Naval Architecture & Ocean Engineering

[2009] [15]

Innovative ship design
- Integral Equation and Approximation-

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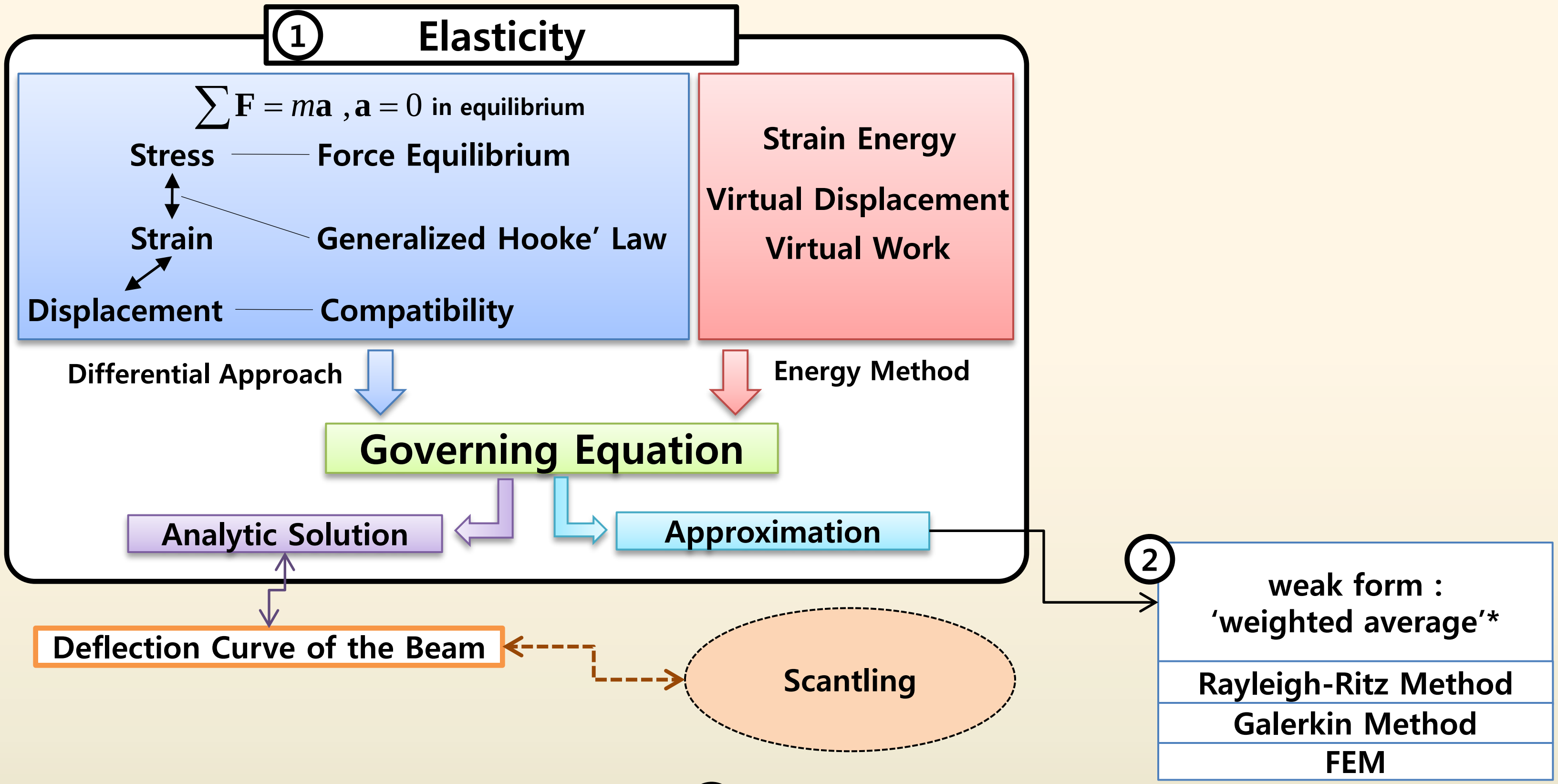

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Summary
Variables and Equations

If we are interested in finding the displacement components in a body, we may reduce the system of equations to three equations with three unknown displacement components.

18 Variables
- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Given : Body force X, Y, Z
Find : Displacement $u,\ v,\ w$

18 Equations

6 Equations of force equilibrium

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \qquad \sum M_x = \tau_{yz} - \tau_{zy} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \qquad \sum M_y = \tau_{xz} - \tau_{zx} = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \qquad \sum M_z = \tau_{xy} - \tau_{yx} = 0$$

6 Relations btw. Strain and Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations btw. 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x \qquad , \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y \qquad , \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_z \qquad , \tau_{zx} = \frac{E}{2(\nu+1)} \gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

1, 2, 3

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0$$

$$(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 v + Y = 0$$

$$(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 w + Z = 0$$

⇒ **3 Variables**
3 Equations

X, Y, Z : bodyforce in x,y, and z direction repectively

$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ $\quad \Theta = \sigma_x + \sigma_y + \sigma_z$

μ, λ : Lame Elastic constant

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$

G : Shear Moldulus

ν : Poisson's Ratio

E : Young's Modulus

Summary

18 Variables → 15 Variables

- 9 Stress $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- 6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- 3 Displacement u, v, w

18 Equations → 15 Equations

If we are interested in finding only the stress components in a body, we may reduce the system of equations to <u>six equations with six unknown stress components</u>

Given : Body force X, Y, Z

Find : Stress $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial X}{\partial x}+\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Y}{\partial y}+\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Z}{\partial z}+\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=0$$

$$\left(\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\right)+\nabla^2\tau_{xy}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y}=0$$

$$\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)+\nabla^2\tau_{yz}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z}=0$$

$$\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)+\nabla^2\tau_{zx}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x}=0$$

⇒ **6 Variables**
6 Equations

X, Y, Z : bodyforce in x,y, and z direction repectively

$e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ $\Theta=\sigma_x+\sigma_y+\sigma_z$

μ, λ : Lame Elastic constant

$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}$

G : Shear Moldulus

ν : Poisson's Ratio

E : Young's Modulus

2

<u>6 Equations</u> <u>of force equilibrium</u>

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0 \qquad \sum M_x=\tau_{yz}-\tau_{zy}=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0 \qquad \sum M_y=\tau_{xz}-\tau_{zx}=0$$

$$\sum F_z=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0 \qquad \sum M_z=\tau_{xy}-\tau_{yx}=0$$

<u>6 Relations</u> btw. <u>Strain and Displacement</u>

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

3

Compatibility equations *<u>3 independent Equations</u>*

$$\begin{cases}\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\\ \frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\\ \frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}\end{cases}$$

or

$$\begin{cases}2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)\\ 2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)\\ 2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)\end{cases}$$

1

<u>6 Relations</u> btw. 6 Strain and 6 Stress

$$\sigma_x=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_x \quad ,\tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_y \quad ,\tau_{yz}=\frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_z \quad ,\tau_{zx}=\frac{E}{2(\nu+1)}\gamma_{zx}$$

$$,e=\varepsilon_x+\varepsilon_y+\varepsilon_z$$

Classification

$$\int_0^1(-u''v)dx=[-u'v]_0^1+\int_0^1(u'v')dx$$

$$\sum_{i=1}^{N}a_i\left(\sum_{j=1}^{N}c_i\left\{\underbrace{\int_0^1\left[\phi_i'(x)\phi_j'(x)+\phi_i(x)\phi_j(x)\right]dx}_{k_{ij}}\right\}\right)=\sum_{i=1}^{N}a_i\underbrace{\int_0^1 x\phi_i(x)\,dx}_{F_i}$$

whenever a smooth 'classical(strong)' solution to a (D.E.) problem exists, it is also the solution of the weak problem[3]

Weak Form[2]

$$\int_0^1(-u''+u-x)v\,dx=0$$

$$u(0)=0,\ u(1)=0$$

multiply v and integration

integration by part and demand the test functions vanish at the endpoints

$$\int_0^1(-u'v'+uv-xv)\,dx=0$$

Approximate Method[4]

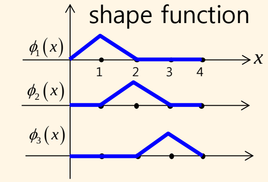
- Collocation
- Least Square
- Galerkin → FEM

$$\sum_{i=1}^{n}c_jk_{ij}=F_i$$

$$u(x)\approx\sum_{j=1}^{n}c_j\phi_j(x)$$

$$v(x)\approx\sum_{i=1}^{n}a_i\phi_i(x)$$

shape function



Differential Equation (ODE/PDE)

Ex.)
$$-u''+u=x,\ 0<x<1,$$
$$u(0)=0,\ u(1)=0$$

Ex.)
$$\frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2y+p=0$$

multiply δy and integration

$$\int_0^l\left(\frac{d}{dx}\left(T\frac{dy}{dx}\right)+\rho\omega^2y+p\right)\delta y\,dx$$

Work and Energy Principle

Variational formulation

$$\delta\int_0^l\left[\frac{1}{2}\rho\omega^2y^2+py-\frac{T}{2}\left(\frac{dy}{dx}\right)^2\right]dx=0$$

integration by part and B/C

Approximate Method

Rayleigh-Ritz

assume:
$$y(x)\approx\phi_0(x)+c_1\phi_1(x)+\cdots+c_n\phi_n(x)$$

- Variation and integration
- Integration and variation

Leibnitz formula[1]

$$\frac{d}{dx}\int_{A(x)}^{B(x)}F(x,\xi)d\xi=\frac{d}{dx}\int_{A(x)}^{B(x)}\frac{\partial F(x,\xi)}{\partial x}d\xi+F[x,B(x)]\frac{dB}{dx}-F[x,A(x)]\frac{dA}{dx}$$

problem of a "hereditary' nature[5]

Integral Equations

Volterra
$$\alpha(x)y(x)=F(x)+\lambda\int_a^xK(x,\xi)y(\xi)d\xi$$

Fredholm
$$\alpha(x)y(x)=F(x)+\lambda\int_a^bK(x,\xi)y(\xi)d\xi$$

Approximate Method

$$\sum_{k=1}^{n}c_ks_k(x)\approx F(x)\ ,\ s_k(x)=\phi_k(x)-\lambda\int_a^bK(x,\xi)y(\xi)d\xi$$

Collocation
$$\sum_{k=1}^{n}c_ks_k(x_i)=F(x_i)$$

Galerkin
$$\sum_{k=1}^{n}c_k\int_a^b\psi_i(x)s_k(x)dx=\int_a^b\psi_i(x)F(x)dx$$
$$,\psi(x)=\sum_{k=1}^{n}a_k\phi_i(x)$$

Least Square
$$\min\int_a^b\left[\sum_{k=1}^{n}c_k\,s_k(x)-F(x)\right]^2dx$$

what is the relationship between 'week form' and 'Variational formulation'?

1) Jerry, A.j., Introduction to Integral Equations with Applications, Marcel Dekker Inc., 1985, p19~25
2) 'variational statement of the problem' -Becker, E.B., et al, Finite Elements An Introduction, Volume 1, Prentice-Hall, 1981, p4
3) Becker, E.B., et al, Finite Elements An Introduction, Volume 1, Prentice-Hall, 1981, p2 . See also Betounes, Partial Differential Equations for Computational Science, Springer, 1988, p408 "…the weak solution is actually a strong (or classical) solution…"
4) some books refer as 'Method of Weighted Residue' from the Finite Element Equation point of view and they have different type depending on how to choose the weight functions. See also Fletcher,C.A.J., "Computational Galerkin Methods", Springer, 1984
5) Jerry, A.j., Introduction to Integral Equations with Applications, Marcel Dekker Inc., 1985, p1 "Problems of a 'hereditary' nature fall under the first category, since the state of the system u(t) at any time t depends by the definition on all the previous states u(t-τ) at the previous time t-τ ,which means that we must sum over them, hence involve them under the integral sign in an integral equation.

Summary : Integral Equations

Integral Equations

An integral equation is an equation in which a function to be determined appears under an *integral sign*

'Fredholm equation' $\alpha(x)y(x) = F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi$

where $\alpha, F \text{ and } K$ are given function and λ, a, b are constant

The given function $K(x,\xi)$, which depends upon the current variable x as well as the auxiliary variable ξ, is known as the *kernel* of the integral equation

The function $y(x)$ is to be determined

'Volterra equation' $\alpha(x)y(x) = F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi,$

upper limit of integral is not a constant

What is the relationship between D.E. and the integral equations?

Differential Equation ⇄ **Integral Equations**

Can you guess what decides the type of integral equation?

How can you transform a D.E. into an integral equation?

Integral Equations

An integral equation is an equation in which a function to be determined appears under an *integral sign*

Differential Equation ⇨ **Integration Equations**

'Fredholm equation' $\alpha(x)y(x) = F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi$

'Volterra equation' $\alpha(x)y(x) = F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi,$

How can you transform a D.E. into an integral equation?

---

it is necessary to make use of the known formula

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

consider the differentiation of the function $I_n(x)$ defined by the equation

$$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

n times differentiation by using the with the formula

we have,

$$\int_a^x \int_a^{x_n} \cdots \int_a^{x_3}\int_a^{x_2} f(x_1)dx_1dx_2\cdots dx_{n-1}dx_n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

What do you think the meaning of this equation is?

