Naval Architecture & Ocean Engineering

학부4학년 교과목 "창의적 선박설계(Innovative ship design)" 강의 교재

[2009] [11]

Innovative ship design

-Elasticity –

Governing Equations

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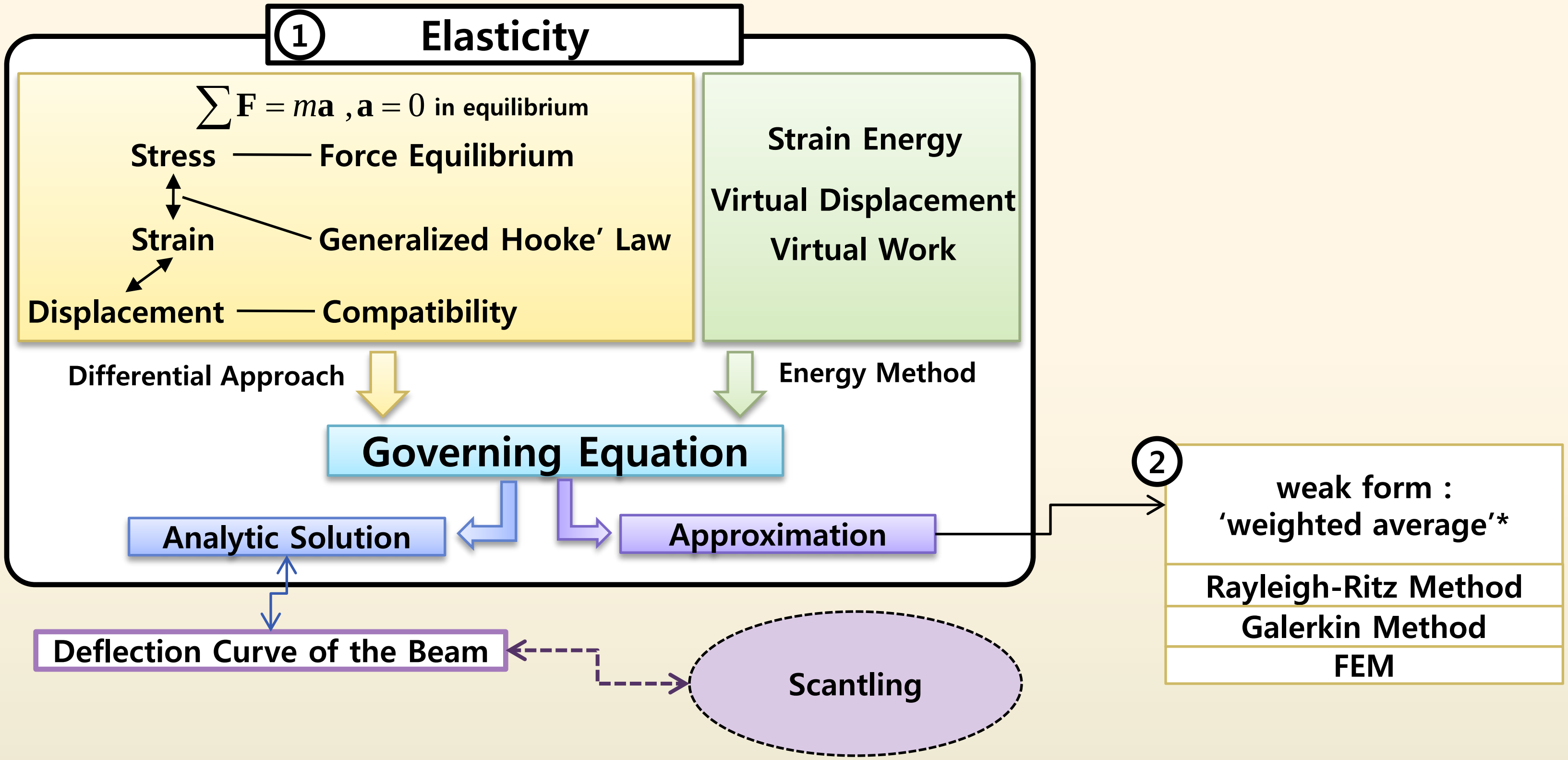

① Elasticity

$\sum \mathbf{F} = m\mathbf{a}$, $\mathbf{a} = 0$ in equilibrium

Stress — Force Equilibrium

Strain — Generalized Hooke' Law

Displacement — Compatibility

Strain Energy

Virtual Displacement

Virtual Work

Differential Approach

Energy Method

Governing Equation

Analytic Solution

Approximation

Deflection Curve of the Beam

Scantling

② weak form : 'weighted average'*

Rayleigh-Ritz Method

Galerkin Method

FEM

①

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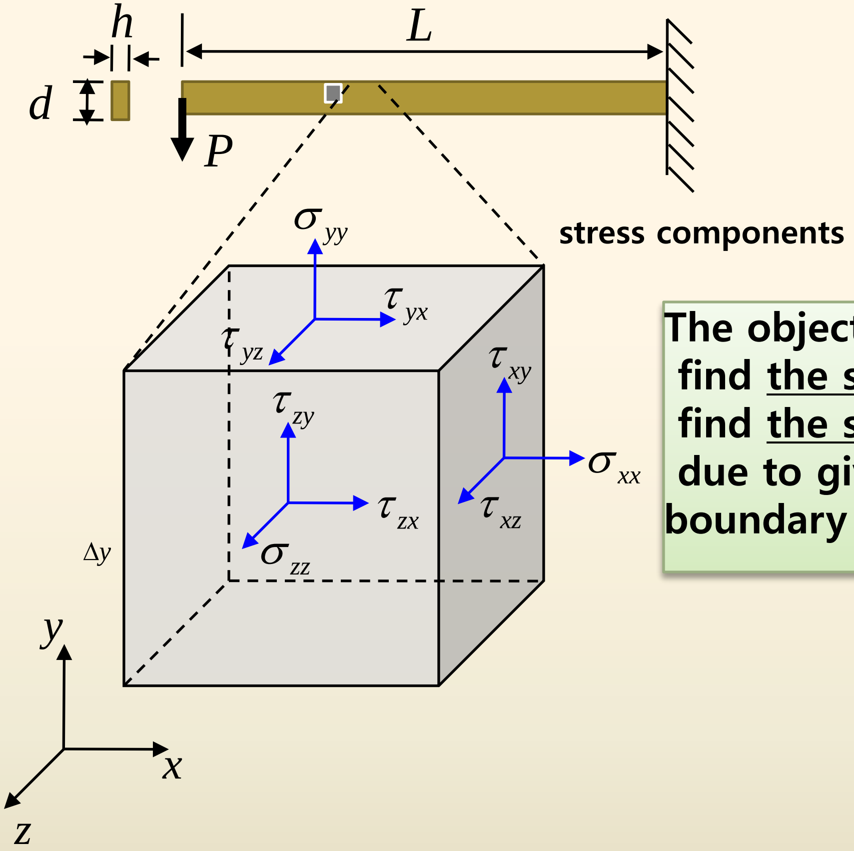
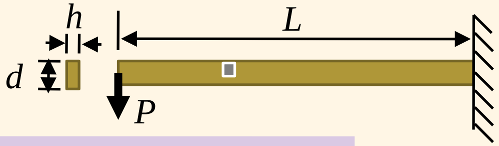
Summary

Problem in Elasticity



The object :
find the stress distribution in an elastic body (or
find the strain at any point)
due to given body forces and given conditions at the boundary of the body

Summary
Variables and Equations



"Displacement"

u, v, w

"Strain"

$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

"Force (Stress)"

$\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$

6 Relations between
Strain and Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\varepsilon_z = \frac{\partial w}{\partial z}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$
$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

- **Generalized Hooke's Law**

6 Relations between
Strain and Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$
$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$
$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$
$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

- **Newton-Euler Equations**

6 Equations of force and moment equilibrium

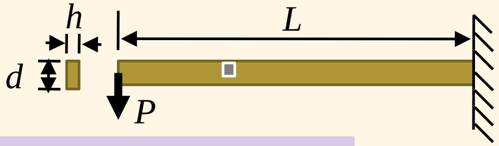
$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$
$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$
$$\sum M_x = \tau_{yz} - \tau_{zy} = 0$$
$$\sum M_y = \tau_{xz} - \tau_{zx} = 0$$
$$\sum M_z = \tau_{xy} - \tau_{yx} = 0$$

$$\frac{2(\nu+1)}{E} = \frac{1}{G}$$

ν : Poisson's Ratio
G : Shear Modulus
E : Young's Modulus

Summary

Variables and Equations



6 Strain

$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement

u, v, w

9 Stress

$\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$

6 Relations between Strain and Displacement

***6 Relations** btw. Strain and Stress*

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

- Generalized Hooke's Law

6 Relations between Strain and Stress

***6 Relations** btw. Strain and Displacement*

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

- Newton-Euler Equations

6 Equations of force and moment equilibrium

***6 Equations** of force and moment equilibrium*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

$$\sum M_x = \tau_{yz} - \tau_{zy} = 0$$

$$\sum M_y = \tau_{xz} - \tau_{zx} = 0$$

$$\sum M_z = \tau_{xy} - \tau_{yx} = 0$$

Problem of

18 Variables, 18 Equations

Summary

Variables and Equations

Problem of **18 Variables, 18 Equations**

$$
\left[\begin{array}{ccccccccc|cccccc|ccc}
\sigma_x & \tau_{yx} & \tau_{zx} & \tau_{xy} & \sigma_y & \tau_{zy} & \tau_{xz} & \tau_{yz} & \sigma_z & \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} & u & v & w \\
\hline
\partial/\partial x & \partial/\partial y & \partial/\partial z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \partial/\partial x & \partial/\partial y & \partial/\partial z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial/\partial x & \partial/\partial y & \partial/\partial z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1/E & 0 & 0 & 0 & \nu/E & 0 & 0 & 0 & \nu/E & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\nu/E & 0 & 0 & 0 & -1/E & 0 & 0 & 0 & \nu/E & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\nu/E & 0 & 0 & 0 & \nu/E & 0 & 0 & 0 & -1/E & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\partial/\partial x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\partial/\partial y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\partial/\partial z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\partial/\partial y & -\partial/\partial x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\partial/\partial z & -\partial/\partial y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\partial/\partial z & 0 & -\partial/\partial x
\end{array}\right]
\begin{bmatrix} \sigma_x \\ \tau_{yx} \\ \tau_{zx} \\ \tau_{xy} \\ \sigma_y \\ \tau_{zy} \\ \tau_{xz} \\ \tau_{yz} \\ \sigma_z \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \\ u \\ v \\ w \end{bmatrix}
=
\begin{bmatrix} -X \\ -Y \\ -Z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Summary

Variables and Equations

Problem of $\boxed{\text{18 Variables, 18 Equations}}$

$$[\sigma_x \quad \tau_{yx} \quad \tau_{zx} \quad \tau_{xy} \quad \sigma_y \quad \tau_{zy} \quad \tau_{xz} \quad \tau_{yz} \quad \sigma_z \quad \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \quad u \quad v \quad w\,]$$

$$\left[\begin{array}{ccccccccc|cccccc|ccc}
\partial/\partial x & \partial/\partial y & \partial/\partial z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \partial/\partial x & \partial/\partial y & \partial/\partial z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial/\partial x & \partial/\partial y & \partial/\partial z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1/E & 0 & 0 & 0 & \nu/E & 0 & 0 & 0 & \nu/E & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\nu/E & 0 & 0 & 0 & -1/E & 0 & 0 & 0 & \nu/E & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\nu/E & 0 & 0 & 0 & \nu/E & 0 & 0 & 0 & -1/E & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\partial/\partial x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\partial/\partial y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\partial/\partial z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\partial/\partial y & -\partial/\partial x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\partial/\partial z & -\partial/\partial y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\partial/\partial z & 0 & -\partial/\partial x
\end{array}\right]
\left[\begin{array}{c}
\sigma_x \\ \tau_{yx} \\ \tau_{zx} \\ \tau_{xy} \\ \sigma_y \\ \tau_{zy} \\ \tau_{xz} \\ \tau_{yz} \\ \sigma_z \\ \hline \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \\ \hline u \\ v \\ w
\end{array}\right]
=
\left[\begin{array}{c}
-X \\ -Y \\ -Z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{array}\right]$$

