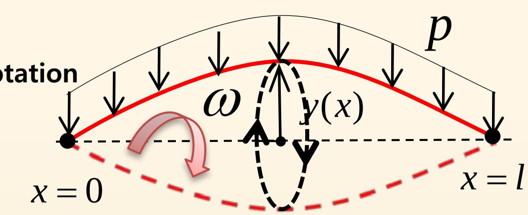

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2 y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx+\left[T\frac{dy}{dx}\delta y\right]_0^l=0$$

if we impose at each of the two ends one of the conditions

$$y=y_0 \quad or \quad T\frac{dy}{dx}=0 \qquad then \quad \left[T\frac{dy}{dx}\delta y\right]_0^l=0$$

$$\therefore \delta\int_0^l\left[\frac{1}{2}\rho\omega^2 y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx=0$$

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Example : the problem of determining small deflection s of a rotating string

❖The governing differential equation

$$\therefore \frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2 y+p=0\cdots(A)$$

❖ The variational form

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2 y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx=0\cdots(B)$$

$$y=y_0 \quad or \quad T\frac{dy}{dx}=0\cdots(B.1)$$

the solution are different ?

it must satisfy (A)	⇐	**if y renders the integral in (B) stationary**

conversely,

if y satisfies (A) and end conditions of the type required in (B.1)	⇒	**it renders the integral in (B) stationary**

the end conditions (B.1) or equivalently $\left[T\frac{dy}{dx}\delta y\right]_0^l=0$

are so-called *'natural boundary conditions'* of the variational problem (B)

y : the displacement of a point from the axis of rotation
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*Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p172

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Example : the problem of determining small deflection s of a rotating string

❖ The variational form (meaning of terms)

$$\delta\int_0^l \left[\frac{1}{2}\rho\omega^2 y^2 + py - \frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx = 0\cdots(B)$$

$$y = y_0 \quad or \quad T\frac{dy}{dx} = 0\cdots(B.1)$$

$$\delta\int_0^l \left[\underbrace{\frac{1}{2}\rho\omega^2 y^2}_{\text{kinetic energy of the string per unit length}} - \left(\underbrace{\frac{T}{2}\left(\frac{dy}{dx}\right)^2}_{\text{potential energy per unit length due to the tension in the string, to a first approximation}} \underbrace{- py}_{\text{potential energy per unit length due to the radial force}}\right)\right]dx = 0\cdots(B)$$

potential energy per unit length due to the radial force

potential energy per unit length due to the tension in the string, to a first approximation

kinetic energy of the string per unit length

$$T\frac{ds - dx}{dx} = T\left\{\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - 1\right\} \qquad \because ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

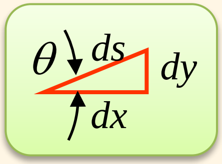
$$= T\left\{1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \cdots - 1\right\} \qquad \leftarrow \text{Taylor series}$$

$$\approx T\frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

*Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p172

Linearization

$$if \quad \theta \ll 1$$



$$ds^2 = dx^2 + dy^2 \rightarrow ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\text{let, } z = \left(\frac{dy}{dx}\right)^2 \text{ then, } \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+z}$$

$$f(z) = \sqrt{1+z}$$

$$f(0) = 1$$

$$f'(0) = \frac{1}{2}(1+z)^{-\frac{1}{2}}\Big|_{z=0} = \frac{1}{2}$$

$$f''(0) = -\frac{1}{4}(1+z)^{-\frac{3}{2}}\Big|_{z=0} = -\frac{1}{4}$$

$$\boxed{\therefore f(z) = 1 + \frac{1}{2}z + \frac{1}{2}\left(-\frac{1}{4}\right)z^2 + \cdots}$$

$$if, \theta \ll 1$$

$$\therefore f(z) = \sqrt{1+z} \approx 1 + \frac{1}{2}z$$

$$\therefore ds \approx 1 + \frac{1}{2}z = 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

The Rayleigh-Ritz method

The Rayleigh-Ritz method

general procedure for obtaining approximate solutions of problems expressed in variational form (Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p181)

In the case when a function y(x) is to be determined,

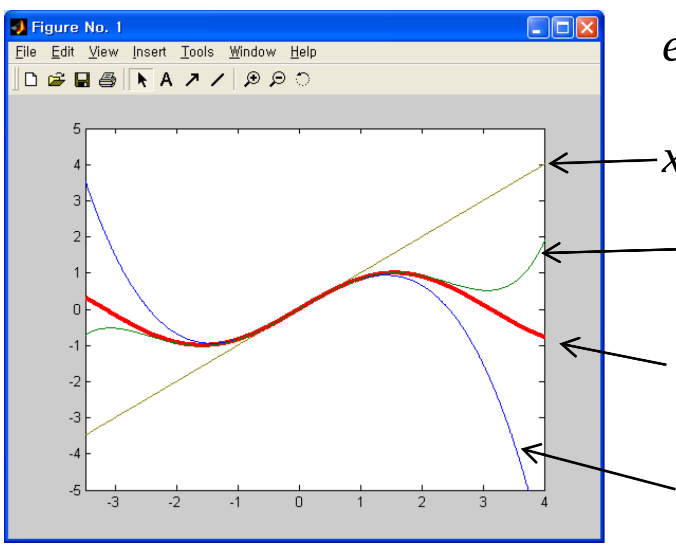
→ assuming that the desired stationary function of a given problem can be approximated by a linear combination of suitably chosen functions, of the form

$$y(x) \approx \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

where $\phi_k(x)\ , k = 0,1,\ldots n$: **(basis) functions chosen that y(x) satisfies the specified boundary conditions for any choice of the c's**

$c_k(x)\ , k = 0,1,\ldots n$: **constants to be determined**

Recall, *



$$ex^{*})\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

x

$x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sin x$

$x - \frac{x^3}{3!}$

*see also Larson R., Hostetler R.P., Edwards B.H., Calculus with Analytic Geometry, English Edition, Houghton Mifflin Company, 2006, p679 example 2

The Rayleigh-Ritz method

general procedure for obtaining approximate solutions of problems expressed in variational form (Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p181)

In the case when a function y(x) is to be determined,

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Example : the problem of determining small static $\omega = 0$ **deflections of a string fixed at the points** $(0,0)$ **and** (l,h) **,** $p(x) = -qx/l$ **and subject to a transverse loading**

corresponding variational problem

$$\omega = 0$$
$$p(x) = -qx/l$$

recall, the variational form

$$\delta\int_0^l \left[\frac{1}{2}Ty'^2 + q\frac{x}{l}y\right]dx = 0 \Leftarrow \delta\int_0^l \left[-q\frac{x}{l}y - \frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx = 0 \Leftarrow \delta\int_0^l \left[\frac{1}{2}\rho\omega^2 y^2 + py - \frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx = 0 \cdots (B)$$

$$y(0) = 0,\ y(l) = h$$

which means δy vanish at end points

$$y = y_0 \quad or \quad T\frac{dy}{dx} = 0 \cdots (B.1)$$

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where $\phi_k(x)\ , k = 0,1,\ldots n$ **: functions chosen that y(x) satisfies the specified boundary conditions for any choice of the c's**

$c_k(x)\ , k = 0,1,\ldots n$ **: constants to be determined**

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variational problem

$$\delta\int_0^l \left[\frac{1}{2}Ty'^2 + q\frac{x}{l}y\right]dx = 0 \quad , y(0) = 0,\ \ y(l) = h$$

The function $\phi_0(x)$ **then is to satisfy the end conditions, whereas the other coordinate functions** $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ **are to vanish at the both ends.**

If polynomials are to be used, for convenience, among the simplest choices, the functions are

$$\phi_0(x) = \frac{h}{l}x,\ \ \phi_1(x) = x(x-l),\ \ \phi_2(x) = x^2(x-l), \cdots, \phi_n(x) = x^n(x-l)$$

Which correspond to an approximation of the form

$$y(x) \approx \frac{h}{l}x + c_1x(x-l) + c_2x^2(x-l) + \cdots + c_nx^n(x-l)$$

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variational problem

$$\delta\int_0^l \left[\frac{1}{2}Ty'^2 + q\frac{x}{l}y\right]dx = 0 \quad , y(0) = 0,\ y(l) = h$$

$$y(x) \approx \frac{h}{l}x + c_1 x(x-l) + c_2 x^2(x-l) + \cdots + c_n x^n (x-l)$$

For simplicity, we consider here only the one-parameter approximation, n=1

$$y(x) \approx \frac{h}{l}x + c_1 x(x-l)$$

can it be the solution of the relevant differential equation?

$$Ty'' - q\frac{x}{l} = 0 \quad , y(0) = 0,\ y(l) = h$$

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variational problem

$$\delta\int_0^l \left[\frac{1}{2}Ty'^2 + q\frac{x}{l}y\right]dx = 0 \quad , y(0)=0,\ \ y(l) = h$$

replacing $\longleftarrow$ $y(x) \approx \dfrac{h}{l}x + c_1 x(x-l)$

$$\delta\int_0^l \left[\frac{1}{2}Ty'^2 + q\frac{x}{l}y\right]dx = \delta\int_0^l \left[\frac{1}{2}T\left(\frac{h}{l} + c_1(2x-l)\right)^2 + q\frac{x}{l}\left(\frac{h}{l}x + c_1 x(x-l)\right)\right]dx$$

The Rayleigh-Ritz method

variational problem

$$\delta\int_0^l\left[\frac{1}{2}Ty'^2+q\frac{x}{l}y\right]dx=0\quad ,y(0)=0,\ y(l)=h$$

$\xleftarrow{\text{replacing}}$ $y(x)\approx\dfrac{h}{l}x+c_1x(x-l)$

$$\delta\int_0^l\left[\frac{1}{2}Ty'^2+q\frac{x}{l}y\right]dx=\delta\int_0^l\left[\frac{1}{2}T\left(\frac{h}{l}+c_1(2x-l)\right)^2+q\frac{x}{l}\left(\frac{h}{l}x+c_1x(x-l)\right)\right]dx$$

$$=\delta\int_0^l\left[\frac{1}{2}T\left(\left(\frac{h}{l}\right)^2+2c_1\frac{h}{l}(2x-l)+c_1^2(4x^2-4xl+l^2)\right)+q\frac{1}{l}\left(\frac{h}{l}x^2+c_1(x^3-lx^2)\right)\right]dx$$

$$=\delta\left[\frac{1}{2}T\left(\left(\frac{h}{l}\right)^2x+2c_1\frac{h}{l}(x^2-lx)+c_1^2(\frac{4}{3}x^3-2x^2l+l^2x)\right)+q\frac{1}{l}\left(\frac{h}{l}\frac{1}{3}x^3+c_1(\frac{1}{4}x^4-\frac{l}{3}x^3)\right)\right]_0^l$$

$$=\delta\left(\frac{1}{2}T\left(\frac{h^2}{l}+c_1^2(\frac{4}{3}l^3-2l^3+l^3)\right)+q\frac{1}{l}\left(\frac{h}{l}\frac{1}{3}l^3+c_1(\frac{1}{4}l^4-\frac{1}{3}l^4)\right)\right)$$

$$=\delta\left(\frac{1}{2}T\left(\frac{h^2}{l}+\frac{1}{3}l^3c_1^2\right)+q\left(\frac{h}{2}l-c_1\frac{1}{12}l^3\right)\right)$$

$$=\left(\frac{1}{2}T\frac{1}{3}l^3 2c_1\delta c_1-q\frac{1}{12}l^3\delta c_1\right)\quad \text{,since here only } c_1 \text{ is varied}$$

$$=l^3\left(T\frac{1}{3}c_1-q\frac{1}{12}\right)\delta c_1\text{ , since } c_1 \text{ is arbitrary}$$

$$\therefore c_1=\frac{q}{4T}$$

The Rayleigh-Ritz method

general procedure for obtaining approximate solutions of problems expressed in variational form (Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p181)

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variational problem

$$\delta\int_0^l \left[\frac{1}{2}Ty'^2 + q\frac{x}{l}y\right]dx = 0 \quad , y(0)=0,\ y(l) = h \xleftarrow{\text{replacing}} y(x) \approx \frac{h}{l}x + c_1 x(x-l)$$

$$\downarrow$$

$$c_1 = \frac{q}{4T}$$

$$\therefore y(x) = \frac{h}{l}x + \frac{q}{4T}x(x-l)$$

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Example : the problem of determining small static $\omega = 0$ **deflections of a string fixed at the points** $(0,0)$ **and** (l,h) **,** $p(x) = -qx/l$ **and subject to a transverse loading**

variational problem

$$\delta \int_0^l \left[\frac{1}{2} T y'^2 + q \frac{x}{l} y \right] dx = 0 \quad , y(0) = 0, \ \ y(l) = h$$

$$y(x) \approx \frac{h}{l} x + c_1 x(x-l) \quad \Longrightarrow \quad y(x) = \frac{h}{l} x + \frac{q}{4T} x(x-l)$$

(approximation)

what about the differential equation?

let $F = \frac{1}{2} T y'^2 + q \frac{x}{l} y$ **then, by the Euler equation** $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y'} F \right) - \frac{\partial}{\partial y} F = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y'}\left(\frac{1}{2} T y'^2 + q \frac{x}{l} y \right) \right) - \frac{\partial}{\partial y}\left(\frac{1}{2} T y'^2 + q \frac{x}{l} y \right) = 0$$

$$\frac{\partial}{\partial x}(T y') - q \frac{x}{l} = 0$$

$$T y'' - q \frac{x}{l} = 0$$

Solution of D.E. : Integration

Rayleigh-Ritz solution: $y(x)=\frac{h}{l}x+\frac{q}{4T}x(x-l)$

$$Ty''-q\frac{x}{l}=0 \qquad , y(0)=0,\ \ y(l)=h$$

Integration twice

$$y''=q\frac{x}{Tl}$$

$$y'=\frac{q}{2Tl}x^2+C$$

$$y=\frac{q}{6Tl}x^3+Cx+D$$

$$y(0)=0$$
$$y(0)=D$$
$$\therefore D=0$$

$$y(l)=h$$
$$y(l)=\frac{q}{6T}l^2+Cl$$
$$\frac{q}{6T}l^2+Cl=h$$
$$\therefore C=\frac{h}{l}-\frac{q}{6T}l$$

$$\therefore y=\frac{q}{6Tl}x^3+\frac{h}{l}x-\frac{q}{6T}lx$$

$$\rightarrow \boxed{y=\frac{h}{l}x+\frac{q}{6Tl}x(x^2-l^2)}$$

Solution of D.E. : Integration

Rayleigh-Ritz solution: $y(x)=\frac{h}{l}x+\frac{q}{4T}x(x-l)$

$$Ty''-q\frac{x}{l}=0 \qquad , y(0)=0,\ y(l)=h$$

Homogeneous Solution

$y''=0$

let $y=e^{\lambda x}$

then

$\lambda^2 T e^{\lambda x}=0$

$\therefore \lambda^2=0$ (double roots)

$y_1=C$

$y_2=Cx$

$\therefore y_h=C_1C+C_2Cx$

$=c_1+c_2x$

Nonhomogeneous Solution

$y''=\frac{q}{Tl}x$

$try\ \ y=A+Bx$

it is associate with y_h

then, Let's assume particular solution as

$y=Ax+Bx^2$

it is also associate with y_h

So, Let's assume particular solution as

$y=Ax^2+Bx^3$

$y''=2A+6Bx$

$A=0$

$B=\frac{q}{6Tl}$

$\therefore y=y_h+y_p=c_1+c_2x+\frac{q}{6Tl}x^3$

$y(0)=c_1=0$

$y(l)=c_2l+\frac{q}{6T}l^2=h$

$\therefore c_2=\frac{h}{l}-\frac{q}{6T}l$

$\therefore y=(\frac{h}{l}-\frac{q}{6T}l)x+\frac{q}{6Tl}x^3$

$or\ \ y=\frac{h}{l}x+\frac{q}{6Tl}x(x^2-l^2)$

Solution of D.E. : series solution

Rayleigh-Ritz solution: $y(x)=\frac{h}{l}x+\frac{q}{4T}x(x-l)$

D.E : $y=\frac{h}{l}x+\frac{q}{6Tl}x(x^2-l^2)$

$$Ty''-q\frac{x}{l}=0 \quad ,y(0)=0,\ y(l)=h$$

assume $$y(x)=\sum_{n=0}^{\infty}c_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}n\cdot c_n x^{n-1}$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)\cdot c_n x^{n-2}$$

$$T\sum_{n=2}^{\infty}n(n-1)\cdot c_n x^{n-2}-\frac{q}{l}x=0$$

$$T(2\cdot 1\cdot c_2 x^0+3\cdot 2\cdot c_3 x^1+4\cdot 3\cdot c_4 x^2+\cdots)-\frac{q}{l}x=0$$

$$(T\cdot 2\cdot 1\cdot c_2)x^0+\left(T\cdot 3\cdot 2\cdot c_3-\frac{q}{l}\right)x^1+(T\cdot 4\cdot 3\cdot c_4)x^2+\cdots=0$$

$$c_2=0, c_4=0, c_5=0,\cdots \quad ,c_3=\frac{q}{6Tl}$$

$$y(0)=0$$
$$y(0)=c_0$$
$$\therefore c_0=0$$

$$y(l)=h$$
$$y(l)=\frac{q}{6Tl}l^3+c_1 l$$
$$\frac{q}{6T}l^2+c_1 l=h$$
$$\therefore c_1=\frac{h}{l}-\frac{q}{6T}l$$

$$\rightarrow y(x)=c_0x^0+c_1x^1+\frac{q}{6Tl}x^3$$

$$\therefore y=\left(\frac{h}{l}-\frac{q}{6T}l\right)x+\frac{q}{6Tl}x^3$$

$$=\frac{h}{l}x+\frac{q}{6Tl}x(x^2-l^2)$$

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general procedure for obtaining approximate solutions of problems expressed in variational form (Hildebrand, F.B., Methods of Applied Mathematics, second edition, Dover, 1965, p181)

Example : the problem of determining small static $\omega = 0$ **deflections of a string fixed at the points** $(0,0)$ **and** (l,h) **,** $p(x) = -qx/l$ **and subject to a transverse loading**

Variational problem

$$\delta \int_0^l \left[\frac{1}{2} T y'^2 + q \frac{x}{l} y \right] dx = 0 \quad , y(0) = 0, \ y(l) = h$$

$$y(x) \approx \frac{h}{l} x + c_1 x(x-l) \implies y(x) = \frac{h}{l} x + \frac{q}{4T} x(x-l)$$

(approximation)

Differential Equation

$$Ty'' - q\frac{x}{l} = 0 \qquad , y(0) = 0, \ y(l) = h$$

(exact solution)

$$y = \frac{h}{l} x + \frac{q}{6Tl} x(x^2 - l^2)$$

what is difference or common?

what happens if we use two-parameter approximation?

The Rayleigh-Ritz method

Differential Equation

$$Ty'' - q\frac{x}{l} = 0 \quad , y(0)=0,\ y(l)=h$$

(exact solution)

$$y(x) = \left(\frac{h}{l} - \frac{q}{6T}l\right)x + \frac{q}{6Tl}x^3$$

$$y(0)=0$$

$$y(l)=0$$

$$y\left(\frac{1}{2}l\right) = \frac{h}{l}\frac{l}{2} + \frac{q}{6Tl}\frac{l}{2}\left(\frac{l^2}{4} - l^2\right)$$

$$= \frac{h}{2} - \frac{q}{6T}\frac{1}{2}\frac{3l^2}{4} = \frac{h}{2} - \frac{ql^2}{16T}$$

basis functions (from characteristic equation)

$$1, \quad x$$

which satisfy the homogeneous equation

$$y'' = 0$$

Variational problem

$$\delta\int_0^l \left[\frac{1}{2}Ty'^2 + q\frac{x}{l}y\right]dx = 0 \quad , y(0)=0,\ y(l)=h$$

(approximation solution)

$$y(x) = \frac{h}{l}x + \frac{q}{4T}x(x-l)$$

$$y(0)=0$$

$$y(l)=0$$

$$y\left(\frac{1}{2}l\right) = \frac{h}{2} - \frac{ql^2}{16T}$$

basis functions (with assumed coefficient)

$$\frac{h}{l}x, \quad x(x-l)$$

which satisfy the boundary conditions

$$, y(0)=0,\ y(l)=h$$

same values at

$$x = 0, l, \frac{1}{2}l$$

The Rayleigh-Ritz method

variational problem $\delta\int_0^l\left[\frac{1}{2}Ty'^2+q\frac{x}{l}y\right]dx=0\quad,y(0)=0,\ \ y(l)=h$

replacing

$$\longleftarrow\ y(x)\approx\frac{h}{l}x+c_1x(x-l)+c_2x^2(x-l)$$

$$\delta\int_0^l\left[\frac{1}{2}Ty'^2+q\frac{x}{l}y\right]dx=\delta\int_0^l\left[\frac{1}{2}T\left(\frac{h}{l}+c_1(2x-l)+c_2x(3x-2l)\right)^2+q\frac{x}{l}\left(\frac{h}{l}x+c_1x(x-l)+c_2x^2(x-l)\right)\right]dx$$

$$=\delta\int_0^l\left[\frac{1}{2}T\left(\left(\frac{h}{l}\right)^2+c_1^2(4x^2-4xl+l^2)+c_2{}^2(9x^4-12x^3l+4l^2x^2)+2c_1\frac{h}{l}(2x-l)+2c_1c_2(6x^3-7lx^2+2l^2x)+2c_2\frac{h}{l}(3x^2-2lx)\right)+q\frac{1}{l}\left(\frac{h}{l}x^2+c_1(x^3-lx^2)+c_2(x^4-lx^3)\right)\right]dx$$

$$=\delta\left[\frac{1}{2}T\left(\left(\frac{h}{l}\right)^2x+c_1^2(\frac{4}{3}x^3-2x^2l+l^2x)+c_2{}^2(\frac{9}{5}x^5-3x^4l+\frac{4}{3}l^2x^3)+2c_1\frac{h}{l}(x^2-lx)+2c_1c_2(\frac{6}{4}x^4-\frac{7}{3}lx^3+l^2x^2)+2c_2\frac{h}{l}(x^3-lx^2)\right)+q\frac{1}{l}\left(\frac{h}{l}\frac{x^3}{3}+c_1(\frac{x^4}{4}-l\frac{x^3}{3})+c_2(\frac{x^5}{5}-l\frac{x^4}{4})\right)\right]_0^l$$

$$=\delta\left[\frac{1}{2}T\left(\left(\frac{h}{l}\right)^2l+c_1^2(\frac{4}{3}l^3-2l^3+l^3)+c_2{}^2(\frac{9}{5}l^5-3l^5+\frac{4}{3}l^5)+2c_1\frac{h}{l}(l^2-l^2)+2c_1c_2(\frac{6}{4}l^4-\frac{7}{3}l^4+l^4)+2c_2\frac{h}{l}(l^3-l^3)\right)+q\frac{1}{l}\left(h\frac{l^2}{3}+c_1(\frac{l^4}{4}-\frac{l^4}{3})+c_2(\frac{l^5}{5}-\frac{l^5}{4})\right)\right]$$

$$=\delta\left[\frac{1}{2}T\left(\frac{h^2}{l}+c_1^2\frac{1}{3}l^3+c_2{}^2\frac{2}{15}l^5+c_1c_2\frac{1}{3}l^4\right)+q\left(h\frac{l}{3}-c_1\frac{l^3}{12}-c_2\frac{l^4}{20}\right)\right]$$

$$=\left(c_1T\frac{1}{3}l^3\delta c_1+c_2T\frac{2}{15}l^5\delta c_2+c_2T\frac{1}{6}l^4\delta c_1+c_1T\frac{1}{6}l^4\delta c_2\right)+\left(-q\frac{l^3}{12}\delta c_1-q\frac{l^4}{20}\delta c_2\right),\text{since here only } c_1 \text{ and } c_2 \text{ are varied}$$

$$=l^3\left(c_1\frac{1}{3}T+c_2\frac{1}{6}lT-\frac{1}{12}q\right)\delta c_1+l^4\left(c_2\frac{2}{15}lT+c_1\frac{1}{6}T-\frac{1}{20}q\right)\delta c_2\quad,\text{since } c_1,c_2 \text{ are arbitrary}$$

$$=0$$

$$\therefore c_1\frac{1}{3}T+c_2\frac{1}{6}lT=\frac{1}{12}q$$

$$c_1\frac{1}{6}T+c_2\frac{2}{15}lT=\frac{1}{20}q,\qquad\qquad \therefore c_1=\frac{q}{6T},\ \ c_2=\frac{q}{6lT}$$

$$y(x)\approx\frac{h}{l}x+\frac{q}{6T}x(x-l)+\frac{q}{6lT}x^2(x-l)$$

The Abbreviated Procedure

differential equation

$$\frac{d}{dx}\left(T\frac{dy}{dx}\right)-\frac{qx}{l}=0 \Longrightarrow \int_0^l\left(\frac{d}{dx}\left(T\frac{dy}{dx}\right)-\frac{qx}{l}\right)\delta y\,dx \Longrightarrow \delta\int_0^l\left[\frac{1}{2}Ty'^2+q\frac{x}{l}y\right]dx=0 \quad ,y(0)=0,\ y(l)=h$$

variational problem

multiply δy and integrate

integrating by part
and two end conditions

approximation

$$y(x)\approx\frac{h}{l}x+c_1x(x-l)$$

approximation

$$y(x)\approx\frac{h}{l}x+c_1x(x-l)$$

variation and then integration
(the abbreviated procedure)

$$\int_0^l\left[\left(\frac{d}{dx}\left(T\left(\frac{h}{l}+c_1(2x-1)\right)\right)-\frac{qx}{l}\right)x(x-l)\delta c_1\right]dx=0$$

$$\Downarrow$$

$$\int_0^l\left[\left(2Tc_1-\frac{qx}{l}\right)x(x-l)\delta c_1\right]dx=0$$

$$\Downarrow$$

$$\delta c_1\int_0^l\left(2Tc_1(x^2-lx)-\frac{q}{l}(x^3-lx^2)\right)dx=0$$

$$\delta c_1\left(Tc_1(-\frac{1}{3}l^3)+\frac{q}{12}l^3\right)=0$$

$$\therefore c_1=\frac{q}{4T}$$

this procedure frequently involves a reduced amount of calculation

the abbreviated procedure

integration and then variation

$$\delta\int_0^l\left[\frac{1}{2}Ty'^2+q\frac{x}{l}y\right]dx=0$$

$$\delta\int_0^l\left[\frac{1}{2}Ty'^2+q\frac{x}{l}y\right]dx=\delta\int_0^l\left[\frac{1}{2}T\left(\frac{h}{l}+c_1(2x-l)\right)^2+q\frac{x}{l}\left(\frac{h}{l}x+c_1x(x-l)\right)\right]dx$$

$$=\delta\int_0^l\left[\frac{1}{2}T\left(\left(\frac{h}{l}\right)^2+2c_1\frac{h}{l}(2x-l)+c_1^2(4x^2-4xl+l^2)\right)+q\frac{1}{l}\left(\frac{h}{l}x^2+c_1(x^3-lx^2)\right)\right]dx$$

$$=\delta\left[\frac{1}{2}T\left(\left(\frac{h}{l}\right)^2x+2c_1\frac{h}{l}(x^2-lx)+c_1^2(\frac{4}{3}x^3-2x^2l+l^2x)\right)+q\frac{1}{l}\left(\frac{h}{l}\frac{1}{3}x^3+c_1(\frac{1}{4}x^4-\frac{l}{3}x^3)\right)\right]_0^l$$

$$=\delta\left(\frac{1}{2}T\left(\frac{h^2}{l}+c_1^2(\frac{4}{3}l^3-2l^3+l^3)\right)+q\frac{1}{l}\left(\frac{h}{l}\frac{1}{3}l^3+c_1(\frac{1}{4}l^4-\frac{1}{3}l^4)\right)\right)$$

$$=\delta\left(\frac{1}{2}T\left(\frac{h^2}{l}+\frac{1}{3}l^3c_1^2\right)+q\left(\frac{h}{2}l-c_1\frac{1}{12}l^3\right)\right)$$

$$=\left(\frac{1}{2}T\frac{1}{3}l^3 2c_1\delta c_1-q\frac{1}{12}l^3\delta c_1\right)\quad \text{,since here only } c_1 \text{ is varied}$$

$$=l^3\left(T\frac{1}{3}c_1-q\frac{1}{12}\right)\delta c_1\text{ , since } c_1 \text{ is arbitrary}$$

$$\therefore c_1=\frac{q}{4T}$$

The Abbreviated Procedure

$$\int_0^l \left(\frac{d}{dx}\left(T\frac{dy}{dx} \right) - \frac{qx}{l} \right) \delta y \, dx$$

approximation

$$y(x) \approx \frac{h}{l}x + c_1 x(x-l)$$

$$\frac{dy}{dx} = \frac{h}{l} + c_1(2x-1)$$

$$\delta y = x(x-l)\delta c_1$$

$$\int_0^l \left[\left(\frac{d}{dx}\left(T\left(\boxed{\frac{h}{l} + c_1(2x-1)} \right) \right) - \frac{qx}{l} \right) \boxed{x(x-l)\delta c_1} \right] dx = 0$$

$$\int_0^l \left[\left(2Tc_1 - \frac{qx}{l} \right) x(x-l)\delta c_1 \right] dx = 0$$

$$\int_0^l \left[\left(2Tc_1(x^2 - lx) - \frac{q}{l}(x^3 - lx^2) \right) \delta c_1 \right] dx = 0$$

$$\delta c_1 \int_0^l \left(2Tc_1(x^2 - lx) - \frac{q}{l}(x^3 - lx^2) \right) dx = 0$$

$$\delta c_1 \left[\left(2Tc_1\left(\frac{1}{3}x^3 - \frac{1}{2}lx^2\right) - \frac{q}{l}\left(\frac{1}{4}x^4 - \frac{l}{3}x^3\right) \right) \right]_0^1 = 0$$

$$\delta c_1 \left(2Tc_1\left(\frac{1}{3}l^3 - \frac{1}{2}l^3\right) - \frac{q}{l}\left(\frac{1}{4}l^4 - \frac{1}{3}l^3\right) \right) = 0$$

$$\delta c_1 \left(Tc_1\left(-\frac{1}{3}l^3\right) + \frac{q}{12}l^3 \right) = 0$$

$$\therefore c_1 = \frac{q}{4T}$$

Strain Energy

Strain Energy

When a body is deformed by external forces, work is done by these forces.