Integral Equations

An integral equation is an equation in which a function to be determined appears under an *integral sign*

Differential Equation ⇨ **Integration Equations**

'Fredholm equation' $\alpha(x)y(x)=F(x)+\lambda\int_a^b K(x,\xi)y(\xi)d\xi$

'Volterra equation' $\alpha(x)y(x)=F(x)+\lambda\int_a^x K(x,\xi)y(\xi)d\xi,$

How can you transform a D.E. into an integral equation?

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it is necessary to make use of the known formula

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi=\int_{A(x)}^{B(x)}\frac{\partial F(x,\xi)}{\partial x}d\xi+F[x,B(x)]\frac{dB}{dx}-F[x,A(x)]\frac{dA}{dx}$$

consider the differentiation of the function $I_n(x)$ defined by the equation

$$I_n(x)=\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

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we have,

$$\int_a^x\int_a^{x_n}\cdots\int_a^{x_3}\int_a^{x_2} f(x_1)dx_1dx_2\cdots dx_{n-1}dx_n=\frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

What do you think the meaning of this equation is?

1) if you have a function f

2) and integrate it n times

3) you have this

Integral Equations

An integral equation is an equation in which a function to be determined appears under an *integral sign*

Differential Equation ⇨ **Integration Equations**

'Fredholm equation' $\alpha(x)y(x) = F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi$

'Volterra equation' $\alpha(x)y(x) = F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi,$

How can you transform a D.E. into an integral equation?

$$\int_a^x\int_a^{x_n}\cdots\int_a^{x_3}\int_a^{x_2} f(x_1)dx_1dx_2\cdots dx_{n-1}dx_n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

1) if you have a function f

2) and integrate it n times

3) you have this

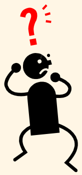
$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(0) = 1,\ y(l) = 0$$

⇩

$$y(x) = \lambda\int_0^l K(x,\xi)y(\xi)d\xi$$

$$,K(x,\xi) = \begin{cases} \frac{\xi}{l}(l-x) & when\ \ \xi < x \\ \frac{x}{l}(l-\xi) & when\ \ \xi > x \end{cases}$$

Differential Equation and Integral Equations
(in undergraduate school)

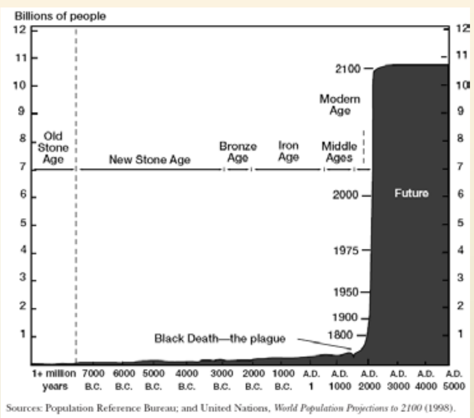


How to solve a Differential Equation?

↳Integration!

Ex) Population dynamics

$$\frac{dP(t)}{dt} = kP(t)$$



Integration!!

L.H.S:

$$\int \frac{dP(t)}{dt} dx = P(t) + C$$

R.H.S:

$$\int kP(t)dt = k\int P(t)dt$$

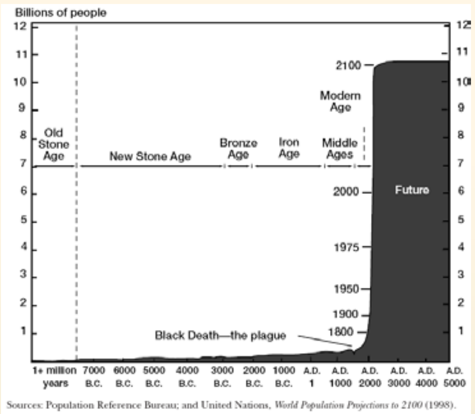
$$\therefore P(t) + C = k\int P(t)d\,x$$

solved?

Then, how?

Differential Equation and Integral Equations
(in undergraduate school)

Ex) Population dynamics



$$\frac{dP(x)}{dx} = kP(x)$$

Integration!!

L.H.S:

$$\int \frac{dP(x)}{dx} = P(x) + C$$

R.H.S:

$$\int kP(x) = k\int P(x)$$

$$\therefore P(x) + C = k\int P(x)dx$$

solved?

transform →

$$\frac{dP(x)}{P} = kdx$$

Separable Variables

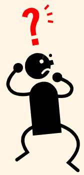
$$\ln|P(x)| = kx + c$$

$$P(x) = e^{kx+c}$$

$$= \tilde{c}e^{kx}$$

$$where, \begin{cases} \tilde{c} > 0 \;\; if \;\; P(x) > 0 \\ \tilde{c} < 0 \;\; if \;\; P(x), 0 \end{cases}$$

Differential Equation and Integral Equations
(in graduate school)



How to solve a Differential Equation?

↳ **Integration!**

Ex) Population dynamics

$$\frac{dP(t)}{dt} = kP(t)$$

for instance

$$P'(t) - kP(t) = 0 \quad , P(0) = 1$$

$$let \; P'(t) = u(t)$$

integration both sides

$$\int_0^t P'(s)ds = \int_0^t u(s)ds$$

$$P(t) - P(0) = \int_0^t u(s)ds$$

$$\therefore P(t) = 1 + \int_0^t u(s)ds$$

$$P'(t) - kP(t) = 0$$

⇩

$$u(t) - k\left(1 + \int_0^t u(s)ds\right) = 0$$

Integral Equation

$$u(t) = k\left(1 + \int_0^t u(s)ds\right) \cdots (1)$$

Then, how to solve?

By using decomposition methods*

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \qquad \cdots (2)$$

Substituting (2) into (1)

$$u_0(t) + u_1(t) + u_2(t) + \cdots = k\left(1 + \int_0^t (u_0(s) + u_1(s) + u_2(s) + \cdots)d\right)$$

⇩

$$u_0(t) = k$$

$$u_1(t) = k\int_0^t u_0(s)ds \;\Rightarrow\; u_1(t) = k\int_0^t kds = k^2[s]_0^t = k^2 t$$

$$u_2(t) = k\int_0^t u_1(s)ds \;\Rightarrow\; u_2(t) = k\int_0^t k^2 s\,ds = \frac{k^3}{2}\left[s^2\right]_0^t = \frac{k^3}{2}t^2$$

⋮

$$\Rightarrow u(t) = k + k\cdot kt + k\frac{1}{2}(kt)^2 + k\frac{1}{2\cdot 3}(kt)^3 + \cdots = k\left[1 + kt + \frac{1}{2}(kt)^2 + \frac{1}{2\cdot 3}(kt)^3 + \cdots\right]$$

$$\therefore u(t) = ke^{kt}$$

* Wazwaz, A.M., A First Course in Integral Equations, World Scientific, 1997, ch3.2

Differential Equation and Integral Equations

Ex) Population dynamics

$$\frac{dP(x)}{dx} = kP(x)$$

for instance

$$P'(t) - kP(t) = 0 \quad , P(0) = 1$$

Differential Equation (Separable Variables)

$$\frac{dP(x)}{P} = kdx \Longrightarrow P(x) = \tilde{c}e^{kx} \qquad \therefore P(x) = e^{kx}$$

$$P(0) = \tilde{c}$$

$$\therefore \tilde{c} = 1$$

same solution

Integral Equation

$$u(t) = k\left(1 + \int_0^t u(s)ds\right) \Longrightarrow \therefore u(t) = ke^{kt} \qquad \therefore P(x) = e^{kx}$$

$$, P'(t) = u(t)$$

$$P'(t) = ke^{kt}$$

integration

$$P(t) = e^{kt} + c$$

$$P(0) = 1 + c$$

$$\therefore c = 0$$

Differential Equation and Integral Equations

Example. $y'' + \lambda y = 0 \quad ,0 < x < 1$

$y(0) = 0, \quad y(1) = 0$

$\Rightarrow \quad u(x) + \lambda\int_0^1 K(x,t)u(t)dt = 0$

$$let\ y''(x) = u(x)$$

integration both sides

$$\int_0^x y''(t)dt = \int_0^x u(t)dt$$

$$y'(x) - y'(0) = \int_0^x u(t)dt$$

$$y'(x) = y'(0) + \int_0^t u(t)dt$$

$$\int_0^x y'(t)dt = \int_0^x y'(0)dt + \int_0^x\int_0^x u(t)dt$$

$$y(x) - y(0) = y'(0)[t]_0^x + \int_0^x (x-t)u(t)dt$$

$$y(x) = y'(0)x + \int_0^x (x-t)u(t)dt$$

$$y(1) = y'(0) + \int_0^1 (1-t)u(t)dt$$

$$0 = y'(0) + \int_0^1 (1-t)u(t)dt$$

$$\therefore y'(0) = -\int_0^1 (1-t)u(t)dt$$

$$y(x) = y'(0)x + \int_0^x (x-t)u(t)dt$$

$$y(x) = -x\int_0^1 (1-t)u(t)dt + \int_0^x (x-t)u(t)dt$$

$$y(x) = -x\int_0^x (1-t)u(t)dt - x\int_x^1 (1-t)u(t)dt + \int_0^x (x-t)u(t)dt$$

$$y(x) = \int_0^x (-x+tx+x-t)u(t)dt - x\int_x^1 (1-t)u(t)dt$$

$$y(x) = \int_0^x (tx-t)u(t)dt - x\int_x^1 (1-t)u(t)dt$$

$$y(x) = \int_0^x t(x-1)u(t)dt + \int_x^1 x(t-1)u(t)dt$$

$$y(x) = \int_0^1 K(x,t)u(t)d\,t \quad ,K(x,t)\begin{cases} t(x-1), & t < x \\ x(t-1), & x < t \end{cases}$$

Integral Equations : Introduction

Integral Equations
- Introduction

An integral equation : an equation in which a function to be determined appears under an integral sign

'Volterra equation'

$$\alpha(x)y(x) = F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi$$

'Fredholm equation'

$$\alpha(x)y(x) = F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi$$

α, F, K : given functions and continuous in (a,b)

λ, a, b : constants

$y(x)$: function is to be determined which is continuous in (a,b)

$K(x,\xi)$: the kernel of the integral equation

$\alpha = 0$

Volterra equation of the first kind

$$F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi = 0$$

Fredholm equation of the first kind

$$F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi = 0$$

$\alpha = 1$

Volterra equation of the second kind

$$y(x) = F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi$$

Fredholm equation of the second kind

$$y(x) = F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi$$

Integral Equations
- Introduction

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$y(x)$: function is to be determined which is continuous in (a,b)

$K(x,\xi)$: the kernel of the integral equation

'Fredholm equation'

In particular, when function $\alpha(x)$ is positive through out (a,b)

$$\sqrt{\alpha(x)}y(x) = \frac{F(x)}{\sqrt{\alpha(x)}} + \lambda\int_a^b \frac{K(x,\xi)}{\sqrt{\alpha(x)\alpha(\xi)}}\sqrt{\alpha(\xi)}y(\xi)d\xi,$$

Fredholm integral equation of second kind in the unknown function $\sqrt{\alpha(x)}y(x)$,with modified kernel.

two-dimensional Fredholm integral equations

$$\alpha(x,y)\,w(x,y) = F(x,y) + \lambda\iint_{\Re} K(x,y;\xi,\eta)\,w(\xi,\eta)\,d\xi\,d\eta$$

Integral Equations
- Introduction

An integral equation : an equation in which a function to be determined appears under an integral sign

'Volterra equation'

$$\alpha(x)y(x) = F(x) + \lambda\int_a^x K(x,\xi)y(\xi)d\xi$$

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$$\alpha(x)y(x) = F(x) + \lambda\int_a^b K(x,\xi)y(\xi)d\xi$$

α, F, K : given functions and continuous in (a,b)

λ, a, b : constants

$y(x)$: function is to be determined which is continuous in (a,b)

$K(x,\xi)$: the kernel of the integral equation

In general, an integral equation comprises the complete formulation of the problem, in the sense that additional conditions need not and cannot be specified.
That is, auxiliary conditions are, in a sense, already written into the equation.*

* Hildebrand, F.B., Methods of Applied Mathematics, Second edition, Dover, 1952, p223

Integral Equations
- Introduction

differential equation ⇔ **integral equation**

Certain integral equations can be deduced from or reduced to differential equations. It is frequently necessary to make us of the known formula.

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x} d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

$where\ F, \frac{\partial F}{\partial x}, \frac{dB}{dx}, \frac{dA}{dx} : continous$

This is a generalization of the fundamental theorem of integral calculus*

$$\frac{d}{dx}\int_a^x F(y)dy = F(x)$$

Proof*)

let $\phi(\alpha,\beta,x) = \int_{\alpha(x)}^{\beta(x)} F(x,y)dy$ and $\frac{\partial f}{\partial y}(x,y) = F(x,y)$

then
$$\phi(\alpha,\beta,x) = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial y}(x,y)dy$$
$$= \left[f(x,y) \right]_{\alpha(x)}^{\beta(x)}$$
$$= f(x,\beta(x)) - f(x,\alpha(x))$$

by the total derivatives $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial \beta}d\beta + \frac{\partial \phi}{\partial \alpha}d\alpha$

$$\therefore \frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial \beta}\frac{d\beta}{dx} + \frac{\partial \phi}{\partial \alpha}\frac{d\alpha}{dx}$$

* Jerry, A.J., Introduction to Integral Equations with Applications, Marcel Dekker, 1985

Integral Equations
- Introduction

differential equation ⟺ **integral equation**

Certain integral equations can be deduced from or reduced to differential equations. It is frequently necessary to make us of the known formula.