Stress

Stress-Strain

Strain- Displacement

Summary

Variables and Equations

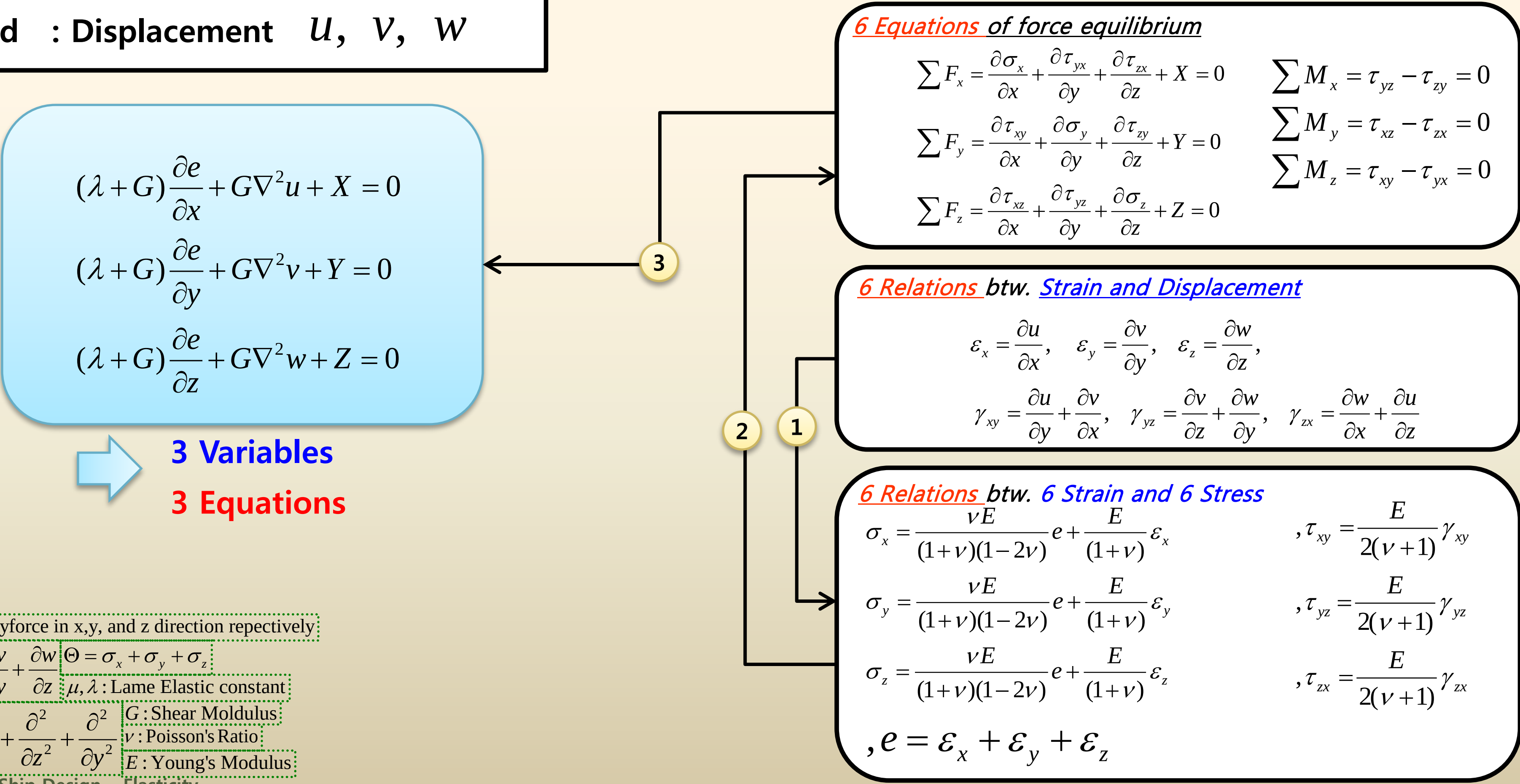
If we are interested in finding the displacement components in a body, we may reduce the system of equations to <u>three equations with three unknown displacement components.</u>

18 Variables

- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Given : Body force X, Y, Z

Find : Displacement u, v, w



18 Equations

6 Equations of force equilibrium

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \qquad \sum M_x = \tau_{yz} - \tau_{zy} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \qquad \sum M_y = \tau_{xz} - \tau_{zx} = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \qquad \sum M_z = \tau_{xy} - \tau_{yx} = 0$$

6 Relations btw. Strain and Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations btw. 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x \qquad , \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y \qquad , \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_z \qquad , \tau_{zx} = \frac{E}{2(\nu+1)} \gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

③ ② ①

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0$$

$$(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 v + Y = 0$$

$$(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 w + Z = 0$$

3 Variables

3 Equations

X, Y, Z : bodyforce in x,y, and z direction repectively

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$\Theta = \sigma_x + \sigma_y + \sigma_z$

μ, λ : Lame Elastic constant

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

G : Shear Moldulus

ν : Poisson's Ratio

E : Young's Modulus

Summary : Compatibility Equations

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

known : u, v, w ⇒ **find :** $\varepsilon_x, \quad \varepsilon_y, \quad \varepsilon_z, \quad \gamma_{xy}, \quad \gamma_{yz}, \quad \gamma_{zx}$

known : $\varepsilon_x, \quad \varepsilon_y, \quad \varepsilon_z, \quad \gamma_{xy}, \quad \gamma_{yz}, \quad \gamma_{zx}$ ⇒ **find :** u, v, w

this system of equations does not, in general, possess a solution for u,v and w unless the six strain components are somehow related.

the strain components must satisfy these expressions in order that solutions for the displacement components exist*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

or

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Chou, P.C., Elasticity, Van Nostard Company, 1967, p39-41*

Summary

If we are interested in finding only the stress components in a body, we may reduce the system of equations to six equations with six unknown stress components.

18 Variables

- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Given : Body force X, Y, Z

Find : Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$

18 Equations

6 Equations of force equilibrium

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

$$\sum M_x = \tau_{yz} - \tau_{zy} = 0$$

$$\sum M_y = \tau_{xz} - \tau_{zx} = 0$$

$$\sum M_z = \tau_{xy} - \tau_{yx} = 0$$

6 Relations btw. *Strain and Displacement*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations btw. *6 Strain and 6 Stress*

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x \qquad , \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y \qquad , \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_z \qquad , \tau_{zx} = \frac{E}{2(\nu+1)} \gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

X, Y, Z: bodyforce in x,y, and z direction repectively

$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

$\Theta = \sigma_x + \sigma_y + \sigma_z$

μ, λ: Lame Elastic constant

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$

G: Shear Moldulus

ν: Poisson's Ratio

E: Young's Modulus

Summary

18 Variables → **15 Variables**

- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

If we are interested in finding only the stress components in a body, we may reduce the system of equations to <u>six equations with six unknown stress components</u>

Given : Body force X, Y, Z

Find : Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial X}{\partial x}+\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Y}{\partial y}+\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Z}{\partial z}+\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=0$$

$$\left(\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\right)+\nabla^2\tau_{xy}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y}=0$$

$$\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)+\nabla^2\tau_{yz}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z}=0$$

$$\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)+\nabla^2\tau_{zx}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x}=0$$

6 Variables
6 Equations

X, Y, Z : bodyforce in x,y, and z direction repectively

$e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$

$\Theta=\sigma_x+\sigma_y+\sigma_z$

μ, λ : Lame Elastic constant

$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}$

G : Shear Moldulus

ν : Poisson's Ratio

E : Young's Modulus

18 Equations → 15 Equations

2

<u>6 Equations</u> <u>of force equilibrium</u>

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

$$\sum F_z=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0$$

$$\sum M_x=\tau_{yz}-\tau_{zy}=0$$

$$\sum M_y=\tau_{xz}-\tau_{zx}=0$$

$$\sum M_z=\tau_{xy}-\tau_{yx}=0$$

<u>6 Relations</u> btw. <u>Strain and Displacement</u>

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

3

Compatibility equations *<u>3 independent Equations</u>*

$$\begin{cases}\dfrac{\partial^2\varepsilon_x}{\partial y^2}+\dfrac{\partial^2\varepsilon_y}{\partial x^2}=\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y}\\[2ex] \dfrac{\partial^2\varepsilon_y}{\partial z^2}+\dfrac{\partial^2\varepsilon_z}{\partial y^2}=\dfrac{\partial^2\gamma_{yz}}{\partial y\partial z}\\[2ex] \dfrac{\partial^2\varepsilon_z}{\partial x^2}+\dfrac{\partial^2\varepsilon_x}{\partial z^2}=\dfrac{\partial^2\gamma_{zx}}{\partial z\partial x}\end{cases}\quad or\quad \begin{cases}2\dfrac{\partial^2\varepsilon_x}{\partial y\partial z}=\dfrac{\partial}{\partial x}\left(-\dfrac{\partial\gamma_{yz}}{\partial x}+\dfrac{\partial\gamma_{zx}}{\partial y}+\dfrac{\partial\gamma_{xy}}{\partial z}\right)\\[2ex] 2\dfrac{\partial^2\varepsilon_y}{\partial z\partial x}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial\gamma_{yz}}{\partial x}-\dfrac{\partial\gamma_{zx}}{\partial y}+\dfrac{\partial\gamma_{xy}}{\partial z}\right)\\[2ex] 2\dfrac{\partial^2\varepsilon_z}{\partial x\partial y}=\dfrac{\partial}{\partial z}\left(\dfrac{\partial\gamma_{yz}}{\partial x}+\dfrac{\partial\gamma_{zx}}{\partial y}-\dfrac{\partial\gamma_{xy}}{\partial z}\right)\end{cases}$$

1

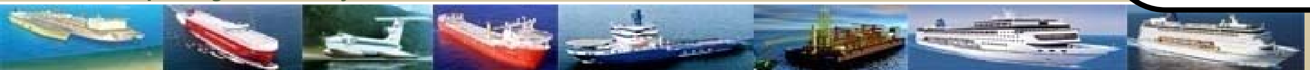
<u>6 Relations</u> btw. 6 Strain and 6 Stress

$$\sigma_x=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_x\quad ,\tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$

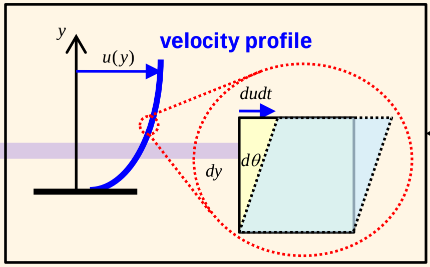
$$\sigma_y=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_y\quad ,\tau_{yz}=\frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_z\quad ,\tau_{zx}=\frac{E}{2(\nu+1)}\gamma_{zx}$$

$$,e=\varepsilon_x+\varepsilon_y+\varepsilon_z$$



Summary

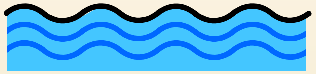


Equation of Motion

- **dynamics** $\sum \mathbf{F} \rightarrow$ cause motion $\rightarrow m\dfrac{d\mathbf{V}}{dt}$
- **static** $\sum \mathbf{F} \rightarrow$ without motion $\rightarrow$ 'internal change' $\rightarrow$ strain

$$\boxed{m\frac{d\mathbf{V}}{dt}} = \sum \mathbf{F} = \mathbf{F}_{Body} + \boxed{\mathbf{F}_{Surface}}$$

Fluid Equation of Motion
Navier Stokes Equation



Total Derivative of Velocity Vector

$$\boxed{\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)} = \rho g_x - \boxed{\frac{\partial P}{\partial x}+\frac{\mu}{3}\frac{\partial}{\partial x}(\Theta)+\mu\nabla^2 u}$$

$\mathbf{V}$: velocity vector

$\mathbf{V} = (u(x,y,z;t), v(x,y,z;t), w(x,y,z;t))$

μ : viscosity , $\left[\dfrac{\mathrm{N}\cdot\mathrm{s}}{m^2}\right]$

$\Theta = \nabla \bullet \mathbf{V} = \dfrac{du}{dx}+\dfrac{dv}{dy}+\dfrac{dw}{dz}$

Newtonian fluid

$\tau \propto \dfrac{du}{dy}, \ \tau_{xy} = \mu\left(\dfrac{du}{dy}+\dfrac{dv}{dx}\right)$

Linear Elastic Equation of Motion*
(also called *the Navier equation* :
principal equation for the motion)

$$\boxed{\rho\frac{\partial^2 \bar{u}}{\partial t^2}} = \rho F + \boxed{(\lambda+\mu)\frac{\partial^2 \bar{u}}{\partial x^2} + \mu\nabla^2 \bar{u}}$$

$\bar{u} = \bar{u}(x,t)$: displacement

ν : Poisson's Ratio
G : Shear Modulus
E : Young's Modulus

μ, λ : Lame Elastic constant

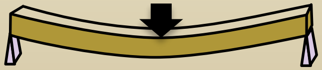
$\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}, \left[\dfrac{N}{m^2}\right]$

$\mu = G, \left[\dfrac{N}{m^2}\right]$

$\tau_{xy} = \mu\gamma_{xy} = \mu\left(\dfrac{\partial \bar{v}}{\partial x}+\dfrac{\partial \bar{u}}{\partial y}\right)$

$\varepsilon = \dfrac{\delta L}{L}, \ \varepsilon_x = \dfrac{\partial \bar{u}}{\partial x}$

Formulation of Elasticity Problems (Static) in displacement components



$$\boxed{0} = X + \boxed{(\lambda+G)\frac{\partial^2 \bar{u}}{\partial x^2} + G\nabla^2 \bar{u}}$$