The energy absorbed in the body due to this external work is called *strain energy*.

If the body behaves elastically, the strain energy can be recovered completely when the body is returned to its unstrained state.

reference : Chou.p.c , Pagano N.J., Elasticity , d. van nostrand, 1967

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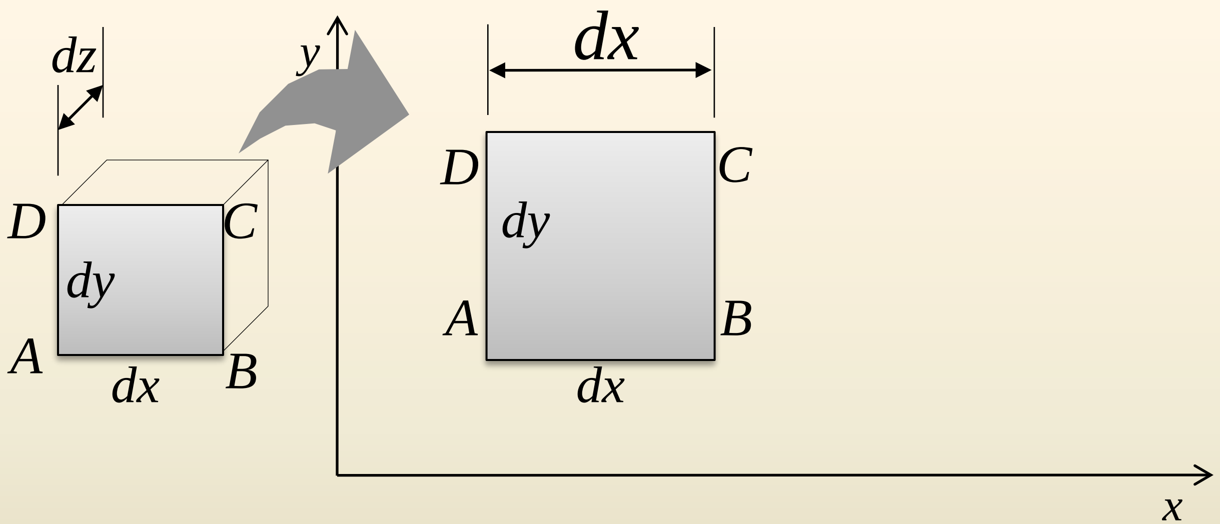
✓Strain Energy due to a uniaxial stress

u : displacement



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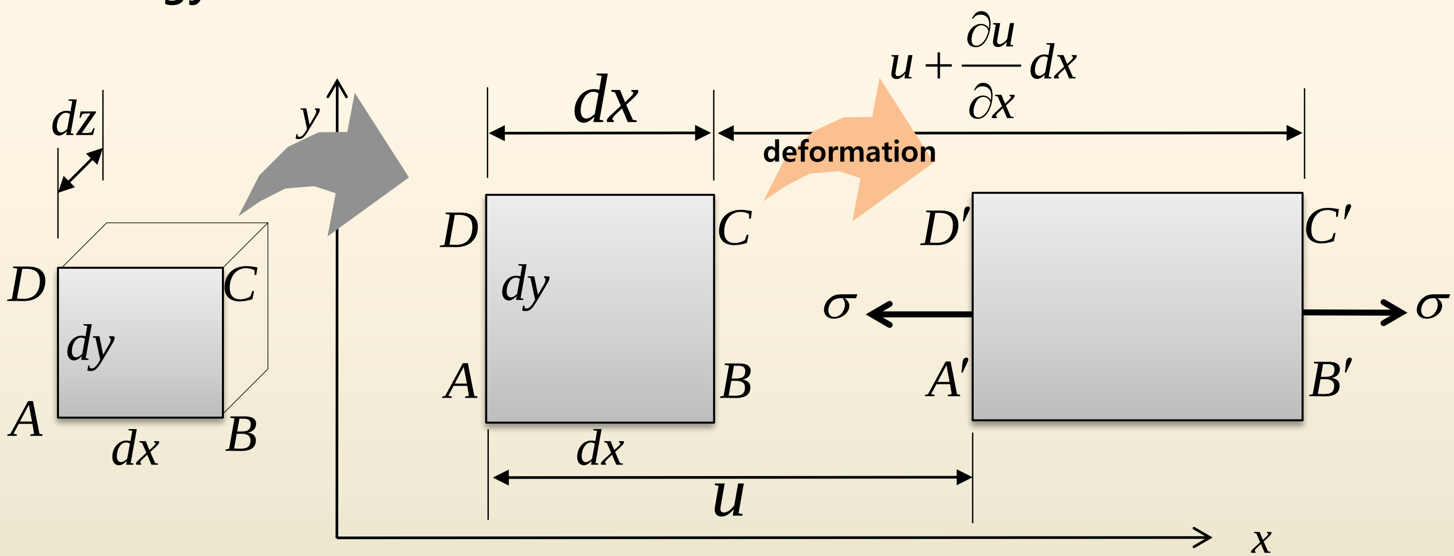

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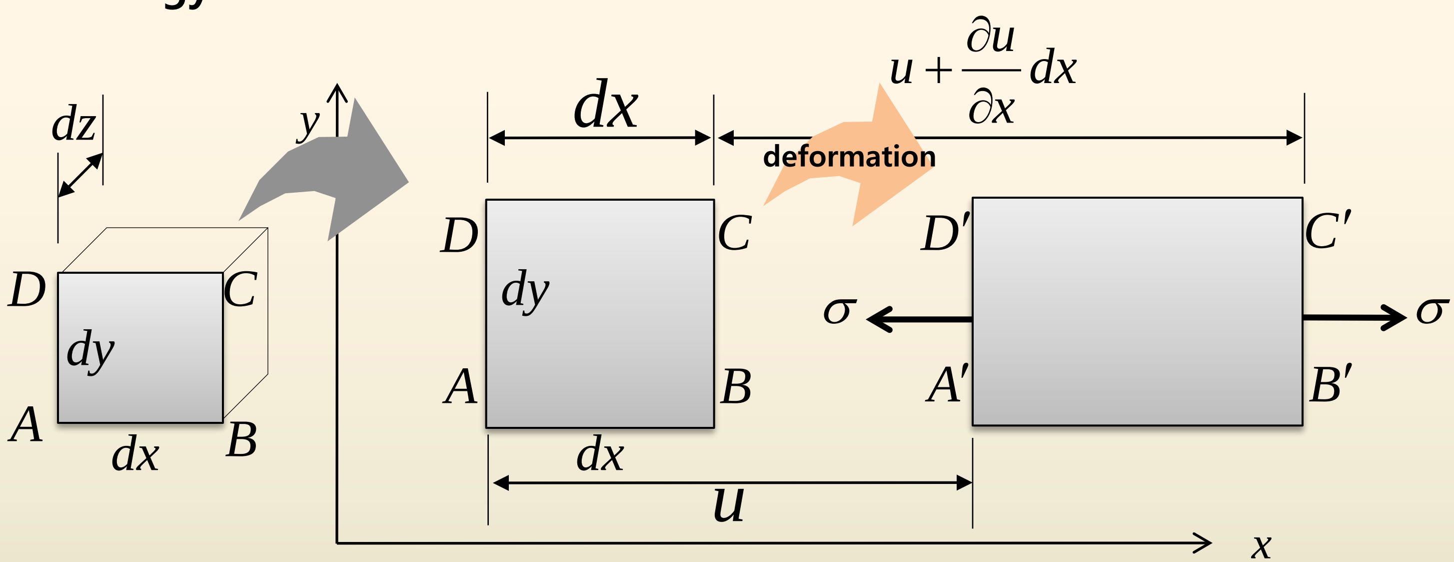

on the plane DA, the stress vector acts in the opposite direction of the displacement u

the work done by σ on DA is negative

the work done by σ on BC is positive

the normal stress σ increase from zero to σ_x

'no contribution is made to the strain energy by v and w as σ_y and σ_z are assumed to be zero'

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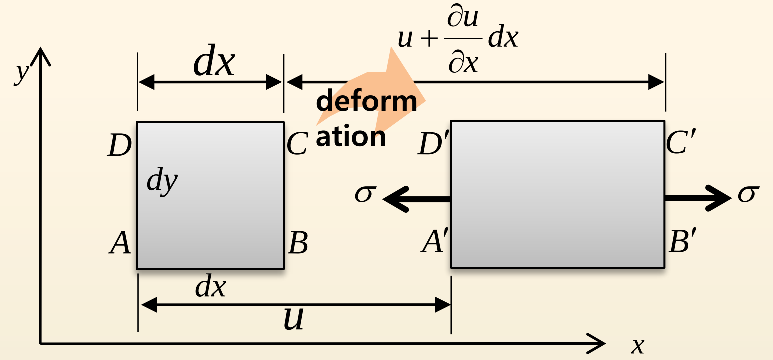

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the net work done on the element

$$\int_{\sigma=0}^{\sigma=\sigma_x}\sigma d\left(u+\frac{\partial u}{\partial x}dx\right)dydz-\int_{\sigma=0}^{\sigma=\sigma_x}\sigma dudydz$$

$$=\int_{\sigma=0}^{\sigma=\sigma_x}\sigma\left(du+d\frac{\partial u}{\partial x}dx-du\right)dydz$$

$$=\int_{\sigma=0}^{\sigma=\sigma_x}\sigma d\frac{\partial u}{\partial x}dxdydz$$

Strain Energy

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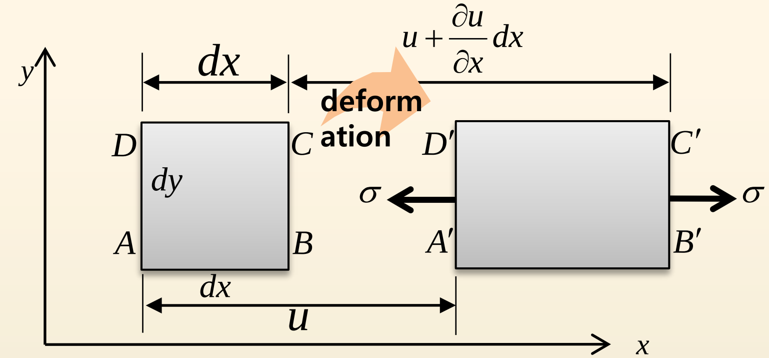

the net work done on the element

$$\int_{\sigma=0}^{\sigma=\sigma_x}\sigma d\left(u+\frac{\partial u}{\partial x}dx\right)dydz-\int_{\sigma=0}^{\sigma=\sigma_x}\sigma dudydz$$

$$=\int_{\sigma=0}^{\sigma=\sigma_x}\sigma d\left(\frac{\partial u}{\partial x}\right)dxdydz$$

$$=\int_{\sigma=0}^{\sigma=\sigma_x}\frac{1}{E}\sigma\, d\sigma\, dxdydz$$

$$\frac{\partial u}{\partial x}=\varepsilon$$ **definition of strain**

$$\varepsilon=\frac{\sigma}{E}$$ **Hooke's law**

$$d\left(\frac{\partial u}{\partial x}\right)=\frac{1}{E}d\sigma$$

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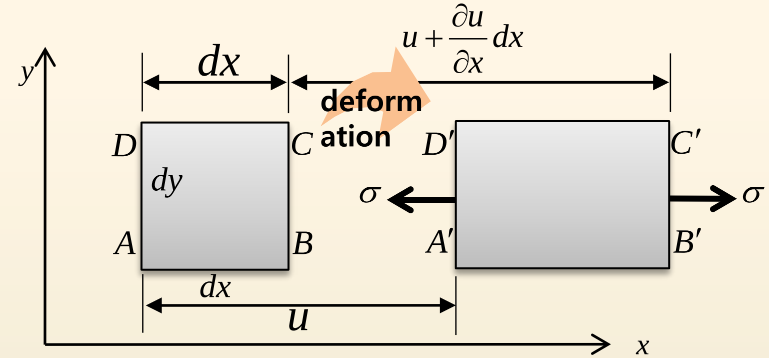

the net work done on the element

, <u>which is equal to the strain energy</u> dU

$$dU=\int_{\sigma=0}^{\sigma=\sigma_x}\frac{1}{E}\sigma\, d\sigma\, dxdydz$$

$\varepsilon=\frac{\sigma}{E}$ **:Hooke's law**

$$\therefore dU=\frac{1}{2E}\sigma_x^{\ 2}dxdydz$$

$$\text{or } dU=\frac{1}{2}\sigma_x\varepsilon_x dxdydz=\frac{1}{2}E\varepsilon_x^2 dxdydz$$

$,\varepsilon_x$ direction

note that $\sigma,\ u,\ \frac{\partial u}{\partial x}$ **are variables and**

the integration is with respect to the displacement gradient $\frac{\partial u}{\partial x}$

dx, dy, dz **are constant for this integration**

$$\int_{\sigma=0}^{\sigma=\sigma_x}\frac{1}{E}\sigma\, d\sigma\, dxdydz=\frac{1}{E}\left[\frac{1}{2}\sigma^2\right]_0^{\sigma_x}dxdydz$$

$$=\frac{1}{2E}\sigma_x^{\ 2}dxdydz$$

Strain Energy

- engineering elastic constant

✓Modulus of Elasticity 'E' $\varepsilon_x = \frac{1}{E}\sigma_x$

✓Poisson's ratio 'ν' $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$

✓Modulus of elasticity in shear 'G' $\gamma_{xy} = \frac{1}{G}\tau_{xy}$

ε_x direction

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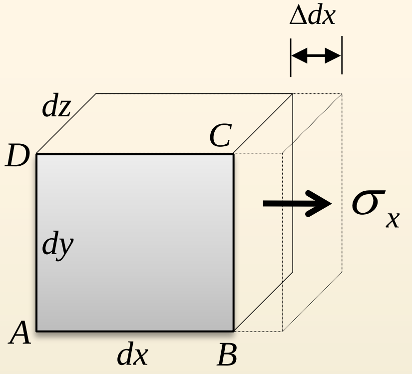
✓Strain Energy due to a uniaxial stress

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{\Delta dx}{dx}$$

net displacement of the element



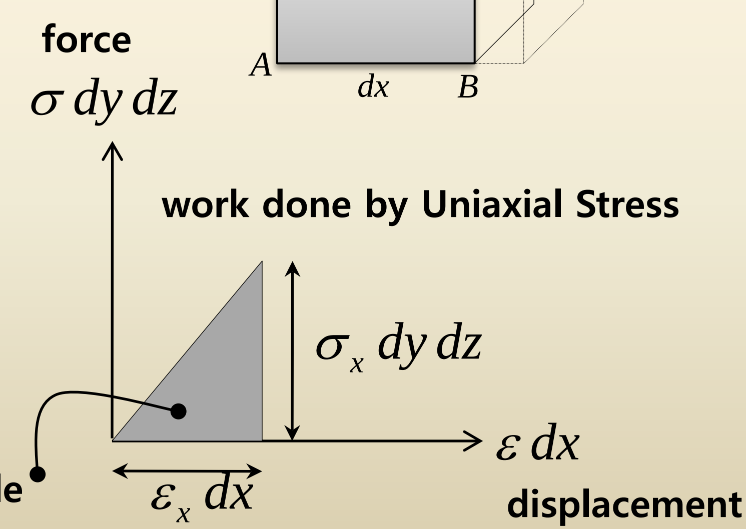
the net work done on the element on the element , which is equal to the strain energy

$$dU = \frac{1}{2E}\sigma_x^{\,2}\,dxdydz$$

The strain energy per unit volume, (the strain energy density)

$$U_0 = \frac{1}{2E}\sigma_x^{\,2} \quad \text{or} \quad U_0 = \frac{1}{2}\sigma_x\varepsilon_x$$

$$U_0 = \frac{1}{2}E\varepsilon_x^2$$



For a linear stress-strain relation, the force-displacement curve is a straight line and the work done is equal to the area of the shaded triangle

Strain Energy

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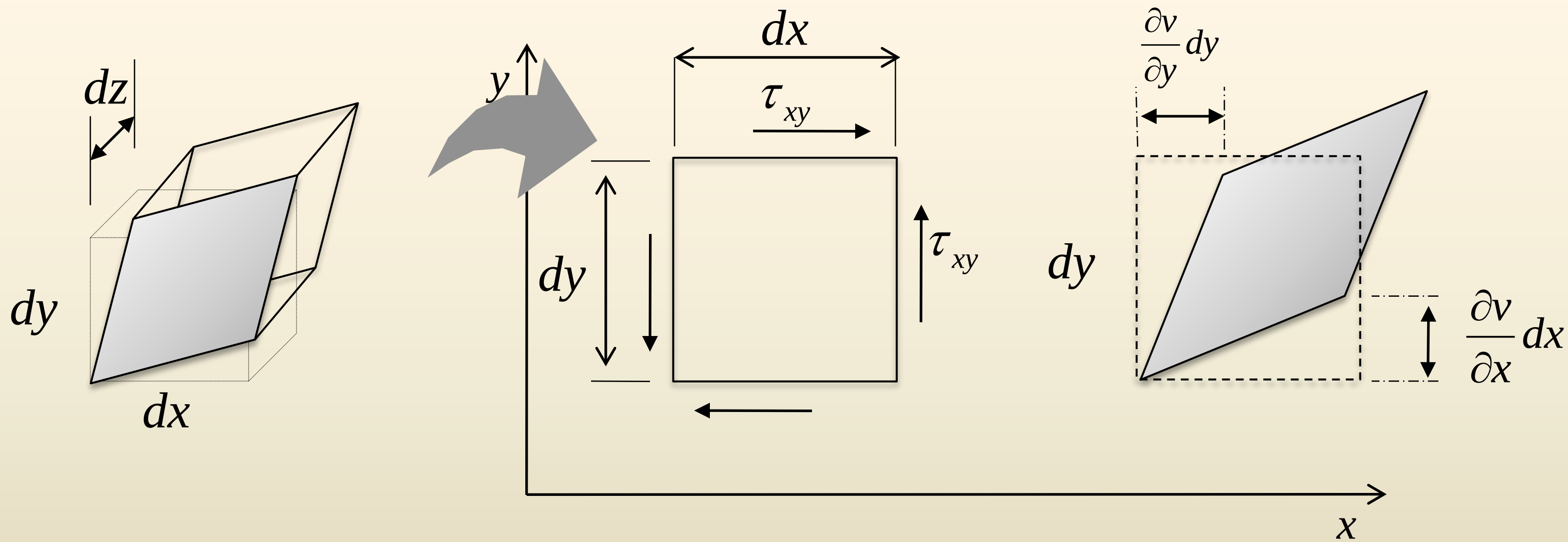
✓Strain Energy due to the shear stress components τ_{xy} and τ_{yx}

u, v :displacement



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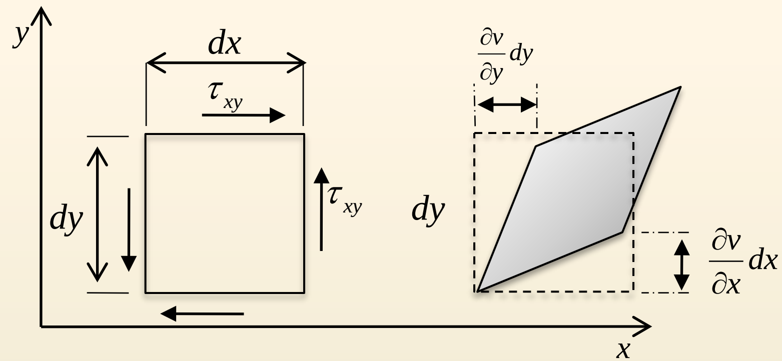
u, v :displacement

$$dU = \frac{1}{2}\left[(\tau_{xy}dydz)\left(\frac{\partial v}{\partial x}dx\right) + (\tau_{yx}dxdz)\left(\frac{\partial u}{\partial y}dy\right)\right]$$

$$= \frac{1}{2}\left[\tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdydz\right]$$

$$= \frac{1}{2}\left(\tau_{xy}\gamma_{xy}\right)dxdydz \qquad \because \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$



The shear strain energy per unit volume, (the shear strain energy density)

$$U_0 = \frac{1}{2}\tau_{xy}\gamma_{xy} \qquad \text{or } U_0 = \frac{1}{2}\frac{\tau_{xy}^2}{G}$$

$$U_0 = \frac{1}{2}G\gamma_{xy}^2$$

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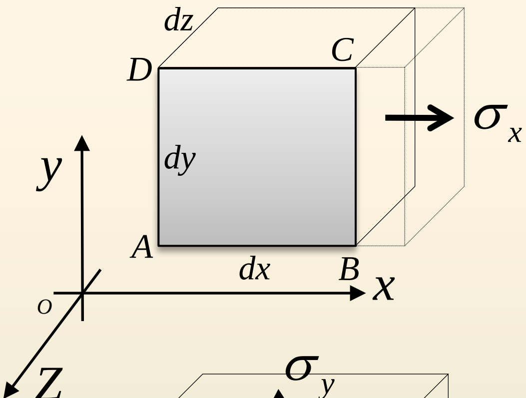
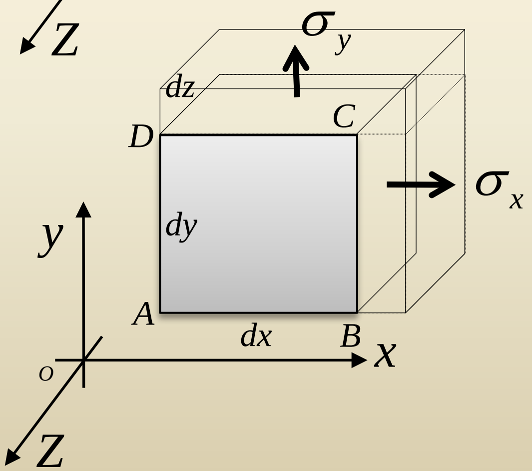
✓Strain Energy under the action of both σ_x and σ_y

u, v :displacement

Let the action take place in the following order

Step	σ_x	ε_x	σ_y	ε_y
1	increase zero to σ_x	$0 \rightarrow \frac{\sigma_x}{E}$	σ_y remains zero	$0 \rightarrow -\frac{\nu}{E}\sigma_x$
2	σ_x remains constant	$\frac{\sigma_x}{E} \rightarrow \frac{\sigma_x - \nu\sigma_y}{E}$	increase zero to σ_y	$\frac{\sigma_y}{E}$





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1	increase zero to σ_x	$0 \to \frac{\sigma_x}{E}$	σ_y remains zero	$0 \to -\frac{\nu}{E}\sigma_x$
2	σ_x remains constant	$\frac{\sigma_x}{E} \to \frac{\sigma_x - \nu\sigma_y}{E}$	increase zero to σ_y	$\frac{\sigma_y}{E}$

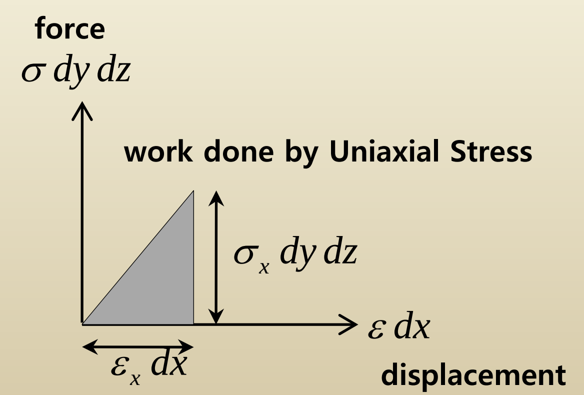
The work done by σ_x

$$dU_1 = \frac{1}{2}(\sigma_x dydz)\left(\frac{\sigma_x}{E}dx\right)$$

$$= \frac{1}{2E}\sigma_x^2 dxdydz$$

recall,

$$dU = \frac{1}{2}\sigma_x\varepsilon_x dxdydz = E\varepsilon_x^2 dxdydz = \frac{1}{2E}\sigma_x^2 dxdydz$$



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Let the action take place in the following order

$$\boxed{dU_1 = \frac{1}{2E}\sigma_x^2 dxdydz}$$

Step	σ_x	ε_x	σ_y	ε_y
1	increase zero to σ_x	$0 \to \frac{\sigma_x}{E}$	σ_y remains zero	$0 \to -\frac{\nu}{E}\sigma_x$
2	σ_x remains constant	$\frac{\sigma_x}{E} \to \frac{\sigma_x - \nu\sigma_y}{E}$	increase zero to σ_y	$\frac{\sigma_y}{E}$

The work done by σ_y

$$dU_2 = \frac{1}{2}\left(\sigma_y dxdz\right)\left(-\frac{\nu\sigma_x}{E}dy\right)$$

$$= 0$$

recall,

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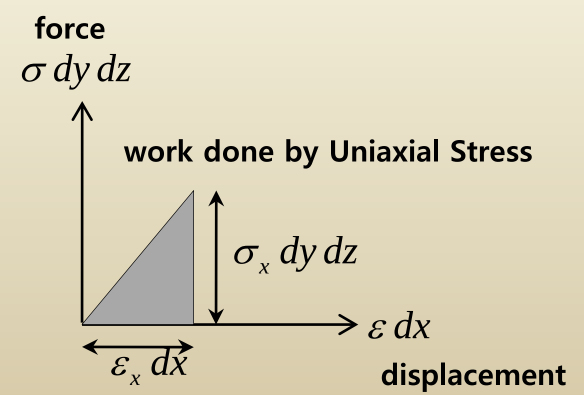