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x} d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

where $F, \frac{\partial F}{\partial x}, \frac{dB}{dx}, \frac{dA}{dx}$: continous

Proof*)

$$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial\beta}\frac{d\beta}{dx} + \frac{\partial\phi}{\partial\alpha}\frac{d\alpha}{dx}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\int_{\alpha(x)}^{\beta(x)} F(x,y)dy = \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} F(x,y)dy$$

$$\frac{\partial\phi(\alpha,\beta,x)}{\partial\beta} = \frac{\partial}{\partial\beta}\left[f(x,\beta(x)) - f(x,\alpha(x))\right] = \frac{\partial f(x,\beta)}{\partial\beta} - 0 = F(x,\beta)$$

$$\frac{\partial\phi(\alpha,\beta,x)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left[f(x,\beta(x)) - f(x,\alpha(x))\right] = 0 - \frac{\partial f(x,\alpha)}{\partial\alpha} = -F(x,\alpha)$$

$$\frac{d\phi}{dx} = \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} F(y)dy + F(x,\beta)\frac{d\beta}{dx} - F(x,\alpha)\frac{d\alpha}{dx}$$

$$\therefore \frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} F(x,y)dy = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F(x,y)}{\partial x} dy + F[x,\beta]\frac{d\beta}{dx} - F[x,\alpha]\frac{d\alpha}{dx}$$

This is a generalization of the fundamental theorem of integral calculus

$$\frac{d}{dx}\int_a^x F(y)dy = F(x)$$

let

$$\phi(\alpha,\beta,x) = \int_{\alpha(x)}^{\beta(x)} F(x,y)dy, \quad \frac{\partial f}{\partial y}(x,y) = F(x,y)$$

then

$$\phi(\alpha,\beta,x) = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial y}(x,y)dy = \left[f(x,y)\right]_{\alpha(x)}^{\beta(x)}$$

$$= f(x,\beta(x)) - f(x,\alpha(x))$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial\beta}d\beta + \frac{\partial\phi}{\partial\alpha}d\alpha$$

* Jerry, A.J., Introduction to Integral Equations with Applications, Marcel Dekker, 1985

Integral Equations
- Introduction

differential equation ⇔ **integral equation**

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

Multiple integrals Reduced to Single Integrals

consider the differentiation of the function $I_n(x)$ defined by the equation

$$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

$$where,\ \ F(x,\xi) = (x-\xi)^{n-1} f(\xi)$$

n: positive integer

differentiation with respect to x

$$\frac{dI_n}{dx} = \frac{d}{dx}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

$$= \int_a^x \frac{\partial}{\partial x}\left[(x-\xi)^{n-1} f(\xi)\right]d\xi + \left[(x-\xi)^{n-1} f(\xi)\right]_{\xi=x}\cancelto{1}{\frac{dx}{dx}} - \left[(x-\xi)^{n-1} f(\xi)\right]_{\xi=a}\cancelto{0}{\frac{da}{dx}}$$

$$= (n-1)\int_a^x (x-\xi)^{n-2} f(\xi)d\xi + \left[(x-\xi)^{n-1} f(\xi)\right]_{\xi=x}$$

Integral Equations
- Introduction

differential equation ⟺ **integral equation**

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x} d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

Multiple integrals Reduced to Single Integrals

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differentiation with respect to x

$$\frac{dI_n}{dx} = \frac{d}{dx}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

$$= (n-1)\int_a^x (x-\xi)^{n-2} f(\xi)d\xi + \left[(x-\xi)^{n-1} f(\xi)\right]_{\xi=x}$$

Hence, if n>1, there follows

$$\frac{dI_n}{dx} = (n-1)\int_a^x (x-\xi)^{n-2} f(\xi)d\xi + \left[(x-\xi)^{n-1} f(\xi)\right]_{\xi=x}$$

(the bracketed term → 0)

$$\therefore \frac{dI_n}{dx} = (n-1)I_{n-1} \quad ,n>1$$

While if n=1, we have

$$\frac{dI_n}{dx} = (n-1)\int_a^x (x-\xi)^{n-2} f(\xi)d\xi + \left[(x-\xi)^{n-1} f(\xi)\right]_{\xi=x}$$

(the term $(n-1)\int\ldots$ → 0; $(x-\xi)^{n-1}$ → $0^0 = 1$[1])

$$\therefore \frac{dI_1}{dx} = f(x)$$

1) Donald E. Knuth, Two notes on notation, Amer. Math. Monthly 99 no. 5 (May 1992), 403–422, see also http://en.wikipedia.org/wiki/Exponentiation

Integral Equations
- Introduction

differential equation ⟺ **integral equation**

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x} d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

Multiple integrals Reduced to Single Integrals

consider the differentiation of the function $I_n(x)$ defined by the equation

$$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)d\xi \Rightarrow \frac{dI_1}{dx} = f(x), \quad \frac{dI_n}{dx} = (n-1)I_{n-1} \quad ,n>1$$

where, $F(x,\xi) = (x-\xi)^{n-1} f(\xi)$

n: positive integer

and $I_n(a) = 0$

$n=1$

$$\frac{dI_1}{dx} = f(x)$$

$$\int_a^x \frac{dI_1}{dx_1} dx_1 = \int_a^x f(x_1)dx_1$$

$$I_1(x) - I_1(a) = \int_a^x f(x_1)dx_1$$

$$I_1(x) = \int_a^x f(x_1)dx_1$$

$n=2$

$$\frac{dI_2}{dx} = (2-1)I_1$$

$$\int_a^x \frac{dI_2}{dx_2} dx_2 = \int_a^x I_1(x_2)dx_2$$

$$I_2(x) - I_2(a) = \int_a^x I(x_2)dx_2$$

$$I_2(x) = \int_a^x I(x_2)dx_2$$

$x \to x_2$

$$I_1(x_2) = \int_a^{x_2} f(x_1)dx_1$$

$$I_2(x) = \int_a^x \int_a^{x_2} f(x_1)dx_1 dx_2$$

Integral Equations
- Introduction

differential equation ⇔ **integral equation**

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

Multiple integrals Reduced to Single Integrals

consider the differentiation of the function $I_n(x)$ defined by the equation

$$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)d\xi \Rightarrow \frac{dI_1}{dx} = f(x), \quad \frac{dI_n}{dx} = (n-1)I_{n-1} \quad ,n>1$$

where, $F(x,\xi) = (x-\xi)^{n-1} f(\xi)$

n: positive integer

and $I_n(a) = 0$

$n=1$

$$\frac{dI_1}{dx} = f(x)$$

$$\int_a^x \frac{dI_1}{dx_1}dx_1 = \int_a^x f(x_1)dx_1$$

$$I_1(x) - I_1(a) = \int_a^x f(x_1)dx_1$$

$$I_1(x) = \int_a^x f(x_1)dx_1$$

$n=2$

$$\frac{dI_2}{dx} = (2-1)I_1$$

$$\int_a^x \frac{dI_2}{dx_2}dx_2 = \int_a^x I_1(x_2)dx_2$$

$$I_2(x) - I_2(a) = \int_a^x I(x_2)dx_2$$

$$I_2(x) = \int_a^x I(x_2)dx_2$$

$n=3$

$$\frac{dI_3}{dx} = (3-1)I_1$$

$$\int_a^x \frac{dI_3}{dx_3}dx_3 = 2\int_a^x I_1(x_3)dx_3$$

$$I_3(x) - I_3(a) = 2\int_a^x I(x_3)dx_3$$

$$I_3(x) = 2\int_a^x I(x_3)dx_3$$

$$I_2(x) = \int_a^x \int_a^{x_2} f(x_1)dx_1 dx_2$$

$x \to x_3$

$$I_2(x_3) = \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1 dx_2$$

$$I_3(x) = 2\cdot 1\int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1 dx_2 dx_3$$

Integral Equations
- Introduction

differential equation ⟺ **integral equation**

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

Multiple integrals Reduced to Single Integrals

where, $F(x,\xi) = (x-\xi)^{n-1} f(\xi)$

n: positive integer

consider the differentiation of the function $I_n(x)$ defined by the equation

and $I_n(a) = 0$

$$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)d\xi \quad \Rightarrow \quad \frac{dI_1}{dx} = f(x), \quad \frac{dI_n}{dx} = (n-1)I_{n-1} \quad, n>1$$

$n=1$

$$\frac{dI_1}{dx} = f(x)$$

$$\int_a^x \frac{dI_1}{dx_1}dx_1 = \int_a^x f(x_1)dx_1$$

$$I_1(x) - I_1(a) = \int_a^x f(x_1)dx_1$$

$$I_1(x) = \int_a^x f(x_1)dx_1$$

$n=2$

$$\frac{dI_2}{dx} = (2-1)I_1$$

$$\int_a^x \frac{dI_2}{dx_2}dx_2 = \int_a^x I_1(x_2)dx_2$$

$$I_2(x) - I_2(a) = \int_a^x I(x_2)dx_2$$

$$I_2(x) = \int_a^x I(x_2)dx_2$$

$$I_2(x) = \int_a^x \int_a^{x_2} f(x_1)dx_1dx_2$$

$n=3$

$$\frac{dI_3}{dx} = (3-1)I_1$$

$$\int_a^x \frac{dI_3}{dx_3}dx_3 = 2\int_a^x I_1(x_3)dx_3$$

$$I_3(x) - I_3(a) = 2\int_a^x I(x_3)dx_3$$

$$I_3(x) = 2\int_a^x I(x_3)dx_3$$

$$I_3(x) = 2\cdot 1\int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1dx_2dx_3$$

$\cdots$

$$I_n(x) = (n-1)!\int_a^x \int_a^{x_n} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1dx_2\cdots dx_{n-1}dx_n$$

Integral Equations
- Introduction

differential equation ⟺ **integral equation**

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

Multiple integrals Reduced to Single Integrals

where, $F(x,\xi) = (x-\xi)^{n-1} f(\xi)$

n: positive integer

consider the differentiation of the function $I_n(x)$ defined by the equation

and $I_n(a) = 0$

$$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)d\xi \Rightarrow \frac{dI_1}{dx} = f(x) \;,\; \frac{dI_n}{dx} = (n-1)I_{n-1} \quad, n>1$$

$$I_n(x) = (n-1)!\int_a^x \int_a^{x_n} \cdots \int_a^{x_3}\int_a^{x_2} f(x_1)dx_1 dx_2 \cdots dx_{n-1}dx_n$$

⇩

$$\int_a^x \int_a^{x_n} \cdots \int_a^{x_3}\int_a^{x_2} f(x_1)dx_1 dx_2 \cdots dx_{n-1}dx_n = \frac{1}{(n-1)!}I_n(x)$$

⇩

$$\int_a^x \int_a^{x_n} \cdots \int_a^{x_3}\int_a^{x_2} f(x_1)dx_1 dx_2 \cdots dx_{n-1}dx_n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

differential equation ⟺ **integral equation**

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x} d\xi + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

Multiple integrals Reduced to Single Integrals

consider the differentiation of the function $I_n(x)$ defined by the equation

$$I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

where, $F(x,\xi) = (x-\xi)^{n-1} f(\xi)$

n: positive integer

$$\int_a^x \int_a^{x_n} \cdots \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1 dx_2 \cdots dx_{n-1}dx_n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

1) if you have a function f

2) and integrate it n times

3) you have this

What do you think the meaning of this equation is?