$\bar{u} = \bar{u}(x)$: displacement

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*Betounes D., Partial Differential Equations for Computational Science, Springer, 1998, p343

Contents

Summary : Generalized Hooke's Law

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = G$$

ν : Poisson's Ratio
G : Shear Modulus
E : Young's Modulus

✓Idealization of material

Assume that the material is *perfectly elastic*, i.e., we shall limit our attention to the behavior of the material before the elastic limit is reached

✓Material Assumption

Continuous : Material having the nature of a structureless mass
Homogeneous : In case the elastic properties are the same throughout the body, i.e., are independent of the location
Isotropic : In case the elastic properties of the body are the same in all directions about any given point

✓Generalized Hooke's Law

- **Hooke's Law : "Extension is proportional to force"** $\sigma = E\varepsilon$
- **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

$$\begin{aligned}
\sigma_x &= c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\
\sigma_y &= c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\
\sigma_z &= c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\
\tau_{xy} &= c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\
\tau_{yz} &= c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\
\tau_{zx} &= c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx}
\end{aligned}$$

36 constants?

the principal directions and isotropy

reduced to only *two independent constants*

$$\begin{cases}
\sigma_1 = \lambda e + 2\mu\varepsilon_1 \\
\sigma_2 = \lambda e + 2\mu\varepsilon_2 \\
\sigma_3 = \lambda e + 2\mu\varepsilon_3
\end{cases}$$

μ, λ : Lame Elastic constant

the principal strains and principal stress occur in the same directions[1)]

The relation between
the principal stress components $\sigma_1,\ \sigma_2,\ \sigma_3$ **and**
the principal strain components $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$

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1) Gere, J.M., Mechanics of Materials, Sixth Edition, Thomson, 2006, p517

l, m, n : directional cosine

Summary : Transformation of Stress

stress ↔ **principal stress**

① $\sigma_x = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3$

② $\sigma_y = l_2^2\sigma_1 + m_2^2\sigma_2 + n_2^2\sigma_3$

③ $\sigma_z = l_3^2\sigma_1 + m_3^2\sigma_2 + n_3^2\sigma_3$

④ $\tau_{xy} = l_1 l_2\sigma_1 + m_1 m_2\sigma_2 + n_1 n_2\sigma_3$

⑤ $\tau_{yz} = l_2 l_3\sigma_1 + m_2 m_3\sigma_2 + n_2 n_3\sigma_3$

⑥ $\tau_{zx} = l_3 l_1\sigma_1 + m_3 m_1\sigma_2 + n_3 n_1\sigma_3$

the transformation of *stress* between principal direction 1,2,3 and arbitrary x,y,z system

$$\sigma_x = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3$$

redefine x'→x

$$\sigma_{x'} = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3$$

$$\sigma_{x'} = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3 + 2l_1m_1\cancelto{}{\tau_{12}} + 2m_1n_1\cancelto{}{\tau_{23}} + 2n_1l_1\cancelto{}{\tau_{31}}$$

redefine
x, y, z → 1,2,3 (principal direction)

$$\sigma_{x'} = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3 + 2l_1m_1\tau_{12} + 2m_1n_1\tau_{23} + 2n_1l_1\tau_{31}$$

ex.) ①

the transformation of *stress* between arbitrary x,y,z and arbitrary x',y',z' system

① $\sigma_{x'} = l_1^2\sigma_x + m_1^2\sigma_y + n_1^2\sigma_z + 2l_1m_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zx}$

② $\sigma_{y'} = l_2^2\sigma_x + m_2^2\sigma_y + n_2^2\sigma_z + 2l_2m_2\tau_{xy} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{zx}$

③ $\sigma_{z'} = l_3^2\sigma_x + m_3^2\sigma_y + n_3^2\sigma_z + 2l_3m_3\tau_{xy} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx}$

④ $\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\sigma_z + (l_1m_2 + m_1l_2)\tau_{xy} + (m_1n_2 + n_1m_2)\tau_{yz} + (n_1l_2 + l_1n_2)\tau_{zx}$

⑤ $\tau_{y'z'} = l_2l_3\sigma_x + m_2m_3\sigma_y + n_2n_3\sigma_z + (l_2m_3 + m_2l_3)\tau_{xy} + (m_2n_3 + n_2m_3)\tau_{yz} + (n_2l_3 + l_2n_3)\tau_{zx}$

⑥ $\tau_{z'x'} = l_3l_1\sigma_x + m_3m_1\sigma_y + n_3n_1\sigma_z + (l_3m_1 + m_3l_1)\tau_{xy} + (m_3n_1 + n_3m_1)\tau_{yz} + (n_3l_1 + l_3n_1)\tau_{zx}$

-The maximum and minimum *normal* stresses, called the principal stress.[1)]

-Principal planes : the planes on which the principal stress act.[2)]

-An element that is oriented to the principal directions of stress has no shear stresses acting on its faces[3)]

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1)2)Gere, J.M., Mechanics of Materials, Sixth Edition, Thomson, 2006, p474,
3) Gere, J.M., Mechanics of Materials, Sixth Edition, Thomson, 2006, p517

l, m, n : directional cosine

Summary : Transformation of Strain

strain ⇔ **principal strain**

① $\varepsilon_x = l_1{}^2\varepsilon_1 + m_1{}^2\varepsilon_2 + n_1{}^2\varepsilon_3$

② $\varepsilon_y = l_2{}^2\varepsilon_1 + m_2{}^2\varepsilon_2 + n_2{}^2\varepsilon_3$

③ $\varepsilon_z = l_3{}^2\varepsilon_1 + m_3{}^2\varepsilon_2 + n_3{}^2\varepsilon_3$

④ $\gamma_{xy} = 2l_1l_2\varepsilon_1 + 2m_1m_2\varepsilon_2 + 2n_1n_2\varepsilon_3$

⑤ $\gamma_{yz} = 2l_2l_3\varepsilon_1 + 2m_2m_3\varepsilon_2 + 2n_2n_3\varepsilon_3$

⑥ $\gamma_{zx} = 2l_3l_1\varepsilon_1 + 2m_3m_1\varepsilon_2 + 2n_3n_1\varepsilon_3$

the transformation of *strain* between principal direction 1,2,3 and arbitrary x,y,z system

$$\therefore \varepsilon_x = l_1{}^2\varepsilon_1 + m_1{}^2\varepsilon_2 + n_1{}^2\varepsilon_3$$

⇧ **redefine x'→x**

$$\therefore \varepsilon_{x'} = l_1{}^2\varepsilon_1 + m_1{}^2\varepsilon_2 + n_1{}^2\varepsilon_3$$

⇧

$$\varepsilon_x = l_1{}^2\varepsilon_1 + m_1{}^2\varepsilon_2 + n_1{}^2\varepsilon_3 + l_1m_1\cancel{\gamma}_{12} + m_1n_1\cancel{\gamma}_{23} + n_1l_1\cancel{\gamma}_{31}$$

⇧ **redefine**
x, y, z → 1,2,3 (principal direction)

$$\varepsilon_x = l_1{}^2\varepsilon_1 + m_1{}^2\varepsilon_2 + n_1{}^2\varepsilon_3 + l_1m_1\gamma_{12} + m_1n_1\gamma_{23} + n_1l_1\gamma_{31}$$

ex.) ①

- An element that is oriented to the principal directions of stress has no shear stresses acting on its faces. Therefore, the shear strain for this element zero.[1)]

the transformation of *strain* between arbitrary x,y,z and arbitrary x',y',z' system

① $\varepsilon_{x'} = l_1{}^2\varepsilon_x + m_1{}^2\varepsilon_y + n_1{}^2\varepsilon_z + l_1m_1\gamma_{xy} + m_1n_1\gamma_{yz} + n_1l_1\gamma_{zx}$

② $\varepsilon_{y'} = l_2{}^2\varepsilon_x + m_2{}^2\varepsilon_y + n_2{}^2\varepsilon_z + l_2m_2\gamma_{xy} + m_2n_2\gamma_{yz} + n_2l_2\gamma_{zx}$

③ $\varepsilon_{z'} = l_3{}^2\varepsilon_x + m_3{}^2\varepsilon_y + n_3{}^2\varepsilon_z + l_3m_3\gamma_{xy} + m_3n_3\gamma_{yz} + n_3l_3\gamma_{zx}$

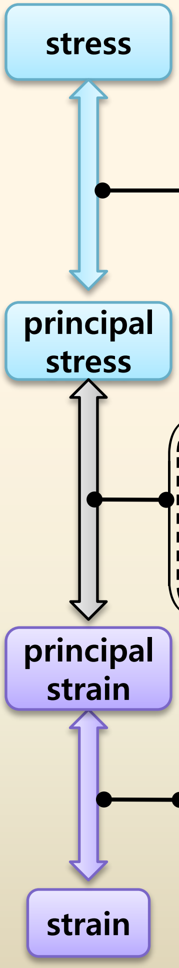
④ $\gamma_{x'y'} = 2l_1l_2\varepsilon_x + 2m_1m_2\varepsilon_y + 2n_1n_2\varepsilon_z + (l_1m_2 + m_1l_2)\gamma_{xy} + (m_1n_2 + n_1m_2)\gamma_{yz} + (n_1l_2 + l_1n_2)\gamma_{zx}$

⑤ $\gamma_{y'z'} = 2l_2l_3\varepsilon_x + 2m_2m_3\varepsilon_y + 2n_2n_3\varepsilon_z + (l_2m_3 + m_2l_3)\gamma_{xy} + (m_2n_3 + n_2m_3)\gamma_{yz} + (n_2l_3 + l_2n_3)\gamma_{zx}$

⑥ $\gamma_{z'x'} = 2l_3l_1\varepsilon_x + 2m_3m_1\varepsilon_y + 2n_3n_1\varepsilon_z + (l_3m_1 + m_3l_1)\gamma_{xy} + (m_3n_1 + n_3m_1)\gamma_{yz} + (n_3l_1 + l_3n_1)\gamma_{zx}$

[1)] Gere, J.M., Mechanics of Materials, Sixth Edition, Thomson, 2006, p517

Summary : Stress-Strain Relation



① $\sigma_x = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3$

② $\sigma_y = l_2^2\sigma_1 + m_2^2\sigma_2 + n_2^2\sigma_3$

③ $\sigma_z = l_3^2\sigma_1 + m_3^2\sigma_2 + n_3^2\sigma_3$

④ $\tau_{xy} = l_1 l_2\sigma_1 + m_1 m_2\sigma_2 + n_1 n_2\sigma_3$

⑤ $\tau_{yz} = l_2 l_3\sigma_1 + m_2 m_3\sigma_2 + n_2 n_3\sigma_3$

⑥ $\tau_{zx} = l_3 l_1\sigma_1 + m_3 m_1\sigma_2 + n_3 n_1\sigma_3$

$$\sigma_1 = \lambda e + 2\mu\varepsilon_1$$
$$\sigma_2 = \lambda e + 2\mu\varepsilon_2 \quad , e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$
$$\sigma_3 = \lambda e + 2\mu\varepsilon_3$$

① → **ex.)** $\sigma_x = l_1^2(\lambda e + 2\mu\varepsilon_1) + m_1^2(\lambda e + 2\mu\varepsilon_2) + n_1^2(\lambda e + 2\mu\varepsilon_3) \quad , e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$$= (l_1^2 + m_1^2 + n_1^2)\lambda e + 2\mu(l_1^2\varepsilon_1 + m_1^2\varepsilon_2 + n_1^2\varepsilon_3)$$

⇩ since*, $\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$$= (l_1^2 + m_1^2 + n_1^2)\lambda e + 2\mu(l_1^2\varepsilon_1 + m_1^2\varepsilon_2 + n_1^2\varepsilon_3) \quad , e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

⇩ since**, $l_1^2 + m_1^2 + n_1^2 = 1$

$$\sigma_x = \lambda e + 2\mu(l_1^2\varepsilon_1 + m_1^2\varepsilon_2 + n_1^2\varepsilon_3)$$

① $\varepsilon_x = l_1^2\varepsilon_1 + m_1^2\varepsilon_2 + n_1^2\varepsilon_3$

② $\varepsilon_y = l_2^2\varepsilon_1 + m_2^2\varepsilon_2 + n_2^2\varepsilon_3$

③ $\varepsilon_z = l_3^2\varepsilon_1 + m_3^2\varepsilon_2 + n_3^2\varepsilon_3$

④ $\gamma_{xy} = 2l_1 l_2\varepsilon_1 + 2m_1 m_2\varepsilon_2 + 2n_1 n_2\varepsilon_3$

⑤ $\gamma_{yz} = 2l_2 l_3\varepsilon_1 + 2m_2 m_3\varepsilon_2 + 2n_2 n_3\varepsilon_3$

⑥ $\gamma_{zx} = 2l_3 l_1\varepsilon_1 + 2m_3 m_1\varepsilon_2 + 2n_3 n_1\varepsilon_3$