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$$\boxed{dU_1 = \frac{1}{2E}\sigma_x^2 dxdydz}$$

$$\boxed{dU_2 = 0}$$

The work done by σ_y

$$dU_3 = \frac{1}{2}\left(\sigma_y dxdz\right)\left(\frac{\sigma_y}{E}dy\right)$$

$$= \frac{1}{2E}\sigma_y^2 dxdydz$$

recall,

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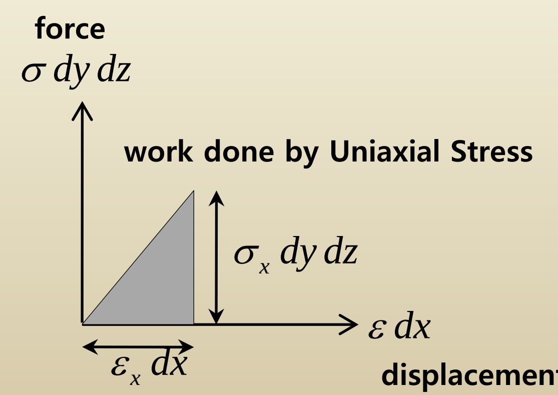

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$$dU_1 = \frac{1}{2E}\sigma_x^2 dxdydz$$

$$dU_2 = 0$$

$$dU_3 = \frac{1}{2E}\sigma_y^2 dxdydz$$

The work done by $\sigma_x = const$

$$dU_4 = (\sigma_x dydz)\left(\frac{-\nu\sigma_y}{E}dx\right)$$

$$= \frac{-\nu}{E}\sigma_x\sigma_y dxdydz$$

the factor ½ is not included since $\sigma_x = const$

$$dU_4 = \boxed{\frac{1}{2}}(\sigma_x dydz)\left(\frac{-\nu\sigma_y}{E}dx\right)$$

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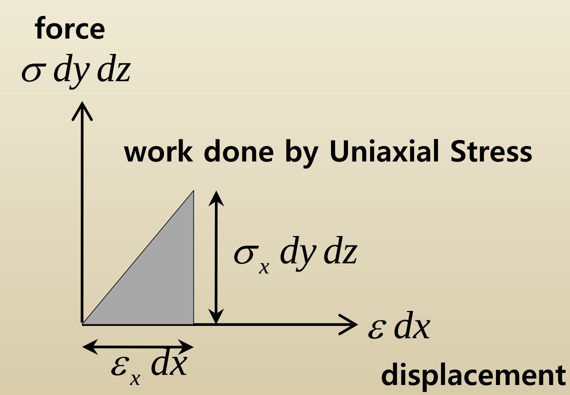

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2	σ_x remains constant	$\frac{\sigma_x}{E} \to \frac{\sigma_x - \nu\sigma_y}{E}$	increase zero to σ_y	$\frac{\sigma_y}{E}$

$$dU_1 = \frac{1}{2E}\sigma_x^2 dxdydz$$

$$dU_2 = 0$$

$$dU_3 = \frac{1}{2E}\sigma_y^2 dxdydz$$

$$dU_4 = \frac{-\nu}{E}\sigma_x\sigma_y dxdydz$$

Total strain energy accumulated in the element

$$dU = dU_1 + dU_2 + dU_3 + dU_4$$

$$= \left(\frac{1}{2E}\sigma_x^2 dxdydz\right) + (0) + \left(\frac{1}{2E}\sigma_y^2 dxdydz\right) + \left(\frac{-\nu}{E}\sigma_x\sigma_y dxdydz\right)$$

$$= \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y\right) dxdydz$$

Strain Energy

- engineering elastic constant

✓**Modulus of Elasticity 'E'** $\varepsilon_x = \frac{1}{E}\sigma_x$

✓**Poisson's ratio 'ν'** $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$

✓**Modulus of elasticity in shear 'G'** $\gamma_{xy} = \frac{1}{G}\tau_{xy}$

Total strain energy accumulated in the element

$$dU = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y\right)dxdydz$$

$$= \frac{1}{2E}\left(\left(\frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)\right)^2 + \left(\frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)\right)^2 - 2\nu\left(\frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)\right)\left(\frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)\right)\right)dxdydz$$

$$= \frac{1}{2E}\left(\frac{E}{1-\nu^2}\right)^2\left[\left(\varepsilon_x^2 + 2\nu\varepsilon_x\varepsilon_y + \nu^2\varepsilon_y^2\right) + \left(\varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y + \nu^2\varepsilon_x^2\right) - 2\nu\left(\varepsilon_x\varepsilon_y + \nu\varepsilon_x^2 + \nu\varepsilon_y^2 + \nu^2\varepsilon_x\varepsilon_y\right)\right]dxdydz$$

$$= \frac{1}{2}\frac{E}{(1-\nu^2)^2}\left[\varepsilon_x^2 + 2\nu\varepsilon_x\varepsilon_y + \nu^2\varepsilon_y^2 + \varepsilon_y^2 + \cancel{2\nu\varepsilon_x\varepsilon_y} + \nu^2\varepsilon_x^2 - \cancel{2\nu\varepsilon_x\varepsilon_y} - 2\nu^2\varepsilon_x^2 - 2\nu^2\varepsilon_y^2 - 2\nu^3\varepsilon_x\varepsilon_y\right]dxdydz$$

$$= \frac{1}{2}\frac{E}{(1-\nu^2)^2}\left[\varepsilon_x^2 + 2\nu\varepsilon_x\varepsilon_y + \boxed{\nu^2\varepsilon_y^2} + \boxed{\varepsilon_y^2} + \nu^2\varepsilon_x^2 - 2\nu^2\varepsilon_x^2 \boxed{- 2\nu^2\varepsilon_y^2} - 2\nu^3\varepsilon_x\varepsilon_y\right]dxdydz$$

$$= \frac{1}{2}\frac{E}{(1-\nu^2)^2}\left[(1-\nu^2)\varepsilon_x^2 + \boxed{(1-\nu^2)\varepsilon_y^2} + 2\nu(1-\nu^2)\varepsilon_x\varepsilon_y\right]dxdydz$$

$$= \frac{1}{2}\frac{E}{(1-\nu^2)}\left[\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y\right]dxdydz$$

$$= \frac{1}{2}\frac{E}{(1-\nu^2)}\left[\varepsilon_x + \nu\varepsilon_x\varepsilon_y + \varepsilon_y^2 + \nu\varepsilon_x\varepsilon_y\right]dxdydz$$

$$= \frac{1}{2}\frac{E}{(1-\nu^2)}\left[\varepsilon_x(\varepsilon_x + \nu\varepsilon_y) + \varepsilon_y(\varepsilon_y + \nu\varepsilon_x)\right]dxdydz$$

recall,

6 Relations between *6 Strain and 6 Stress*

In case of "Plane Stress State", $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

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Generalized Hooke's Law for "Plane Stress State"

Strain-stress relation for "Plane Stress State"

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

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Strain Energy

- engineering elastic constant

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When a body is deformed by external forces, work is done by these forces. The energy absorbed in the body due to this external work is called *strain energy*. If the body behaves elastically, the strain energy can be recovered completely when the body is returned to its unstrained state.

✓ Strain Energy under the action of both σ_x and σ_y

Let the action take place in the following order

Step	σ_x	ε_x	σ_y	ε_y
1	increase zero to σ_x	$0 \to \frac{\sigma_x}{E}$	σ_y remains zero	$0 \to -\frac{\nu}{E}\sigma_x$
2	σ_x remains constant	$\frac{\sigma_x}{E} \to \frac{\sigma_x - \nu\sigma_y}{E}$	increase zero to σ_y	$\frac{\sigma_y}{E}$

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Total strain energy accumulated in the element

$$dU = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y\right)dxdydz$$

$$\therefore dU = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y\right]dxdydz$$

recall,

6 Relations between 6 Strain and 6 Stress

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$$dU_1 = \frac{1}{2E}\sigma_x^2 dxdydz$$

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Total strain energy accumulated in the element

$$dU = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y\right]dxdydz$$

If we assume that the stresses are applied in a different order, we can prove that the strain energy stored in a body will be exactly same.

That is, *<u>the strain energy depends on the final state of the stress</u>* and is independent of the manner in which the stresses are applied

recall,

<u>6 Relations</u> between 6 Strain and 6 Stress

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Strain Energy

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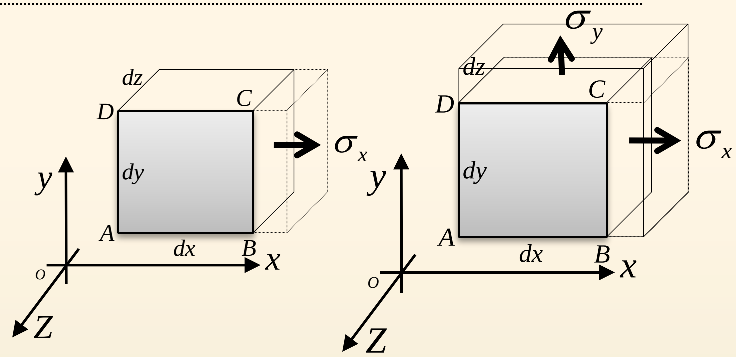
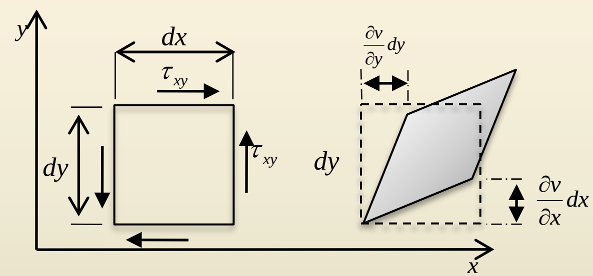


✓**Strain Energy due to a uniaxial stress**

$$dU = \frac{1}{2E}\sigma_x^{\,2} dxdydz$$

✓**Strain Energy under the action of both** σ_x **and** σ_y

$$dU = \frac{1}{2}\left[\sigma_x \varepsilon_x + \sigma_y \varepsilon_y\right] dxdydz$$



✓**Strain Energy due to the shear stress components** τ_{xy} **and** τ_{yx}

$$dU = \frac{1}{2}\left(\tau_{xy}\gamma_{xy}\right) dxdydz$$

✓**The Strain Energy stored in the element under a general three dimensional stress system can be found in a similar way,**

$$dU = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) dxdydz$$

Strain Energy

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✓**The Strain Energy Density**

$$U_0 = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

Strain Energy

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by using generalized Hooke's law,
it may be expressed in terms of
the stress components* or *strain components

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad \gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

=

6 Relations between 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

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$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Strain Energy

- engineering elastic constant

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6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

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$$U_0 = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

$$= \frac{1}{2}\left(\sigma_x\frac{1}{E}[\sigma_x - \nu(\sigma_y+\sigma_z)] + \sigma_y\frac{1}{E}[\sigma_y - \nu(\sigma_z+\sigma_x)] + \sigma_z\frac{1}{E}[\sigma_z - \nu(\sigma_x+\sigma_y)] + \tau_{xy}\frac{2(\nu+1)}{E}\tau_{xy} + \tau_{yz}\frac{2(\nu+1)}{E}\tau_{yz} + \tau_{zx}\frac{2(\nu+1)}{E}\tau_{zx}\right)$$

$$= \frac{1}{2}\left(\frac{1}{E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{2\nu}{E}(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z + \sigma_y\sigma_x + \sigma_z\sigma_x + \sigma_z\sigma_y) + \frac{1}{G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right)$$

$$\therefore U_0 = \frac{1}{2}\left[\frac{1}{E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{2\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]$$

Strain Energy

● engineering elastic constant

✓<u>Modulus of Elasticity 'E'</u> $\varepsilon_x = \frac{1}{E}\sigma_x$

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✓<u>Modulus of elasticity in shear 'G'</u> $\gamma_{xy} = \frac{1}{G}\tau_{xy}$

$$\boxed{G = \frac{E}{2(\nu+1)}}$$

$$\boxed{\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}}$$

When a body is deformed by external forces, work is done by these forces. The energy absorbed in the body due to this external work is called *strain energy*. If the body behaves elastically, the strain energy can be recovered completely when the body is returned to its unstrained state.

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6 Relations between 6 Strain and 6 Stress

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$$U_0 = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

$$= \frac{1}{2}\Big((\lambda e + 2G\varepsilon_x)\varepsilon_x + (\lambda e + 2G\varepsilon_y)\varepsilon_y + (\lambda e + 2G\varepsilon_z)\varepsilon_z + G\gamma_{xy}\gamma_{xy} + G\gamma_{yz}\gamma_{yz} + G\gamma_{zx}\gamma_{zx}\Big)$$

$$= \frac{1}{2}\Big(\lambda e(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)\Big)$$

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$$U_0 = \frac{1}{2}\left[\frac{1}{E}({\sigma_x}^2 + {\sigma_y}^2 + {\sigma_z}^2) - \frac{2\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{G}({\tau_{xy}}^2 + {\tau_{yz}}^2 + {\tau_{zx}}^2)\right]$$

6 Relations between 6 Strain and 6 Stress

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||

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in terms of *strain components*

$$U_0 = \frac{1}{2}\left(\lambda e^2 + 2G(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)\right)$$

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$$U_0 = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

in terms of *<u>the stress components</u>*

$$U_0 = \frac{1}{2}\left[\frac{1}{E}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) - \frac{2\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]$$

in terms of *<u>strain components</u>*

$$U_0 = \frac{1}{2}\left(\lambda e^2 + 2G(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)\right)$$

✓**We observe that the derivative of U_0 with respect to any stress components is equal to the corresponding strain component, and the reverse is true, i.e.,**

$$\frac{\partial U_0(\sigma_x, \sigma_y, \ldots, \tau_{xz})}{\partial \sigma_x} = \frac{1}{2}\left(\frac{1}{E}2\sigma_x - \frac{2\nu}{E}(\sigma_y + \sigma_z)\right)$$

$$= \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right)$$

$$= \varepsilon_x$$

<u>6 Relations</u> between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad \gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Strain Energy

- engineering elastic constant

✓**Modulus of Elasticity 'E'** $\varepsilon_x = \frac{1}{E}\sigma_x$

✓**Poisson's ratio 'ν'** $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$

✓**Modulus of elasticity in shear 'G'** $\gamma_{xy} = \frac{1}{G}\tau_{xy}$

$$G = \frac{E}{2(\nu+1)}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

When a body is deformed by external forces, work is done by these forces. The energy absorbed in the body due to this external work is called *strain energy*. If the body behaves elastically, the strain energy can be recovered completely when the body is returned to its unstrained state.

✓**The Strain Energy Density under a general three dimensional stress system can be found in a similar way,**

$$U_0 = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

in terms of *the stress components*

$$U_0 = \frac{1}{2}\left[\frac{1}{E}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) - \frac{2\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]$$

in terms of *strain components*

$$U_0 = \frac{1}{2}\left(\lambda e^2 + 2G(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)\right)$$

✓**We observe that the derivative of U_0 with respect to any stress components is equal to the corresponding strain component, and the reverse is true, i.e.,**

$$\frac{\partial U_0(\varepsilon_x, \varepsilon_y, \ldots, \gamma_{xz})}{\partial \varepsilon_x} = \frac{1}{2}\left(\lambda 2e + 2G\cdot 2\varepsilon_x\right)$$

$$= \left(\lambda e + 2G\varepsilon_x\right)$$

$$= \sigma_x$$

6 Relations between 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Strain Energy

When a body is deformed by external forces, work is done by these forces.
The energy absorbed in the body due to this external work is called *strain energy*.
If the body behaves elastically, the strain energy can be recovered completely when the body is returned to its unstrained state.

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in terms of *the stress components*

$$U_0 = \frac{1}{2}\left[\frac{1}{E}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) - \frac{2\nu}{E}(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + \frac{1}{G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]$$

in terms of *strain components*

$$U_0 = \frac{1}{2}\left(\lambda e^2 + 2G(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)\right)$$

✓ **The total Strain Energy absorbed in an elastic body is found by the integral**

$$U = \iiint_V U_0 \, dx \, dy \, dz$$

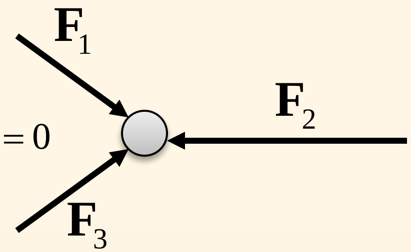
Principle of Virtual Work

✓**Virtual Displacement**

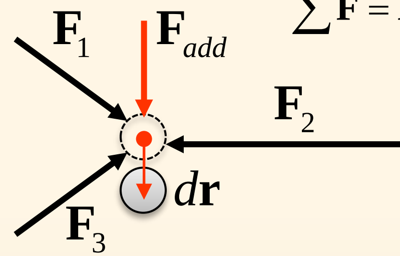
a particle which is acted upon by a system of forces, assume the particle is *at rest*

: the resultant of all the forces acting on it is zero

$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$$

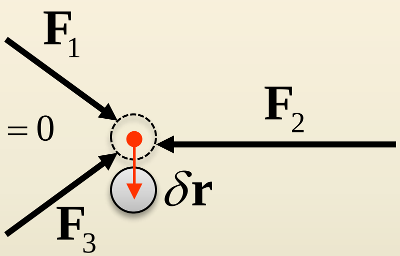


$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_{add} \neq 0$$



if we want to move this particle to a new position or to give it a small displacement, additional force is required and the original force system must be altered

$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$$



now, we shall consider a "virtual displacement", defined as an arbitrary displacement which does not affect the force system acting on the particle

during the process of it each of the forces acting on the particle remains constant in magnitude and direction

✓**Virtual Work**

the work done by the forces acting on the particle during a virtual displacement is called the *virtual work* : the virtual work done is zero if the particle is in equilibrium, since the resultant force vanishes

$$\left(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\right) \bullet \delta\mathbf{r} = 0$$

$$\left(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_{add}\right) \bullet d\mathbf{r} \neq 0$$

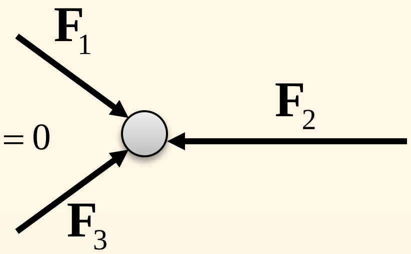
Principle of Virtual Work

✓Virtual Displacement

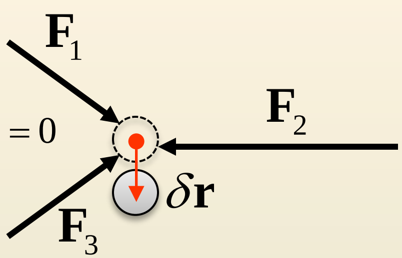
a particle which is acted upon by a system of forces, assume the particle is <u>*at rest*</u>

: <u>the resultant of all the forces acting on it is zero</u>

$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$$



$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$$



a <u>"virtual displacement"</u>, defined as an arbitrary displacement which <u>does not affect the force system</u> acting on the particle

during the process of it each of the forces acting on the particle remains constant in magnitude and direction

✓Virtual Work

the work done by the forces acting on the particle during a virtual displacement is called the *virtual work :*

the virtual work done is zero if the particle is in equilibrium, since the resultant force vanishes

$$\left(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\right) \bullet \delta \mathbf{r} = 0$$

it is also evident that

if the virtual work vanishes, the force system acting on the article must be in equilibrium

In the case of single rigid particle, it is easy to write the equilibrium equations governing the forces, and it seems that the virtual work does not contribute much to the problem.

<u>*For more complicated problems, however, it is sometimes more convenient to require the virtual work corresponding to a certain virtual displacement to vanish*</u> than to write down and solve the equilibrium equations

Principle of Virtual Work

✓**Virtual Displacement / Strain Field**
in discussing the principle of virtual work for an elastic body, we must introduce a virtual displacement field and virtual strain filed

$$\delta u = \delta u(x,y,z), \quad \delta v = \delta v(x,y,z), \quad \delta w = \delta w(x,y,z)$$

$$\delta\varepsilon_x = \delta\varepsilon_x(x,y,z),\ \delta\varepsilon_y = \delta\varepsilon_y(x,y,z),\ \delta\varepsilon_z = \delta\varepsilon_z(x,y,z),$$
$$\delta\gamma_{xy} = \delta\gamma_{xy}(x,y,z),\ \delta\gamma_{yz} = \delta\gamma_{yz}(x,y,z),\ \delta\gamma_{zx} = \delta\gamma_{zx}(x,y,z)$$

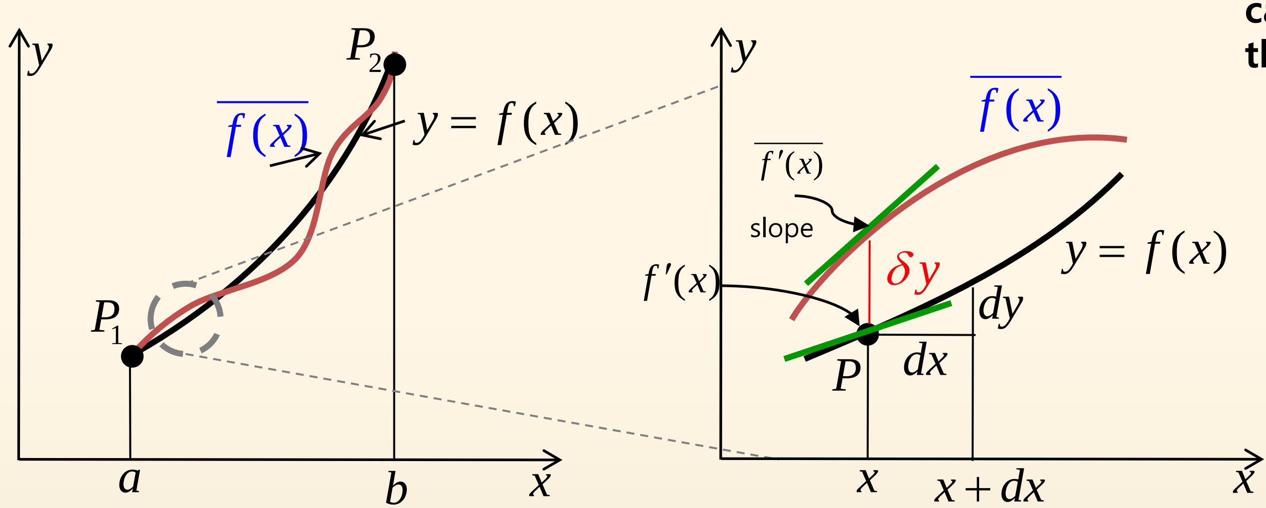
all of which take place after the body has reached its equilibrium configuration

We define the virtual strain components in a manner analogous to the definition of real strain components

$$\delta\varepsilon_x = \delta\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(\delta u),\quad \delta\varepsilon_y = \frac{\partial}{\partial y}(\delta v),\ \delta\varepsilon_z = \frac{\partial}{\partial z}(\delta w),$$

$$\delta\gamma_{xy} = \delta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}(\delta u) + \frac{\partial}{\partial x}(\delta v),\ \delta\gamma_{yz} = \frac{\partial}{\partial y}(\delta w) + \frac{\partial}{\partial z}(\delta v),\ \delta\gamma_{zx} = \frac{\partial}{\partial z}(\delta u) + \frac{\partial}{\partial x}(\delta w)$$

The commutative properties of the δ-process



The modified function $\overline{f(x)}$ can be written making use of the variable parameter ε

$$\overline{f(x)} = f(x) + \varepsilon\phi(x)$$

$$\delta y = \overline{f(x)} - f(x) = \varepsilon\phi(x)$$

- **The derivative of the variation**

$$\frac{d}{dx}\delta y = \frac{d}{dx}\left[\overline{f(x)} - f(x)\right] = \frac{d}{dx}\left[\varepsilon\phi(x)\right] = \varepsilon\phi'(x)$$

- **The variation of the derivative**

$$\delta\frac{d}{dx}y = \left[\overline{f'(x)} - f'(x)\right] = \left(f'(x) + \varepsilon\phi'\right) - f'(x) = \varepsilon\phi'(x)$$

$$\therefore \frac{d}{dx}\delta y = \delta\frac{d}{dx}y$$

Lanczos C., The Variational Mechanics, Fourth Edition, Dover, 1970, p56-57

Virtual Work in Elastic Body

✓**Virtual Work**

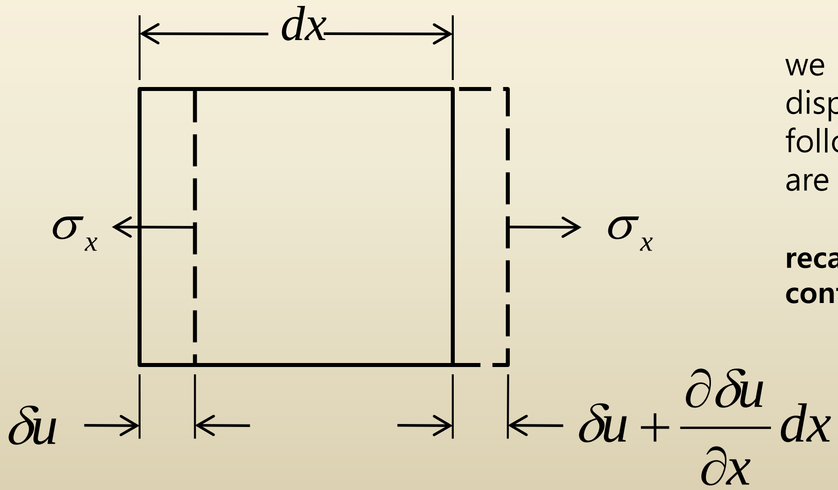
δW : **the virtual work done by the external (surface and body) forces**

δU : **the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

Consider an elastic body subjected to a force system causing actual displacement u, v, w

then let the body be subjected to a virtual displacement field with components $\delta u, \delta v, \delta w$

in order to determine the virtual strain energy, we first consider the virtual work done by σ_x



we are not concerned with the work done while the actual displacements occur; we assume that u, v and w occur first and following this we imagine that the virtual displacement components are applied

recall "all of which take place after the body has reached its equilibrium configuration"

Virtual Work in Elastic Body

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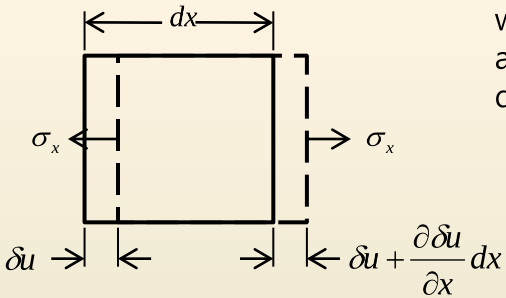

we are not concerned with the work done while the actual displacements occur; we assume that u, v and w occur first and following this we imagine that the virtual displacement components are applied

the virtual work done by σ_x **is therefore,**

$$\sigma_x \left(\delta u + \frac{\partial \delta u}{\partial x} dx \right) dy\, dz - \sigma_x (\delta u)\, dy\, dz$$

$$= \sigma_x \frac{\partial \delta u}{\partial x} dx dy\, dz$$

Virtual Work in Elastic Body

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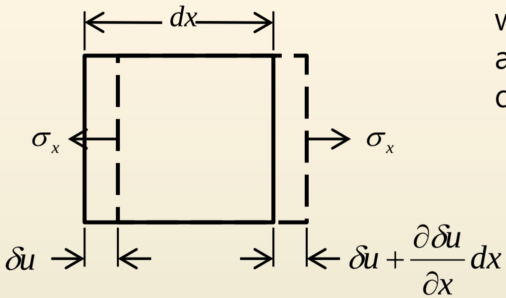

we are not concerned with the work done while the actual displacements occur; we assume that u, v and w occur first and following this we imagine that the virtual displacement components are applied

the virtual work done by σ_x **is therefore,** $\sigma_x \dfrac{\partial \delta u}{\partial x} dx\,dy\,dz$

the virtual work done per unit volume δU_0

$$\delta U_0 = \sigma_x \frac{\partial \delta u}{\partial x}$$

$$= \sigma_x \delta \frac{\partial u}{\partial x} = \sigma_x \delta \varepsilon_x$$

$$\therefore \delta U_0 = \sigma_x \delta \varepsilon_x$$

Virtual Work in Elastic Body

✓The Strain Energy Density

$$U_0 = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

✓Virtual Work

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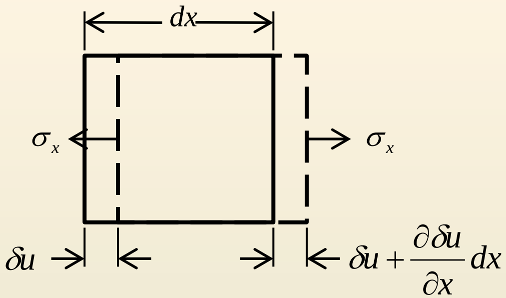

the virtual work done by σ_x is $\sigma_x \dfrac{\partial \delta u}{\partial x} dx\,dy\,dz$

the virtual work done per unit volume δU_0

$$\delta U_0 = \sigma_x \delta\varepsilon_x$$

under a general stress condition it can be shown that the virtual strain energy is given by

$$\delta U_0 = \sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}$$

the total virtual strain energy

$$\delta U = \iiint_V \delta U_0 \, dx\, dy\, dz$$

the factor ½ is not included since stresses are constant during the virtual displacement

$$U_0 = \boxed{\frac{1}{2}}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

Virtual Work in Elastic Body

✓The Strain Energy Density

$$U_0 = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

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δW : the virtual work done by the external (surface and body) forces