Integral Equations : Relation between differential and integral equations

Relation between differential and integral equations

Linear second order differential equation **I.V.P**

$$\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=f(x)$$

Initial condition : $y(a)=y_0,\ y'(a)=y'_0$

Integrate with respect to x_1 over the interval (a, x)

$$\int_a^x y''(x_1)dx_1+\int_a^x A(x_1)y'(x_1)dx_1+\int_a^x B(x_1)y(x_1)dx_1=\int_a^x f(x_1)dx_1$$

⇩

$$[y'(x_1)]_a^x+\int_a^x A(x_1)y'(x_1)dx_1+\int_a^x B(x_1)y(x_1)dx_1=\int_a^x f(x_1)dx_1$$

⇩

$$y'(x)-y'(a)=-\int_a^x A(x_1)y'(x_1)dx_1-\int_a^x B(x_1)y(x_1)dx_1+\int_a^x f(x_1)dx_1$$

⇩

$$y'(x)-y'_0=-\int_a^x A(x_1)y'(x_1)dx_1-\int_a^x B(x_1)y(x_1)dx_1+\int_a^x f(x_1)dx_1$$

after integrating the first term on the right by parts,

$$y'(x)=[-A(x_1)y(x_1)]_a^x+\int_a^x A'(x_1)y(x_1)dx_1-\int_a^x B(x_1)y(x_1)dx_1+\int_a^x f(x_1)dx_1+y'_0$$

⇩

$$y'(x)=-A(x)y(x)+A(a)y(a)-\int_a^x [B(x_1)-A'(x_1)]y(x_1)dx_1+\int_a^x f(x_1)dx_1+y'_0$$

⇩

$$y'(x)=-A(x)y(x)-\int_a^x [B(x_1)-A'(x_1)]y(x_1)dx_1+\int_a^x f(x_1)dx_1+A(a)y_0+y'_0$$

Relation between differential and integral equations

Linear second order differential equation **I.V.P**

$$\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=f(x)$$

Initial condition : $\boxed{y(a)=y_0}, \quad y'(a)=y_0'$

$$y'(x)=-A(x)y(x)-\int_a^x[B(x_1)-A'(x_1)]y(x_1)dx_1+\int_a^x f(x_1)dx_1+A(a)y_0+y_0'$$

Integrate again over the interval (a, x)

$$\int_a^x y'(x_2)dx_2=\int_a^x\left\{-A(x)y(x)-\int_a^x[B(x_1)-A'(x_1)]y(x_1)dx_1+\int_a^x f(x_1)dx_1+\left(A(a)y_0+y_0'\right)\right\}dx_2$$

⇩

$$y(x)-\boxed{y(a)}=-\int_a^x A(x_1)y(x_1)dx_1-\int_a^x\int_a^{x_2}[B(x_1)-A'(x_1)]y(x_1)dx_1dx_2+\int_a^x\int_a^{x_2} f(x_1)dx_1dx_2+\left(A(a)y_0+y_0'\right)[x_2]_a^x$$

⇩

$$y(x)-y_0=-\int_a^x A(x_1)y(x_1)dx_1-\int_a^x\int_a^{x_2}[B(x_1)-A'(x_1)]y(x_1)dx_1dx_2+\int_a^x\int_a^{x_2} f(x_1)dx_1dx_2+\left(A(a)y_0+y_0'\right)(x-a)$$

Relation between differential and integral equations

Linear second order differential equation **I.V.P**

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Initial condition : $y(a) = y_0,\ \ y'(a) = y'_0$

Integrate twice over the interval (a, x)

$$y(x) - y_0 = -\int_a^x A(x_1)y(x_1)dx_1 - \int_a^x\int_a^{x_2}[B(x_1) - A'(x_1)]y(x_1)dx_1dx_2 + \int_a^x\int_a^{x_2} f(x_1)dx_1dx_2 + \left(A(a)y_0 + y'_0\right)(x-a)$$

recall,

$$\int_a^x\int_a^{x_n}\cdots\int_a^{x_3}\int_a^{x_2} f(x_1)dx_1dx_2\cdots dx_{n-1}dx_n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$$

and for n=2

$$\int_a^x\int_a^{x_2} f(x_1)dx_1dx_2 = \frac{1}{(2-1)!}\int_a^x (x-\xi)^{2-1} f(\xi)d\xi$$

$$\therefore \int_a^x\int_a^{x_2} f(x_1)dx_1dx_2 = \int_a^x (x-\xi) f(\xi)d\xi$$

$$y(x) = -\int_a^x A(\xi)y(\xi)d\xi - \int_a^x (x-\xi)[B(\xi) - A'(\xi)]y(\xi)d\xi + \int_a^x (x-\xi) f(\xi)d\xi + [A(a)y_0 + y'_0](x-a) + y_0$$

$$\therefore y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi + \int_a^x (x-\xi) f(\xi)d\xi + [A(a)y_0 + y'_0](x-a) + y_0$$

Relation between differential and integral equations

Linear second order differential equation **I.V.P**

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Initial condition : $y(a) = y_0, \ \ y'(a) = y_0'$

⇩ **Integrate twice over the interval** (a, x)

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi + \int_a^x (x-\xi)f(\xi)d\xi + [A(a)y_0 + y_0'](x-a) + y_0$$

⇩

$$y(x) = \int_a^x K(x,\xi)y(\xi)d\xi + F(x),$$

Where, $K(x,\xi) = (\xi - x)[B(\xi) - A'(\xi)] - A(\xi)$

: a linear function of the current variable x**.**

$$F(x) = \int_a^x (x-\xi)f(\xi)d\xi + [A(a)y_0 + y_0'](x-a) + y_0$$

This equation is seen to be a ***Volterra equation of the second kind.***

Relation between differential and integral equations

Linear second order differential equation

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Initial condition :

$$y(a) = y_0,\ \ y'(a) = y'_0$$

I.V.P

⇔

Volterra integral equation of second kind

$$y(x) = \int_a^x K(x,\xi)y(\xi)d\xi + F(x),$$

$$where,\ K(x,\xi) = (\xi - x)[B(\xi) - A'(\xi)] - A(\xi)$$

$$F(x) = \int_a^x (x-\xi)f(\xi)d\xi + [A(a)y_0 + y'_0](x-a) + y_0$$

Example : I.V.P)

$$\frac{d^2y}{dx^2} + \lambda y = f(x),\ \ y(0) = 1,\ \ y'(0) = 0$$

Integrate

$$\int_0^x y'' dx_1 + \lambda \int_0^x y dx_1 = \int_0^x f(x)dx_1$$

$$y'(x) - y'(0) = -\lambda \int_0^x y(x_1)dx_1 + \int_0^x f(x_1)dx_1$$

$$y'(x) = -\lambda \int_0^x y(x_1)dx_1 + \int_0^x f(x_1)dx_1$$

Integrate

$$\int_0^x y'(x)dx_2 = -\lambda \int_0^x \int_0^{x_2} y(x_1)dx_1dx_2 + \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2$$

$$y(x) - y(0) = -\lambda \int_0^x \int_0^{x_2} y(x_1)dx_1dx_2 + \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2$$

$$y(x) = -\lambda \int_0^x \int_0^{x_2} y(x_1)dx_1dx_2 + \int_0^x \int_0^{x_2} f(x_1)dx_1dx_2 + 1$$

recall, $\therefore \int_a^x \int_a^{x_2} f(x_1)dx_1dx_2 = \int_a^x (x-\xi)f(\xi)d\xi$

$$\therefore y(x) = \lambda \int_0^x (\xi - x)y(\xi)d\xi - \int_0^x (\xi - x)f(\xi)d\xi + 1$$

Relation between differential and integral equations

Linear second order differential equation

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Initial condition :

$$y(a) = y_0,\ \ y'(a) = y'_0$$

I.V.P

Volterra integral equation of second kind

$$y(x) = \int_a^x K(x,\xi)y(\xi)d\xi + F(x),$$

$$where,\ K(x,\xi) = (\xi - x)[B(\xi) - A'(\xi)] - A(\xi)$$

$$F(x) = \int_a^x (x-\xi) f(\xi)d\xi + [A(a)y_0 + y'_0](x-a) + y_0$$

Example : I.V.P)

$$\frac{d^2y}{dx^2} + \lambda y = f(x),\ \ y(0) = 1,\ y'(0) = 0$$

$$y(x) = \lambda\int_0^x (\xi - x)y(\xi)d\xi - \int_0^x (\xi - x) f(\xi)d\xi + 1$$

$$y(x) = \int_a^x K(x,\xi)y(\xi)d\xi + F(x),$$

check!

$$A(x) = 0, \quad B(x) = \lambda \quad, y_0 = 1, \quad y'_0 = 0$$

$$K(x,\xi) = (\xi - x)[B(\xi) - A'(\xi)] - A(\xi)$$
$$= \lambda(\xi - x)$$

$$F(x) = \int_a^x (x-\xi) f(\xi)d\xi + [A(a)y_0 + y'_0](x-a) + y_0$$
$$= -\int_a^x (\xi - x) f(\xi)d\xi + 1$$

Relation between differential and integral equations

B.V.P

Linear second order differential equation

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Boundary condition : $\boxed{y(a)} = y_a, \; y(b) = y_b$

Integrate with respect to x_1 over the interval (a, x)

$$\int_a^x y''(x_1)dx_1 + \int_a^x A(x_1)y'(x_1)dx_1 + \int_a^x B(x_1)y(x_1)dx_1 = \int_a^x f(x_1)dx_1$$

⇩

$$\left[y'(x_1)\right]_a^x + \int_a^x A(x_1)y'(x_1)dx_1 + \int_a^x B(x_1)y(x_1)dx_1 = \int_a^x f(x_1)dx_1$$

⇩

$$y'(x) - y'(a) + \int_a^x A(x_1)y'(x_1)dx_1 + \int_a^x B(x_1)y(x_1)dx_1 = \int_a^x f(x_1)dx_1$$

⇩

$$y'(x) - y'(a) = -\int_a^x A(x_1)y'(x_1)dx_1 - \int_a^x B(x_1)y(x_1)dx_1 + \int_a^x f(x_1)dx_1$$

after integrating the first term on the right by parts,

$$y'(x) = \left[-A(x_1)y(x_1)\right]_a^x + \int_a^x A'(x_1)y(x_1)dx_1 - \int_a^x B(x_1)y(x_1)dx_1 + \int_a^x f(x_1)dx_1 + y'(a)$$

⇩

$$y'(x) = -A(x)y(x) + A(a)\boxed{y(a)} - \int_a^x \left[B(x_1) - A'(x_1)\right]y(x_1)dx_1 + \int_a^x f(x_1)dx_1 + y'(a)$$

⇩

$$y'(x) = -A(x)y(x) - \int_a^x [B(x_1) - A'(x_1)]y(x_1)dx_1 + \int_a^x f(x_1)dx_1 + A(a)y_a + y'(a)$$

Relation between differential and integral equations

Linear second order differential equation **B.V.P**

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Boundary condition : $y(a) = y_a, \; y(b) = y_b$

$$y'(x) = -A(x)y(x) - \int_a^x [B(x_1) - A'(x_1)]y(x_1)dx_1 + \int_a^x f(x_1)dx_1 + A(a)y_a + y'(a)$$

Integrate again over the interval (a, x)

$$\int_a^x y'(x_2)dx_2 = \int_a^x \left\{ -A(x)y(x) - \int_a^x [B(x_1) - A'(x_1)]y(x_1)dx_1 + \int_a^x f(x_1)dx_1 + \left(A(a)y_a + y'(a)\right) \right\} dx_2$$

⇩

$$y(x) - y(a) = -\int_a^x A(x_1)y(x_1)dx_1 - \int_a^x \int_a^{x_2} [B(x_1) - A'(x_1)]y(x_1)dx_1dx_2 + \int_a^x \int_a^{x_2} f(x_1)dx_1dx_2 + \left(A(a)y_a + y'(a)\right)[x_2]_a^x$$

⇩

$$y(x) - y_a = -\int_a^x A(x_1)y(x_1)dx_1 - \int_a^x \int_a^{x_2} [B(x_1) - A'(x_1)]y(x_1)dx_1dx_2 + \int_a^x \int_a^{x_2} f(x_1)dx_1dx_2 + \left(A(a)y_a + y'(a)\right)(x - a)$$

Relation between differential and integral equations

Linear second order differential equation **B.V.P**

$$\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=f(x)$$

Boundary condition : $y(a)=y_a,\ \ y(b)=y_b$

Integrate twice over the interval (a, x)

$$y(x)-y_a=-\int_a^x A(x_1)y(x_1)dx_1-\int_a^x\int_a^{x_2}[B(x_1)-A'(x_1)]y(x_1)dx_1dx_2+\int_a^x\int_a^{x_2} f(x_1)dx_1dx_2+\left(A(a)y_a+y'(a)\right)(x-a)$$

recall,

$$\int_a^x\int_a^{x_n}\cdots\int_a^{x_3}\int_a^{x_2} f(x_1)dx_1dx_2\cdots dx_{n-1}dx_n=\frac{1}{(n-1)!}\int_a^x(x-\xi)^{n-1}f(\xi)d\xi$$

and for n=2

$$\int_a^x\int_a^{x_2} f(x_1)dx_1dx_2=\frac{1}{(2-1)!}\int_a^x(x-\xi)^{2-1}f(\xi)d\xi$$

$$\therefore \int_a^x\int_a^{x_2} f(x_1)dx_1dx_2=\int_a^x(x-\xi)f(\xi)d\xi$$

$$y(x)=-\int_a^x A(\xi)y(\xi)d\xi-\int_a^x(x-\xi)[B(\xi)-A'(\xi)]y(\xi)d\xi+\int_a^x(x-\xi)f(\xi)d\xi+[A(a)y_a+y'(a)](x-a)+y_a$$

$$\therefore y(x)=-\int_a^x\{A(\xi)+(x-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi+\int_a^x(x-\xi)f(\xi)d\xi+[A(a)y_a+y'(a)](x-a)+y_a$$

Relation between differential and integral equations

Linear second order differential equation **B.V.P**

$$\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=f(x)$$

Boundary condition : $y(a)=y_a,\ y(b)=y_b$

⇩ **Integrate twice over the interval** (a, x)

$$\therefore y(x)=-\int_a^x\{A(\xi)+(x-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi+\int_a^x(x-\xi)f(\xi)d\xi+[A(a)y_a+y'(a)](x-a)+y_a$$

$$y(b)=-\int_a^b\{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi+\int_a^b(b-\xi)f(\xi)d\xi+[A(a)y_a+y'(a)](b-a)+y_a$$

⇩

$$y_b=-\int_a^b\{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi+\int_a^b(b-\xi)f(\xi)d\xi+[A(a)y_a+y'(a)](b-a)+y_a$$

⇩

$$[A(a)y_a+y'(a)](b-a)=\int_a^b\{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi-\int_a^b(b-\xi)f(\xi)d\xi+(y_b-y_a)$$

⇩

$$A(a)y_a+y'(a)=\frac{1}{(b-a)}\int_a^b\{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi-\frac{1}{(b-a)}\int_a^b(b-\xi)f(\xi)d\xi+\frac{(y_b-y_a)}{(b-a)}$$