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

②

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \quad , \tau_{xy} = \mu\gamma_{xy}$$
$$\sigma_y = \lambda e + 2\mu\varepsilon_y \quad , \tau_{yz} = \mu\gamma_{yz}$$
$$\sigma_z = \lambda e + 2\mu\varepsilon_z \quad , \tau_{zx} = \mu\gamma_{zx}$$

ν : Poisson's Ratio
G : Shear Modulus
E : Young's Modulus

$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

μ, λ : Lame Elastic constant

*Wang,C.T., Applied Elasticity , McGRAW-HILL, 1953, p21

*Wang,C.T., Applied Elasticity , McGRAW-HILL, 1953, p9

Summary : Stress-Strain Relation

stress

① $\sigma_x = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3$

② $\sigma_y = l_2^2\sigma_1 + m_2^2\sigma_2 + n_2^2\sigma_3$

③ $\sigma_z = l_3^2\sigma_1 + m_3^2\sigma_2 + n_3^2\sigma_3$

④ $\tau_{xy} = l_1 l_2\sigma_1 + m_1 m_2\sigma_2 + n_1 n_2\sigma_3$

⑤ $\tau_{yz} = l_2 l_3\sigma_1 + m_2 m_3\sigma_2 + n_2 n_3\sigma_3$

⑥ $\tau_{zx} = l_3 l_1\sigma_1 + m_3 m_1\sigma_2 + n_3 n_1\sigma_3$

principal stress

$$\sigma_1 = \lambda e + 2\mu\varepsilon_1$$
$$\sigma_2 = \lambda e + 2\mu\varepsilon_2$$
$$\sigma_3 = \lambda e + 2\mu\varepsilon_3$$

$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

principal strain

strain

① $\varepsilon_x = l_1^2\varepsilon_1 + m_1^2\varepsilon_2 + n_1^2\varepsilon_3$

② $\varepsilon_y = l_2^2\varepsilon_1 + m_2^2\varepsilon_2 + n_2^2\varepsilon_3$

③ $\varepsilon_z = l_3^2\varepsilon_1 + m_3^2\varepsilon_2 + n_3^2\varepsilon_3$

④ $\gamma_{xy} = 2l_1 l_2\varepsilon_1 + 2m_1 m_2\varepsilon_2 + 2n_1 n_2\varepsilon_3$

⑤ $\gamma_{yz} = 2l_2 l_3\varepsilon_1 + 2m_2 m_3\varepsilon_2 + 2n_2 n_3\varepsilon_3$

⑥ $\gamma_{zx} = 2l_3 l_1\varepsilon_1 + 2m_3 m_1\varepsilon_2 + 2n_3 n_1\varepsilon_3$

①② →

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \quad , \tau_{xy} = \mu\gamma_{xy}$$
$$\sigma_y = \lambda e + 2\mu\varepsilon_y \quad , \tau_{yz} = \mu\gamma_{yz}$$
$$\sigma_z = \lambda e + 2\mu\varepsilon_z \quad , \tau_{zx} = \mu\gamma_{zx}$$

ν : Poisson's Ratio
G : Shear Modulus
E : Young's Modulus

$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

μ, λ : Lame Elastic constant

Solving the equations for strain

$$\gamma_{xy} = \frac{1}{\mu}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{\mu}\tau_{yz}, \quad \gamma_{zx} = \frac{1}{\mu}\tau_{zx}$$

$$\varepsilon_x = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_x - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_y+\sigma_z)$$
$$\varepsilon_y = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_y - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_z+\sigma_x)$$
$$\varepsilon_z = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_z - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_x+\sigma_y)$$

in case of tension

$$\varepsilon_x = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_x$$
$$\varepsilon_y = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_x$$
$$\varepsilon_z = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_x$$

Engineering Constants

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

Shear Modulus 'G'

$$\therefore \mu = G$$

$$\therefore \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\varepsilon_x = \frac{1}{E}\sigma_x$$

Young's Modulus 'E'

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

Poisson's ratio 'ν'

*Wang,C.T., Applied Elasticity , McGRAW-HILL, 1953, p21
*Wang,C.T., Applied Elasticity , McGRAW-HILL, 1953, p9

Stress Analysis

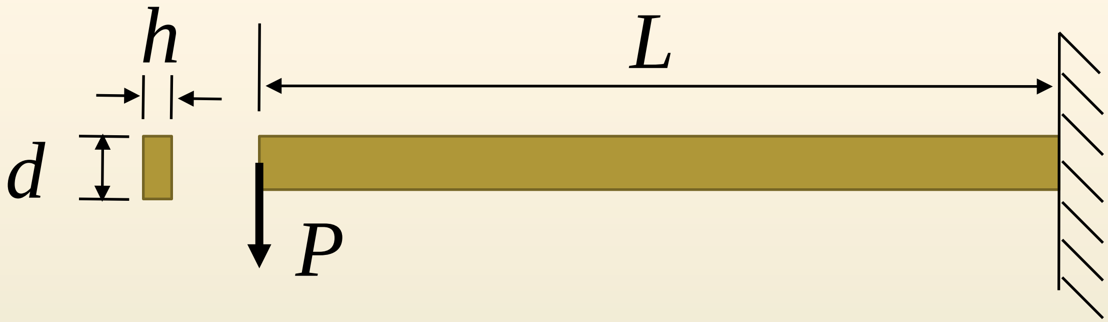
Problem in Elasticity

The object :

find the stress distribution in an elastic body or

find the strain at any point

due to given body forces and given conditions at the boundary of the body



Stress Analysis

18 Variables

9 Stress $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement u, v, w

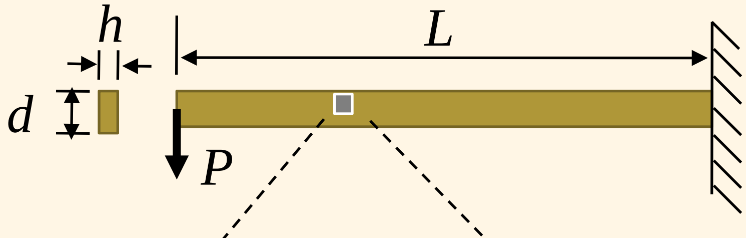
9 Stress distribution in an elastic body

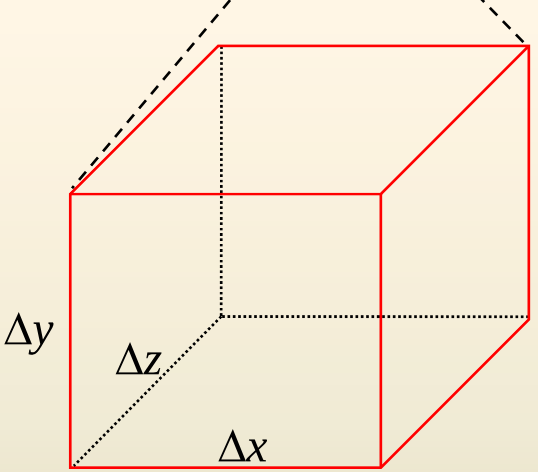
6 Equations of force and moment equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Consider an infinitesimal element to construct a differential equation





What kind of physical law would be applied to describe the motion?

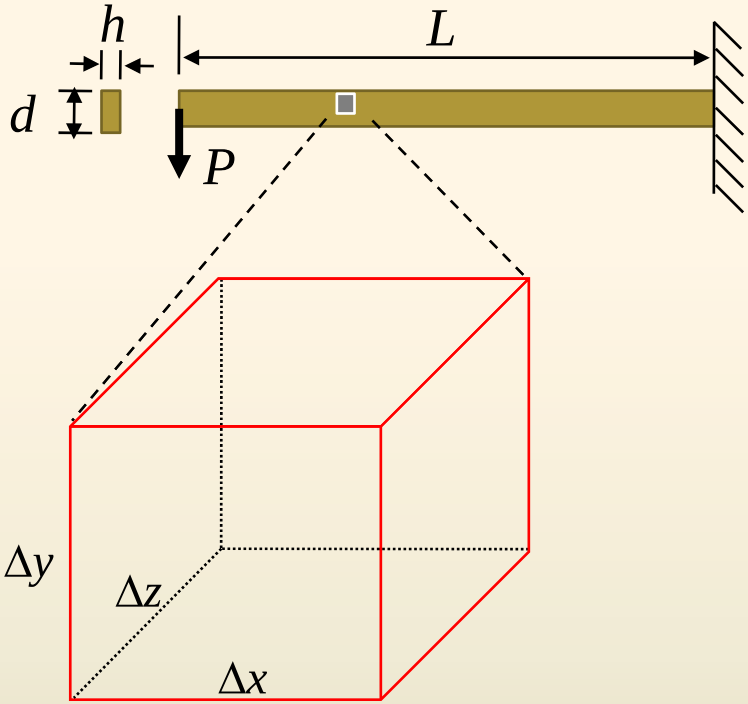
$$m\frac{d\mathbf{V}}{dt}=\sum\mathbf{F}$$

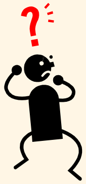
How can we categorize the forces?

$$m\frac{d\mathbf{V}}{dt}=\sum\mathbf{F}=\mathbf{F}_{Body}+\mathbf{F}_{Surface}$$

Stress Analysis

Consider an infinitesimal element to construct a differential equation





How can we categorize the forces?

$$m\frac{d\mathbf{V}}{dt}=\sum\mathbf{F}$$

$$\sum\mathbf{F}=\mathbf{F}_{\text{external}}+\mathbf{F}_{\text{constraint}}$$ **(in Multibody Dynamics)**

$$\sum\mathbf{F}=\mathbf{F}_{\text{body}}+\mathbf{F}_{\text{surface}}$$ **(in Hydrodynamics, Elasticity)**

$$\sum\mathbf{F}=\mathbf{F}_{\text{disturbance}}+\mathbf{F}_{\text{control}}$$ **(in Control)**

Stress Analysis

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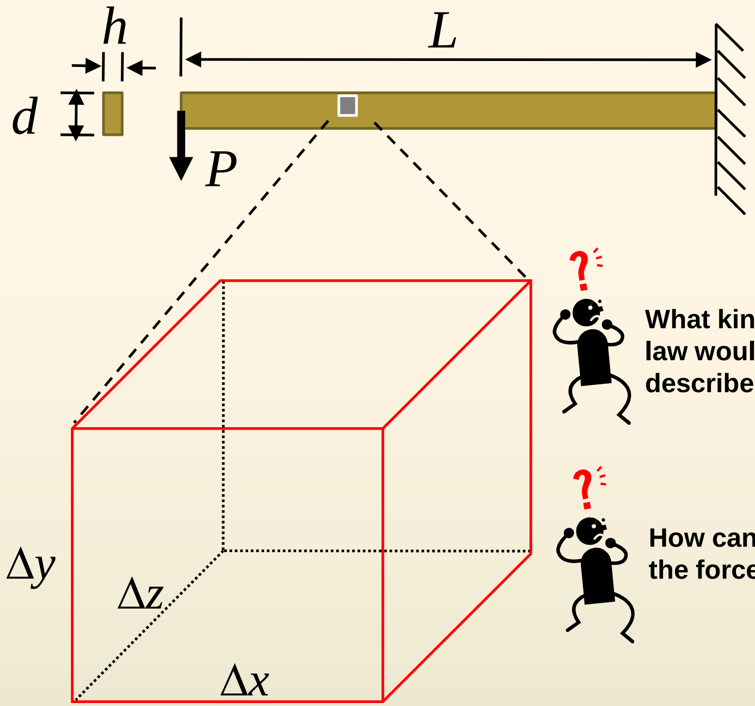

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		Symbol defined
$\mathbf{F}_{Body}$	Such forces as the gravitational and centrifugal forces, which are distributed over the volume of the body, are called body forces. Given in terms of force per unit volume	X, Y, Z x, y, z component of body force
$\mathbf{F}_{surface}$	A force such as the hydrostatic pressure, which is distributed over the surface of the body, are called surface forces. Given in terms of force per unit area	$\bar{X}, \bar{Y}, \bar{Z}$ x, y, z component of surface force

Stress Analysis

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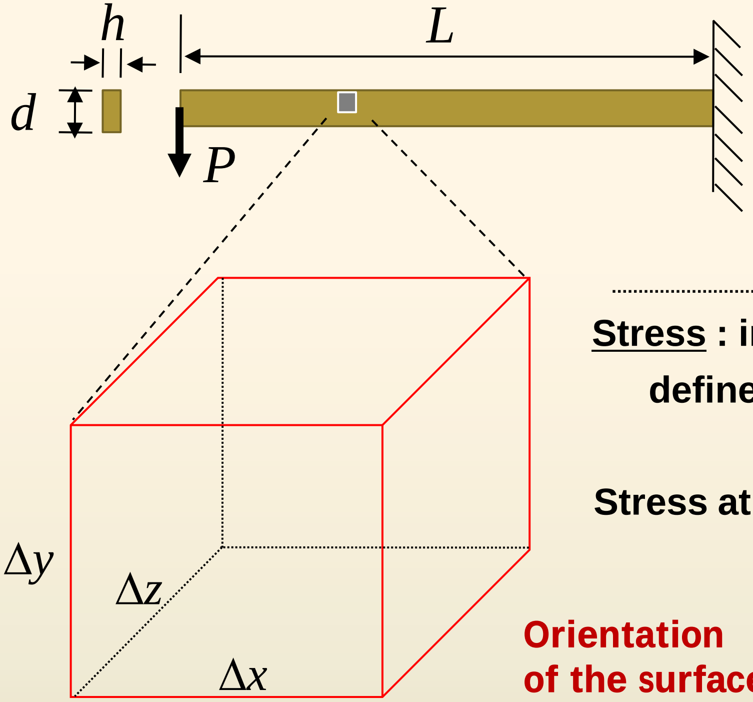