δU : the virtual strain energy
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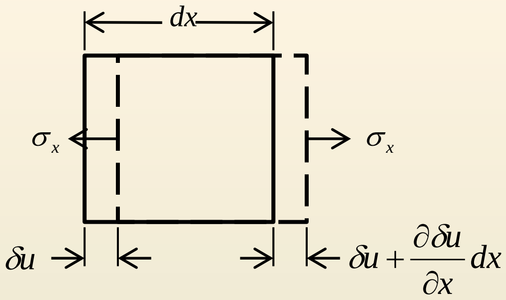
then let the body be subjected to a virtual displacement field with components $\delta u, \delta v, \delta w$



the virtual work done by σ_x is $\sigma_x \dfrac{\partial \delta u}{\partial x} dxdy\,dz$

the virtual work done per unit volume δU_0

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$$\delta U_0 = \sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}$$

the total virtual strain energy

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right) dx\,dy\,dz$$

Virtual Work in Elastic Body

✓**Virtual Work**

δW **: the virtual work done by the external (surface and body) forces**

δU **: the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy} \delta\gamma_{xy} + \tau_{yz} \delta\gamma_{yz} + \tau_{zx} \delta\gamma_{zx} \right) dx\, dy\, dz$$

next, consider the virtual work done by the external work

The virtual work done by the surface forces is $\int_A (T_x^{\mu} \delta u + T_y^{\mu} \delta v + T_z^{\mu} \delta w)\, dA$

dA is an elemental surface area and the integration is taken over the complete boundary surface of the body

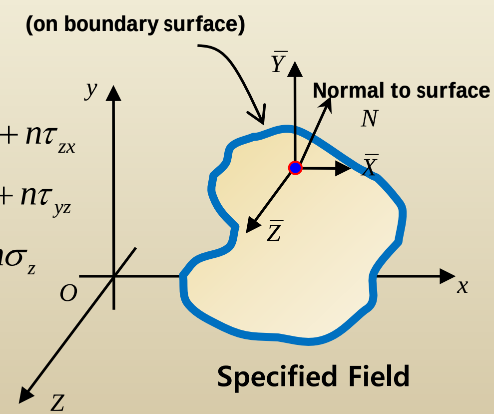
Again, the factor of ½ is not present because the surface force are constant during the virtual displacement

recall,

$$T_x^{\mu} \leftarrow \quad \bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$

$$T_y^{\mu} \leftarrow \quad \bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$$

$$T_z^{\mu} \leftarrow \quad \bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$$



Virtual Work in Elastic Body

$$T_x^{\ \mu} \leftarrow \bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$
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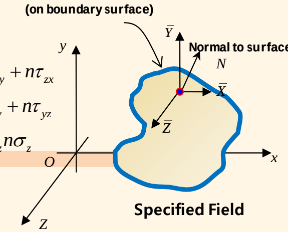
$$T_z^{\ \mu} \leftarrow \bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$$



✓**Virtual Work**

δW **: the virtual work done by the external (surface and body) forces**

δU **: the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy} \delta\gamma_{xy} + \tau_{yz} \delta\gamma_{yz} + \tau_{zx} \delta\gamma_{zx} \right) dx\, dy\, dz$$

Consider next consider the virtual work done by the external work (definition)

The virtual work done by the surface forces is $\int_A (T_x^{\ \mu} \delta u + T_y^{\ \mu} \delta v + T_z^{\ \mu} \delta w)\, dA$

dA is an elemental surface area and the integration is taken over the complete boundary surface of the body

Again, the factor of ½ is not present because the surface force are constant during the virtual displacement

The virtual work done by the body forces is $\int_V (F_x \delta u + F_y \delta v + F_z \delta w)\, dV$

The virtual work done by the external forces

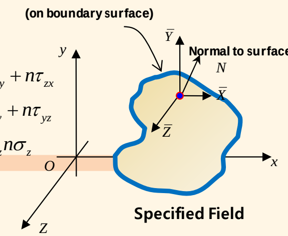
$$\delta W = \int_A \left(T_x^{\ \mu} \delta u + T_y^{\ \mu} \delta v + T_z^{\ \mu} \delta w \right) dA + \int_V \left(F_x \delta u + F_y \delta v + F_z \delta w \right) dA$$

Virtual Work in Elastic Body

$T_x^{\mu} \leftarrow \bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$

$T_y^{\mu} \leftarrow \bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$

$T_z^{\mu} \leftarrow \bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$



✓**Virtual Work**

δW : **the virtual work done by the external (surface and body) forces**

$$\delta W = \int_A \left(T_x^{\mu}\,\delta u + T_y^{\mu}\,\delta v + T_z^{\mu}\,\delta w\right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right) dA$$

δU : **the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right) dx\,dy\,dz$$

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right) dx\,dy\,dz$$

$$\delta U = \int_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right) dV$$

$$\delta U = \int_V \left\{\sigma_x \frac{\partial}{\partial x}(\delta u) + \sigma_y \frac{\partial}{\partial y}(\delta v) + \sigma_z \frac{\partial}{\partial z}(\delta w) + \tau_{xy}\left[\frac{\partial}{\partial y}(\delta u) + \frac{\partial}{\partial x}(\delta v)\right] + \tau_{yz}\left[\frac{\partial}{\partial z}(\delta v) + \frac{\partial}{\partial y}(\delta w)\right] + \tau_{zx}\left[\frac{\partial}{\partial z}(\delta u) + \frac{\partial}{\partial x}(\delta w)\right]\right\} dV$$

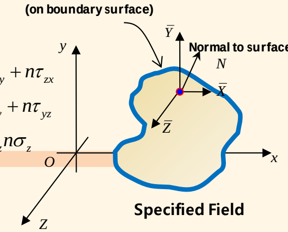
$$\boxed{\int_V \sigma_x \frac{\partial}{\partial x}(\delta u)\, dV}$$

Virtual Work in Elastic Body

$T_x^{\mu} \leftarrow \bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$

$T_y^{\mu} \leftarrow \bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$

$T_z^{\mu} \leftarrow \bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$



✓Virtual Work

δW : the virtual work done by the external (surface and body) forces

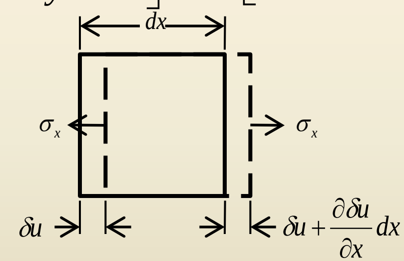
$$\delta W = \int_A \left(T_x^{\mu}\,\delta u + T_y^{\mu}\,\delta v + T_z^{\mu}\,\delta w\right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right) dA$$

δU : the virtual strain energy
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{xy}\,\delta\gamma_{xy} + \tau_{yz}\,\delta\gamma_{yz} + \tau_{zx}\,\delta\gamma_{zx}\right) dx\,dy\,dz$$

$$\delta U = \int_V \left\{ \boxed{\sigma_x \frac{\partial}{\partial x}(\delta u)} + \sigma_y \frac{\partial}{\partial y}(\delta v) + \sigma_z \frac{\partial}{\partial z}(\delta w) + \tau_{xy}\left[\frac{\partial}{\partial y}(\delta u) + \frac{\partial}{\partial x}(\delta v)\right] + \tau_{yz}\left[\frac{\partial}{\partial z}(\delta v) + \frac{\partial}{\partial y}(\delta w)\right] + \tau_{zx}\left[\frac{\partial}{\partial z}(\delta u) + \frac{\partial}{\partial x}(\delta w)\right]\right\} dV$$

let $M = \int_V \sigma_x \frac{\partial}{\partial x}(\delta u)\, dV = \iiint_V \sigma_x \frac{\partial}{\partial x}(\delta u)\, dxdydz$



integrating by part

$$M = \iint_A \left[\sigma_x \delta u\right]_{x_1(y,z)}^{x_2(y,z)} dydz - \iiint_V \delta u \frac{\partial \sigma_x}{\partial x} dV$$

thus it may be written as

$$M = \int_A \sigma_x \delta u\, \mu_x dA - \iiint_V \delta u \frac{\partial \sigma_x}{\partial x} dV$$

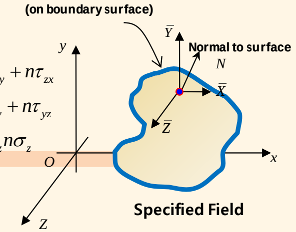
where, $x_2(y,z)$ and $x_1(y,z)$ are the equations of the right and left surfaces respectively of A

we have $dydz = A\mu_x$ on x_1

$dydz = -A\mu_x$ on x_2

Virtual Work in Elastic Body

$$T_x^{\mu} \leftarrow \bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$
$$T_y^{\mu} \leftarrow \bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$$
$$T_z^{\mu} \leftarrow \bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$$



✓**Virtual Work**

δW **: the virtual work done by the external (surface and body) forces**

$$\delta W = \int_A \left(T_x^{\mu}\,\delta u + T_y^{\mu}\,\delta v + T_z^{\mu}\,\delta w\right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right) dA$$

δU **: the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint\limits_V \left(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{xy}\,\delta\gamma_{xy} + \tau_{yz}\,\delta\gamma_{yz} + \tau_{zx}\,\delta\gamma_{zx}\right) dx\,dy\,dz$$

$$\delta U = \int_V \left\{ \boxed{\sigma_x \frac{\partial}{\partial x}(\delta u)} + \sigma_y \frac{\partial}{\partial y}(\delta v) + \sigma_z \frac{\partial}{\partial z}(\delta w) + \tau_{xy}\left[\frac{\partial}{\partial y}(\delta u) + \frac{\partial}{\partial x}(\delta v)\right] + \tau_{yz}\left[\frac{\partial}{\partial z}(\delta v) + \frac{\partial}{\partial y}(\delta w)\right] + \tau_{zx}\left[\frac{\partial}{\partial z}(\delta u) + \frac{\partial}{\partial x}(\delta w)\right] \right\} dV$$

$$M = \int_V \sigma_x \frac{\partial}{\partial x}(\delta u)\,dV = \int_A \sigma_x \delta u\,\mu_x\,dA - \iiint\limits_V \delta u \frac{\partial \sigma_x}{\partial x}\,dV$$

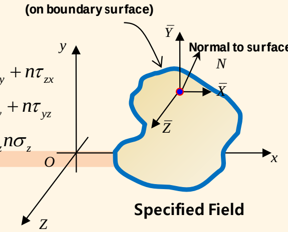
$$\delta U = \int_A [(\sigma_x \mu_x + \tau_{xy}\mu_y + \tau_{zx}\mu_z)\delta u + (\sigma_y \mu_y + \tau_{xy}\mu_x + \tau_{yz}\mu_z)\delta v + (\sigma_z \mu_z + \tau_{yz}\mu_y + \tau_{zx}\mu_x)\delta w]\,dA$$
$$- \int_V \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta u + \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}\right)\delta v + \left(\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial x}\right)\delta w \right] dV$$

Virtual Work in Elastic Body

$T_x^{\mu} \leftarrow \bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$

$T_y^{\mu} \leftarrow \bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$

$T_z^{\mu} \leftarrow \bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$



✓**Virtual Work**

δW : **the virtual work done by the external (surface and body) forces**

$$\delta W = \int_A \left(T_x^{\ \mu}\,\delta u + T_y^{\ \mu}\,\delta v + T_z^{\ \mu}\,\delta w\right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right) dA$$

δU : **the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{xy}\,\delta\gamma_{xy} + \tau_{yz}\,\delta\gamma_{yz} + \tau_{zx}\,\delta\gamma_{zx}\right) dx\,dy\,dz$$

$$\delta U = \int_A [(\sigma_x\mu_x + \tau_{xy}\mu_y + \tau_{zx}\mu_z)\,\delta u + (\sigma_y\mu_y + \tau_{xy}\mu_x + \tau_{yz}\mu_z)\,\delta v + (\sigma_z\mu_z + \tau_{yz}\mu_y + \tau_{zx}\mu_x)\,\delta w]\,dA$$

$$-\int_V \left[\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right)\delta u + \left(\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yz}}{\partial z}\right)\delta v + \left(\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zx}}{\partial x}\right)\delta w\right] dV$$

since,

$$T_x^{\mu} = \sigma_x\mu_x + \tau_{xy}\mu_y + \tau_{zx}\mu_z$$
$$T_y^{\mu} = \sigma_y\mu_y + \tau_{xy}\mu_x + \tau_{yz}\mu_z$$
$$T_z^{\mu} = \sigma_z\mu_z + \tau_{yz}\mu_y + \tau_{zx}\mu_x$$

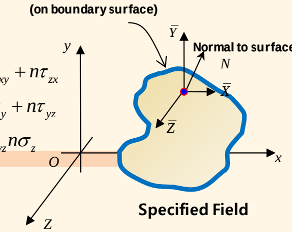
and

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + F_x = 0$$
$$\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yz}}{\partial z} + F_y = 0$$
$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zx}}{\partial x} + F_z = 0$$

$$\delta U = \int_A \left(T_x^{\ \mu}\,\delta u + T_y^{\ \mu}\,\delta v + T_z^{\ \mu}\,\delta w\right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right) dV$$

Virtual Work in Elastic Body

$$T_x^{\mu} \leftarrow \bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$
$$T_y^{\mu} \leftarrow \bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$$
$$T_z^{\mu} \leftarrow \bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$$



✓**Virtual Work**

δW : **the virtual work done by the external (surface and body) forces**

$$\boxed{\delta W = \int_A \left(T_x^{\mu}\,\delta u + T_y^{\mu}\,\delta v + T_z^{\mu}\,\delta w\right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right) dA}$$

δU : **the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right) dx\,dy\,dz$$

$$\delta U = \int_A [(\sigma_x \mu_x + \tau_{xy}\mu_y + \tau_{zx}\mu_z)\delta u + (\sigma_y \mu_y + \tau_{xy}\mu_x + \tau_{yz}\mu_z)\delta v + (\sigma_z \mu_z + \tau_{yz}\mu_y + \tau_{zx}\mu_x)\delta w] dA$$
$$-\int_V \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta u + \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z}\right)\delta v + \left(\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial x}\right)\delta w\right] dV$$

$$\boxed{\delta U = \int_A \left(T_x^{\mu}\,\delta u + T_y^{\mu}\,\delta v + T_z^{\mu}\,\delta w\right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right) dV}$$

if the displacement components satisfy the equilibrium equations, the virtual strain energy is equal to the virtual work done by external force

Virtual Work in Elastic Body

✓**The Strain Energy**

$$U_{strain} = \frac{1}{2}\iiint_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dx\,dy\,dz$$

✓**Virtual Work**

δW **: the virtual work done by the external (surface and body) forces**

$$\delta W = \int_A \left(T_x^{\ \mu}\,\delta u + T_y^{\ \mu}\,\delta v + T_z^{\ \mu}\,\delta w\right)dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right)dA$$

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or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{xy}\,\delta\gamma_{xy} + \tau_{yz}\,\delta\gamma_{yz} + \tau_{zx}\,\delta\gamma_{zx}\right)dx\,dy\,dz$$

$$\delta U = \int_A [(\sigma_x\mu_x + \tau_{xy}\mu_y + \tau_{zx}\mu_z)\delta u + (\sigma_y\mu_y + \tau_{xy}\mu_x + \tau_{yz}\mu_z)\delta v + (\sigma_z\mu_z + \tau_{yz}\mu_y + \tau_{zx}\mu_x)\delta w]dA$$

$$\delta U = \int_A \left(T_x^{\ \mu}\,\delta u + T_y^{\ \mu}\,\delta v + T_z^{\ \mu}\,\delta w\right)dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w\right)dV$$

if the displacement components satisfy the equilibrium equations, the virtual strain energy is equal to the virtual work done by external force

Since the external forces are unchanged during the virtual displacement and the limits of integration are constant, the operator δ may be placed before the integral signs

$$\delta U = \delta\left[\int_A \left(T_x^{\ \mu} u + T_y^{\ \mu} v + T_z^{\ \mu} w\right)dA + \int_V \left(F_x u + F_y v + F_z w\right)dV\right]$$

$$= \delta W \ \ , \ where \ \ W = \int_A \left(T_x^{\ \mu} u + T_y^{\ \mu} v + T_z^{\ \mu} w\right)dA + \int_V \left(F_x u + F_y v + F_z w\right)dV$$

$$\therefore \delta U - \delta W = 0$$

Virtual Work in Elastic Body

✓The Strain Energy

$$U_{strain} = \frac{1}{2}\iiint_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dx\,dy\,dz$$

✓**Virtual Work**

δW : **the virtual work done by the external (surface and body) forces**

$$\delta W = \int_A \left(T_x^{\ \mu}\delta u + T_y^{\ \mu}\delta v + T_z^{\ \mu}\delta w\right)dA + \int_V \left(F_x \delta u + F_y \delta v + F_z \delta w\right)dA$$

δU : **the virtual strain energy or the strain energy absorbed in the body during a virtual displacement**

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right)dx\,dy\,dz$$

$$\delta U - \delta W = 0,\ \ W = \int_A \left(T_x^{\ \mu} u + T_y^{\ \mu} v + T_z^{\ \mu} w\right)dA + \int_V \left(F_x u + F_y v + F_z w\right)dV$$

$$\delta\Pi = 0$$

Defining $\Pi = U - W$ **called** ***<u>the potential energy of the body</u>***

what's this supposed to mean?

Virtual Work in Elastic Body

✓**The Strain Energy**

$$U_{strain}=\frac{1}{2}\iiint_V(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx})dx\,dy\,dz$$

✓**Virtual Work**

δW : **the virtual work done by the external (surface and body) forces**

$$\delta W=\int_A\left(T_x^{\ \mu}\delta u+T_y^{\ \mu}\delta v+T_z^{\ \mu}\delta w\right)dA+\int_V\left(F_x\delta u+F_y\delta v+F_z\delta w\right)dA$$

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$$\delta U=\iiint_V\left(\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\sigma_z\delta\varepsilon_z+\tau_{xy}\delta\gamma_{xy}+\tau_{yz}\delta\gamma_{yz}+\tau_{zx}\delta\gamma_{zx}\right)dx\,dy\,dz$$

$\therefore\ \delta\Pi=\delta(U-W)=0$ **,Defining** $\Pi=U-W$ **called** ***the potential energy of the body***

what's this supposed to mean?

$$\Pi=U-W$$

⇩ 1

$$\Pi=U_{strain}-W$$

⇩ 2

$$\Pi=U_{strain}-2W_{ext}$$

⇩ 3

$$\Pi=U_{strain}-2U_{strain}$$

⇩

$$\therefore\ \Pi=-U_{strain}$$

,where $U_{strain}=\dfrac{1}{2}\iiint_V\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}\right)dx\,dy\,dz$

1 $\delta U_{strain}=\delta\iiint_V\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}\right)dx\,dy\,dz$

,where $W=\int_A\left(T_x^{\ \mu}u+T_y^{\ \mu}v+T_z^{\ \mu}w\right)dA+\int_V\left(F_xu+F_yv+F_zw\right)dA$

$$W_{ext}=\frac{1}{2}\int_A\left(T_x^{\ \mu}u+T_y^{\ \mu}v+T_z^{\ \mu}w\right)dA+\frac{1}{2}\int_V\left(F_xu+F_yv+F_zw\right)dA$$

2 $\therefore\ W_{ext}=\dfrac{1}{2}W$

energy conservation

3 $W_{ext}=U_{strain}$

Virtual Work in Elastic Body

✓**The Strain Energy**

$$U_{strain}=\frac{1}{2}\iiint_V(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx})dx\,dy\,dz$$

$\therefore \delta\Pi=\delta(U-W)=0$,**Defining** $\Pi=U-W$ **called** *the potential energy of the body*

what's this supposed to mean?

$$\Pi=U-W$$

⇩ 1

$$\Pi=U_{strain}-W$$

⇩ 2

$$\Pi=U_{strain}-2W_{ext}$$

⇩ 3

$$\Pi=U_{strain}-2U_{strain}$$

⇩

$$\therefore\Pi=-U_{strain}$$

,where $U_{strain}=\frac{1}{2}\iiint_V\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}\right)dx\,dy\,dz$

1 $\delta U_{strain}=\delta\iiint_V\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}\right)dx\,dy\,dz$

,where $W=\int_A\left(T_x{}^{\mu}u+T_y{}^{\mu}v+T_z{}^{\mu}w\right)dA+\int_V\left(F_xu+F_yv+F_zw\right)dA$

$$W_{ext}=\frac{1}{2}\int_A\left(T_x{}^{\mu}u+T_y{}^{\mu}v+T_z{}^{\mu}w\right)dA+\frac{1}{2}\int_V\left(F_xu+F_yv+F_zw\right)dA$$

2 $\therefore W_{ext}=\frac{1}{2}W$

energy conservation

3 $W_{ext}=U_{strain}$

---

*c.f.)

- Total Potential Energy

$$\Pi=\boxed{\frac{1}{2}\int_V\varepsilon_{ij}\sigma_{ij}dV}\boxed{-\int_V u_ib_idV-\int_S u_iT_idS}$$

1 2

$$\Pi=\boxed{U_{strain}}-\boxed{2W_{ext}}$$

3

means

$$\Pi=U_{strain}-\boxed{2U_{strain}}$$

$$\therefore\Pi=-U_{strain}$$

- Energy Conservation

$$\boxed{\Pi_{ext}=\frac{1}{2}\int_S u_iT_idS+\frac{1}{2}\int_V u_ib_idV}$$

$$=\frac{1}{2}\int_S u_iT_idS-\frac{1}{2}\int_V u_i\frac{\partial\sigma_{ij}}{\partial x_j}dV$$

$$=\frac{1}{2}\int_S u_iT_idS+\frac{1}{2}\int_V\frac{\partial u_i}{\partial x_j}\sigma_{ij}dV-\frac{1}{2}\int_S u_i\sigma_{ij}n_jdS$$

$$=\frac{1}{2}\int_S u_iT_idS+\frac{1}{2}\int_V\frac{\partial u_i}{\partial x_j}\sigma_{ij}dV-\frac{1}{2}\int_S u_iT_idS$$

$$=\frac{1}{2}\int_V\frac{\partial u_i}{\partial x_j}\sigma_{ij}dV=\boxed{\frac{1}{2}\int_V\varepsilon_{ij}\sigma_{ij}dV=\Pi_{int}}$$

2

work done by external forces

W_{ext}

3 $W_{ext}=U_{strain}$ **same**

U_{strain}

strain energy stored in elastic body

1

Virtual Work and Principle of Potential Energy

✓**Virtual Work**

δW **: the virtual work done by the external (surface and body) forces**

$$\delta W = \int_A \left(T_x^{\mu} \delta u + T_y^{\mu} \delta v + T_z^{\mu} \delta w \right) dA + \int_V \left(F_x \delta u + F_y \delta v + F_z \delta w \right) dA$$

δU **: the virtual strain energy**
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy} \delta\gamma_{xy} + \tau_{yz} \delta\gamma_{yz} + \tau_{zx} \delta\gamma_{zx} \right) dx\, dy\, dz$$

$$\delta U = \delta W$$

$$\delta \Pi = \delta (U - W) = 0$$

this implies that
at the *<u>equilibrium</u>* configuration of a body, the *<u>potential energy</u>* assumes a *<u>stationary value</u>*

The principle of potential energy :

<u>Of all the displacement distribution satisfying the conditions of continuity and the prescribed displacement boundary conditions,</u>
<u>the one which actually takes place (or which satisfies the equilibrium equations) is the one which makes the potential energy assume a stationary(minimum) value)</u>

Discussion : Energy

internal force and external force*

Newton's third law asserts that if body *A* exert a force *F* on body *B*,
then body *B* exerts the force *-F* on body *A*.
This law signifies that forces may be mated, 'action' and 'reaction'

The reaction of a given force *F* is understood to act on the body that cause or exert force *F*.

if a force *F* acts on a mechanical system

its reaction *-F* acts on another part of the same system
or it acts on a body outside the system

in the first case, it is called an '<u>internal force</u>'

in the second case, it is called an '<u>external force</u>'

Accordingly, all the forces that act on a mechanical system may be classified as internal or external

Hence the work *W* of all the forces that act on a mechanical system is separated in a sum

$$W = W_e + W_i$$

Where

W_e is the work of the external forces and
W_i is the work of the internal forces

* Langhar, H.L., "Energy Methods in Applied Mechanics", John Wiley and Sons, 1962, p10-11

Discussion : Energy

Law of Kinetic Energy*

Newton's equation for a mass particle is $\mathbf{F} = m\dfrac{d\mathbf{v}}{dt}, \quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt}$

the infinitesimal work dW that the force $\mathbf{F}$ performs on the particle

$$dW = \mathbf{F} \bullet d\mathbf{r}$$

$$dW = \left(m\frac{d\mathbf{v}}{dt} \right) \bullet \left(\mathbf{v}\, dt \right) = m\mathbf{v} \bullet d\mathbf{v}$$

$let \;\; \mathbf{v} = v\mathbf{i}$

$$\therefore dW = \frac{d}{dt}\left(\frac{1}{2} m v^2 \right) dt$$

since, by definition: the kinetic energy of the particle is $T = \dfrac{1}{2} m v^2$

$$\therefore dW = dT$$

consequently, by integration, $W = \Delta T$

where, ΔT is the increment of kinetic energy that result from work W

* Langhar, H.L., "Energy Methods in Applied Mechanics", John Wiley and Sons, 1962, p10-11

Discussion : Energy

Law of Kinetic Energy*

$$W = \Delta T$$

where, ΔT is the increment of kinetic energy that result from work W

This conclusion may be generalized immediately to apply to all finite mechanical systems. The kinetic energy of any mechanical system is defined as the sum of the kinetic energies of its particles.
Consequently, by summing the equation $W = \Delta T$ over all the particles of a system, we obtain the following conclusion : $W = W_e + W_i$

The work of all the forces (internal and external) that act on a mechanical system equals the increase of kinetic energy of the system

This theorem is a modern statement of Leibniz's law of vis viva ;
it is called the 'law of kinetic energy'

$$W_e + W_i = \Delta T$$

* Langhar, H.L., "Energy Methods in Applied Mechanics", John Wiley and Sons, 1962, p10-11

Discussion : Energy

The first law of thermodynamics*

The work that is performed on a mechanical system by external forces plus the heat that flows into the system from the outside equals the increase of kinetic energy plus the increase of internal energy

$$W_e + Q = \Delta T + \Delta U$$

, W_e : the work performed on the system by external forces

, Q : the heat flows into the system

, ΔT : the increase of kinetic energy

, ΔU : the increase of internal energy

and $$W_e + W_i = \Delta T$$ **Where**

W_e **is the work of the external forces and**

W_i **is the work of the internal forces**

$$\therefore W_i = Q - \Delta U$$

if the system is adiabatic $, Q = 0$

$$\therefore W_i = -\Delta U$$

* Langhar, H.L., "Energy Methods in Applied Mechanics", John Wiley and Sons, 1962, p10-11

Discussion : Energy

The first law of thermodynamics : $W_e + Q = \Delta T + \Delta U$

Law of Kinetic Energy : $W = W_e + W_i$

The first law of thermodynamics applied to a deformation process*

If R is any region within a deformable body and S is the surface enclosing region R,
the external forces acting on the material in R consists of the tractive forces due to stresses on S and the body forces acting on material R.