⇩

$$y'(a)=\frac{1}{(b-a)}\int_a^b\{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi-\frac{1}{(b-a)}\int_a^b(b-\xi)f(\xi)d\xi+\frac{(y_b-y_a)}{(b-a)}-A(a)y_a$$

Relation between differential and integral equations

Linear second order differential equation **B.V.P**

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Boundary condition : $y(a) = y_a,\ y(b) = y_b$

⇩ **Integrate twice over the interval** (a, x)

$$\therefore y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi + \int_a^x (x-\xi)f(\xi)d\xi + [A(a)y_a + y'(a)](x-a) + y_a$$

$$y'(a) = \frac{1}{(b-a)}\int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi - \frac{1}{(b-a)}\int_a^b (b-\xi)f(\xi)d\xi + \frac{(y_b - y_a)}{(b-a)} - A(a)y_a$$

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi$$
$$+ \int_a^x (x-\xi)f(\xi)d\xi + [A(a)y_a + \frac{1}{(b-a)}\int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi - \frac{1}{(b-a)}\int_a^b (b-\xi)f(\xi)d\xi + \frac{(y_b - y_a)}{(b-a)} - A(a)y_a](x-a) + y_a$$

⇩

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi$$
$$+ \int_a^x (x-\xi)f(\xi)d\xi + [\frac{1}{(b-a)}\int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\}y(\xi)d\xi - \frac{1}{(b-a)}\int_a^b (b-\xi)f(\xi)d\xi + \frac{(y_b - y_a)}{(b-a)}](x-a)$$
$$+ y_a$$

Relation between differential and integral equations

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi$$
$$+ \int_a^x (x-\xi) f(\xi) d\xi + [\frac{1}{(b-a)}\int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi - \frac{1}{(b-a)}\int_a^b (b-\xi) f(\xi) d\xi + \frac{(y_b - y_a)}{(b-a)}](x-a)$$
$$+ y_a$$

1

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi + \frac{(x-a)}{(b-a)}\int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi$$

2

$$+ \int_a^x (x-\xi) f(\xi) d\xi - \frac{(x-a)}{(b-a)}\int_a^b (b-\xi) f(\xi) d\xi + y_a + \frac{(x-a)}{(b-a)}(y_b - y_a)$$

Relation between differential and integral equations

$$y(x) = \boxed{-\int_a^x \{A(\xi)+(x-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi + \frac{(x-a)}{(b-a)}\int_a^b \{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi}\ ①$$

$$② \boxed{+\int_a^x (x-\xi)f(\xi)d\xi - \frac{(x-a)}{(b-a)}\int_a^b (b-\xi)f(\xi)d\xi + y_a + \frac{(x-a)}{(b-a)}(y_b - y_a)}$$

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①

$$-\int_a^x \{A(\xi)+(x-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi + \frac{(x-a)}{(b-a)}\int_a^b \{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi$$

$$= -\int_a^x \{A(\xi)+(x-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi + \frac{(x-a)}{(b-a)}\int_a^x \{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi + \frac{(x-a)}{(b-a)}\int_x^b \{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi$$

$$= -\int_a^x \{A(\xi)\left[1-\frac{(x-a)}{(b-a)}\right] + \left[(x-\xi)-\frac{(x-a)}{(b-a)}(b-\xi)\right][B(\xi)-A'(\xi)]\}y(\xi)d\xi + \frac{(x-a)}{(b-a)}\int_x^b \{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi$$

$$= -\int_a^x \{A(\xi)\left[\frac{b-a-x+a}{(b-a)}\right] + \left[\frac{(x-\xi)(b-a)-(x-a)(b-\xi)}{(b-a)}\right][B(\xi)-A'(\xi)]\}y(\xi)d\xi + \frac{(x-a)}{(b-a)}\int_x^b \{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi$$

$$= -\int_a^x \{A(\xi)\left[\frac{b-x}{b-a}\right] + \left[\frac{xb-xa-\xi b+\xi a-xb+x\xi+ab-a\xi}{b-a}\right][B(\xi)-A'(\xi)]\}y(\xi)d\xi + \frac{(x-a)}{(b-a)}\int_x^b \{A(\xi)+(b-\xi)[B(\xi)-A'(\xi)]\}y(\xi)d\xi$$

$$= -\int_a^x \{A(\xi)\left(\frac{b-x}{b-a}\right) + \left[\frac{-xa-\xi b+x\xi+ab}{b-a}\right][B(\xi)-A'(\xi)]\}y(\xi)d\xi + \int_x^b \{A(\xi)\left(\frac{x-a}{b-a}\right)+(b-\xi)\left(\frac{x-a}{b-a}\right)[B(\xi)-A'(\xi)]\}y(\xi)d\xi$$

$$= -\int_a^x \{A(\xi)\left(\frac{b-x}{b-a}\right) + \left[\frac{a(b-x)-\xi(b-x)}{b-a}\right][B(\xi)-A'(\xi)]\}y(\xi)d\xi + \int_x^b \{A(\xi)\left(\frac{x-a}{b-a}\right)+\left[\frac{(x-a)(b-\xi)}{b-a}\right][B(\xi)-A'(\xi)]\}y(\xi)d\xi$$

Relation between differential and integral equations

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi + \frac{(x-a)}{(b-a)} \int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi$$ ①

② $$+ \int_a^x (x-\xi) f(\xi) d\xi - \frac{(x-a)}{(b-a)} \int_a^b (b-\xi) f(\xi) d\xi + y_a + \frac{(x-a)}{(b-a)} (y_b - y_a)$$

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①
$$-\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi + \frac{(x-a)}{(b-a)} \int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi$$

$$= -\int_a^x \left\{A(\xi)\left(\frac{b-x}{b-a}\right) + \left[\frac{a(b-x) - \xi(b-x)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi + \int_x^b \left\{A(\xi)\left(\frac{x-a}{b-a}\right) + \left[\frac{(x-a)(b-\xi)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi$$

$$= -\int_a^x \left\{A(\xi)\left(\frac{b-x}{b-a}\right) + \left[\frac{(b-x)(a-\xi)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi + \int_x^b \left\{A(\xi)\left(\frac{x-a}{b-a}\right) + \left[\frac{(x-a)(b-\xi)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi$$

$$= \int_a^x \left\{A(\xi)\left(\frac{x-b}{b-a}\right) + \left[\frac{(x-b)(a-\xi)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi + \int_x^b \left\{A(\xi)\left(\frac{x-a}{b-a}\right) + \left[\frac{(x-a)(b-\xi)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi$$

$$= \int_a^x \left\{A(\xi)\left(\frac{x-b}{b-a}\right) - \left[\frac{(\xi-a)(x-b)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi + \int_x^b \left\{A(\xi)\left(\frac{x-a}{b-a}\right) - \left[\frac{(x-a)(\xi-b)}{b-a}\right][B(\xi) - A'(\xi)]\right\} y(\xi) d\xi$$

$$= \int_a^b K(x,\xi) y(\xi) d\xi, \qquad K(x,\xi) = \begin{cases} A(\xi)\left(\dfrac{x-b}{b-a}\right) - \left[\dfrac{(x-b)(\xi-a)}{b-a}\right][B(\xi) - A'(\xi)], & \xi < x \\ A(\xi)\left(\dfrac{x-a}{b-a}\right) - \left[\dfrac{(\xi-b)(x-a)}{b-a}\right][B(\xi) - A'(\xi)], & x < \xi \end{cases}$$

Relation between differential and integral equations

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi + \frac{(x-a)}{(b-a)} \int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi \quad ①$$

$$② \; + \int_a^x (x-\xi) f(\xi) d\xi - \frac{(x-a)}{(b-a)} \int_a^b (b-\xi) f(\xi) d\xi + y_a + \frac{(x-a)}{(b-a)}(y_b - y_a)$$

$$② \; \int_a^x (x-\xi) f(\xi) d\xi - \frac{(x-a)}{(b-a)} \int_a^b (b-\xi) f(\xi) d\xi + y_a + \frac{(x-a)}{(b-a)}(y_b - y_a)$$

$$= \int_a^x (x-\xi) f(\xi) d\xi + \frac{x-a}{b-a}\left(y_b - y_a - \int_a^b (b-\xi) f(\xi) d\xi\right) + y_a$$

Relation between differential and integral equations

$$y(x) = -\int_a^x \{A(\xi) + (x-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi + \frac{(x-a)}{(b-a)} \int_a^b \{A(\xi) + (b-\xi)[B(\xi) - A'(\xi)]\} y(\xi) d\xi \quad (1)$$

$$(2) \quad + \int_a^x (x-\xi) f(\xi) d\xi - \frac{(x-a)}{(b-a)} \int_a^b (b-\xi) f(\xi) d\xi + y_a + \frac{(x-a)}{(b-a)}(y_b - y_a)$$

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(1) $\int_a^b K(x,\xi) y(\xi) d\xi, \qquad K(x,\xi) = \begin{cases} A(\xi)\left(\frac{x-b}{b-a}\right) - \left[\frac{(x-b)(\xi-a)}{b-a}\right][B(\xi) - A'(\xi)] \ , \ \xi < x \\ A(\xi)\left(\frac{x-a}{b-a}\right) - \left[\frac{(\xi-b)(x-a)}{b-a}\right][B(\xi) - A'(\xi)] \ , \ x < \xi \end{cases}$

(2) $\int_a^x (x-\xi) f(\xi) d\xi + \frac{x-a}{b-a}\left(y_b - y_a - \int_a^b (b-\xi) f(\xi) d\xi\right) + y_a$

$$\therefore y(x) = \int_a^b K(x,\xi) y(\xi) d\xi + F(x),$$

$$, K(x,\xi) = \begin{cases} A(\xi)\left(\frac{x-b}{b-a}\right) - \left[\frac{(x-b)(\xi-a)}{b-a}\right][B(\xi) - A'(\xi)] \ , \ \xi < x \\ A(\xi)\left(\frac{x-a}{b-a}\right) - \left[\frac{(\xi-b)(x-a)}{b-a}\right][B(\xi) - A'(\xi)] \ , \ x < \xi \end{cases}$$

$$, F(x) = \int_a^x (x-\xi) f(\xi) d\xi + \frac{x-a}{b-a}\left(y_b - y_a - \int_a^b (b-\xi) f(\xi) d\xi\right) + y_a$$

This equation is seen to be a ***Fredholm equation of the second kind.****

* Jerry, A.J., Introduction to Integral Equations with Applications, Marcel Dekker Inc., 1985, p67

Relation between differential and integral equations

Linear second order differential equation

$$\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=f(x)$$

B.V.P

Boundary condition :

$$y(a)=y_a,\ y(b)=y_b$$

⇔

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Example : Boundary Value Problem

$$\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)=0,\ y(l)=0$$

$$\int_0^l y''dx_1+\lambda\int_0^l ydx_1=0$$

$$\int_0^l y''dx_1=-\lambda\int_0^l ydx_1$$

$$y'(x)-y'(0)=-\lambda\int_0^x y(x_1)dx_1$$

$$\int_0^x y'dx_2=\int_0^x\left[-\lambda\int_0^x y(x_1)dx_1+y'(0)\right]dx_2$$

$$y(x)-y(0)=-\lambda\int_0^x\int_0^{x_2}y(x_1)dx_1dx_2+\int_0^x y'(0)dx_1$$

recall,

$$\therefore\int_a^x\int_a^{x_2}f(x_1)dx_1dx_2=\int_a^x(x-\xi)f(\xi)d\xi$$

$$y(x)=-\lambda\int_0^x(x-\xi)y(\xi)d\xi+y'(0)\left[x_1\right]_0^x$$

$$y(x)=-\lambda\int_0^x(x-\xi)y(\xi)d\xi+y'(0)\cdot x$$

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$$y(x)=-\lambda\int_0^x(x-\xi)y(\xi)d\xi+y'(0)\cdot x$$

$$y(l)=-\lambda\int_0^l(l-\xi)y(\xi)d\xi+y'(0)\cdot l$$

$$0=-\lambda\int_0^l(l-\xi)y(\xi)d\xi+y'(0)\cdot l$$

$$\therefore y'(0)=\frac{\lambda}{l}\int_0^l(l-\xi)y(\xi)d\xi$$

$$y(x)=-\lambda\int_0^x(x-\xi)y(\xi)d\xi+\frac{\lambda x}{l}\int_0^l(l-\xi)y(\xi)d\xi$$

$$y(x)=-\lambda\int_0^x(x-\xi)y(\xi)d\xi+\frac{\lambda x}{l}\int_0^x(l-\xi)y(\xi)d\xi+\frac{\lambda x}{l}\int_x^l(l-\xi)y(\xi)d\xi$$

$$y(x)=-\lambda\int_0^x\left[(x-\xi)-\frac{x}{l}(l-\xi)\right]y(\xi)d\xi+\frac{\lambda x}{l}\int_x^l(l-\xi)y(\xi)d\xi$$

$$y(x)=-\lambda\int_0^x\left[-\xi+\frac{x}{l}\xi\right]y(\xi)d\xi+\frac{\lambda x}{l}\int_x^l(l-\xi)y(\xi)d\xi$$

$$y(x)=\lambda\int_0^x\frac{\xi}{l}(l-x)y(\xi)d\xi+\lambda\int_x^l\frac{x}{l}(l-\xi)y(\xi)d\xi$$

$$\therefore y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi$$

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$$\therefore\ y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi$$

$$,K(x,\xi)=\begin{cases}\frac{\xi}{l}(l-x) & when\ \ \xi<x \\ \frac{x}{l}(l-\xi) & when\ \ \xi>x\end{cases}$$

check!

$$A(x)=0,\ B(x)=\lambda\ ,y_a=0, y_b=0,\ a=0,\ b=l,\ \ f(x)=0$$

$$K(x,\xi)=\begin{cases}\lambda\frac{\xi}{l}(l-x),\ \xi<x \\ \lambda\frac{x}{l}(l-\xi), x<\xi\end{cases}$$

$$F(x)=0$$

$$\therefore\ y(x)=\int_a^b K(x,\xi)y(\xi)d\xi$$

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Example : Boundary Value Problem

$$\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)=0,\ \ y(l)=0$$

⇩

$$\therefore\ y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi$$

$$,K(x,\xi)=\begin{cases} \dfrac{\xi}{l}(l-x) & when\ \ \xi<x \\ \dfrac{x}{l}(l-\xi) & when\ \ \xi>x \end{cases}$$

from the example, by direct integration.