$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \boxed{\mathbf{F}_{Surface}}$$

Stress : internal force per unit area

defined by its magnitude, direction and the surface upon which it acts

Stress at a point on a surface
- σ **Normal stress : one normal direction**
- τ **Shearing stress : two tangential direction**

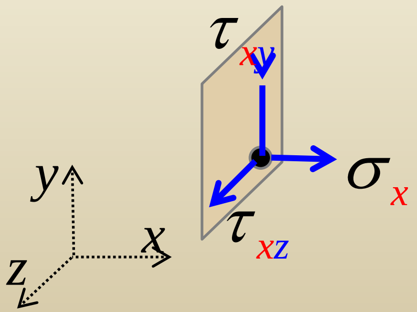
Orientation of the surface → x

Direction of the force → y

In case of Shearing stress τ_{xy}

In case of normal stress σ_{xx} ⇨ σ_x

(same orientation & direction)



CAUTION

$$\sigma_x \neq \frac{\partial \sigma}{\partial x}$$

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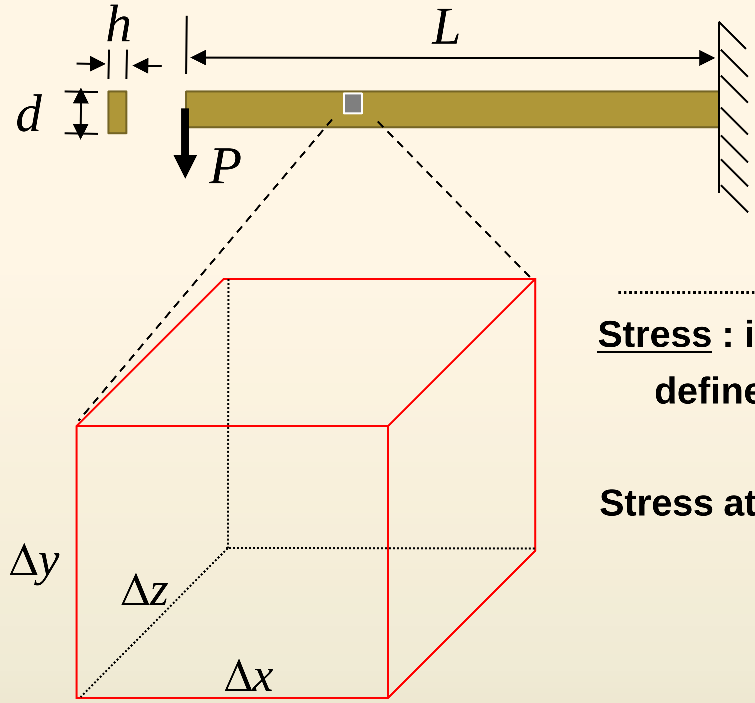

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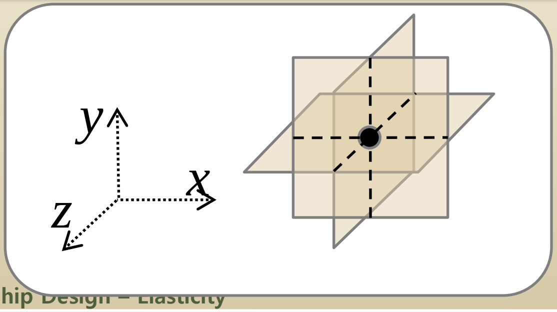
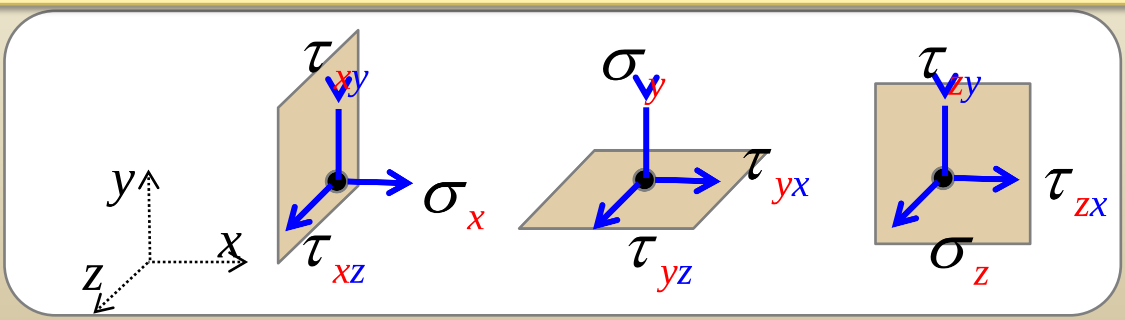
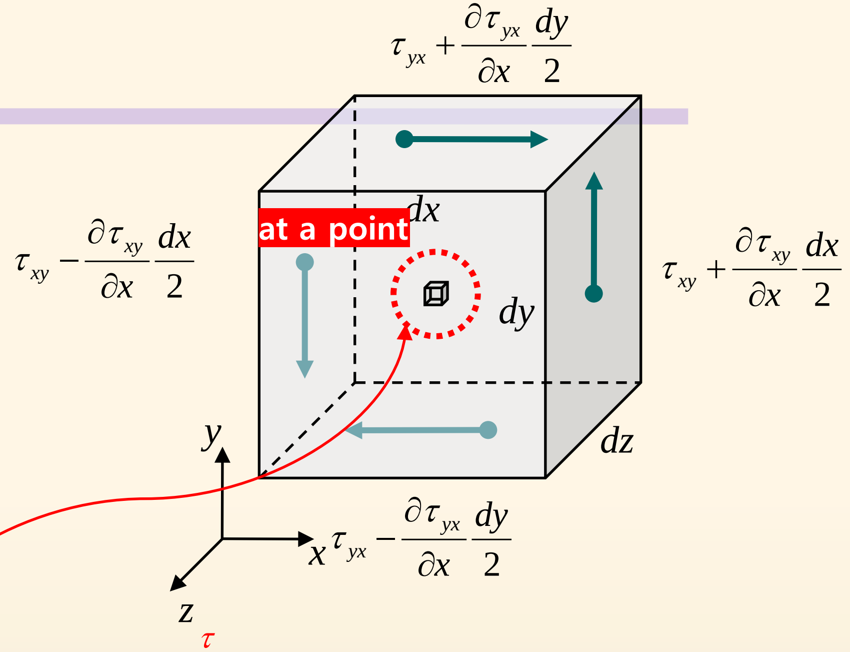
Through *a point* in a body we can construct *three orthogonal coordinate planes* on which we have *nine stress component*





Stress Analysis

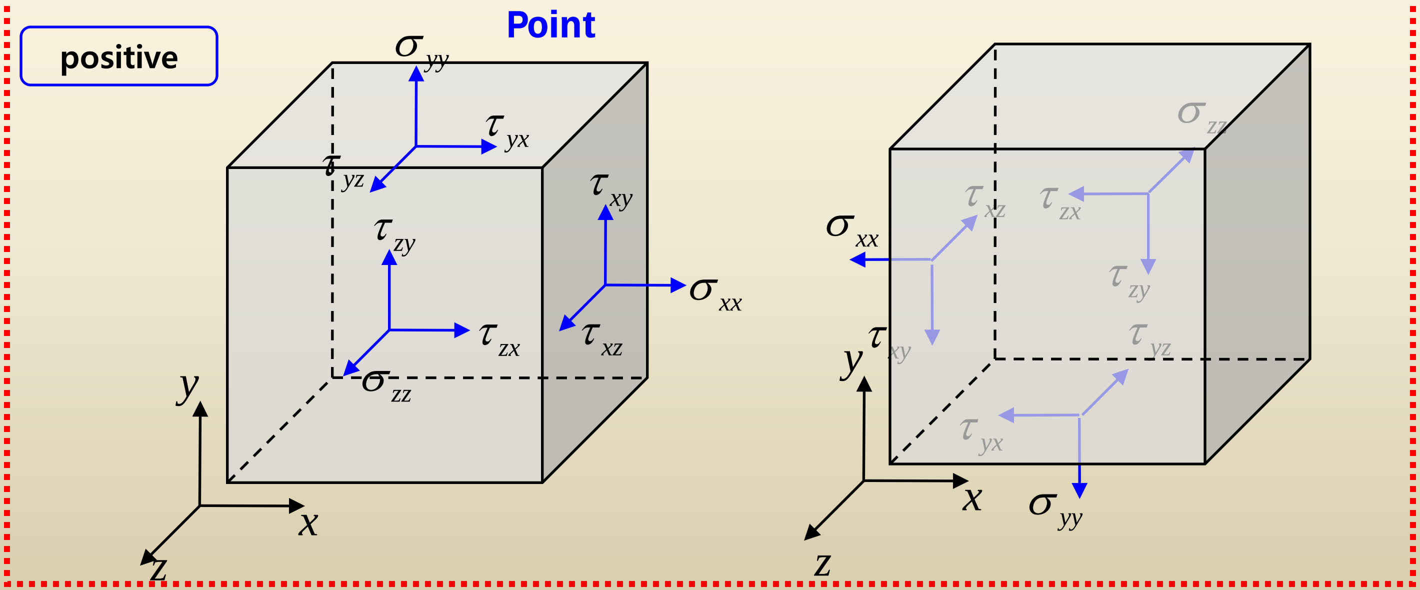


Stress Sign convention*

A normal stress is defined as positive if it is a tensile stress, i.e., if it is directed away from the surface upon which it acts

A shearing stresses are positive if they are in the positive directions of the other two coordinates axes on any surface where the tensile stress is in the positive direction of the coordinate axis

If the tensile stress is opposite to the positive axis, the positive directions of the shearing stresses are also opposite to the positive axes.



*Wang.C.T , Applied Elasticity , McGRAW-HILL, 1953, p2
*Kundu P.K., Cohen I.M., Fulid Mechanics, Fourth Edition, Academic Press, p31

Stress Analysis

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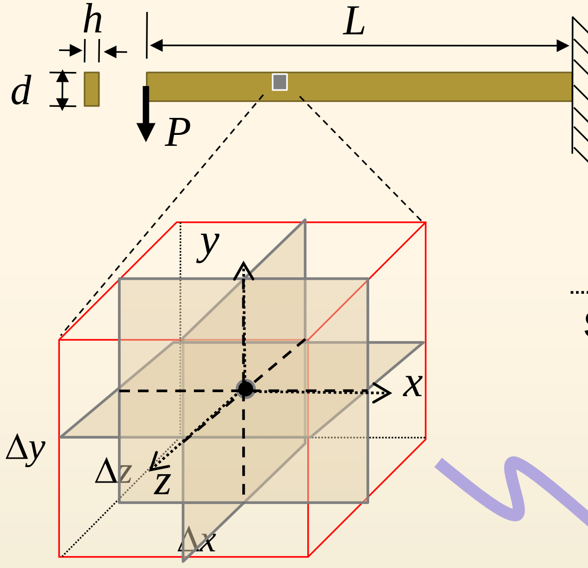
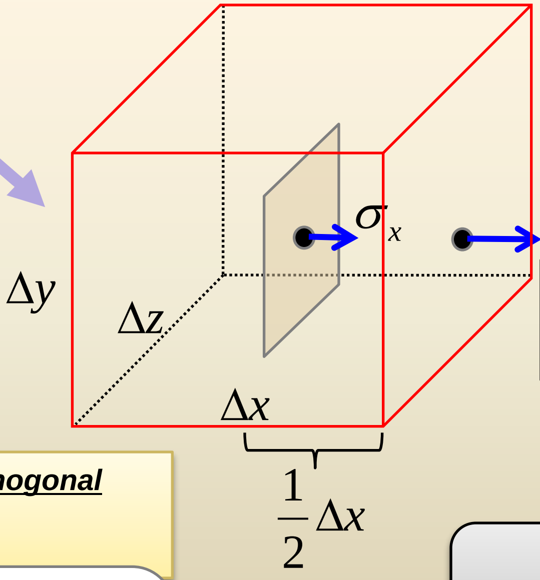


$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \boxed{\mathbf{F}_{Surface}}$$

Stress for the element



How we can guess the stress at the surface (when the stress at the center point is known)?