The work that these forces perform when the virtual displacement is imposed is denoted by δW_e .

It will be supposed that the equilibrium conditions prevail during the displacement, and the kinetic energy is zero.

Then, by the first law of thermodynamics

$$\delta W_e = \delta U - Q$$

where δU is the increase of internal energy in R and

Q is the heat that flows into R while the virtual displacement is being performed

if the deformation is adiabatic, $\delta W_e = \delta U$

since $W_i = -\Delta U$

$$\delta W_e - \delta U = 0$$

$$\delta W_e + \delta W_i = 0$$

$$\therefore \delta W = 0$$

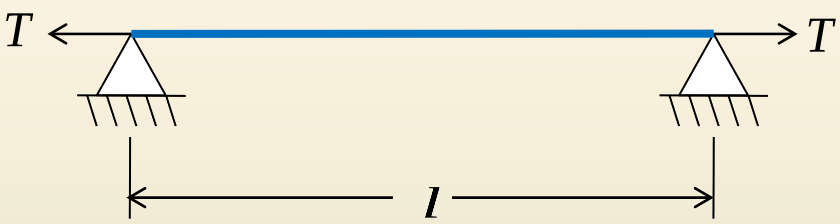
what this means?

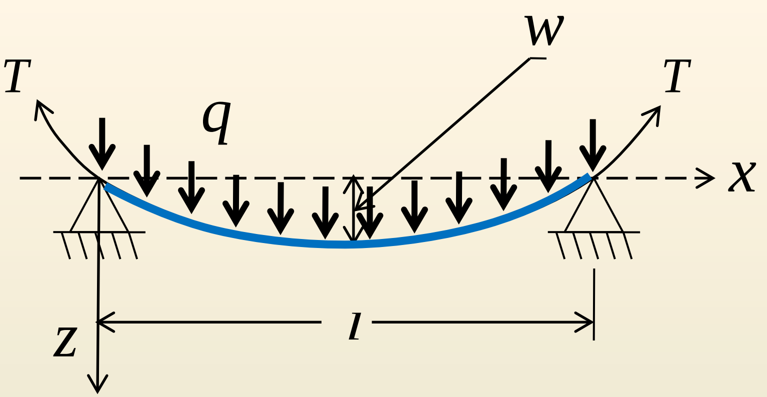
* Langhar, H.L., “Energy Methods in Applied Mechanics”, John Wiley and Sons, 1962, p115-116

Illustrative Problem

Uniform Loaded String : the application of the principle of potential energy

- **initially under a large tensile force** T
- **uniform transverse load** q
- **assume that the application of** q **does not change the magnitude of applied force** T
- **assume that body force is neglected**





Illustrative Problem

δW : the virtual work done by the external (surface and body) forces

$$\delta W = \int_A \left(T_x{}^{\mu} \delta u + T_y{}^{\mu} \delta v + T_z{}^{\mu} \delta w \right) dA + \int_V \left(F_x \delta u + F_y \delta v + F_z \delta w \right) dA$$

δU : the virtual strain energy
or the strain energy absorbed in the body during a virtual displacement

$$\delta U = \iiint_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy} \delta\gamma_{xy} + \tau_{yz} \delta\gamma_{yz} + \tau_{zx} \delta\gamma_{zx} \right) dx\,dy\,dz$$

$$\therefore \delta\Pi = \delta(U - W) = 0$$

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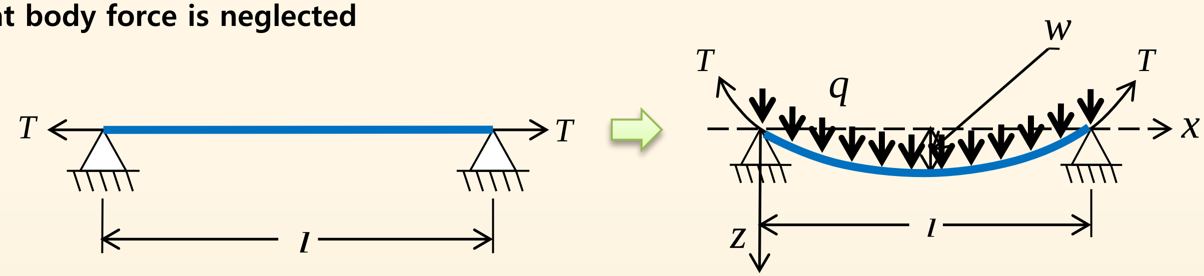

$\Pi = U - W$

① $\Pi = U_{strain} - W$

② $\Pi = U_{strain} - 2W_{ext}$

③ $\Pi = U_{strain} - 2U_{strain}$

$\therefore \Pi = -U_{strain}$

Considering the stretched string under tension T and $q = 0$ as the reference state

$$\Pi = U - W$$

since the surface force per unit length is q and there are no body force

$$W = \int_A \left(T_x{}^{\mu} u + T_y{}^{\mu} v + T_z{}^{\mu} w \right) dA = \int_0^l (qw)\, dx$$

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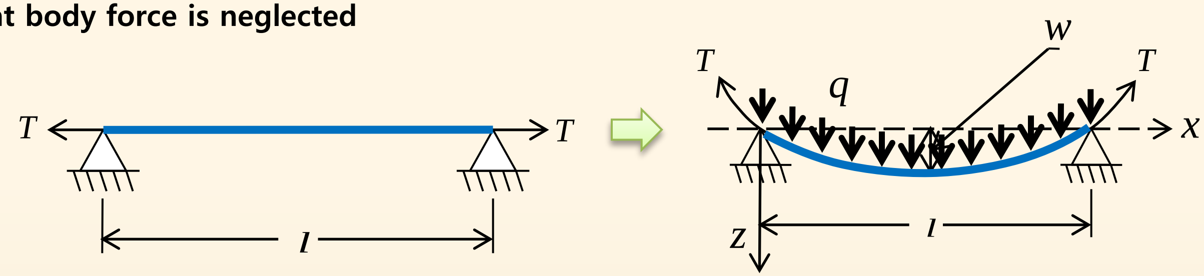

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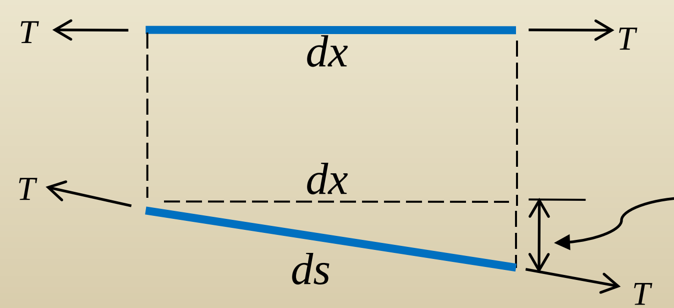
U : strain energy

in order to evaluate U we must determine the change in length of the string caused by the transverse load q



$$dw = \frac{dw}{dx} dx$$

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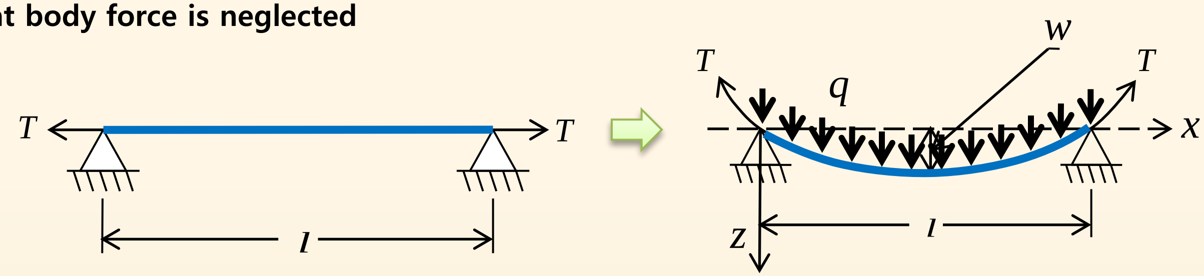

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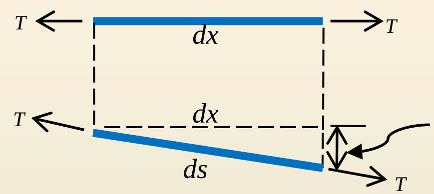

$$dw = \frac{dw}{dx}dx$$

$ds - dx$: the elongation of the element due to the application of q

$T \cdot (ds - dx)$: the internal work done by T on the element

notice that no facto of ½ is introduced, since T is constant during the displacement

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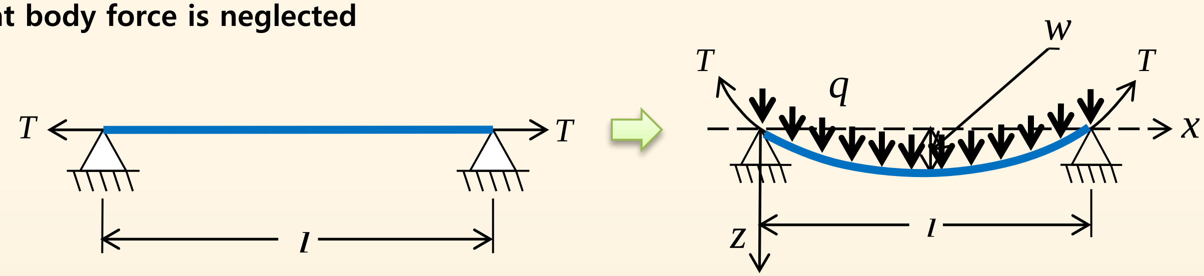

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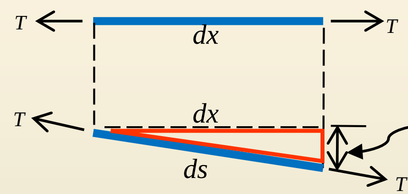

$$dw = \frac{dw}{dx}dx$$

recall, Taylor Series
$$f(x^* + \Delta x) = f(x^*) + f'(x^*)\Delta x + \frac{1}{2} f''(x^*)\Delta x^2 + .$$

$T \cdot (ds - dx)$: the internal work done by T on the element

$$ds = \sqrt{dx^2 + dw^2} = dx\sqrt{1 + (dw/dx)^2}$$

let, $z = \left(\dfrac{dw}{dx}\right)^2$ then, $\sqrt{1 + \left(\dfrac{dw}{dx}\right)^2} = \sqrt{1+z}$

let, $f(z) = \sqrt{1+z}$

$$f(0) = 1$$

$$f'(0) = \frac{1}{2}(1+z)^{-\frac{1}{2}}\Big|_{z=0} = \frac{1}{2}$$

$\vdots$

$$f(z) = 1 + \frac{1}{2}z + \frac{1}{2}\left(-\frac{1}{4}\right)z^2 + \cdots$$

$$f(z) \cong 1 + \frac{1}{2}z$$

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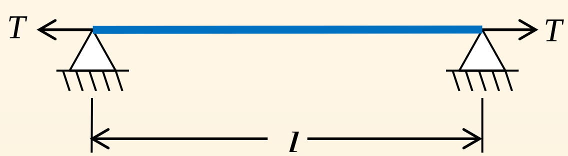

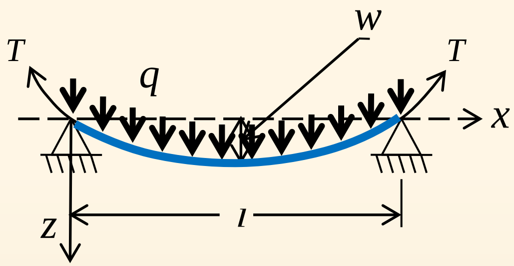

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$$\Pi = U - W \quad , W = \int_0^l (qw)\,dx$$

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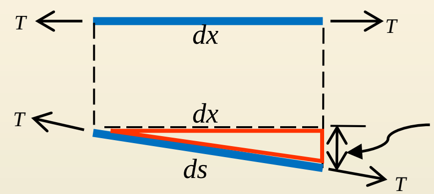

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$T\cdot(ds - dx)$: the internal work done by T on the element

$$ds = \sqrt{dx^2 + dw^2} = dx\sqrt{1 + (dw/dx)^2}$$

$$\therefore ds = dx\sqrt{1 + (dw/dx)^2} \cong dx\left(1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right)$$

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Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots$$

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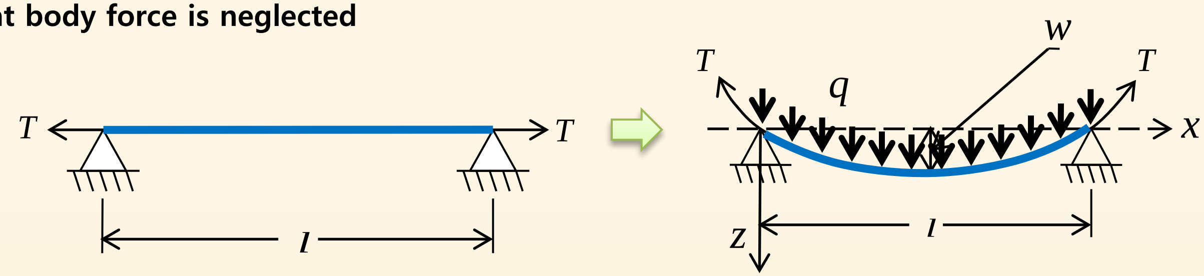

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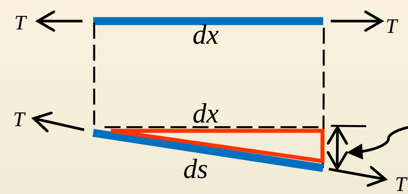

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$$ds = dx\left(1 + \frac{1}{2}\frac{dw}{dx}\right)$$

$$T\cdot(ds - dx) = T\cdot\left(dx\left(1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right) - dx \right)$$

$$= \frac{T}{2}\left(\frac{dw}{dx}\right)^2 dx$$

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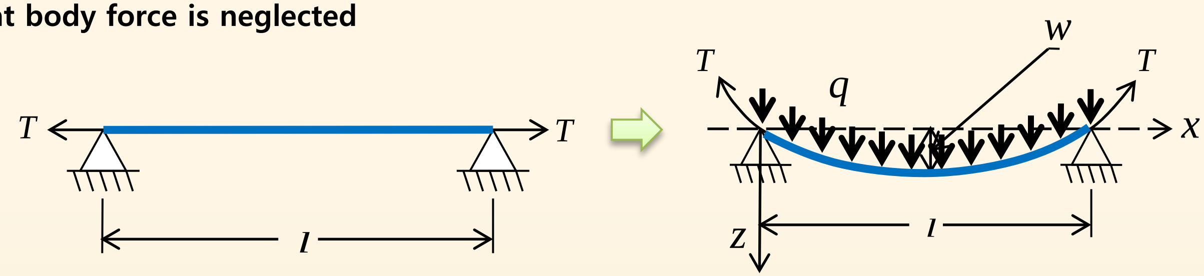

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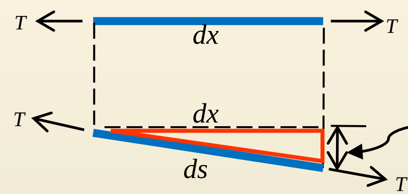

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$$ds = dx\left(1 + \frac{1}{2}\frac{dw}{dx}\right)$$

$$T \cdot (ds - dx) = \frac{T}{2}\left(\frac{dw}{dx}\right)^2 dx$$

$$\therefore U = \int_0^l T\,(ds - dx) = \frac{T}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx$$

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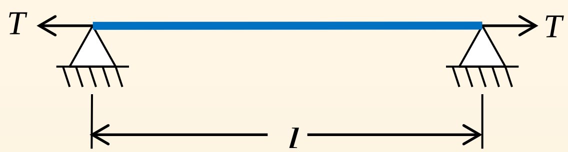

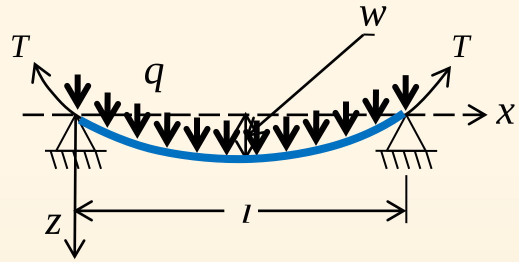

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$$\therefore \Pi = \frac{T}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx - \int_0^l (qw)\,dx$$

variation of Π :

$$\delta\Pi = T\int_0^l \frac{dw}{dx}\,\delta\!\left(\frac{dw}{dx}\right) dx - q\int_0^l \delta w\,dx$$

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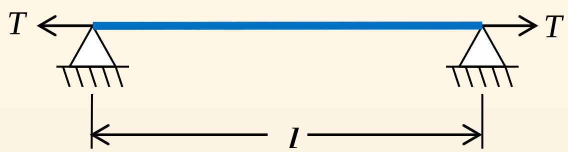

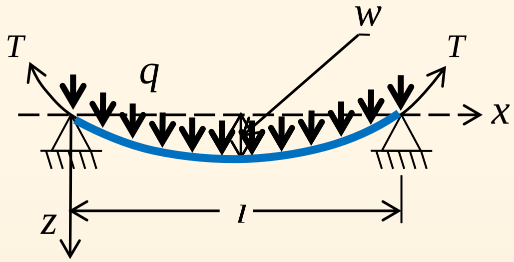

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variation of Π : $\quad \delta\Pi = T\boxed{\int_0^l \frac{dw}{dx}\delta\left(\frac{dw}{dx}\right)}dx - q\int_0^l \delta w\,dx$

$$T\boxed{\int_0^l \frac{dw}{dx}\delta\left(\frac{dw}{dx}\right)}dx = T\int_0^l \frac{dw}{dx}\left(\frac{d\delta w}{dx}\right)dx$$

recall, $\quad \therefore \frac{d}{dx}\delta y = \delta\frac{d}{dx}y$

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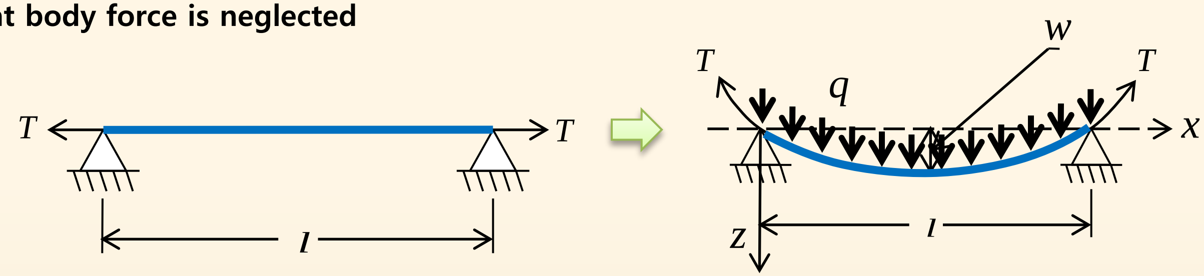

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③
$$\Pi = U_{strain} - 2U_{strain}$$

$$\therefore \Pi = -U_{strain}$$

Considering the stretched string under tension T and $q = 0$ as the reference state

$$\Pi = U - W \qquad \Pi = \frac{T}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx - \int_0^l (qw)\,dx$$

variation of Π :
$$\delta\Pi = T\boxed{\int_0^l \frac{dw}{dx}\delta\left(\frac{dw}{dx}\right)}dx - q\int_0^l \delta w\,dx$$

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since $\delta w = 0 \ at \ x = 0 \ and \ x = l$

Illustrative Problem

δW : the virtual work done by the external (surface and body) forces

$$\delta W = \int_A \left(T_x^{\,\mu}\,\delta u + T_y^{\,\mu}\,\delta v + T_z^{\,\mu}\,\delta w \right) dA + \int_V \left(F_x\,\delta u + F_y\,\delta v + F_z\,\delta w \right) dA$$

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the application of the principle of potential energy

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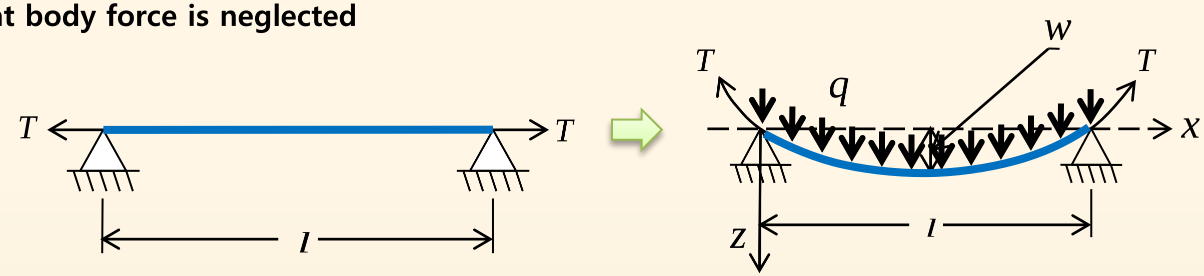

$$\Pi = U - W$$
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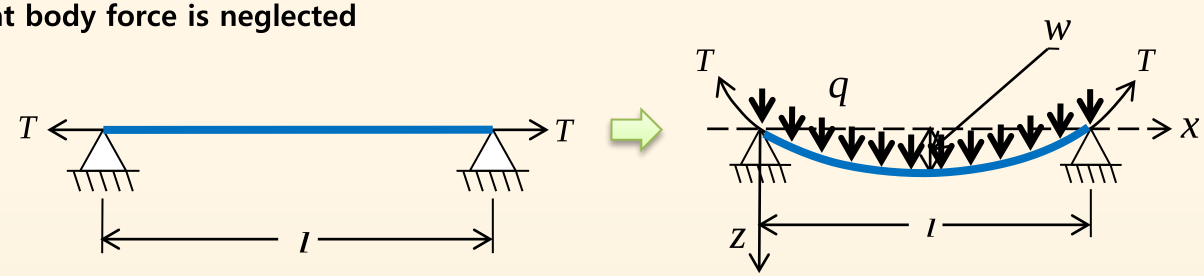

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from $\delta\Pi = 0$ $$-\int_0^l \left(T\frac{d^2w}{dx^2} + q\right)\delta w\,dx = 0$$

since δw is arbitrary $$\boxed{T\frac{d^2w}{dx^2} + q = 0}$$

recall, differential equation

$$\frac{d}{dx}\left(T\frac{dw}{dx}\right) + \rho\omega^2 w + q = 0$$

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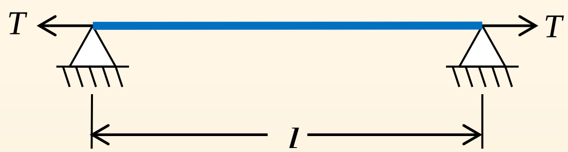

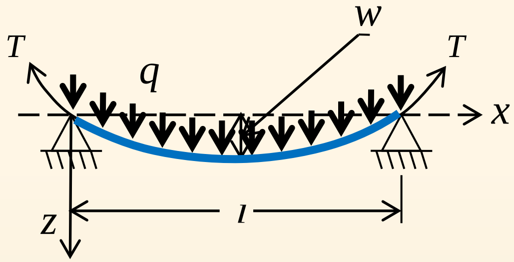

$$\boxed{T\frac{d^2w}{dx^2} + q = 0}$$

recall,

The principle of potential energy :

<u>Of all the displacement distribution satisfying the conditions of continuity and the prescribed displacement boundary conditions,</u>
<u>the one which actually takes place (or which satisfies the equilibrium equations) is the one which makes the potential energy assume a stationary(minimum) value)</u>

<u>**We shall now demonstrate that this stationary value is a minimum**</u>

In order to prove this, we shall show the quantity

$$\Delta\Pi = \Pi(w + \Delta w) - \Pi(w) \quad \text{is always positive}$$

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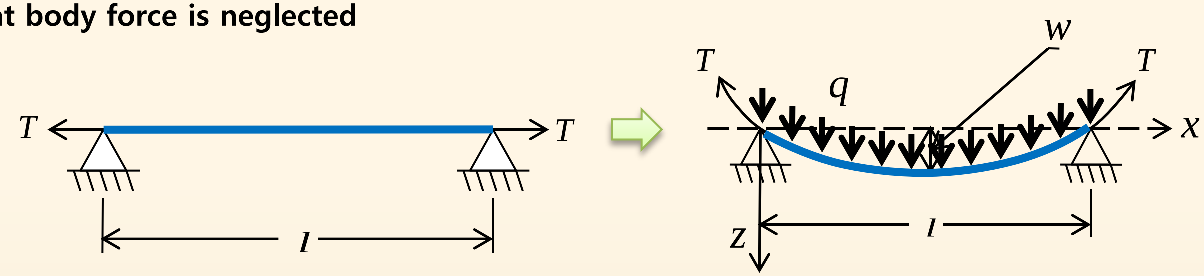

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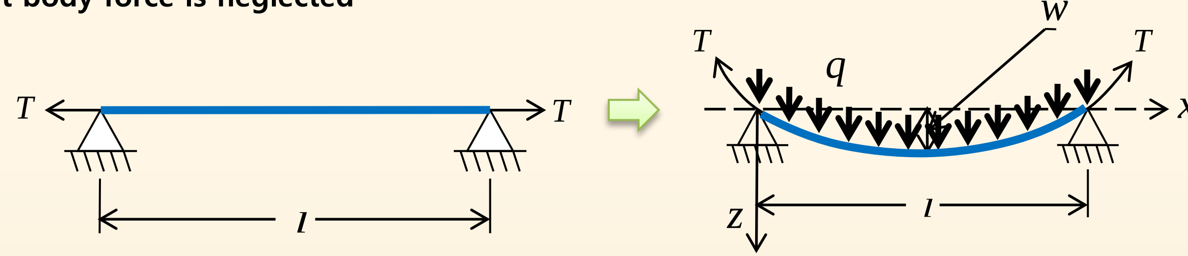

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$$= \left[\frac{T}{2}\int_0^l \left(\frac{dw}{dx} + \frac{d\Delta w}{dx}\right)^2 dx - \int_0^l (qw + q\Delta w)\,dx\right] - \left[\frac{T}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx - \int_0^l (qw)\,dx\right]$$

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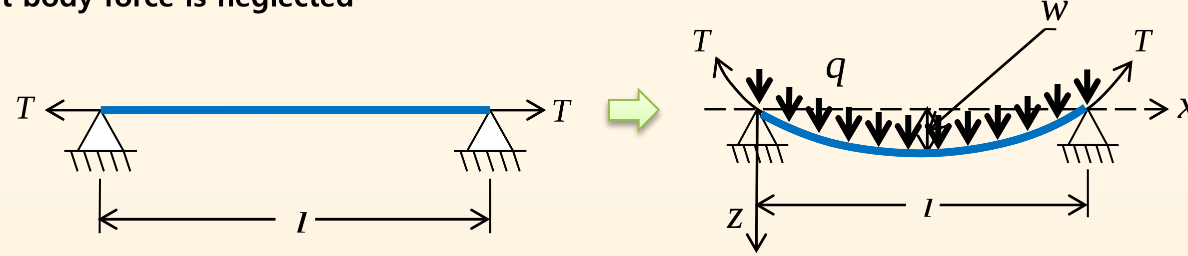