- **What we did is** **to transform D.E of B.V.P into I.E**
- **What we had was** $\dfrac{d^2y}{dx^2}+A(x)\dfrac{dy}{dx}+B(x)y=f(x)$

  in case o f $A(x)=0,\ B(x)=\lambda,\ f(x)=0$
- **What we have is** $y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi$

  we express y in terms of 'Kernel' $,K(x,\xi)=\begin{cases} \dfrac{\xi}{l}(l-x) & when\ \ \xi<x \\ \dfrac{x}{l}(l-\xi) & when\ \ \xi>x \end{cases}$
- **When does this mean?**

what kind of properties?

if we find a 'kernel' of some properties,
we can express y of a D.E as an 'integral' form
which can be a solution or an equation

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Example : Boundary Value Problem

$$\mathscr{L}y+\Phi=0\ \ \textbf{+B/C}\ \Leftrightarrow\ y(x)=\int_0^1 G(x,\xi)\Phi(\xi)d\xi$$

equivalent

$$\mathscr{L}y=\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)-q(x)y$$

$$\mathscr{L}y=\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)-q(x)y\ \Rightarrow$$

$p=1,\ q=0$

$$\frac{d^2y}{dx^2}+\lambda y=0\ ,y(0)=0,\ y(l)=0$$

$$\Rightarrow y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi\ \ ,K(x,\xi)=\begin{cases}\frac{\xi}{l}(l-x) & when\ \ \xi<x\\ \frac{x}{l}(l-\xi) & when\ \ \xi>x\end{cases}$$

$$\mathscr{L}y=\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)-q(x)y\ \Rightarrow$$

$p=x,\ q=-\frac{1}{x}$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda x^2-1)y=0\ ,y(0)=0,\ \ y(1)=0$$

$$\Rightarrow y(x)=\lambda\int_0^1 G(x,\xi)\,\xi\, y(\xi)\,d\xi\ \ ,G(x,\xi)=\begin{cases}\frac{1}{2\xi}(1-\xi^2) & when\ \ x<\xi,\\ \frac{1}{2x}(1-x^2) & when\ \ x>\xi,\end{cases}$$

*

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{1}{x}y+\lambda xy=0$$

* Greenberg, M.D., Application of Green's Functions in Science and Engineering, Prentice-Hall, 1971, p8 : adjoint operator $\mathscr{L}$ consists of the differential operator plus boundary conditions which are such that the boundary terms, arising through the integration by parts, all vanish.

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what kind of properties?

✓Properties of kernel of the example

- **continuous :** when $\xi=x$ two expressions are equivalent

$$\left.\frac{\xi}{l}(l-x)\right|_{x=\xi}=\left.\frac{x}{l}(l-\xi)\right|_{\xi=x}$$

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what kind of properties?

✓Properties of kernel of the example

- **continuous** :when $\xi=x$ two expressions are equivalent
- **discontinuous first derivative** (finite jump) at $\xi=x$

$$\frac{d}{dx}\left(\frac{\xi}{l}(l-x)\right)=-\frac{\xi}{l}$$

$$\frac{d}{dx}\left(\frac{x}{l}(l-\xi)\right)=1-\frac{\xi}{l}$$

$$\therefore\ \frac{d}{dx}\left(\frac{\xi}{l}(l-x)\right)-\frac{d}{dx}\left(\frac{x}{l}(l-\xi)\right)=-\frac{\xi}{l}-1+\frac{\xi}{l}=-1$$

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- **continuous :** when $\xi=x$ two expressions are equivalent
- **discontinuous first derivative** (finite jump) at $\xi=x$
- **linear function of x:**

$$\frac{\partial^2K(x,\xi)}{\partial x^2}=0$$

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- **continuous :** when $\xi=x$ two expressions are equivalent
- **discontinuous first derivative** (finite jump) at $\xi=x$
- **linear function of x:**
- **satisfying B/C**

$$K(0,\xi)=\frac{x}{l}(l-\xi)\bigg|_{x=0}=0\qquad ,x<\xi$$

$$K(l,\xi)=\frac{\xi}{l}(l-x)\bigg|_{x=l}=0\qquad ,\ \xi<x$$

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✓Properties of kernel of the example

- **continuous :when** $\xi=x$ **two expressions are equivalent**
- **discontinuous first derivative (finite jump) at** $\xi=x$
- **linear function of x:**
- **satisfying B/C**
- **symmetry :** $K(x,\xi)=K(\xi,x)$

$K(x,\xi)$ is unchanged if x and ξ are interchanged

can we always get the kernel of these properties?

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- **continuous :when** $\xi=x$ **two expressions are equivalent**
- **discontinuous first derivative (finite jump) at** $\xi=x$
- **linear function of x:**
- **satisfying B/C**
- **symmetry**

can we always get the kernel of these properties?

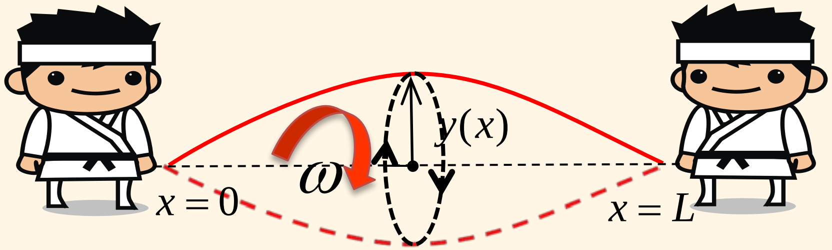
the kernel so obtained usually is *discontinuous* at $\xi=x$ **in the more general second order equation**

however(!), a kernel which is continuous can be obtained, in general

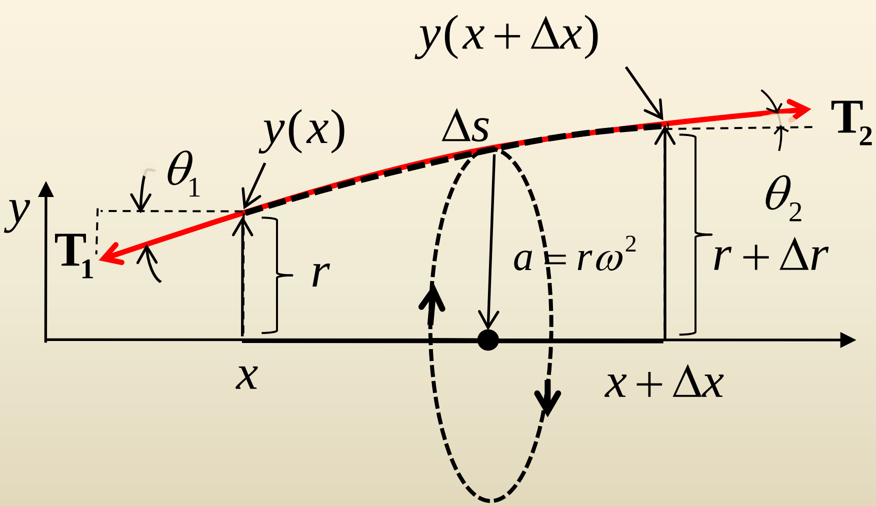
→ **"Green Function"**

Physical Meaning of Green Function

Ex) Rotating String



ρ **: string density**
ω **: string angular velocity**
T **: magnitude of tension**



$$\therefore T\frac{d^2y}{dx^2}+\rho\omega^2 y=0$$

$$\boxed{\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)=0,\ y(l)=0}$$

$$\therefore y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi \quad , K(x,\xi)=\begin{cases}\frac{\xi}{l}(l-x) & \text{when } \xi<x\\ \frac{x}{l}(l-\xi) & \text{when } \xi>x\end{cases}$$

Green function

displacement can be occurred with no external force and homogeneous B/C ?

in this example, string's angular velocity are causing the displacement. If tension is zero, this equation is not valid. With non zero tension, displacement is affected by the string's angular velocity and in the equation it is λ .

Even in the case of homogeneous B/C and no external force (actually, it means the nonhomogeneous term in the equation), there could be 'a source' causing 'motion' of the system in the equation *

* this statement is just writer's note, sea also Morse, P.M, Feshbach, H, Methods of theoretical Physics, McGraw-Hill, 1953, p791~p793

Relation between differential and integral equations

Example : Boundary Value Problem

$$\frac{d^2y}{dx^2}+\lambda y=0, y(0)=0,\ y(l)=0$$

$$\therefore y(x)=\lambda\int_0^l K(x,\xi)y(\xi)d\xi$$

$$,K(x,\xi)=\begin{cases}\frac{\xi}{l}(l-x) & when\ \ \xi<x\\ \frac{x}{l}(l-\xi) & when\ \ \xi>x\end{cases}$$

To recover differential equation from integral equation, differentiate

$$y(x)=\lambda\int_0^x\frac{\xi}{l}(l-x)y(\xi)d\xi+\lambda\int_x^l\frac{x}{l}(l-\xi)y(\xi)d\xi$$

$$\frac{dy}{dx}=\lambda\frac{d}{dx}\int_0^x\frac{\xi}{l}(l-x)y(\xi)d\xi+\lambda\frac{d}{dx}\int_x^l\frac{x}{l}(l-\xi)y(\xi)d\xi$$

by using $\frac{d}{dx}\int_{A(x)}^{B(x)}F(x,\xi)d\xi=\int_{A(x)}^{B(x)}\frac{\partial F(x,\xi)}{\partial x}d\xi+F[x,B(x)]\frac{dB}{dx}-F[x,A(x)]\frac{dA}{dx}$

$$\frac{dy}{dx}=\frac{\lambda}{l}\left[-\int_0^x\xi y(\xi)\,d\xi+x(l-x)y(x)+\int_x^l(l-\xi)y(\xi)d\xi-x(l-x)y(x)\right]$$

$$=\frac{\lambda}{l}\left[-\int_0^x\xi y(\xi)d\xi+\int_x^l(l-\xi)y(\xi)d\xi\right]$$

$$\frac{d^2y}{dx^2}=\frac{\lambda}{l}\left[-x\,y(x)-(l-x)\,y(x)\right]=-\lambda\,y(x)$$

$$\therefore\frac{d^2y}{dx^2}=-\lambda\,y$$

Integral Equations : The green's function

The green's function

Linear second order differential equation

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$$

Initial condition :

$$y(a) = y_0,\; y'(a) = y'_0$$

⟺

Fredholm integral equation of second kind

$$y(x) = \int_a^b K(x,\xi)y(\xi)d\xi + F(x),$$

F, K : given functions and continuous in (a,b)

$y(x)$: function is to be determined which is continuous in (a,b)

⇓ **consider this form**

$$\mathscr{L}y + \Phi(x) = 0$$

$$,where\quad \mathscr{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$$

$$,\Phi(x) = \phi(x, y(x))$$

homogeneous boundary conditions

$$\alpha y + \beta\frac{dy}{dx} = 0$$

For some constant values of α and β, which are imposed at the end points of an interval a≤x≤b.

In order to obtain a convenient reformulation of this problem, we first attempt the determination of a Green's function G which, for a given number ξ,

$$G = \begin{cases} G_1(x) & when\;\; x < \xi \\ G_2(x) & when\;\; x > \xi \end{cases}$$

which has the four following properties.

1. The function G_1 and G_2 satisfy the equation $\mathscr{L}G$=0 in their intervals of definition; that is, $\mathscr{L}G_1$=0 when $x<\xi$, and $\mathscr{L}G_2$=0 when $x>\xi$.

2. The function G satisfies the homogeneous conditions prescribed at the end points $x=a$ and $x=b$; that is, G_1 satisfies the condition prescribed at $x=a$, and G_2 that corresponding to $x=b$.

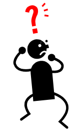
3. The function G is continuous at $x=\xi$; that is, $G_1(\xi)=G_2(\xi)$

4. The derivative of G has a discontinuity of magnitude -1/$p(\xi)$ at the point $x=\xi$; that is, $G_2'(\xi) - G_1'(\xi) = -1/p(\xi)$

When the function $G(x,\xi)$ exists, the original formulation of the problem can be transformed to the relation

$$y(x) = \int_a^b G(x,\xi)\Phi(\xi)d\xi$$

$$\begin{cases} \text{solution} : if\; \Phi = \Phi(x) \\ \text{integral equation} : if\; \Phi = \Phi(x,y) \end{cases}$$



then, how to get G?