Taylor Series

$$f(x^* + \Delta x) = f(x^*) + f'(x^*)\Delta x + \frac{1}{2} f''(x^*)\Delta x^2 + .$$

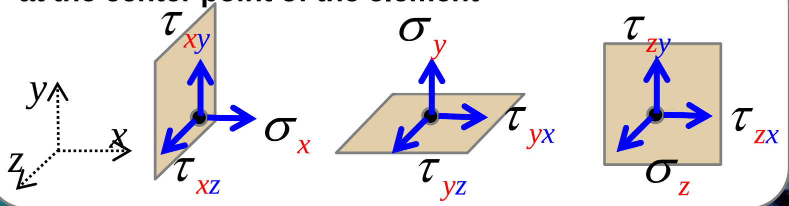
for example, $f \rightarrow \sigma,\ \Delta x \rightarrow \frac{1}{2}\Delta x$

$$\therefore \sigma'_x = \sigma_x + \frac{\partial \sigma_x}{\partial x}\left(\frac{1}{2}\Delta x\right) + \cancel{\frac{1}{2}\frac{\partial^2 \sigma_x}{\partial x^2}\left(\frac{1}{2}\Delta x\right)^2} + \dots$$

Linearlize

Through *a point* in a body we can construct *three orthogonal coordinate planes* on which we have *nine stress component*

at the center point of the element



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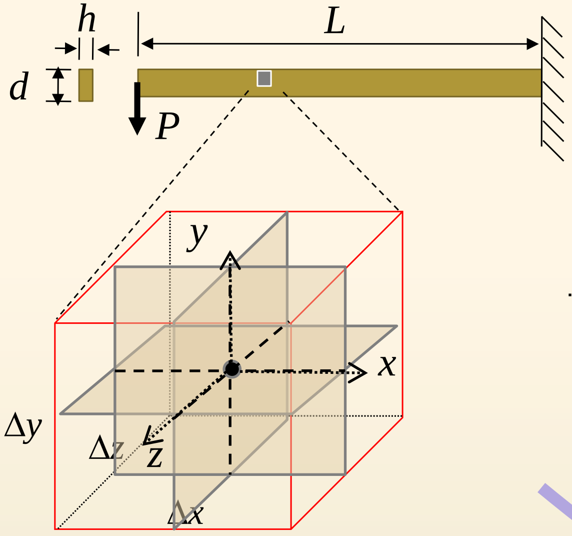

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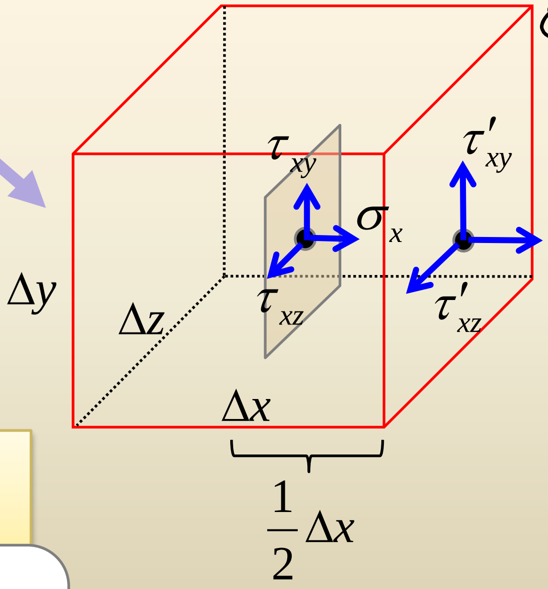

How we can guess the stress at the surface (when the stress at the center point is known)?

By using Taylor series and 'linearization'

$$\sigma'_x = \sigma_x + \frac{\partial \sigma_x}{\partial x}\frac{1}{2}\Delta x$$

In same way

$$\tau'_{xy} = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}\Delta x$$

$$\tau'_{xz} = \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}\frac{1}{2}\Delta x$$

Through *a point* in a body we can construct *three orthogonal coordinate planes* on which we have *nine stress component*

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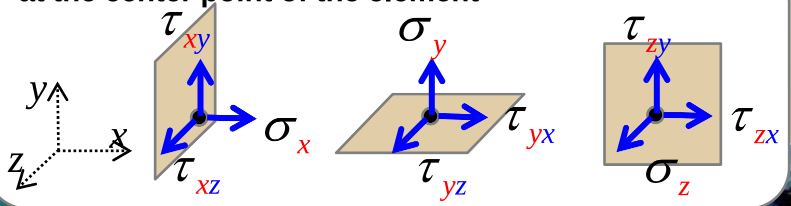

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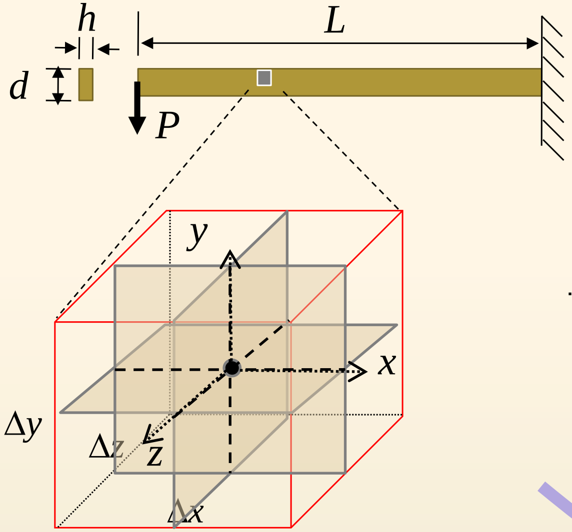

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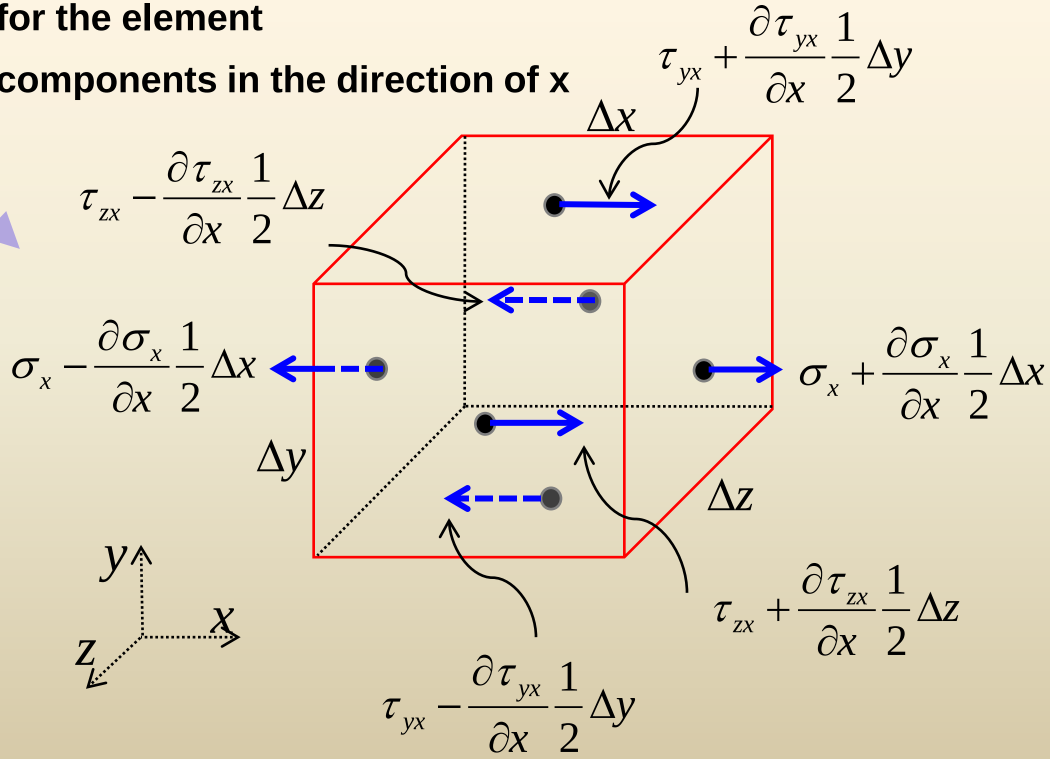
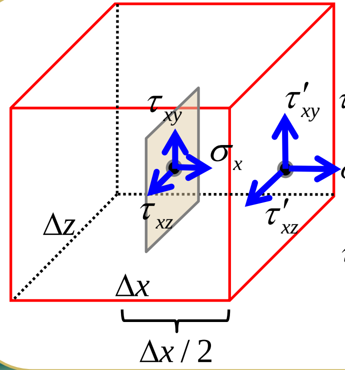
Stress for the element

Stress components in the direction of x





By using Taylor series and 'linearization'

$$\tau'_{xy} = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}\Delta x$$

$$\sigma'_x = \sigma_x + \frac{\partial \sigma_x}{\partial x}\frac{1}{2}\Delta x$$

$$\tau'_{xz} = \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}\frac{1}{2}\Delta x$$

Stress Analysis

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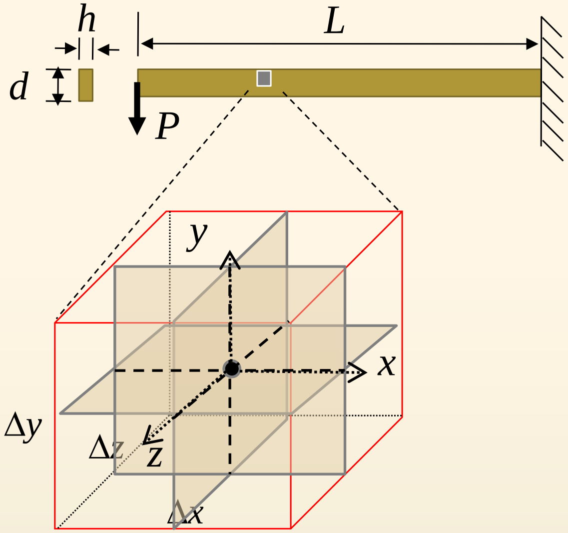


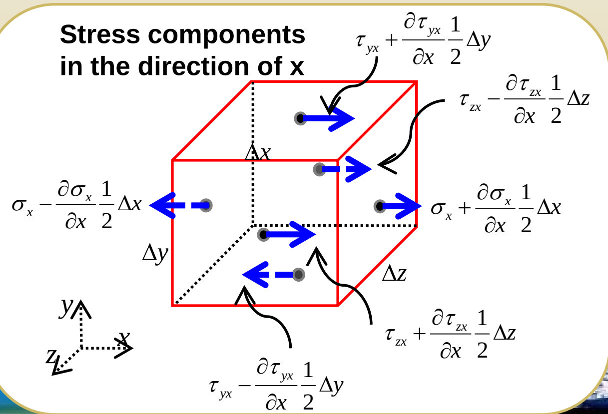
$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \boxed{\mathbf{F}_{Surface}}$$

Stress for the element

Net surface force acting on the element in the direction of x

$$\begin{aligned}
\sum \mathbf{F}_{Surface,x} &= \left[\sigma_x + \frac{\partial \sigma_x}{\partial x}\frac{1}{2}\Delta x - \left(\sigma_x - \frac{\partial \sigma_x}{\partial x}\frac{1}{2}\Delta x\right)\right]\Delta y \Delta z \\
&+ \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\Delta y - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\Delta y\right)\right]\Delta z \Delta x \\
&+ \left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\Delta z - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\Delta z\right)\right]\Delta x \Delta y \\
&= \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\Delta x \Delta y \Delta z
\end{aligned}$$

Stress components in the direction of x



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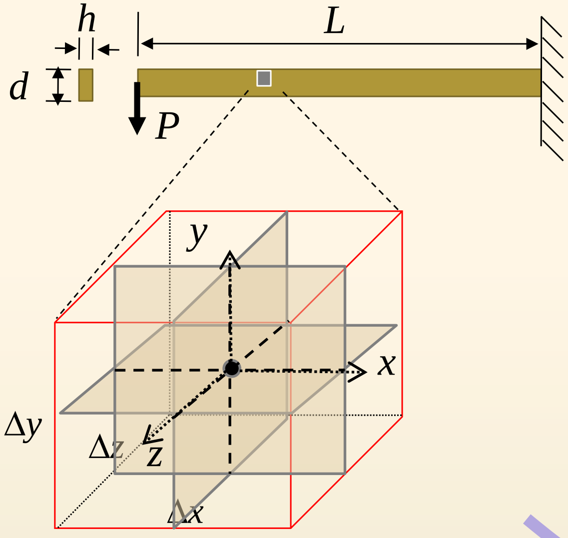
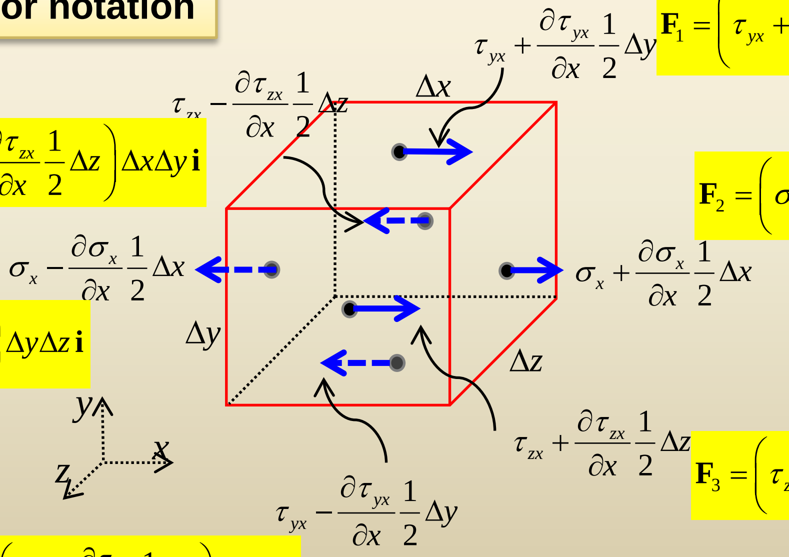