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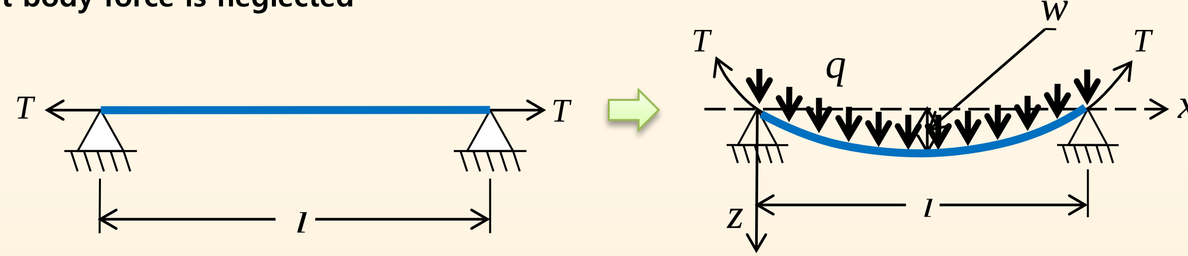

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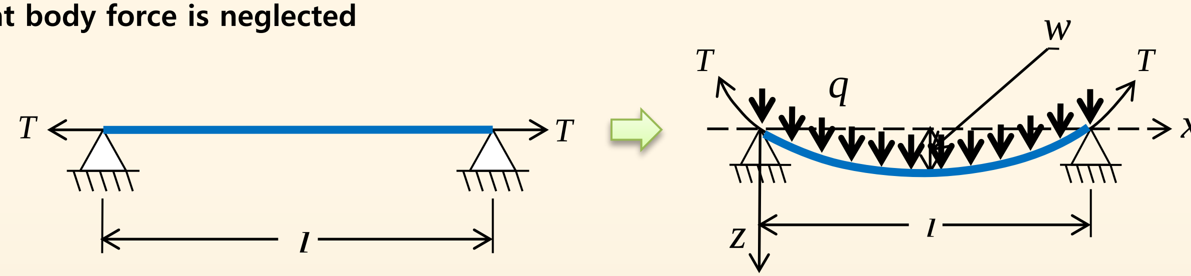

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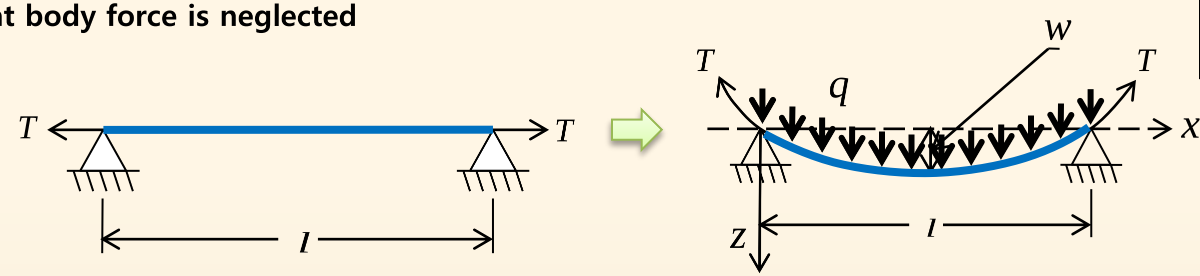

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because of equilibrium condition $T\dfrac{d^2w}{dx^2} + q = 0$

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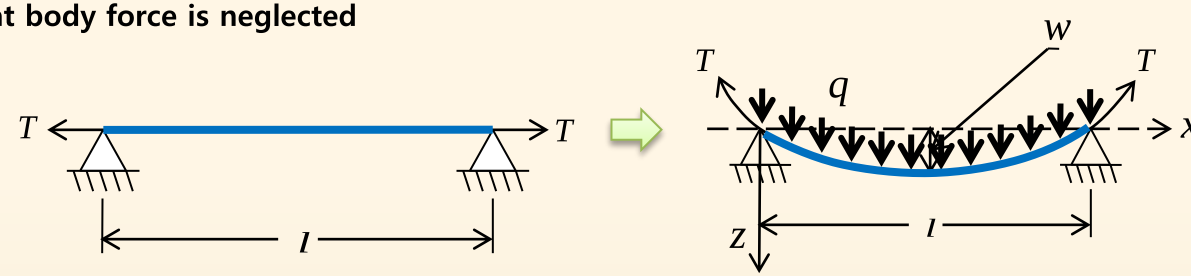

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since the integral cannot be negative $\quad \Delta\Pi \geq 0$

the integral vanishes only in the exceptional case when, $\dfrac{d\Delta w}{dx} = 0$

which only occurs at the equilibrium position

thus we have demonstrated the theorem of minimum potential energy

Illustrative Problem

δW : the virtual work done by the external (surface and body) forces

$$\delta W = \int_A \left(T_x^{\mu}\delta u + T_y^{\mu}\delta v + T_z^{\mu}\delta w\right)dA + \int_V \left(F_x\delta u + F_y\delta v + F_z\delta w\right)dA$$

δU : the virtual strain energy
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$$\delta U = \iiint_V \left(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right)dx\,dy\,dz$$

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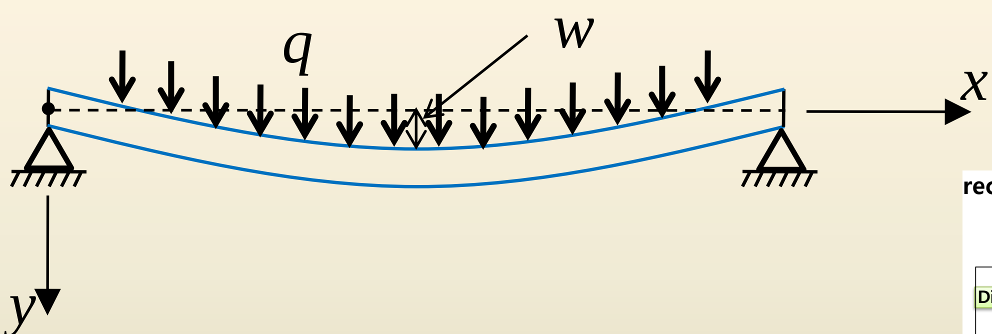

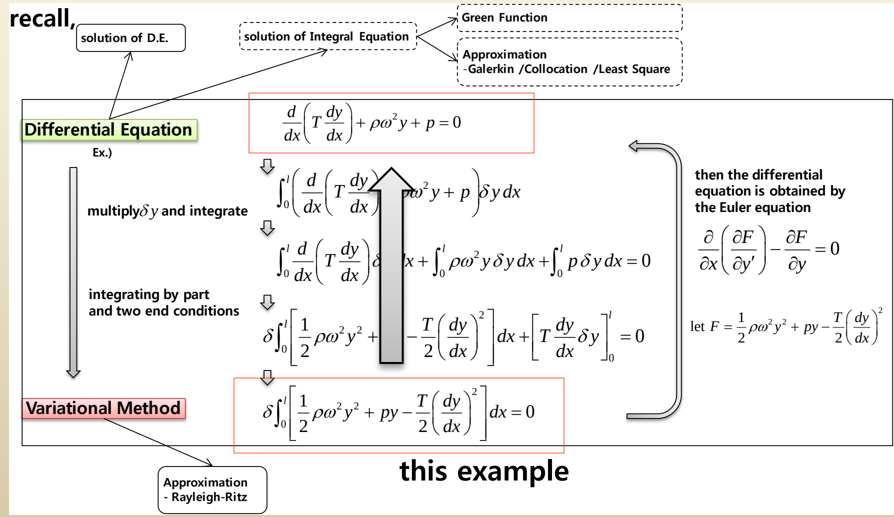

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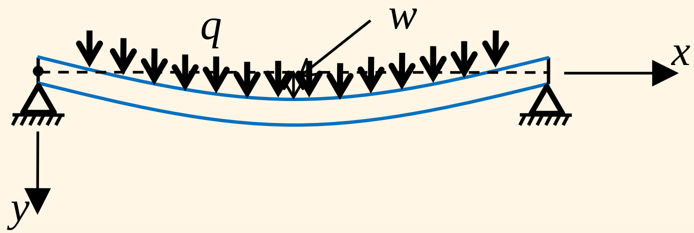
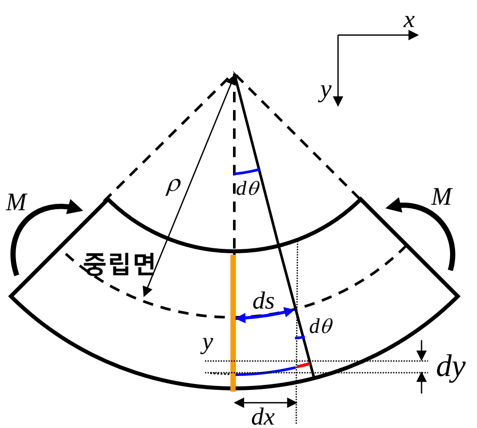


According to the Bernoulli-Euler law in beam theory $\quad M = EI \dfrac{d^2 w}{dx^2}$

recall,



$$\rho \cdot d\theta = ds \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$dF = \sigma dA = E \cdot \frac{y}{\rho} dA$$

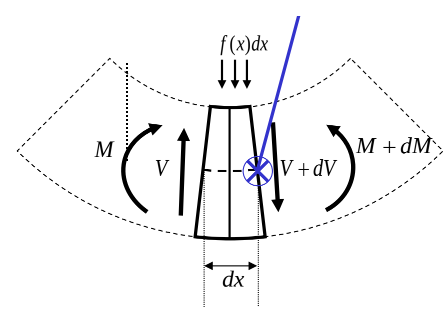
$$\therefore \frac{d\theta}{ds} = -\frac{M}{EI}$$

$$dM = -y\sigma dA \Rightarrow \frac{M}{EI} = -\frac{1}{\rho}$$

⑦ Assume that

$$ds \approx dx,\ \theta \approx \tan(\theta) = \frac{dy}{dx}$$

$$\therefore \frac{d\theta}{ds} = \frac{d^2 y}{dx^2} \Rightarrow \boxed{\frac{d^2 y}{dx^2} = -\frac{M}{EI}}$$



$f(x)$: 분포하중

$$\frac{dV}{dx} = -f(x) \qquad \frac{dM}{dx} = V(x) \qquad \boxed{\therefore EI \frac{d^4 y}{dx^4} = f(x)}$$

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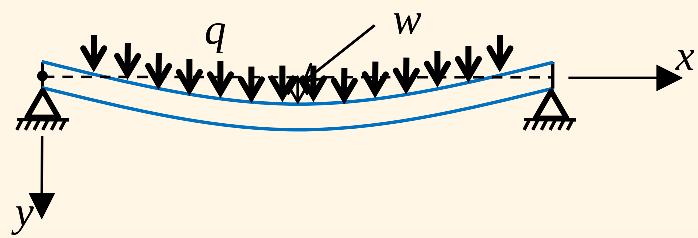

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$$\sigma_x = \frac{M}{I/y} = EI\,\frac{y}{I}\frac{d^2w}{dx^2}$$

$$\therefore \sigma_x = E\frac{d^2w}{dx^2}\,y$$

The strain energy per unit volume,
(the strain energy density)

$$U_0 = \frac{\sigma_x^{\,2}}{2E} = \frac{E^2}{2E}\left(\frac{d^2w}{dx^2}\right)^2 y^2$$

$$\therefore U_0 = \frac{E}{2}\left(\frac{d^2w}{dx^2}\right)^2 y^2$$

recall,

The strain energy per unit volume,
(the strain energy density)

$$U_0 = \frac{1}{2E}\sigma_x^{\,2} \quad \text{or } U_0 = \frac{1}{2}\sigma_x\varepsilon_x$$

$$U_0 = \frac{1}{2}E\varepsilon_x^2$$

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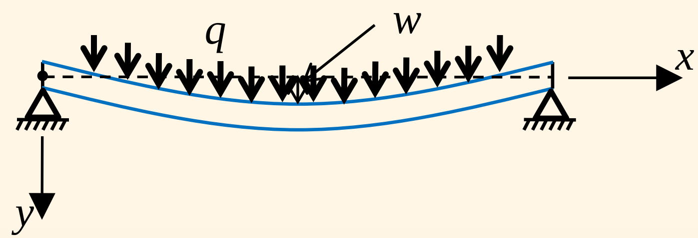

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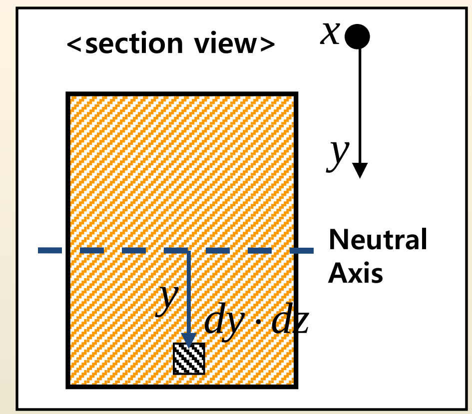

The strain energy per unit volume,
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$$U_0 = \frac{E}{2}\left(\frac{d^2w}{dx^2}\right)^2 y^2$$

The total Strain Energy absorbed in the beam

$$U = \iiint_V U_0\,dx\,dy\,dz = \iiint \frac{E}{2}\left(\frac{d^2w}{dx^2}\right)^2 y^2 dxdydz$$

since $\iint y^2 dydz = I$

$$\therefore U = \int_0^l \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx$$

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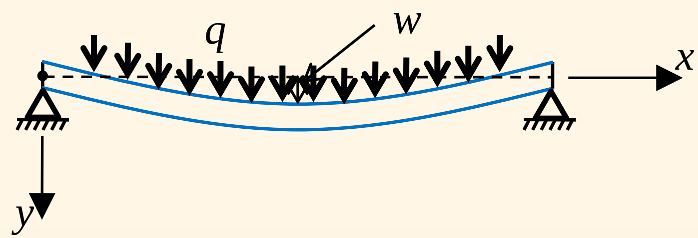

$$\Pi = U - W \qquad ,U = \int_0^l \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx$$

$$,W = \int_0^l qw\,dx$$

recall, uniform loaded string

$$\Pi = \int_0^l \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^l qw\,dx$$

$$\therefore \Pi = \int_0^l \left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 - qw\right]dx$$

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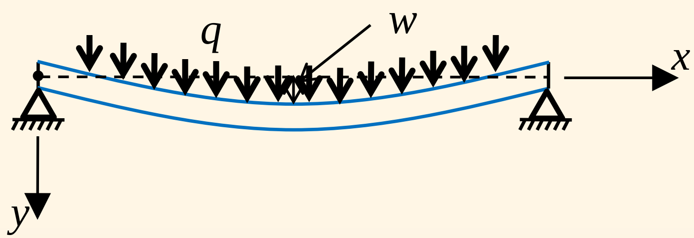

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variation of Π :

$$\delta\Pi = \frac{EI}{2}\int_0^l 2\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}dx - \int_0^l q\delta w dx$$

integrating by part

$$\delta\Pi = EI\boxed{\int_0^l \frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}dx} - \int_0^l q\delta w dx$$

$$\delta\Pi = EI\int_0^l \frac{d^4w}{dx^4}\delta w dx - \int_0^l q\delta w dx$$

integrating by part

$$\boxed{\int_0^l \frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}dx}$$

boundary condition
simple support :

$$\frac{d^2w}{dx^2} = 0 \ at \ x = 0 \text{ and } x = l$$

$$\int_0^l \frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}dx = \left[\frac{d^2w}{dx^2}\frac{d\delta w}{dx}\right]_0^l - \int_0^l \frac{d^3w}{dx^3}\frac{d\delta w}{dx}dx$$

$$= -\int_0^l \frac{d^3w}{dx^3}\frac{d\delta w}{dx}dx$$

$$= -\left\{\left[\frac{d^4w}{dx^4}\delta w\right]_0^l - \int_0^l \frac{d^4w}{dx^4}\delta w dx\right\}$$

$$= \int_0^l \frac{d^4w}{dx^4}\delta w dx$$

since

$$\delta w = 0 \ at \ x = 0 \ and \ x = l$$

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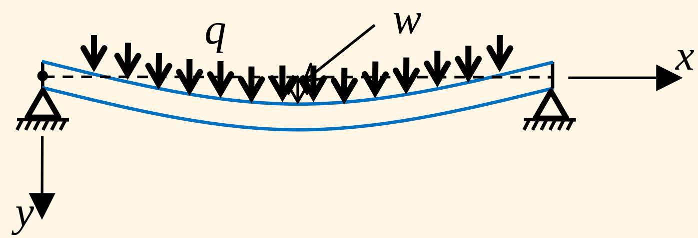

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variation of Π :

$$\delta\Pi = EI\int_0^l \frac{d^4w}{dx^4}\delta w dx - \int_0^l q\delta w dx \quad \Rightarrow \quad \therefore \delta\Pi = \int_0^l \left[EI\frac{d^4w}{dx^4} - q\right]\delta w dx$$

from $\delta\Pi = 0$ $\quad \int_0^l \left[EI\dfrac{d^4w}{dx^4} - q\right]\delta w dx = 0$

since δw is arbitrary

$$\boxed{EI\frac{d^4w}{dx^4} - q = 0}$$

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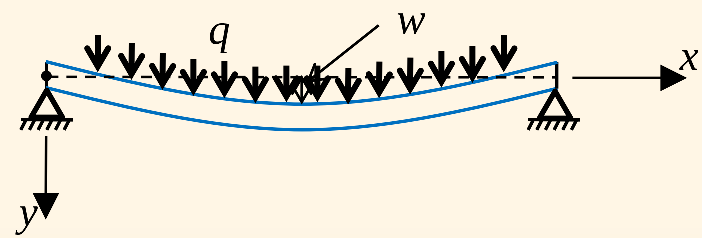

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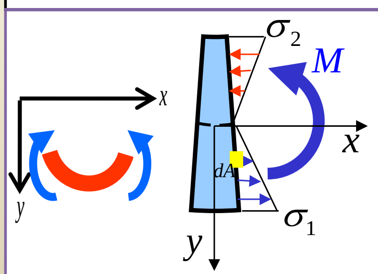
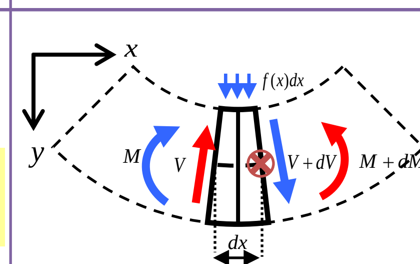
임상전 편저, 재료역학, 2002년 ,문운당 (Timoshenko S., Young D.H., Elements of strength of materials, 5th edition, Van Nostrand, 1968)

	B.M.	K =1/ρ	y	ε	σ	check	dM= yσdA	dM	$M=\int_A dM$	relation btw V, M, f(x)	
	$\oplus$ ≠	-	-	κy or $\dfrac{y}{\rho}$	-	comp.	$\oplus$	$\sigma y dA$	$\dfrac{d^2y}{dx^2} = \oplus\dfrac{M}{EI}$		$-V + f(x)dx + (V+dV) = 0$ $\Rightarrow \dfrac{dV}{dx} = -f(x)$
			+		+	tension			$\dfrac{d^2y}{dx^2} = \ominus\dfrac{M}{EI}$		$M-(M+dM)+Vdx-f(x)dx\cdot\dfrac{1}{2}dx=0$ $\Rightarrow \dfrac{dM}{dx} = V(x)$ $\boxed{EI\dfrac{d^4y}{dx^4} = f(x)}$

match

modify considering the curvature

all sign convention is same
except y-axis in opposite direction

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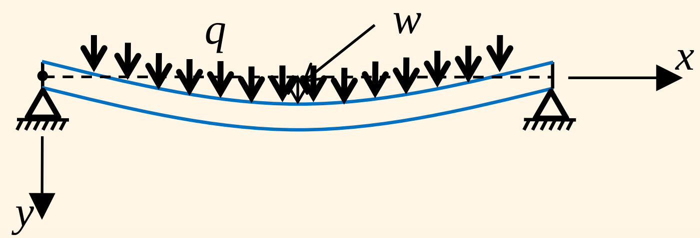
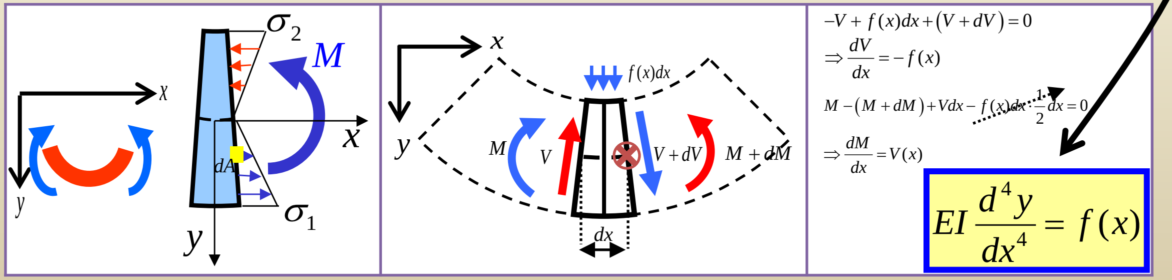


$$\delta\Pi = \int_0^l \left[EI\frac{d^4w}{dx^4} - q\right]\delta w dx \quad \text{from} \quad \delta\Pi = 0 \quad \text{since} \quad \delta w \text{ is arbitrary} \quad \boxed{EI\frac{d^4w}{dx^4} - q = 0}$$

The condition $\delta\Pi = 0$ leads directly to the governing equilibrium equation in terms of displacement



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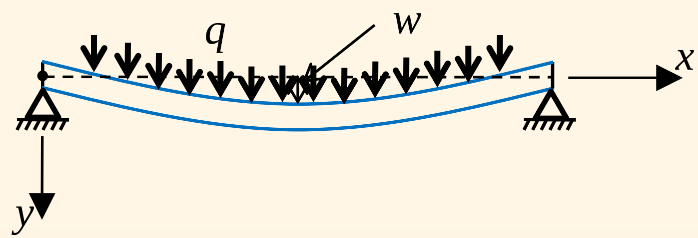
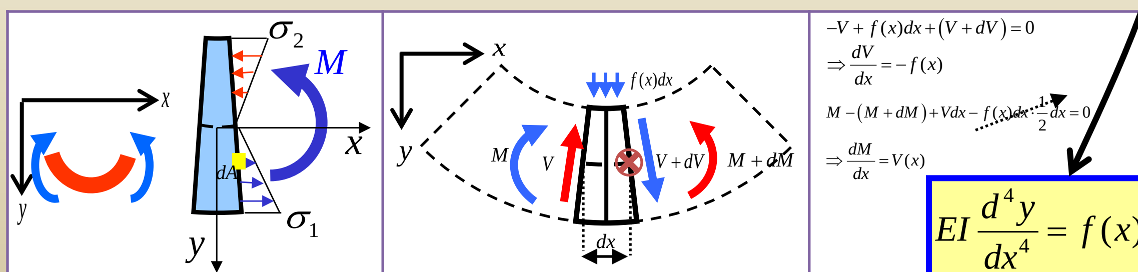


$$\delta\Pi = \int_0^l \left[EI \frac{d^4 w}{dx^4} - q \right] \delta w dx$$ from $$\delta\Pi = 0$$ since δw is arbitrary $$\boxed{EI \frac{d^4 w}{dx^4} - q = 0}$$

If it is difficult to find a solution for the equilibrium equations, we can find an approximate solution which satisfies the equation $\delta\Pi = 0$

approximation method



CALCULUS OF VARIATION
- EQUATION OF EULER-LAGRANGE
- HAMILTON'S PRINCIPLE

Calculus of Variation

- Mechanical System의 운동 방정식 유도

Newton's 2nd law

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

D'Alembert's Principle

Virtual work

$$\delta W = \sum_{i=1}^{N} (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$$

Hamilton's Principle

- Mechanical System에 가해지는 힘은 보존력임
- System의 초기 상태와 최종 상태는 정의되어 있음

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad define: L = T - V$$

(T: Kinetic energy, V: Potential energy)

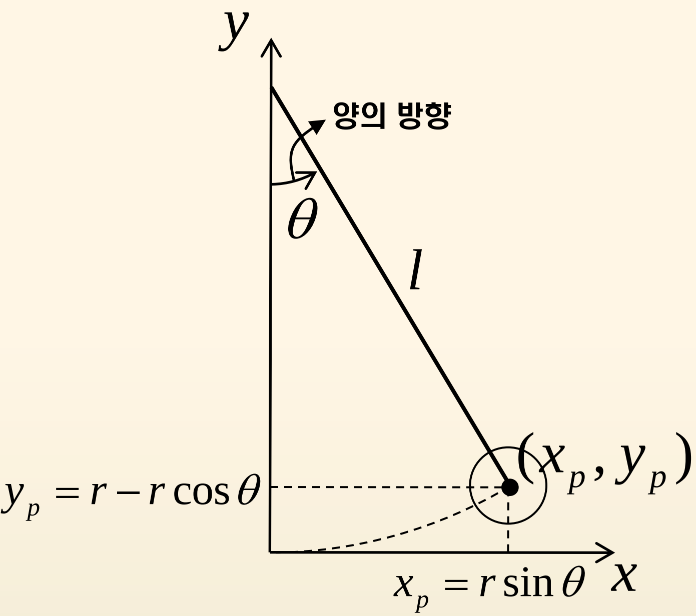
Equation of Euler-Lagrange

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

보존력이 작용하는 Mechanical System의
Kinetic energy(T)**와** Potential energy(V)**의 차이**(L)를
Equation of Euler-Lagrange**에 대입**후 정리하면
Mechanical System의 **운동 방정식을 유도**할 수 있음

진자(Pendulum)의 운동 방정식 유도 - 예시

보존력이 작용하는 Mechanical System의 Kinetic energy(T)와 Potential energy(V)의 차이(L)를 Equation of Euler-Lagrange에 대입후 정리하면 Mechanical System의 운동 방정식을 유도할 수 있음



$$define: L = T - V$$

$$T = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}mr^2\dot{\theta}^2, \quad V = mg(l - l\cos\theta)$$

$$\dot{x}_p = l\cos\theta\cdot\dot{\theta}$$

$$\dot{y}_p = l\sin\theta\cdot\dot{\theta}$$

$$\dot{\mathbf{r}}^2 = \dot{x}_p^2 + \dot{y}_p^2 = l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta = l^2\dot{\theta}^2$$

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mg(l - l\cos\theta) \text{----} ①$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \text{----} ②$$

(단, $q = \theta$)

식①은 ②를 만족함

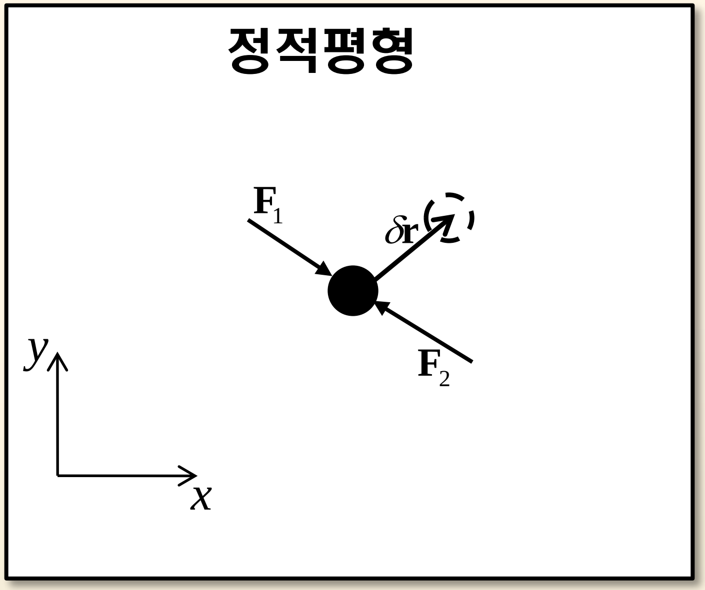
$$-mgl\sin\theta - \frac{d}{dt}ml^2\dot{\theta} = 0$$

$$-mgl\sin\theta - ml^2\ddot{\theta} = 0$$

$$-g\sin\theta - l\ddot{\theta} = 0$$

$$\boxed{-\frac{g}{l}\sin\theta = \ddot{\theta}}$$

Virtual work and D'Alembert's Principle



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$$

힘의 합력이 0이므로 물체는 정지해 있음
"**정적 평형 상태**"

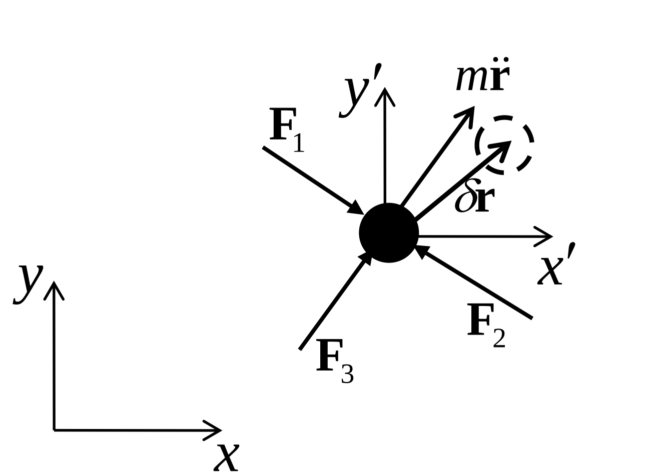
물체를 미소 변위$\delta\mathbf{r}$ 만큼 움직인다고 가정 하면
질점에 작용하는 모든 힘이 한 일은 다음과 같다.

$$\begin{aligned}\delta W &= \mathbf{F}\cdot\delta\mathbf{r}\\ &= \mathbf{F}_1\cdot\delta\mathbf{r} + \mathbf{F}_2\cdot\delta\mathbf{r}\\ &= 0\end{aligned}$$

가상일의 원리

Virtual work and D'Alembert's Principle

동적평형



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = m\ddot{\mathbf{r}}$$

해석

- 질량과 가속도의 곱은 물체에 가해지는 힘과 같다.
- 물체에 F라는 힘이 가해져서 m의 질량을 가진 물체가 가속도 운동을 한다.