The green's function

$$\mathscr{L} y + \Phi(x) = 0 \quad , B/C : \alpha y + \beta \frac{dy}{dx} = 0$$

$$, where \quad \mathscr{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$$

$$, \Phi(x) = \phi(x, y(x))$$

$$y(x) = \int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

1. The function G_1 and G_2 satisfy the equation $\mathscr{L}G=0$ in their intervals of definition; that is, $\mathscr{L}G_1=0$ when $x<\xi$, and $\mathscr{L}G_2=0$ when $x>\xi$.

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4. The derivative of G has a discontinuity of magnitude $-1/p(\xi)$ at the point $x=\xi$; that is, $G_2'(\xi) - G_1'(\xi) = -1/p(\xi)$

then, how to get G?

For the purpose of determining G,

let $y=u(x)$ be a nontrivial solution of the associated equation $\mathscr{L}y=0$ which satisfies the prescribed homogeneous condition at $x=a$, and

let $y=v(x)$ be a nontrivial solution of that equation which satisfies the condition prescribed at $x=b$.

If we write $G_1=c_1u(x)$ and $G_2=c_2v(x)$, condition 1 and 2 are satisfied.

$$G = \begin{cases} c_1 u(x) & when \quad x<\xi, \\ c_2 v(x) & when \quad x>\xi, \end{cases}$$

Condition 3 , determine c_1 and c_2 in terms of the value of ξ since condition 3 requires that

$$c_2 v(\xi) - c_1 u(\xi) = 0$$

The green's function

$$\mathscr{L}y + \Phi(x) = 0 \quad ,B/C: \alpha y + \beta\frac{dy}{dx} = 0$$

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then, how to get G?

from condition 1, 2 :

$$G = \begin{cases} c_1 u(x) & when \quad x<\xi, \\ c_2 v(x) & when \quad x>\xi, \end{cases}$$

from condition 3 : $c_2 v(\xi) - c_1 u(\xi) = 0 \cdots (a)$

for condition 4 : $c_2 v'(\xi) - c_1 u'(\xi) = -\dfrac{1}{p(\xi)} \cdots (b)$

$$\Rightarrow \begin{bmatrix} -u(\xi) & v(\xi) \\ -u'(\xi) & v'(\xi) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{1}{p(\xi)} \end{bmatrix}$$

We can get the value of c_1, c_2 **as** $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \dfrac{1}{-u(\xi)v'(\xi)+u'(\xi)v(\xi)}\begin{bmatrix} v'(\xi) & -v(\xi) \\ u'(\xi) & -u(\xi) \end{bmatrix}\begin{bmatrix} 0 \\ -\dfrac{1}{p(\xi)} \end{bmatrix}$

only when $-u(\xi)v'(\xi)+u'(\xi)v(\xi) \neq 0$ **means the functions u and v are linearly independent**

how to get G when they are linearly dependent?*

* Hildebrand, F.B., Methods of Applied Mathematics, Second edition, Dover, 1985, p300

The green's function

$$\mathscr{L}y + \Phi(x) = 0 \quad ,B/C: \alpha y + \beta\frac{dy}{dx} = 0$$

$$, where \quad \mathscr{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$$

$$,\Phi(x) = \phi(x, y(x))$$

$$y(x) = \int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

1. The function G_1 and G_2 satisfy the equation $\mathscr{L}G$=0 in their intervals of definition; that is, $\mathscr{L}G_1$=0 when $x<\xi$, and $\mathscr{L}G_2$=0 when $x>\xi$.

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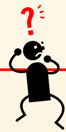

then, how to get G?

from condition 1, 2 :

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$$\begin{bmatrix} -u(\xi) & v(\xi) \\ -u'(\xi) & v'(\xi) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{1}{p(\xi)} \end{bmatrix}$$

We can get the value of c_1, c_2 **as**

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-u(\xi)v'(\xi) + u'(\xi)v(\xi)}\begin{bmatrix} v'(\xi) & -v(\xi) \\ u'(\xi) & -u(\xi) \end{bmatrix}\begin{bmatrix} 0 \\ -\dfrac{1}{p(\xi)} \end{bmatrix}$$

only when $u(\xi)v'(\xi) - u'(\xi)v(\xi) \neq 0$ **means the functions u and v are <u>linearly independent</u>**

by the condition 1,

$$\mathscr{L}G_1 = (p c_1 u')' + q c_1 u = 0 \rightarrow (p u')' + q u = 0$$
$$\mathscr{L}G_2 = (p c_2 v')' + q c_2 v = 0 \rightarrow (p v')' + q v = 0$$
$$at \quad x = \xi$$

The green's function

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$$at\ x = \xi$$

⇩

$$(pu')' + qu = 0 \cdots (a)$$
$$(pv')' + qv = 0 \cdots (b)$$

⇩

$$v \cdot (a): \quad v(pu')' + vqu = 0$$
$$u \cdot (b): \quad u(pv')' + uqv = 0$$

⇨

$$u \cdot (b) - v \cdot (a):$$
$$u(pv')' - v(pu')' = 0$$
$$u p' v' + u p v'' - v p' u' - v p u'' = 0$$
$$p'(uv' - u'v) + p(uv'' - vu'') = 0$$
$$p'(uv' - u'v) + p(uv'' + u'v' - v'u' - vu'') = 0$$
$$p'(uv' - u'v) + p\left[(uv')' - (vu')'\right] = 0$$
$$p'(uv' - u'v) + p\left[uv' - vu'\right]' = 0$$
$$[p(uv' - vu')]' = 0$$

⇩

$$\therefore\ p(uv' - vu') = A, \quad A: const$$
$$\therefore u(\xi)v'(\xi) - v(\xi)u'(\xi) = \frac{A}{p(\xi)}$$

* Hildebrand, F.B., Methods of Applied Mathematics, Second edition, Dover, 1985, p300

The green's function

$$\mathscr{L}y+\Phi(x)=0 \quad ,B/C:\alpha y+\beta\frac{dy}{dx}=0$$

$$,where \quad \mathscr{L}=\frac{d}{dx}\left(p\frac{d}{dx}\right)+q=p\frac{d^2}{dx^2}+\frac{dp}{dx}\frac{d}{dx}+q$$

$$,\Phi(x)=\phi(x,y(x))$$

$$y(x)=\int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

1. The function G_1 and G_2 satisfy the equation $\mathscr{L}G=0$ in their intervals of definition; that is, $\mathscr{L}G_1=0$ when $x<\xi$, and $\mathscr{L}G_2=0$ when $x>\xi$.

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$$G=\begin{cases} c_1u(x) & when \quad x<\xi, \\ c_2v(x) & when \quad x>\xi, \end{cases}$$

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$$at \quad x=\xi$$

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$$\therefore \begin{bmatrix}c_1\\ c_2\end{bmatrix}=-\frac{p(\xi)}{A}\begin{bmatrix}v'(\xi) & -v(\xi)\\ u'(\xi) & -u(\xi)\end{bmatrix}\begin{bmatrix}0\\ -\dfrac{1}{p(\xi)}\end{bmatrix}=-\frac{p(\xi)}{A}\begin{bmatrix}\dfrac{v(\xi)}{p(\xi)}\\ \dfrac{u(\xi)}{p(\xi)}\end{bmatrix}=\begin{bmatrix}-\dfrac{v(\xi)}{A}\\ -\dfrac{u(\xi)}{A}\end{bmatrix}$$

only when $u(\xi)v'(\xi)-u'(\xi)v(\xi)\neq 0$

$$u(\xi)v'(\xi)-v(\xi)u'(\xi)=\frac{A}{p(\xi)}, \quad A:const$$

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$$\therefore G(x,\xi)=\begin{cases} -\dfrac{1}{A}u(x)v(\xi) & when \quad x<\xi \\ -\dfrac{1}{A}v(x)u(\xi) & when \quad x>\xi \end{cases}$$

only when $u(\xi)v'(\xi)-u'(\xi)v(\xi)\neq 0$

$$u(\xi)v'(\xi)-v(\xi)u'(\xi)=\frac{A}{p(\xi)}, \quad A:const$$

where A is a constant, independent of x, ξ

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The green's function

Linear second order differential equation

$$\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=f(x)$$

Initial condition :

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$$\mathscr{L}y+\Phi(x)=0$$

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When the function $G(x,\xi)$ exists, the original formulation of the problem can be transformed to the relation

$$y(x)=\int_a^b G(x,\xi)\Phi(\xi)d\xi \qquad \begin{cases}\text{solution : } if\ \Phi=\Phi(x)\\ \text{integral equation : } if\ \Phi=\Phi(x,y)\end{cases}$$

in explicit form

$$y(x)=-\frac{1}{A}\left[\int_a^x u(\xi)v(x)\Phi(\xi)d\xi+\int_x^b u(x)v(\xi)\Phi(\xi)d\xi\right]$$

$$\therefore G(x,\xi)=\begin{cases}-\frac{1}{A}u(x)v(\xi) & when\quad x<\xi\\ -\frac{1}{A}v(x)u(\xi) & when\quad x>\xi\end{cases}$$

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The green's function

$$\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q \quad , G(x,\xi) = \begin{cases} -\frac{1}{A}u(x)v(\xi) & when \quad x < \xi \\ -\frac{1}{A}v(x)u(\xi) & when \quad x > \xi \end{cases}$$

$$A = p(\xi)\left[u(\xi)v'(\xi) - v(\xi)u'(\xi)\right]$$

Linear second order differential equation

$$\mathcal{L}y + \Phi(x) = 0 \quad , \Phi(x) = \phi(x, y(x))$$

homogeneous B/C $\alpha y + \beta \frac{dy}{dx} = 0$

⇔

When the function $G(x,\xi)$ exists, the original formulation of the problem can be transformed to the relation

$$y(x) = \int_a^b G(x,\xi)\Phi(\xi)d\xi \quad \begin{cases} \text{solution} : if\ \Phi = \Phi(x) \\ \text{integral equation} : if\ \Phi = \Phi(x,y) \end{cases}$$

$$y(x) = -\frac{1}{A}\left[\int_a^x u(\xi)v(x)\Phi(\xi)d\xi + \int_x^b u(x)v(\xi)\Phi(\xi)d\xi\right]$$

Show that $y(x) = \int_a^b G(x,\xi)\Phi(\xi)d\xi$

implies the differential equation $\mathcal{L}y + \Phi(x) = 0$

differentiate by using $\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_{A(x)}^{B(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + F[x, B(x)]\frac{dB}{dx} - F[x, A(x)]\frac{dA}{dx}$

$$y'(x) = -\frac{1}{A}\left[\int_a^x v'(x)u(\xi)\Phi(\xi)d\xi + \int_x^b u'(x)v(\xi)\Phi(\xi)d\xi\right]$$

$$\Rightarrow p'(x)y'(x) = -\frac{1}{A}\left[\int_a^x p'(x)v'(x)u(\xi)\Phi(\xi)d\xi + \int_x^b p'(x)u'(x)v(\xi)\Phi(\xi)d\xi\right]$$

$$y''(x) = -\frac{1}{A}\left[\int_a^x v''(x)u(\xi)\Phi(\xi)d\xi + \int_x^b u''(x)v(\xi)\Phi(\xi)d\xi\right] - \frac{1}{A}[v'(x)u(x) - u'(x)v(x)]\Phi(x)$$

$$\Rightarrow p(x)y''(x) = -\frac{1}{A}\left[\int_a^x p(x)v''(x)u(\xi)\Phi(\xi)d\xi + \int_x^b p(x)u''(x)v(\xi)\Phi(\xi)d\xi\right] - \frac{p(x)}{A}[v'(x)u(x) - u'(x)v(x)]\Phi(x)$$

The green's function

$$\mathscr{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q \quad , G(x,\xi) = \begin{cases} -\frac{1}{A}u(x)v(\xi) & when \quad x < \xi \\ -\frac{1}{A}v(x)u(\xi) & when \quad x > \xi \end{cases}$$

$$A = p(\xi)\left[u(\xi)v'(\xi) - v(\xi)u'(\xi)\right]$$

Linear second order differential equation

$$\mathscr{L}y + \Phi(x) = 0 \quad , \Phi(x) = \phi(x, y(x))$$

homogeneous B/C $\alpha y + \beta \frac{dy}{dx} = 0$

⇔

When the function $G(x,\xi)$ exists, the original formulation of the problem can be transformed to the relation

$$y(x) = \int_a^b G(x,\xi)\Phi(\xi)\,d\xi \quad \begin{cases} \text{solution} : if\ \Phi = \Phi(x) \\ \text{integral equation} : if\ \Phi = \Phi(x,y) \end{cases}$$

$$y(x) = -\frac{1}{A}\left[\int_a^x u(\xi)v(x)\Phi(\xi)\,d\xi + \int_x^b u(x)v(\xi)\Phi(\xi)\,d\xi\right]$$

In particular, $\Phi(x) = \lambda r(x)y(x) - f(x)$

$$\mathscr{L}y(x) + \lambda r(x)y(x) = f(x)$$

⇔

$$y(x) = \lambda\int_a^b G(x,\xi)r(\xi)\,y(\xi)\,d\xi - \int_a^b G(x,\xi)\,f(\xi)\,d\xi \quad \cdots(4\ \ 0$$

where *G* is relevant Green's function.