$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \boxed{\mathbf{F}_{Surface}}$$

Stress for the element

Stress components in the direction of x

in vector notation



$$\mathbf{F}_1 = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial x}\frac{1}{2}\Delta y\right)\Delta x \Delta z\,\mathbf{i}$$

$$\mathbf{F}_4 = -\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial x}\frac{1}{2}\Delta z\right)\Delta x \Delta y\,\mathbf{i}$$

$$\mathbf{F}_2 = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\frac{1}{2}\Delta x\right)\Delta y \Delta z\,\mathbf{i}$$

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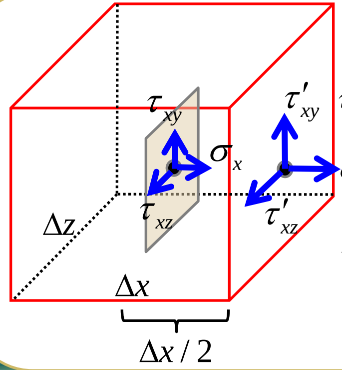

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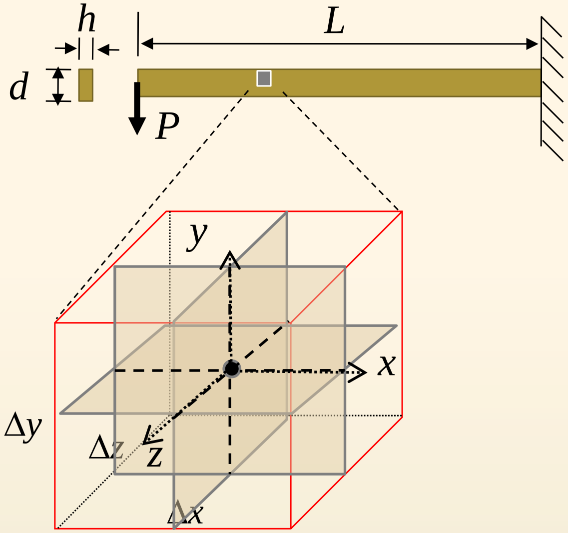

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Stress for the element

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in vector notation

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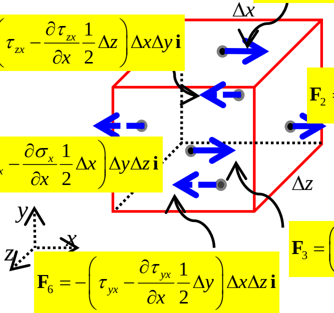
$$\begin{aligned}\sum \mathbf{F}_{Surface,x} = & \left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\Delta z - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\Delta z\right)\right]\Delta x \Delta y\,\mathbf{i} \\ & + \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\Delta y - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\Delta y\right)\right]\Delta z \Delta x\,\mathbf{i} \\ & + \left[\sigma_x + \frac{\partial \sigma_x}{\partial x}\frac{1}{2}\Delta x - \left(\sigma_x - \frac{\partial \sigma_x}{\partial x}\frac{1}{2}\Delta x\right)\right]\Delta y \Delta z\,\mathbf{i} \\ & = \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\Delta x \Delta y \Delta z\,\mathbf{i}\end{aligned}$$

Stress components in the direction of x



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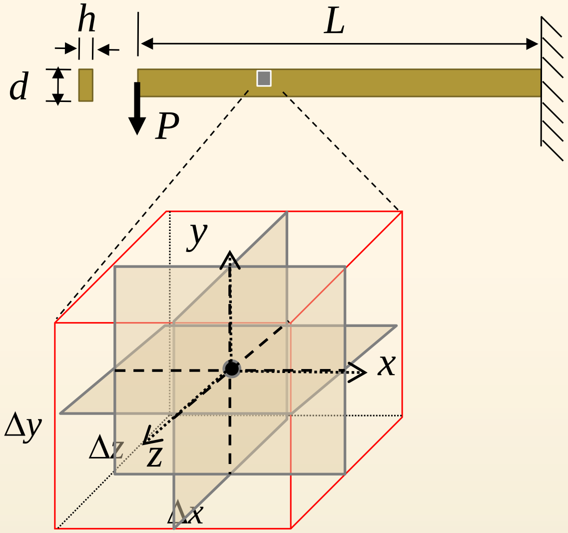
6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Consider an infinitesimal element to construct a differential equation



$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \boxed{\mathbf{F}_{Surface}}$$

Stress for the element

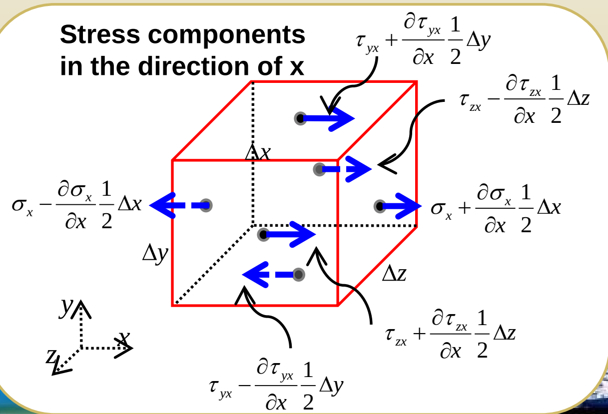
In same way, net surface force acting on the element in the direction of x,y,z

$$\sum \mathbf{F}_{Surface,x} = \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] \Delta x \Delta y \Delta z\, \mathbf{i}$$

$$\sum \mathbf{F}_{Surface,y} = \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] \Delta x \Delta y \Delta z\, \mathbf{j}$$

$$\sum \mathbf{F}_{Surface,z} = \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right] \Delta x \Delta y \Delta z\, \mathbf{k}$$

Stress components in the direction of x



Stress Analysis

18 Variables
- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
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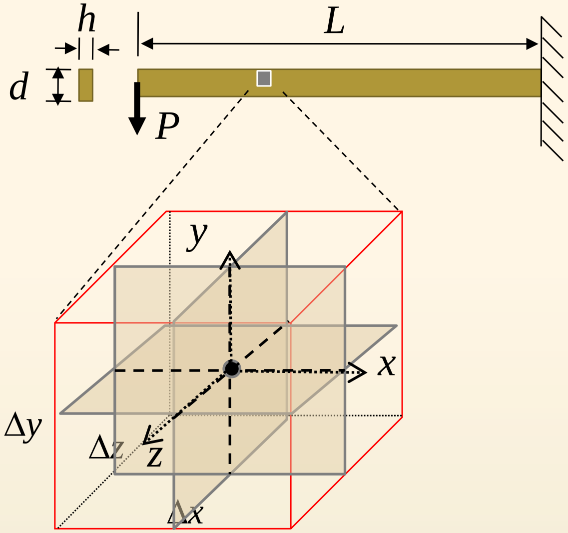

$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \boxed{\mathbf{F}_{Body} + \mathbf{F}_{Surface}}$$

X, Y, Z

x, y, z component of body force

Stress for the element

Net force acting on the element in the direction of x

$$\sum \mathbf{F}_x = \mathbf{F}_{Body,x} + \mathbf{F}_{Surface,x} = X\Delta x\Delta y\Delta z\,\mathbf{i} + \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z\,\mathbf{i}$$

$$= \left[X + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z\,\mathbf{i}$$

In same way

$$\sum \mathbf{F}_y = \mathbf{F}_{Body,y} + \mathbf{F}_{Surface,y} = \left[Y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right]\Delta x\Delta y\Delta z\,\mathbf{j}$$

$$\sum \mathbf{F}_z = \mathbf{F}_{Body,z} + \mathbf{F}_{Surface,z} = \left[Z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right]\Delta x\Delta y\Delta z\,\mathbf{k}$$

$$\sum \mathbf{F}_{Surface,x} = \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z$$

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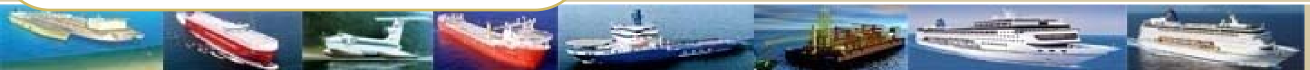

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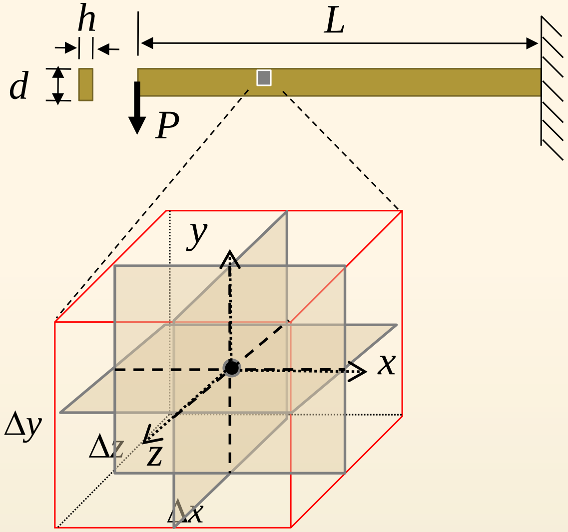

$$\boxed{m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}}$$

Newton's 2nd law for the element

$$\rho\Delta x\Delta y\Delta z\frac{d^2u}{dt^2} = \left[X + \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z$$

$$\rho\Delta x\Delta y\Delta z\frac{d^2v}{dt^2} = \left[Y + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}\right]\Delta x\Delta y\Delta z$$

$$\rho\Delta x\Delta y\Delta z\frac{d^2w}{dt^2} = \left[Z + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z}\right]\Delta x\Delta y\Delta z$$

$$\mathbf{V} = (\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt})$$

$$m = \rho\Delta x\Delta y\Delta z$$

u, v, w : displacement

ρ : density

$$\sum\mathbf{F}_x = \left[X + \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z\,\mathbf{i}$$

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$$\sum\mathbf{F}_z = \left[Z + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z}\right]\Delta x\Delta y\Delta z\,\mathbf{k}$$

$$\rho\frac{d^2u}{dt^2} = \left[X + \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right]$$

$$\rho\frac{d^2v}{dt^2} = \left[Y + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}\right]$$

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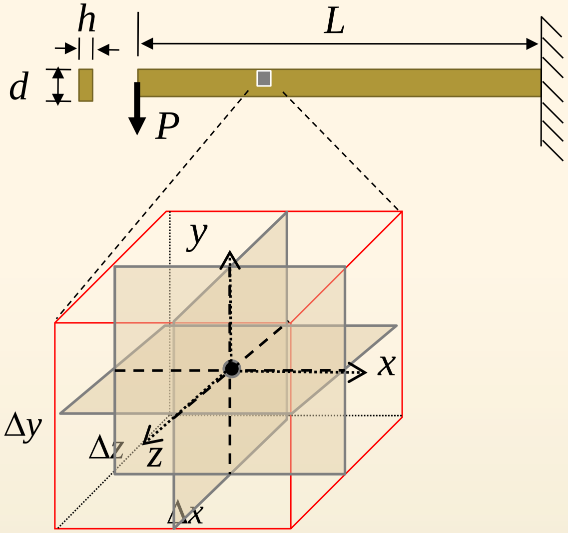

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in equilibrium

$$X + \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = 0$$

$$Y + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} = 0$$

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Equilibrium : under the action of external forces, it is at rest or moving in straight line with constant velocity

9 stress and 3 equations?