만일 물체에 고정되어 있는 좌표계(x',y')에서 물체를 바라본다면?

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 - m\ddot{\mathbf{r}} = 0$$

해석

물체에 외력 이외에 **관성력($-m\ddot{\mathbf{r}}$)**이 가해져서 힘의 합력은 0이고, 움직이지 않는다.
"**동적 평형 상태**"

D'Alembert's Principle

동적 평형 상태에 가상일의 원리를 적용

$$\delta W = (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 - m\ddot{\mathbf{r}})\delta\mathbf{r}$$
$$= (\mathbf{F} - m\ddot{\mathbf{r}})\cdot\delta\mathbf{r} = 0$$

외력과 관성력이 한 가상 일(Virtual work)의 총 합은 0이다.

$$\delta W = (\mathbf{F} - m\ddot{\mathbf{r}})\delta\mathbf{r} = 0$$

Hamilton's Principle (1/3)

부분 적분

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$
$$= (\text{적분})(\text{그대로}) - \int (\text{적분})(\text{미분})dx$$

$$\delta W = \sum_{i=1}^{N} (\mathbf{F}_i - m\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i = 0 \quad \Leftarrow$$

From D'Alembert's principle

양변을 t1 부터 t2까지 적분하면

$$\int_{t_1}^{t_2} \delta W\, dt = \int_{t_1}^{t_2} \sum_{i=1}^{N} (\mathbf{F}_i - m\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i\, dt$$
$$= \boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} \mathbf{F}_i \cdot\delta\mathbf{r}_i\, dt} - \boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} m\ddot{\mathbf{r}}_i \cdot\delta\mathbf{r}_i\, dt}$$

$$\boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} \mathbf{F}_i \cdot\delta\mathbf{r}_i\, dt} = \int_{t_1}^{t_2} \delta U\, dt = -\int_{t_1}^{t_2} \delta V\, dt$$

작용하는 외력(F)**이 보존력이라면** U = -V**가 성립한다.** ▷
U: Work function, V: Potential energy

$$\boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} m\ddot{\mathbf{r}}_i \cdot\delta\mathbf{r}_i\, dt} = \cancel{\left[\sum_{i=1}^{N} m\dot{\mathbf{r}}_i \cdot\delta\mathbf{r}_i\right]_{t_1}^{t_2}} - \int_{t_1}^{t_2} \sum_{i=1}^{N} m\dot{\mathbf{r}}_i \cdot \frac{d}{dt}(\delta\mathbf{r}_i)\, dt$$

(부분 적분 적용)

$$\because\ \delta\mathbf{r}_i(t_1) = \delta\mathbf{r}_i(t_2) = 0$$

시스템의 초기 상태와 마지막 상태는 정의되어 있으며, 변분에 의해 변화하지 않는다.

Hamilton's Principle (2/3)

$$\delta W = \sum_{i=1}^{N} (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$$

양변을 t1 부터 t2까지 적분하면

$$\int_{t_1}^{t_2} \delta W \, dt = \int_{t_1}^{t_2} \sum_{i=1}^{N} (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i \, dt$$

$$= \boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i \, dt} - \boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \, dt}$$

$$= -\delta \int_{t_1}^{t_2} V \, dt + \delta \int_{t_1}^{t_2} T \, dt$$

$$\boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i \, dt} = \int_{t_1}^{t_2} \delta U \, dt = -\int_{t_1}^{t_2} \delta V \, dt$$

작용하는 외력(F)**이 보존력이라면** U = -V**가 성립한다.**
U: Work function, V: Potential energy

$$\boxed{\int_{t_1}^{t_2} \sum_{i=1}^{N} m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \, dt} = -\int_{t_1}^{t_2} \sum_{i=1}^{N} m\dot{\mathbf{r}}_i \cdot \frac{d}{dt}(\delta \mathbf{r}_i) \, dt = -\int_{t_1}^{t_2} \sum_{i=1}^{N} m\dot{\mathbf{r}}_i \cdot \delta \dot{\mathbf{r}}_i \, dt = -\int_{t_1}^{t_2} \sum_{i=1}^{N} m \frac{1}{2} \delta(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \, dt$$

$$\because \delta(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) = (\delta \dot{\mathbf{r}}_i) \cdot \dot{\mathbf{r}}_i + \dot{\mathbf{r}}_i \cdot (\delta \dot{\mathbf{r}}_i) = 2\dot{\mathbf{r}}_i \cdot (\delta \dot{\mathbf{r}}_i)$$

$$= -\delta \int_{t_1}^{t_2} \sum_{i=1}^{N} \frac{1}{2} m \dot{\mathbf{r}}_i^2 \, dt = -\delta \int_{t_1}^{t_2} T \, dt$$

T: Kinetic energy

Hamilton's Principle (3/3)

$$\delta W = \sum_{i=1}^{N} (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$$

양변을 t1 부터 t2까지 적분하면

$$\int_{t_1}^{t_2} \delta W \, dt = \int_{t_1}^{t_2} \sum_{i=1}^{N} (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i \, dt$$

$$= \int_{t_1}^{t_2} \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i \, dt - \int_{t_1}^{t_2} \sum_{i=1}^{N} m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \, dt$$

$$= -\delta \int_{t_1}^{t_2} V \, dt + \delta \int_{t_1}^{t_2} T \, dt$$

$$= \delta \int_{t_1}^{t_2} T - V \, dt = 0$$

$$= \delta \int_{t_1}^{t_2} L \, dt = 0$$

$L = T - V$ 라고 정의 하면

V: Potential energy
T: Kinetic energy

참고(Equation of Euler-Lagrange)

•Given: $I = \int_a^b F(y, y', x) dx$

$f(a) = \alpha, \quad f(b) = \beta$

•Find: 적분 I가 stationary value를 갖도록 하는 $y = f(x)$

• F의 정적분값의 변화율(δI)이 0이 되도록 하는 $y = f(x)$를 찾아야 함

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

초기 상태와 최종 상태가 정의되어 있는 Mechanical System의 Potential energy와 Kinetic energy의 차이를 L이라 정의하면 L의 정적분값은 stationary value가 된다.
(L의 정적분값의 변화율은 0이 된다.)

The Calculus of Variation
- Equation of Euler-Lagrange (1/6)

$$\therefore \delta \int_a^b F(x)dx = \int_a^b \delta F(x)dx$$

The stationary value of a definite integral treated by the calculus of variation

•Given: $$I = \int_a^b F(y, y', x)dx$$

$$\text{where } \; y = f(x), \;\; f(a) = \alpha, \;\; f(b) = \beta$$

•Find: 적분 I가 stationary value를 갖도록 하는 $y = f(x)$

• $y = f(x)$가 I 가 stationary value를 갖도록 하는 함수라 가정한다.

•y의 변분(variation) δy를 고려한다.

•y가 $y + \delta y$로 이동했을 때, 적분 I의 변분 δI 를 살펴본다.
- •함수 F 의 변분 δF의 값을 살펴본다.
- •δI의 계산식을 구한다.

•적분 I의 변화율$(\delta I / \varepsilon)$이 0이 되도록 하는 조건을 만족하는 $y = f(x)$ 가 적분 I가 stationary value를 갖도록 한다.

The Calculus of Variation
- Equation of Euler-Lagrange (2/6)

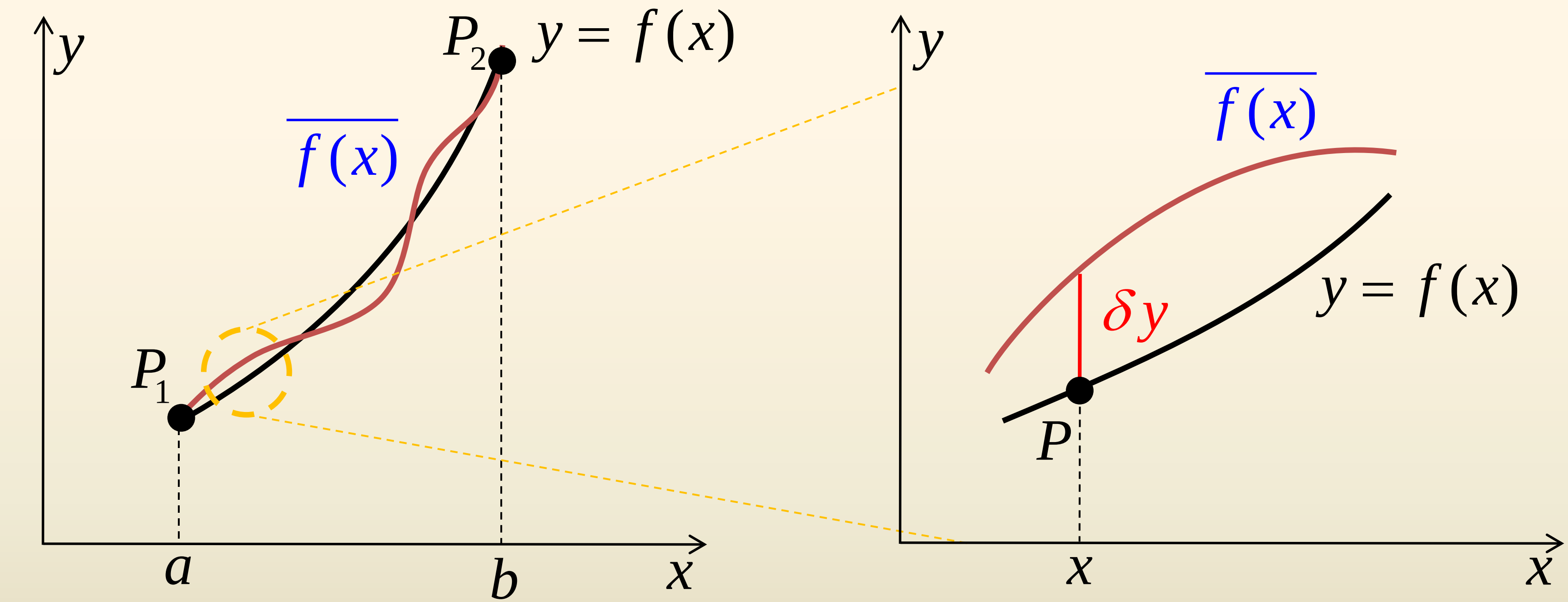
•Given: $I = \int_a^b F(y, y', x)dx$

$$f(a) = \alpha, \quad f(b) = \beta$$

•Find: 적분 I가 stationary value를 갖도록 하는 $y = f(x)$

• $y = f(x)$가 I 가 stationary value를 갖도록 하는 함수라 가정한다.

• y의 변분(variation) δy를 고려한다.



The given function: $y = f(x)$

The modified function: $\overline{f(x)} = f(x) + \varepsilon\phi(x)$

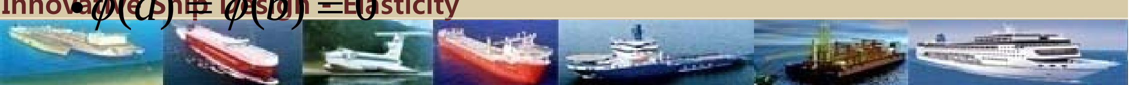
• $\Phi(x)$: arbitrary new function, continuous and differentiable.

• $\phi(a) = \phi(b) = 0$

변분 δy :

$$\delta y = \overline{f(x)} - f(x) = \varepsilon\phi(x)$$

•parameter ε 이 감소함에 따라 0으로 접근(infinitesimal change), 임의의 방향으로 변경(virtual change)

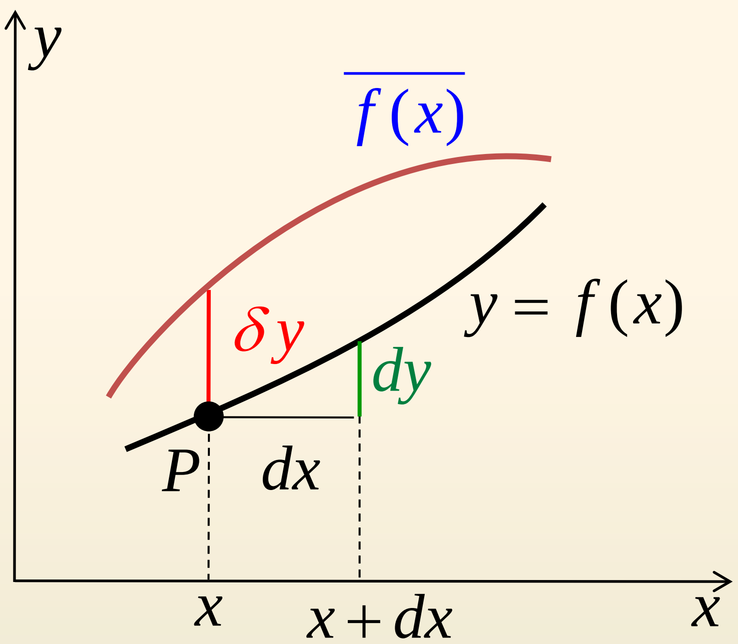


The Calculus of Variation
- Equation of Euler-Lagrange (3/6)

•Given: $I = \int_a^b F(y, y', x)dx$

$$f(a) = \alpha, \quad f(b) = \beta$$

•Find: 적분 I가 stationary value를 갖도록 하는 $y = f(x)$



•dy 와 δy

•dy : 주어진 함수 $y = f(x)$의 독립변수 x가 미소변위 dx만큼 이동하여 생기는, 함수 $f(x)$ 의 변화량

•δy : 주어진 함수 $y = f(x)$에 새로운 함수를 더함으로써 생긴 변화량

The Calculus of Variation
- Equation of Euler-Lagrange (4/6)

•Given: $I = \int_a^b F(y, y', x)dx$

$$f(a) = \alpha, \quad f(b) = \beta$$

•Find: 적분 I가 stationary value를 갖도록 하는 $y = f(x)$

$$\left.\begin{aligned} y &= f(x) \\ \overline{f}(x) &= f(x) + \varepsilon\phi(x) \end{aligned}\right\}$$

$$\delta y = \varepsilon\phi(x)$$

$$\phi(a) = \phi(b) = 0$$

•y가 $y + \delta y$로 이동했을 때, 적분 I의 변분 δI 를 살펴본다.
- •함수 F 의 변분 δF의 값을 살펴본다.
- •δI의 계산식을 구한다.

•함수 F 의 변분 δF

$$\delta F(y, y', x) = F(y + \varepsilon\phi, y' + \varepsilon\phi', x) - F(y, y', x)$$

$\Downarrow$ Taylor series expansion 이용

$$= \frac{\partial F}{\partial y}\varepsilon\phi + \frac{\partial F}{\partial y'}\varepsilon\phi' + \cancel{\frac{1}{2}\left(\frac{\partial^2 F}{\partial y^2}\varepsilon\phi + 2\frac{\partial^2 F}{\partial y \partial y'}\varepsilon\phi\varepsilon\phi' + \frac{\partial^2 F}{\partial y'^2}\varepsilon\phi'\right)} + \cdots$$

ε이 작으므로, 고차항은 무시함

$$= \varepsilon\left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right)$$

The Calculus of Variation - Equation of Euler-Lagrange (5/6)

•Given: $I = \int_a^b F(y, y', x)dx$

$f(a) = \alpha, \quad f(b) = \beta$

•Find: 적분 I가 stationary value를 갖도록 하는 $y = f(x)$

•함수 F 의 변분 δF:

$$\delta F(y, y', x) = \varepsilon\left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right)$$

•y가 $y + \delta y$로 이동했을 때, 적분 I의 변분 δI 를 살펴본다.
- •함수 F 의 변분 δF의 값을 살펴본다.
- •δI의 계산식을 구한다.

•적분 I 의 변분 δI

$$\delta I = \delta\int_a^b F dx = \int_a^b \delta F dx = \varepsilon\int_a^b\left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right)dx$$

•적분 I의 변화율(양변을 ε으로 나눔)

$$\frac{\delta I}{\varepsilon} = \int_a^b\left(\frac{\partial F}{\partial y}\phi + \underline{\frac{\partial F}{\partial y'}\phi'}\right)dx$$

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$
$$= (\text{적분})(\text{그대로}) - \int(\text{적분})(\text{미분})dx$$

부분 적분을 이용하여 구함.

$$\int_a^b \frac{\partial F}{\partial y'}\phi' dx = \left[\frac{\partial F}{\partial y'}\phi\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\phi dx$$

0 $(\because \phi(a) = \phi(b) = 0)$

대입

$$\therefore \frac{\delta I}{\varepsilon} = \int_a^b\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\phi dx$$

The Calculus of Variation
- Equation of Euler-Lagrange (6/6)

•적분 I의 변화율: $$\frac{\delta I}{\varepsilon}=\int_a^b\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}\right)\phi dx$$

•적분 I의 변화율(δI)이 0이 되도록 하는 $y=f(x)$를 찾아야 함

•Given: $I=\int_a^b F(y,y',x)dx$

$$f(a)=\alpha,\quad f(b)=\beta$$

•Find: 적분 I가 stationary value를 갖도록 하는 $y=f(x)$

$$\therefore\ \frac{\delta I}{\varepsilon}=\int_a^b\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}\right)\phi dx=0$$

• 함수 $\Phi(x)$는 임의의 함수이므로, 위 식이 항상 0이 되기 위해서는 괄호 안의 식이 0이 되어야 한다.

$$\therefore\ \frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0$$

적분 I가 stationary value를 갖게 하는 필요충분조건

• 따라서 적분 I가 stationary value를 갖도록 하는 $y=f(x)$는 위 미분방정식을 만족하는 함수이다.

(참고) 곱의 미분

$$\frac{df}{dx} = f'(x), \quad \frac{dg}{dx} = g'(x)$$

$$\frac{d}{dx}\big(f(x)g(x)\big) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x+h)g(x+h) - f(x)g(x+h)\} + \{f(x)g(x+h) - f(x)g(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\cdot g(x+h) + \lim_{h\to 0} f(x)\cdot\frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

(참고) 부분 적분

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

Integral with respect to x

$$\int \left(f(x)g(x)\right)' dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$f(x)g(x) - \int f(x)g'(x)dx = \int f'(x)g(x)dx$$

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$
$$= (\text{적분})(\text{그대로}) - \int (\text{적분})(\text{미분})dx$$

$$\int u'v dx = uv - \int uv' dx \,, \qquad \left(where\ u = f(x),\ v = g(x)\right)$$

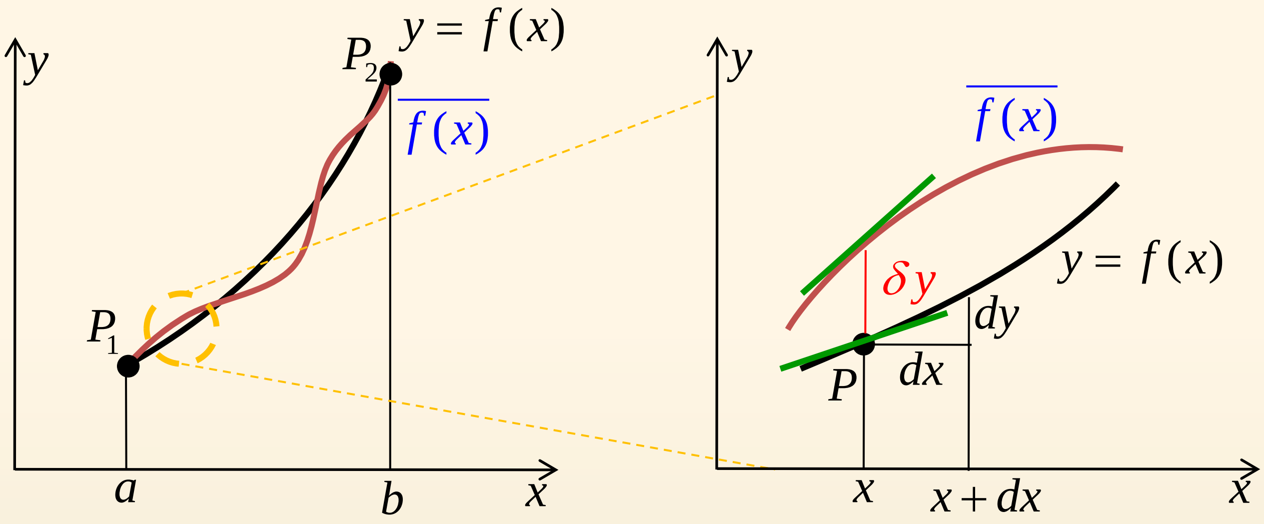
u'으로 가정하는 순서 :지수함수, 삼각함수, 다항함수, 로그함수

The commutative properties of the δ-process (1)

$$y = f(x)$$

$$\overline{f(x)} = f(x) + \varepsilon\phi(x)$$

$$\delta y = \overline{f(x)} - f(x) = \varepsilon\phi(x)$$



•The derivative of the variation

$$\frac{d}{dx}\delta y = \frac{d}{dx}\left[\overline{f(x)} - f(x)\right] = \frac{d}{dx}\left[\varepsilon\phi(x)\right] = \varepsilon\phi'(x)$$

$f(x)$와 $\overline{f(x)}$의 차이의 변화율(기울기)

$$\therefore \frac{d}{dx}\delta y = \delta\frac{d}{dx}y$$

•The variation of the derivative

$$\delta\frac{d}{dx}y = \left[\overline{f'(x)} - f'(x)\right] = \left(y' + \varepsilon\phi'\right) - y' = \varepsilon\phi'(x)$$

$f(x)$와 $\overline{f(x)}$의 변화율(기울기)의 차이

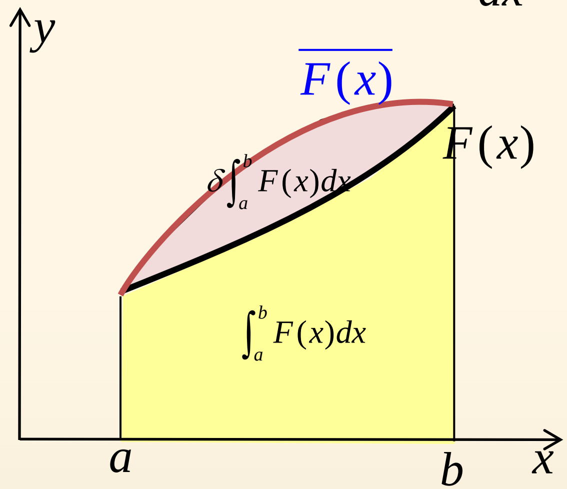
The commutative properties of the δ-process (2)

$$y = f(x)$$
$$\overline{f(x)} = f(x) + \varepsilon\phi(x)$$
$$\therefore \frac{d}{dx}\delta y = \delta \frac{d}{dx} y$$

The variation of a definite integral

- The given integrand: $F(x)$
- The modified integrand: $\overline{F(x)} = F(x) + \delta F(x)$
- The variation of a definite integral

$$\delta\int_a^b F(x)dx = \int_a^b \overline{F(x)}dx - \int_a^b F(x)dx$$
$$= \int_a^b \left[\overline{F(x)} - F(x)\right]dx = \int_a^b \delta F(x)dx$$



$$\therefore \delta\int_a^b F(x)dx = \int_a^b \delta F(x)dx$$

The δ-process reveals two characteristic properties:
(a) Variation and differentiation are permutable processes.
(b) Variation and integration are permutable processes.

Fourier Series(2)
: Sturm-Liouville Problem

Sturm-Liouville Problem

- **Review**

Linear Equations	**General solutions**
$y' + \alpha y = 0$	$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0 \quad \alpha > 0$	$y = c_1 \cos \alpha x + c_2 \sin \alpha x$
$y'' - \alpha^2 y = 0 \quad \alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, or \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x) \end{cases}$ ← When x is an infinite or half finite interval; ← When x is a finite interval

Cauchy-Euler Equation	**General solutions** $x > 0$
$x^2 y'' + xy' - \alpha^2 y = 0 \quad \alpha \geq 0$	$\begin{cases} y = c_1 x^{-\alpha} + c_2 x^{\alpha}, \alpha \neq 0 \\ y = c_1 + c_2 \ln x, \quad \alpha = 0 \end{cases}$

Linear Equations

Sturm-Liouville Problem

- **Review**

Parametric Bessel equation $\nu = 0$	General solutions $x > 0$
$x^2 y'' + y' + \alpha^2 x^2 y = 0$	$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$

Legendre's equation $n = 0,1,2,\ldots$	Particular solutions are polynomials
$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	$y = P_0(x) = 1,$ $y = P_1(x) = x,$ $y = P_2(x) = \frac{1}{2}(3x^2 - 1),$ $\vdots$

Sturm-Liouville Problem

- Eigenvalues and Eigenfunctions

Recall example 2 of section 3.9

$$y''+\lambda y=0,\quad y(0)=0,\ y(L)=0$$

When $\lambda>0$ (Case III)

Write $\lambda=\alpha^2,\ \alpha>0$

Then roots of auxiliary equation is

$$m_1=i\alpha,\ m_2=-i\alpha$$

$$y=c_1\cos\alpha x+c_2\sin\alpha x$$

Eigenvalues

$$y(0)=0 \Rightarrow c_1=0$$

$$y(L)=0 \Rightarrow c_2=0 \text{ or } \alpha L=n\pi,\quad \lambda_n=\alpha_n^{\ 2}=\left(\frac{n\pi}{L}\right)^2$$

Eigenfunctions

(nontrivial solution) $y_n=c_2\sin\frac{n\pi}{L}x$

Sturm-Liouville Problem

- **Eigenvalues and Eigenfunctions**

$$y'' + \lambda y = 0, \quad y(0) = 0, \;\; y(L) = 0$$

Eigenvalues

$$\lambda_n = \alpha_n^{\;2} = (\frac{n\pi}{L})^2$$

Eigenfunctions

$$y_n = c_2 \sin \frac{n\pi}{L} x$$

It is important to recognize the set of functions generated by this B.V.P the orthogonal set of functions on the interval $(0, L)$ *used as the basis for the Fourier sine series*

$$y'' + \lambda y = 0, \quad y'(0) = 0, \;\; y'(L) = 0 \longrightarrow$$ **the Fourier cosine series**

Sturm-Liouville Problem

☑ Example 1 Eigenvalues and Eigenfunctions

It is left as an exercise to show, by considering the three possible cases for the parameter λ (zero, negative, or positive; that is, $\lambda = 0,\ \lambda = -\alpha^2 < 0,\ \alpha > 0,\ and\ \lambda = \alpha^2 > 0,\ \alpha > 0$) that the eigenvalues and eigenfunctions for the boundary-value problem

$$y'' + \lambda y = 0,\ \ y'(0) = 0,\ \ y'(L) = 0$$

are, respectively, $\lambda_n = {\alpha_n}^2 = n^2\pi^2 / L^2,\ n = 0,1,2,\cdots,$ $y = c_1 \cos(n\pi x / L),\ c_1 \neq 0.$ $\lambda_0 = 0$ is an eigenvalue for this BVP and $y = 1$ is the corresponding eigenfunction. The latter comes from solving $y'' = 0$ subject to the same boundary conditions $y'(0) = 0,\ y'(L) = 0$. Note also that $y = 1$ can be incorporated into the family $y = \cos(n\pi x / L)$ by permitting $n = 0$. The set $\{\cos(n\pi x / L)\},$ $n = 0,1,2,3,\cdots,$ is orthogonal on the interval [0,L].