Kernel $K(x,\xi)$ is actually the product $G(x,\xi)r(\xi)$, and is not symmetric unless $r(x)$ is a constant. However if we write $\sqrt{r(x)}y(x) = Y(x)$

Under the assumption that r(x) is nonnegative over (a,b), the equation can be written in the form

$$Y(x) = \lambda\int_a^b \tilde{K}(x,\xi)Y(\xi)d\xi - \int_a^b \tilde{K}(x,\xi)\frac{f(\xi)}{\sqrt{r(\xi)}}d\xi$$

Where $\tilde{K}$ is defined by the relation $\tilde{K}(x,\xi) = \sqrt{r(x)r(\xi)}\,G(x,\xi)$

Hence possesses the same symmetry as *G*. $\left(\tilde{K}(x,\xi) = \tilde{K}(\xi,x)\right)$

The green's function

$$\mathscr{L}=\frac{d}{dx}\left(p\frac{d}{dx}\right)+q=p\frac{d^2}{dx^2}+\frac{dp}{dx}\frac{d}{dx}+q \quad ,G(x,\xi)=\begin{cases}-\frac{1}{A}u(x)v(\xi) & when \quad x<\xi \\ -\frac{1}{A}v(x)u(\xi) & when \quad x>\xi\end{cases}$$

$$A=p(\xi)\left[u(\xi)v'(\xi)-v(\xi)u'(\xi)\right]$$

Linear second order differential equation

$$\mathscr{L}y+\Phi(x)=0 \quad ,\Phi(x)=\phi(x,y(x))$$

homogeneous B/C $\alpha y+\beta\frac{dy}{dx}=0$

⇔

When the function $G(x,\xi)$ exists, the original formulation of the problem can be transformed to the relation

$$y(x)=\int_a^b G(x,\xi)\Phi(\xi)\,d\xi \qquad \begin{cases}\text{solution} : if\ \Phi=\Phi(x) \\ \text{integral equation} : if\ \Phi=\Phi(x,y)\end{cases}$$

$$y(x)=-\frac{1}{A}\left[\int_a^x u(\xi)v(x)\Phi(\xi)\,d\xi+\int_x^b u(x)v(\xi)\Phi(\xi)\,d\xi\right]$$

In particular, $\Phi(x)=\lambda r(x)y(x)-f(x)$

$$\mathscr{L}y(x)+\lambda r(x)y(x)=f(x) \quad \Leftrightarrow \quad y(x)=\lambda\int_a^b G(x,\xi)r(\xi)y(\xi)\,d\xi-\int_a^b G(x,\xi)f(\xi)\,d\xi \quad \cdots(4\ \ 0$$

where G is relevant Green's function.

In the special case when the operator $\mathscr{L}$ and the associated end conditions are such that

$$\mathscr{L}y\equiv y'',\ \ y(0\ \ =)y(l)=0$$

it is readily verified that the relevant Green's function G is identified with the kernel K defined by

$$G(x,\xi)=\begin{cases}\frac{x}{l}(l-\xi) & (x<\xi), \\ \frac{\xi}{l}(l-x) & (x>\xi).\end{cases}$$

Thus, in particular, the solution of the problem

$$y''=f(x) \\ ,y(0)=y(l)=0 \quad \Leftrightarrow \quad y(x)=-\int_0^l G(x,\xi)\,f(\xi)\,d\xi,$$

The green's function

$$\mathscr{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q \quad , G(x,\xi) = \begin{cases} -\frac{1}{A}u(x)v(\xi) & \text{when } x < \xi \\ -\frac{1}{A}v(x)u(\xi) & \text{when } x > \xi \end{cases}$$

$$A = p(\xi)\left[u(\xi)v'(\xi) - v(\xi)u'(\xi)\right]$$

Linear second order differential equation

$$\mathscr{L}y + \Phi(x) = 0 \quad , \Phi(x) = \phi(x, y(x))$$

homogeneous B/C $\quad \alpha y + \beta \frac{dy}{dx} = 0$

⇔

When the function $G(x,\xi)$ exists, the original formulation of the problem can be transformed to the relation

$$y(x) = \int_a^b G(x,\xi)\Phi(\xi)\,d\xi \quad \begin{cases} \text{solution} : if\ \Phi = \Phi(x) \\ \text{integral equation} : if\ \Phi = \Phi(x,y) \end{cases}$$

$$y(x) = -\frac{1}{A}\left[\int_a^x u(\xi)v(x)\Phi(\xi)\,d\xi + \int_x^b u(x)v(\xi)\Phi(\xi)\,d\xi\right]$$

In particular, $\Phi(x) = \lambda r(x)y(x) - f(x)$

$$\mathscr{L}y(x) + \lambda r(x)y(x) = f(x) \quad \Leftrightarrow \quad y(x) = \lambda\int_a^b G(x,\xi)r(\xi)y(\xi)\,d\xi - \int_a^b G(x,\xi)f(\xi)\,d\xi \quad \cdots(4\ \ 0$$

where *G* is relevant Green's function.

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whereas(!) the problem

$$y'' + \lambda r y = f(x), \quad y(0) = y(l) = 0$$

is equivalent to the integral equation

$$y(x) = \lambda\int_0^l G(x,\xi)r(\xi)y(\xi)\,d\xi - \int_0^l G(x,\xi)f(\xi)\,d\xi$$

$$\mathscr{L}y \equiv y'', \quad y(0\ {=})y(l) = 0$$

Thus, in particular, the solution of the problem

$$y'' = f(x), \ y(0) = y(l) = 0$$

is

$$y(x) = -\int_0^l G(x,\xi)f(\xi)\,d\xi,$$

The green's function

$$\mathcal{L}y+\Phi(x)=0 \iff y(x)=\int_a^b G(x,\xi)\Phi(\xi)d\xi$$

$$\mathcal{L}=\frac{d}{dx}\left(p\frac{d}{dx}\right)+q=p\frac{d^2}{dx^2}+\frac{dp}{dx}\frac{d}{dx}+q$$

homogeneous B/C $\alpha y+\beta\dfrac{dy}{dx}=0$

In particular, $\Phi(x)=\lambda r(x)y(x)-f(x)$

$$\mathcal{L}y(x)+\lambda r(x)y(x)=f(x) \iff y(x)=\lambda\int_a^b G(x,\xi)r(\xi)y(\xi)d\xi-\int_a^b G(x,\xi)f(\xi)d\xi \quad \cdots(4\ \ 0$$

where *G* is relevant Green's function.

When the prescribed end condition are not homogeneous, $\left(y(a)=f(x),\ y(b)=g(x)\right)$
a modified procedure is needed.

In this case,
we denote by $G(x,\xi)$ the **Green's function** corresponding the associated **homogeneous end conditions**,
and attempt to determine a function $P(x)$ such that the relation

$$y(x)=P(x)+\int_a^b G(x,\xi)\Phi(\xi)d\xi$$

is equivalent to the differential equation

$$\mathcal{L}y(x)+\Phi(x)=0$$

together with the prescribed **nonhomogeneous** end conditions.

The requirement imply $\mathcal{L}\left(P(x)+\int_a^b G(x,\xi)\Phi(\xi)d\xi\right)+\Phi(x)=0$

$$\mathcal{L}P(x)+\boxed{\mathcal{L}\int_a^b G(x,\xi)\Phi(\xi)d\xi+\Phi(x)}=0$$

zero with homogeneous end condition

⇩

$\mathcal{L}P(x)=0$ **nonhomogeneous end condition**

The green's function

$$\mathscr{L}y + \Phi(x) = 0 \quad \Longleftrightarrow \quad y(x) = \int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

$$\mathscr{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$$

homogeneous B/C $\alpha y + \beta \dfrac{dy}{dx} = 0$

In particular, $\Phi(x) = \lambda r(x)y(x) - f(x)$

$$\mathscr{L}y(x) + \lambda r(x)y(x) = f(x) \quad \Longleftrightarrow \quad y(x) = \lambda\int_a^b G(x,\xi)r(\xi)y(\xi)\,d\xi - \int_a^b G(x,\xi)f(\xi)\,d\xi \quad \cdots (4\ 0$$

where G is relevant Green's function.

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example) $y'' + xy = 1, \quad y(0) = 0, \; y(l) = 1$

$$,G(x,\xi) = \begin{cases} \dfrac{\xi}{l}(l-x) & (\xi < x) \\ \dfrac{x}{l}(l-\xi) & (\xi > x) \end{cases}$$

attempt to determine,

$$y(x) = P(x) + \int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

with Green function to the homogeneous end conditions $y(0) = 0, \; y(l) = 0$

and $P(x)$ satisfying the nonhomogeneous end conditions

in the problem, $\Phi(x) = xy(x) - 1, \quad \mathscr{L}y \equiv y''$

$$\therefore \mathscr{L}y + \Phi(x) = 0$$

$$\mathscr{L}P(x) + \boxed{\mathscr{L}\int_a^b G(x,\xi)\Phi(\xi)\,d\xi + \Phi(x)} = 0$$

⇩ **zero with homogeneous end condition**

$\mathscr{L}P(x) = 0$ **nonhomogeneous end condition** ⇨ $P''(x) = 0, \; P(0) = 0, \; P(l) = 1,$

The green's function

$$\mathscr{L}y+\Phi(x)=0 \iff y(x)=\int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

$$\mathscr{L}=\frac{d}{dx}\left(p\frac{d}{dx}\right)+q=p\frac{d^2}{dx^2}+\frac{dp}{dx}\frac{d}{dx}+q$$

homogeneous B/C $\alpha y+\beta\frac{dy}{dx}=0$

In particular, $\Phi(x)=\lambda r(x)y(x)-f(x)$

$$\mathscr{L}y(x)+\lambda r(x)y(x)=f(x) \iff y(x)=\lambda\int_a^b G(x,\xi)r(\xi)y(\xi)\,d\xi-\int_a^b G(x,\xi)f(\xi)\,d\xi \quad \cdots(4\ 0$$

where G is relevant Green's function.

example) $y''+xy=1,\quad y(0)=0,\ y(l)=1$

$$,G(x,\xi)=\begin{cases}\frac{\xi}{l}(l-x) & (\xi<x)\\ \frac{x}{l}(l-\xi) & (\xi>x)\end{cases}$$

attempt to determine,

$$y(x)=P(x)+\int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

with Green function to the homogeneous end conditions $y(0)=0,\ y(l)=0$

and $P(x)$ satisfying the nonhomogeneous end conditions

in the problem, $\Phi(x)=xy(x)-1,\quad \mathscr{L}y\equiv y''$

$$P''(x)=0,\ P(0)=0,\ P(l)=1,$$

$$P'(x)=c_1,\quad P(x)=c_1x+c_2 \ \Rightarrow\ \text{by the B/C}\ \Rightarrow\ c_2=0,\ c_1=\frac{1}{l}\ \Rightarrow\ \therefore P(x)=\frac{x}{l}$$

The green's function

$$\mathscr{L} y + \Phi(x) = 0 \quad \Longleftrightarrow \quad y(x) = \int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

$$\mathscr{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q = p\frac{d^2}{dx^2} + \frac{dp}{dx}\frac{d}{dx} + q$$

homogeneous B/C $\alpha y + \beta \dfrac{dy}{dx} = 0$

In particular, $\Phi(x) = \lambda r(x) y(x) - f(x)$

$$\mathscr{L} y(x) + \lambda r(x) y(x) = f(x) \quad \Longleftrightarrow \quad y(x) = \lambda \int_a^b G(x,\xi) r(\xi) y(\xi)\,d\xi - \int_a^b G(x,\xi) f(\xi)\,d\xi \quad \cdots (4\ \ 0$$

where G is relevant Green's function.

---

example) $y'' + xy = 1, \quad y(0) = 0, \; y(l) = 1$

attempt to determine,

$$y(x) = P(x) + \int_a^b G(x,\xi)\Phi(\xi)\,d\xi$$

$$\Phi(x) = xy(x) - 1, \qquad P(x) = \frac{x}{l}$$

$$\therefore\ y(x) = \frac{x}{l} + \int_0^l G(x,\xi)[\xi\, y(\xi) - 1]\,d\xi$$

$$-\int_0^l G(x,\xi)\,d\xi = -\int_0^x \frac{\xi}{l}(l-x)\,d\xi - \int_x^l \frac{x}{l}(l-\xi)\,d\xi$$

$$= -\left[\frac{1}{2}\frac{\xi^2}{l}(l-x)\right]_0^x - \left[\frac{x}{l}(l - \frac{1}{2}\xi^2)\right]_x^l$$

$$= -\frac{1}{2}\frac{x^2}{l}(l-x) - \frac{x}{l}(l - \frac{1}{2}l^2) + \frac{x}{l}(l - \frac{1}{2}x^2)$$

$$= -\frac{1}{2}x^2 + \frac{1}{2}\frac{x^3}{l} - x + \frac{1}{2}xl + x - \frac{1}{2}\frac{x^3}{l}$$

$$= -\frac{1}{2}x^2 + \frac{1}{2}xl$$

$$= -\frac{1}{2}x(x-l)$$

and reduces to the form

$$y(x) = \frac{x}{l} - \frac{x}{2}(l-x) + \int_0^l G(x,\xi)\,\xi\, y(\xi)\,d\xi \quad , where \quad G(x,\xi) = \begin{cases} \dfrac{\xi}{l}(l-x) & (\xi < x) \\ \dfrac{x}{l}(l-\xi) & (\xi > x) \end{cases}$$