Sturm-Liouville Problem

▪Regular Sturm-Liouville Problem **B.V.P**

Solve : $$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

p, q, r, r'

real-valued functions continuous on an interval $[a,b]$

$r(x) > 0,\ p(x) > 0$

for every x **in the interval** $[a,b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

Special case

$$p(x) = 1, q(x) = 0, r(x) = 1$$

$$A_1 = 1, B_1 = 0, A_2 = 1, B_2 = 0, a = 0, b = L$$

$$\Rightarrow\ y'' + \lambda y = 0, \quad y(0) = 0, \ y(L) = 0$$

$$A_1 = 0, B_1 = 1, A_2 = 0, B_2 = 1, a = 0, b = L$$

$$\Rightarrow\ y'' + \lambda y = 0, \quad y'(0) = 0, \ y'(L) = 0$$

Sturm-Liouville Problem :
Homogeneous Boundary Value problem

-> Trivial solution y=0

-> goal : find nontrivial solution y
(Eigenvalues, Eigenfunctions)

Sturm-Liouville Problem

▪Regular Sturm-Liouville Problem **B.V.P**

Solve : $$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

p, q, r, r'

real-valued functions continuous on an interval $[a,b]$

$r(x) > 0,\ p(x) > 0$

for every x **in the interval** $[a,b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

"Homogeneous"

⇨ **Homogeneous D.E.+**

Homogeneous B/C

Nonhomogeneous B/C

$$A_1 y(a) + B_1 y'(a) = C_2, C_2 : nonzero$$

12.5 Sturm-Liouville Problem

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

p, q, r, r'

real-valued functions continuous on an interval $[a, b]$

$r(x) > 0,\ p(x) > 0$

for every x **in the interval** $[a, b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

"trivial solution is not our interest"

Homogeneous B.V.P always possesses the trivial solution $y = 0$

Sturm-Liouville Problem

Theorem 12.3

Properties of the Regular Sturm-Liouville Problem

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \lambda_n \rightarrow \infty$ such that $n \rightarrow \infty$ as

(b) For each eigenvalues there is only one eigenfunction (except for nonzero constant multiples)

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a, b]$

Solve : $$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$\frac{d}{dx}[r(x)y_m'] + [q(x) + \lambda_m p(x)]y_m = 0 \cdots (1)$$

$$\frac{d}{dx}[r(x)y_n'] + [q(x) + \lambda_n p(x)]y_n = 0 \cdots (2)$$

$$(1)\times y_n - (2)\times y_m : \; y_n \frac{d}{dx}[r(x)y_m'] - y_m \frac{d}{dx}[r(x)y_n'] + (\lambda_m - \lambda_n)p(x)y_n y_m = 0$$

$$(\lambda_n - \lambda_m)p(x)y_n y_m = y_n \frac{d}{dx}[r(x)y_m'] - y_m \frac{d}{dx}[r(x)y_n']$$

$$= y_n \frac{d}{dx}[r(x)y_m'] \underset{r(x)y_n' y_m'}{\boxed{+[r(x)y_m']\frac{d}{dx}y_n}} - y_m \frac{d}{dx}[r(x)y_n'] \underset{-r(x)y_n' y_m'}{\boxed{-[r(x)y_n']\frac{d}{dx}y_m}}$$

$$= \frac{d}{dx}\left(y_n[r(x)y_m']\right) - \frac{d}{dx}\left(y_m[r(x)y_n']\right)$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $$\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$$

Integrating $$(\lambda_n - \lambda_m)p(x)y_n y_m = \frac{d}{dx}\left(y_n[r(x)y_m']\right) - \frac{d}{dx}\left(y_m[r(x)y_n']\right)$$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = \int_a^b \left(\frac{d}{dx}\left(y_n[r(x)y_m']\right) - \frac{d}{dx}\left(y_m[r(x)y_n']\right)\right)dx$$

$$= y_n(b)[r(b)y_m'(b)] - y_n(a)[r(a)y_m'(a)]$$

$$-\left(y_m(b)[r(b)y_n'(b)] - y_m(a)[r(a)y_n'(a)]\right)$$

$$= r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)]$$

$$-r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_1 y(a) + B_1 y'(a) = 0 \longrightarrow \begin{cases} A_1 y_m(a) + B_1 y_m'(a) = 0 \cdots (3) \\ A_1 y_n(a) + B_1 y_n'(a) = 0 \cdots (4) \end{cases}$$

$$A_2 y(b) + B_2 y'(b) = 0 \longrightarrow \begin{cases} A_2 y_m(b) + B_2 y_m'(b) = 0 \cdots (5) \\ A_2 y_n(b) + B_2 y_m'(b) = 0 \cdots (6) \end{cases}$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $$\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_1 y_m(a) + B_1 y_m'(a) = 0 \cdots (3)$$
$$A_1 y_n(a) + B_1 y_n'(a) = 0 \cdots (4)$$

$$\Rightarrow \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As A_1, B_1 are not both zero

$$\det\begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} = y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_2 y_m(b) + B_2 y_m'(b) = 0 \cdots (5)$$
$$A_2 y_n(b) + B_2 y_n'(b) = 0 \cdots (6)$$

$$\Rightarrow \begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As A_2, B_2 **are not both zero**

$$\det\begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix} = y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $$\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)\overbrace{[y_m'(b)y_n(b) - y_m(b)y_n'(b)]}^{\text{zero}}$$
$$- r(a)\overbrace{[y_m'(a)y_n(a) - y_m(a)y_n'(a)]}^{\text{zero}}$$

From Boundary Condition:

$$y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$
$$y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

$$\therefore (\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = 0$$

$$\int_a^b p(x)y_n y_m dx = 0, \; \lambda_n \neq \lambda_m$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ **on interval** $[a,b]$

Proof of (d) $\int_a^b p(x) y_m(x) y_n(x) dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = r(b)[\underbrace{y_m'(b) y_n(b) - y_m(b) y_n'(b)}_{\text{zero}}] - r(a)[\underbrace{y_m'(a) y_n(a) - y_m(a) y_n'(a)}_{\text{zero}}]$$

From Boundary Condition:

$$y_m(a) y_n'(a) - y_m'(a) y_n(a) = 0$$

$$y_m(b) y_n'(b) - y_m'(b) y_n(b) = 0$$

$$\therefore (\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Sturm-Liouville Problem

☑ Example 2
A Regular Sturm-Liouville Problem

Slove the boundary-value problem

$$y'' + \lambda y = 0,\ \ y(0) = 0,\ \ y(1) + y'(1) = 0.$$

$$\lambda = 0 \ \ and \ \ \lambda = -\alpha^2 < 0,\ \ where \ \ \alpha > 0,$$

$$the \ \ trivial \ \ solution \ \ y = 0$$

$$\lambda = \alpha^2 > 0,\ \ \alpha > 0,$$

$$the \ \ general \ \ solution \ \ of \ \ y'' + \alpha^2 y = 0$$

$$is \ \ y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

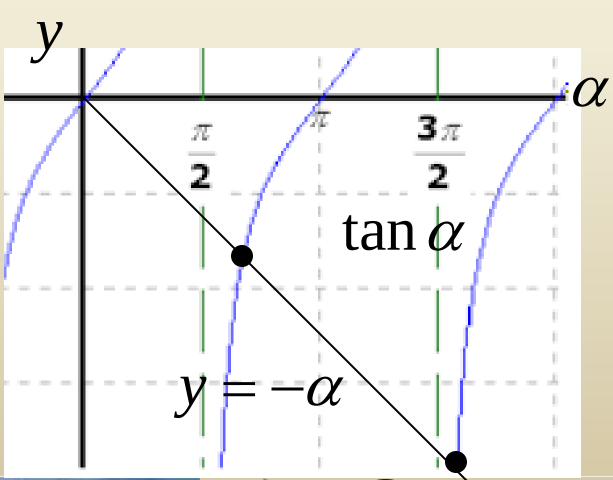
$$y(0) = c_1 = 0 \ \ \therefore y = c_2 \sin \alpha x$$

The second boundary condition $y(1) + y'(1) = 0$ **is satisfied if**

$$c_2 \sin \alpha + c_2 \alpha \cos \alpha = c_2 (\sin \alpha + \alpha \cos \alpha) = 0$$

Choosing $c_2 \neq 0$ **, we see that the last equation is equivalent to** $\tan \alpha = -\alpha$

The eigenvalues of problem are then $\lambda_n = \alpha_n^2$**, where** α_n **,** $n = 1,2,3,\cdots$ **, are the consecutive positive roots** $\alpha_1, \alpha_2, \alpha_3, \cdots$ **of** $\tan \alpha = -\alpha$



Sturm-Liouville Problem

☑ Example 2 A Regular Sturm-Liouville Problem

Slove the boundary-value problem

$$y'' + \lambda y = 0, \;\; y(0) = 0, \;\; y(1) + y'(1) = 0.$$

$\lambda = 0$ *and* $\lambda = -\alpha^2 < 0$, *where* $\alpha > 0$,

the trivial solution $y = 0$

$\lambda = \alpha^2 > 0, \;\; \alpha > 0,$

the general solution of $y'' + \alpha^2 y = 0$

is $y = c_1 \cos\alpha x + c_2 \sin\alpha x$

$y(0) = c_1 = 0 \;\; \therefore y = c_2 \sin\alpha x$

The second boundary condition $y(1) + y'(1) = 0$ is satisfied if

$$c_2 \sin\alpha + c_2 \alpha\cos\alpha = c_2(\sin\alpha + \alpha\cos\alpha) = 0$$

Choosing $c_2 \neq 0$, we see that the last equation is equivalent to $\tan\alpha = -\alpha$

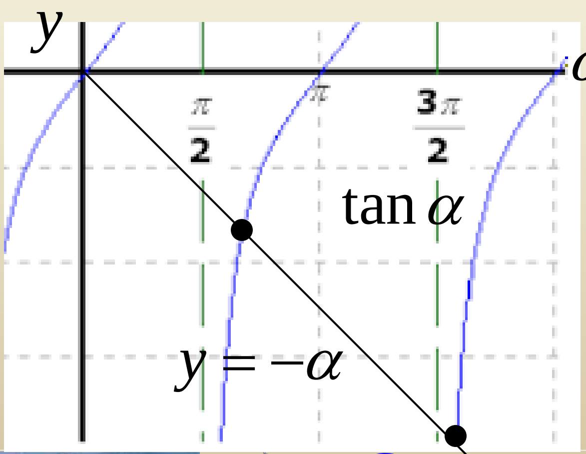
$\alpha_1 = 2.0288, \;\; \alpha_2 = 4.9132, \;\; \alpha_3 = 7.9787, \;\; \alpha_4 = 11.0855,$

and the corresponding solutions are

$y_1 = \sin 2.0288x, \;\; y_2 = \sin 4.9132x, \;\; y_3 = \sin 7.9787x,$

$y_4 = \sin 11.0855x$

In general, the eigenfunctions of the problem are $\{\sin\alpha_n\}, \;\; n = 1, 2, 3, \cdots$



Sturm-Liouville Problem

☑ Example 2 A Regular Sturm-Liouville Problem

Slove the boundary-value problem

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\dfrac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$

Subject to: $A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

$$r(x) = 1, q(x) = 0, p(x) = 1$$

$$A_1 = 1, B_1 = 0, A_2 = 1, B_2 = 1$$

In general, the eigenfunctions of the problem are $\{\sin \alpha_n\}, \; n = 1,2,3,\cdots$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Regular Sturm-Liouville Problem

$\{\sin \alpha_n\}, \; n = 1,2,3,\cdots$ **is an orthogonal set with respect to the weight function** $p(x) = 1$ **on the interval [0,1].**

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = \boxed{r(b)[y_m'(b) y_n(b) - y_m(b) y_n'(b)] - r(a)[y_m'(a) y_n(a) - y_m(a) y_n'(a)]}$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \;\; \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- *In some circumstances, we can prove the orthogonality of the solutions of* $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$ *without the necessity of specifying a boundary condition at x=a and at x=b*

→ Singular Sturm-Liouville Problem

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a, b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = \boxed{\begin{aligned} & r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] \\ & - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)] \end{aligned}}$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- **Singular Sturm-Liouville Problem**

If $r(a) = 0$ **then** $x = a$ **may be a singular and the equation**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

may become unbounded as $x \to a$ $, (a, b]$

however

zero

$$r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - \cancel{r(a)}\underline{[y_m'(a)y_n(a) - y_m(a)y_n'(a)]}$$

Orthogonal relation hold on $[a, b]$ ⇐ **dropped from the problem**
: no boundary condition at $x = a$

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = \boxed{\begin{aligned} & r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] \\ & - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)] \end{aligned}}$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- **Singular Sturm-Liouville Problem**

If $r(b) = 0$ **then** $x = b$ **may be a singular and the equation**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

may become unbounded as $x \to b$ $,[a,b]$

however

$$\overset{\text{zero}}{\cancel{r(b)}}\underline{[y_m'(b)y_n(b) - y_m(b)y_n'(b)]} - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

dropped from the problem

: no boundary condition at $x = b$

➡ **Orthogonal relation hold on** $[a,b]$

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \ \lambda_n \neq \lambda_m$$

▪Singular Sturm-Liouville Problem

example*) Legendre's equation is a Sturm-Liouville equation

$$\left[(1-x^2)y'\right]' + \lambda y = 0 \iff \underbrace{(1-x^2)}_{r(x)} y'' - 2xy' + \lambda y = 0 \qquad \lambda = n(n+1)$$

Since $r(\pm 1) = 0$ **need no boundary conditions, but have a singular Sturm-Liouville problem on the interval** $-1 \le x \le 1$ **. We know that , the Legendre polynomials** $P_n(x)$ **are solutions of the problem for** $n = 0,1,2,\ldots (\lambda = 0, 1\cdot 2, 2\cdot 3, \ldots)$

Hence these are the eigenfunctions. They are orthogonal on the interval

$$\int_{-1}^{1} p_m(x) p_n(x) dx = 0, \ (m \neq n)$$

Sturm-Liouville Problem

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- **Periodic Sturm-Liouville Problem**

If $r(a) = r(b)$ then

$$r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$

$$= r(a)\left[(y'_m(b)y_n(b) - y'_m(a)y_n(a)) + (y_m(a)y'_n(a) - y_m(b)y'_n(b))\right]$$

$\therefore$ Orthogonal relation hold on $[a,b]$ with $y(a) = y(b), \ y'(a) = y'(b)$

Sturm-Liouville Problem

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx$$

$$= \boxed{\begin{aligned} &r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] \\ &-r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)] \end{aligned}}$$

▪**Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

▪**By assuming the solution (y) are bounded on the closed interval [a,b] , then**

Orthogonal relation hold on.. [a,b]

$$\int_a^b p(x) y_n y_m dx = 0, \;\; \lambda_n \neq \lambda_m \;\; [a,b]$$

- **Regular** $r(x) \neq 0$ **with boundary Condition:** $A_1 y(a) + B_1 y'(a) = 0$

  $A_2 y(b) + B_2 y'(b) = 0$

- **Singular** $r(a) = 0$ **without B/C at x=a** $A_2 y(b) + B_2 y'(b) = 0$

  $r(b) = 0$ **without B/C at x=b** $A_1 y(a) + B_1 y'(a) = 0$

- **Periodic** $r(a) = r(b)$ **with** $y(a) = y(b), \; y'(a) = y'(b)$

Sturm-Liouville Problem

▪Self-Adjoint Form

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

If the coefficient are continuous and $a(x) \neq 0$ for all x in some interval, then

any second-order differential equation

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

can be recast into the so-called '**self-adjoint form**'.

Recall, ch. 2.3 integrating factor

$$a_1(x)y' + a_0(x)y = 0 \longrightarrow \frac{d}{dx}[\mu y] = 0$$

$$\mu = e^{\int p(x)dx},\ p(x) = \frac{a_0(x)}{a_1(x)}$$

Sturm-Liouville Problem

- **Self-Adjoint Form**

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

$$y'' + \frac{b(x)}{a(x)}y' + \left(\frac{c(x)}{a(x)} + \lambda\frac{d(x)}{a(x)}\right)y = 0$$

divided by $a(x)$

multiply $e^{\int \frac{b(x)}{a(x)}dx}$

$$e^{\int \frac{b(x)}{a(x)}dx}y'' + \frac{b(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}y' + \left(e^{\int \frac{b(x)}{a(x)}dx}\frac{c(x)}{a(x)} + \lambda e^{\int \frac{b(x)}{a(x)}dx}\frac{d(x)}{a(x)}\right)y = 0$$

$$\frac{d}{dt}\left[\underbrace{e^{\int \frac{b(x)}{a(x)}dx}}_{r(x)}y'\right] + \left(\underbrace{\frac{c(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}}_{q(x)} + \lambda\underbrace{\frac{d(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}}_{p(x)}\right)y = 0$$

Sturm-Liouville Problem

▪**Self-Adjoint Form**

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

$$r(x) = e^{\int \frac{b(x)}{a(x)}dx}$$

$$q(x) = \frac{c(x)}{a(x)} e^{\int \frac{b(x)}{a(x)}dx}$$

$$p(x) = \frac{d(x)}{a(x)} e^{\int \frac{b(x)}{a(x)}dx}$$

Example) $3y'' + 6y' + \lambda y = 0$

$$y'' + \frac{6}{3}y' + \frac{\lambda}{3}y = 0$$

$$e^{\int \frac{b(x)}{a(x)}dx} = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y'' + 2e^{2x}y' + \frac{\lambda}{3}e^{2x}y = 0$$

$$\frac{d}{dx}\left[e^{2x}y'\right] + \lambda \frac{e^{2x}}{3} y = 0$$

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

$$\int_a^b p(x) y_n y_m dx = 0, \ \ \lambda_n \neq \lambda_m$$

$[a,b]$

- **Self-Adjoint Form**

Ex.)Parametric Bessel Series*

$$x^2 y'' + xy' + (\alpha^2 x^2 - n^2)y = 0, \quad n = 0,1,2,\ldots$$

General solution $y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$

$J_n(x)$ converges on $[0,\infty)$ when $n \geq 0$

$Y_n(x)$ converges on $(0,\infty)$

---

divided by x^2

$$y'' + \frac{1}{x}y' + (\alpha^2 - \frac{n^2}{x^2})y = 0 \longrightarrow xy'' + y' + (x\alpha^2 - \frac{n^2}{x})y = 0$$

multiply $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$x > 0$

$$\frac{d}{dx}[\underbrace{x}_{r(x)}y'] + (\underbrace{-\frac{n^2}{x}}_{q(x)} + \underbrace{\alpha^2}_{\lambda}\underbrace{x}_{p(x)})y = 0$$

*Zill & Cullen, Advanced Engineering Mathematics, 3rd edition, John and Bartlett, 2006, p263

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \qquad [a,b]$$
$$\int_a^b p(x) y_n y_m dx = 0, \ \lambda_n \neq \lambda_m$$

- **Singular** $r(a)=0$: **Orthogonal relation holds on without B/C at x=a**

▪**Self-Adjoint Form**

Ex.)Parametric Bessel Series*

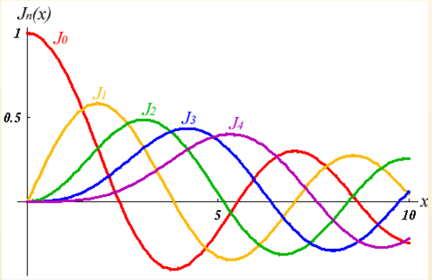
$$x^2 y'' + xy' + (\alpha^2 x^2 - n^2)y = 0, \quad n = 0,1,2,...$$

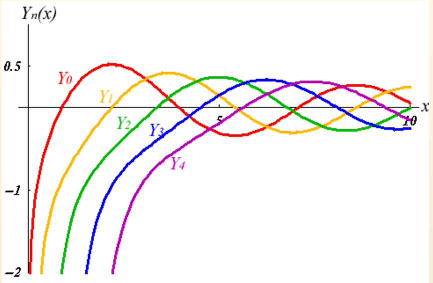
General solution $y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$ $J_n(x)$ converges on $[0,\infty)$ when $n \geq 0$

$Y_n(x)$ converges on $(0,\infty)$

$$\frac{d}{dx}[\underline{x}y'] + (\underline{-\frac{n^2}{x}} + \underline{\alpha^2}\,\underline{x})y = 0$$

$r(x)$ $q(x)$ λ $p(x)$





$r(0) = 0$

only $J_n(\alpha x)$ is bounded at $x = 0$

of the two solutions $J_n(\alpha x), Y_n(\alpha x)$

$(Y_n \to -\infty \ as \ x \to 0)$

Recall, singular Sturm-Liouville Problem, the set $\{J_n(\alpha_i x)\}, i = 1,2,3...,$ is orthogonal with respect to the weight function $p(x) = x$ on an interval $[0,b]$

The orthogonality relation is

$$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x) dx = 0, \ \lambda_i \neq \lambda_j \ (\lambda = \alpha^2)$$

*Zill & Cullen, Advanced Engineering Mathematics, 3rd edition, John and Bartlett, 2006, p263

Sturm-Liouville Problem

- **Self-Adjoint Form**

Ex.)Parametric Bessel Series*

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \ \lambda_n \neq \lambda_m$$

$$\frac{d}{dx}[\underbrace{x}_{r(x)} y'] + (\underbrace{-\frac{n^2}{x}}_{q(x)} + \underbrace{\alpha^2}_{\lambda} \underbrace{x}_{p(x)})y = 0$$

$\{J_n(\alpha_i x)\}, i = 1, 2, 3...,$ **: orthogonal set**

orthogonal relation

$$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x) dx = 0, \ \lambda_i \neq \lambda_j \ (\lambda = \alpha^2)$$

Provided the α_i, and hence the eigenvalues $\lambda_i = \alpha_i^2, \ i = 1, 2, 3, \cdots$ are defined by means of a boundary condition at $x = b$ of the type given in $A_2 y(b) + B_2 y'(b) = 0$:

$$A_2 J_n(\boxed{\alpha b}) + B_2 \alpha J_n{}'(\boxed{\alpha b}) = 0$$

→ **Roots** $x_i (= \alpha_i b)$

→ **Eigenvalues** $\lambda_i = (\alpha_i)^2 = \left(\frac{x_i}{b}\right)^2$

Sturm-Liouville Problem

- **Self-Adjoint Form**

Ex.) Legendre's Equation*

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

$$\left[(1-x^2)y'\right]' + \lambda y = 0 \quad \Leftrightarrow \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$q(x) = 0, \quad p(x) = 1$$

$$\lambda = n(n+1)$$

Legendre's D.E.
→ polynomial solutions $P_n(x) \qquad n = 0,1,2,\cdots$

As $P_n(x)$ is the only solutions of the equation that are bounded on the closed interval [-1,1], and $r(-1) = r(1) = 0$ (no boundary condition required) that the set $P_n(x)$ is orthogonal with respect to the weight function on [-1,1], The orthogonality relation is $\int_{-1}^{1} 1 \cdot P_m(x) P_n(x) dx = 0, \quad m \neq n$

참고자료

제약 최적화 문제를 비제약 최적화 문제로 변환하는 방법
- Lagrange Multiplier 사용

제약 최적화 문제

Minimize $f(\mathbf{x})$

Subject to $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ 등호 제약 조건

$\mathbf{g}(\mathbf{x}) \le \mathbf{0}$ 부등호 제약 조건

Lagrange 함수를 이용한 비제약 최적화 문제로의 변환

$$L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)$$

국부적 후보 최적성 조건인 $\nabla L=0$으로부터 $\mathbf{u}, \mathbf{v}$를 계산해야 함

1) 현재의 설계점에서 제약 조건을 만족하는 경우

등호 제약 조건의 경우: $\mathbf{h}(\mathbf{x}) = \mathbf{0}$

부등호 제약 조건의 경우: $\mathbf{u} = \mathbf{0}$ (설계점이 제약 조건의 경계에 있지 않을 때)

$\mathbf{s} = \mathbf{0} \Rightarrow \mathbf{g}(\mathbf{x}) = \mathbf{0}$ (설계점이 제약 조건의 경계 상에 있을 때)

따라서 $L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2) \Rightarrow f(\mathbf{x})$

➡ 제약 조건을 만족할 때 Lagrange 함수가 원래의 목적 함수와 동일함

2) 현재의 설계점에서 제약 조건을 위배하는 경우

등호 제약 조건의 경우: $\mathbf{v}^T \mathbf{h}(\mathbf{x}) \neq \mathbf{0}$

부등호 제약 조건의 경우: $\mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2) > \mathbf{0}$

따라서 $L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) = f(\mathbf{x}) + \underline{\mathbf{v}^T \mathbf{h}(\mathbf{x}) + \mathbf{u}^T (\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)}$

➡ 제약 조건을 위배할 때 원래의 목적 함수에 벌칙 값을 더한 것으로 생각할 수 있다.

제약 최적화 문제

$$\text{Minimize} \quad f(\mathbf{x})$$

$$\text{Subject to} \quad \mathbf{h}(\mathbf{x}) = 0$$ 등호 제약 조건

$$\mathbf{g}(\mathbf{x}) \le 0$$ 부등호 제약 조건

1968년에 Fiacco와 McCormick이
제약 조건의 위배량을 원래 목적 함수에 더한 수정된 목적 함수를 이용하여
제약 최적화 문제를 비제약 최적화 문제로 변환하는 방법을 제안함.
- SUMT: Sequential Unconstrained Minimization Technique

$$\Phi(\mathbf{x}, r_k) = f(\mathbf{x}) - r_k \sum_{j=1}^{m} \frac{1}{g_j(\mathbf{x})}$$

여기서, r_k는 문제에서 주어지는 양의 상수로서
Iteration이 진행될수록 그 값이 커짐

설계점이 Feasible region에서 부등호 제약 조건의 경계로 접근하면

$g_j(\mathbf{x}) \le 0$ 이며, 절대값이 작아짐

$-r_k \dfrac{1}{g_j(\mathbf{x})} > 0$ 이며, 절대값이 커짐

따라서 설계점이 부등호 제약 조건의 경계로 접근 할 때
수정된 목적 함수의 값이 증가하게 되고,
이는 제약 조건 식을 위배하는 것을 방지한다.