Stress Analysis

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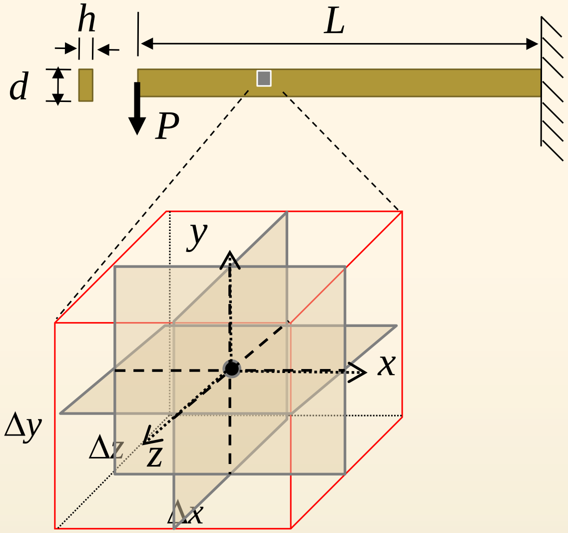


$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}$$

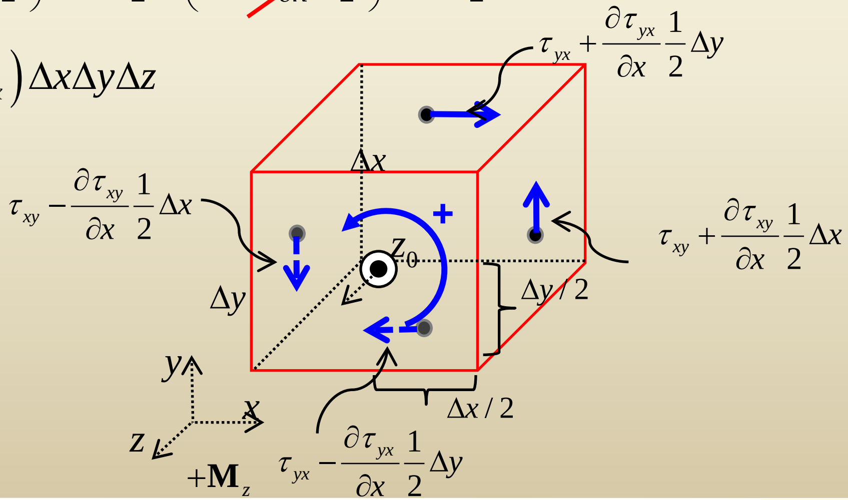
Euler's equation $I\dot{\omega} = \sum \mathbf{M}$ $, \omega = \dfrac{d\theta}{dt}$

Moment about z$_0$-axis

$$I\dot{\omega} = \sum \mathbf{M}_{z_0} = \left(\tau_{xy} + \cancel{\frac{\partial \tau_{xy}}{\partial x}\frac{\Delta x}{2}}\right)\Delta y \Delta z \frac{\Delta x}{2} - \left(\tau_{yx} + \cancel{\frac{\partial \tau_{yx}}{\partial x}\frac{\Delta y}{2}}\right)\Delta x \Delta z \frac{\Delta y}{2}$$

$$+\left(\tau_{xy} - \cancel{\frac{\partial \tau_{xy}}{\partial x}\frac{\Delta x}{2}}\right)\Delta y \Delta z \frac{\Delta x}{2} - \left(\tau_{yx} - \cancel{\frac{\partial \tau_{yx}}{\partial x}\frac{\Delta y}{2}}\right)\Delta x \Delta z \frac{\Delta y}{2}$$

$$= \left(\tau_{xy} - \tau_{yx}\right)\Delta x \Delta y \Delta z$$



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$$X + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

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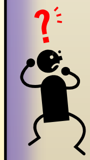

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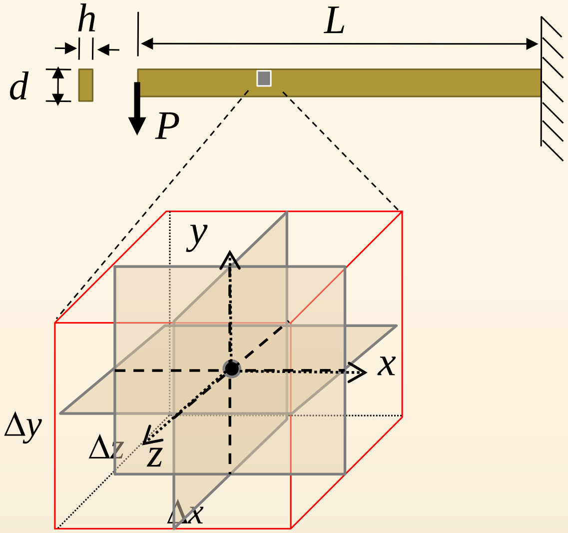

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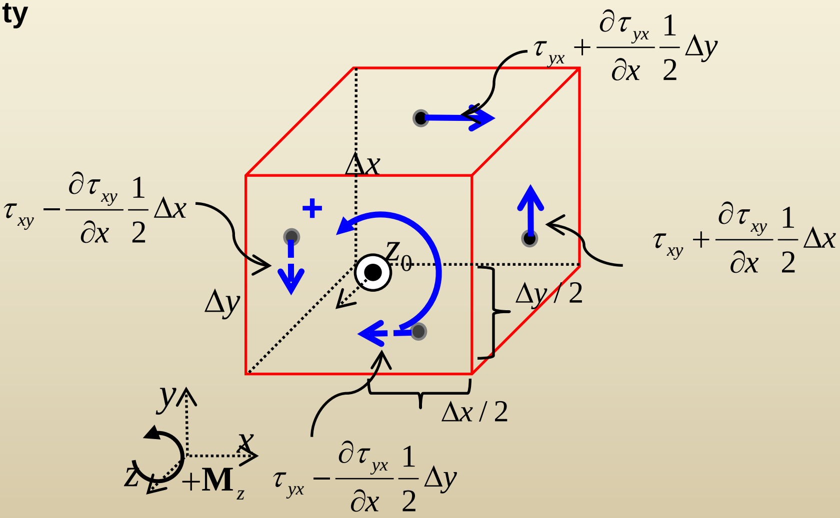
in equilibrium $\omega = 0$

Equilibrium : under the action of external forces, it is at rest or moving in straight line with constant velocity

$$\therefore \tau_{xy} = \tau_{yx}$$



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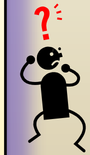

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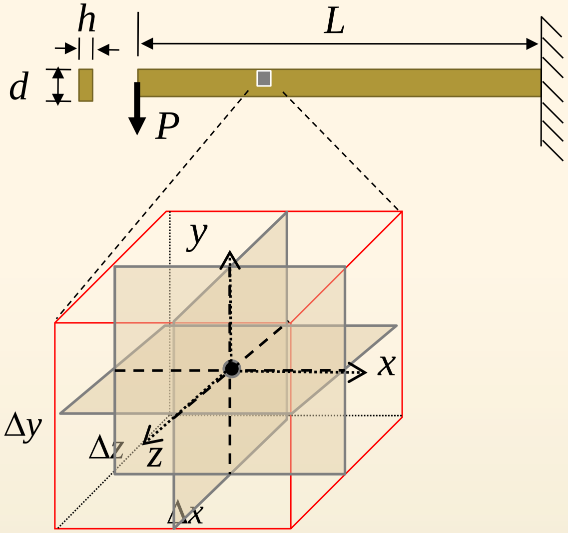

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$$\therefore \tau_{xy} = \tau_{yx}$$

In same way, $\tau_{zx} = \tau_{xz} \quad , \tau_{yz} = \tau_{zy}$

in equilibrium

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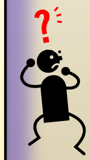

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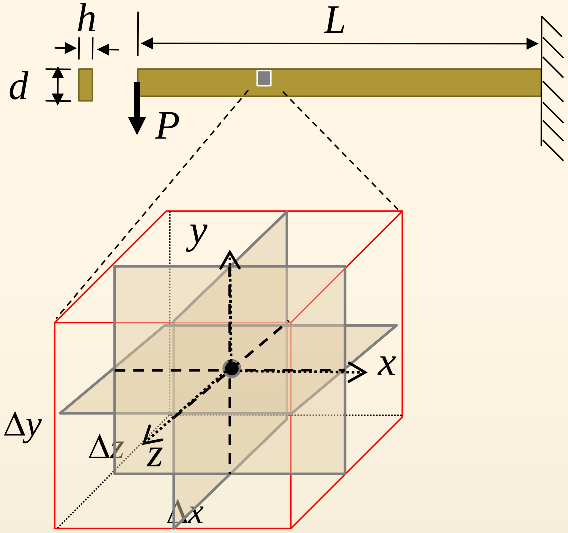

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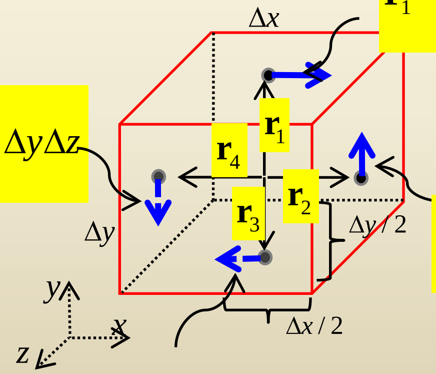
Euler's equation $I\dot{\omega} = \sum \mathbf{M}$ $, \omega = \dfrac{d\theta}{dt}$

Moment about z_0-axis **in vector notation**

$$I\dot{\omega} = \sum \mathbf{M}_{z_0} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4$$



$$\mathbf{F}_1 = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial x}\frac{1}{2}\Delta y\right)\Delta x \Delta z\, \mathbf{i}$$

$$\mathbf{r}_1 = \frac{1}{2}\Delta y\, \mathbf{j}$$

$$\mathbf{F}_4 = -\left(\tau_{xy} - \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}\Delta x\right)\Delta y \Delta z$$

$$\mathbf{r}_4 = -\frac{1}{2}\Delta x\, \mathbf{i}$$

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$$\mathbf{r}_3 = -\frac{1}{2}\Delta y\, \mathbf{j}$$

in equilibrium

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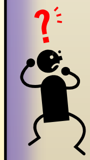

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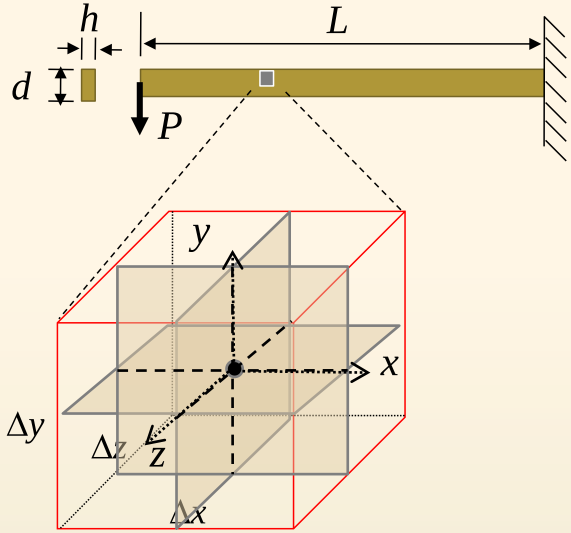

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in vector notation

$$I\dot{\omega} = \sum \mathbf{M}_{z_0} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4$$

$$= \left(\frac{\Delta y}{2}\mathbf{j}\right) \times \left(\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial x}\frac{\Delta y}{2}\right)\Delta x \Delta z\,\mathbf{i}\right) + \left(\frac{\Delta x}{2}\mathbf{i}\right) \times \left(\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z\,\mathbf{j}\right)$$

$$+\left(-\frac{\Delta y}{2}\mathbf{j}\right) \times \left(-\left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial x}\frac{\Delta y}{2}\right)\Delta x \Delta z\mathbf{i}\right) + \left(-\frac{\Delta x}{2}\mathbf{i}\right) \times \left(-\left(\tau_{xy} - \frac{\partial \tau_{xy}}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z\mathbf{j}\right)$$

$$= \left(-\left(\tau_{yx} + \cancel{\frac{\partial \tau_{yx}}{\partial x}\frac{\Delta y}{2}}\right)\frac{\Delta x \Delta y \Delta z}{2}\mathbf{k}\right) + \left(\left(\tau_{xy} + \cancel{\frac{\partial \tau_{xy}}{\partial x}}\frac{\Delta x}{2}\right)\frac{\Delta x \Delta y \Delta z}{2}\mathbf{k}\right)$$

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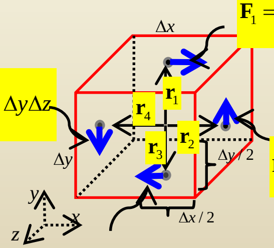
$$= \left(\tau_{xy} - \tau_{yx}\right)\Delta x \Delta y \Delta z\,\mathbf{k}$$

in equilibrium

$$\omega = 0$$

$$\therefore \tau_{xy} = \tau_{yx}$$



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$$\mathbf{r}_4 = -\frac{1}{2}\Delta x\,\mathbf{i}$$

$$\mathbf{F}_2 = \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}\Delta x\right)\Delta y \Delta z\,\mathbf{i}$$

$$\mathbf{r}_2 = \frac{1}{2}\Delta x\,\mathbf{i}$$

$$\mathbf{F}_3 = -\left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial x}\frac{1}{2}\Delta y\right)\Delta x \Delta z\,\mathbf{i}$$

$$\mathbf{r}_3 = -\frac{1}{2}\Delta y\,\mathbf{j}$$

Stress Analysis

18 Variables
- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

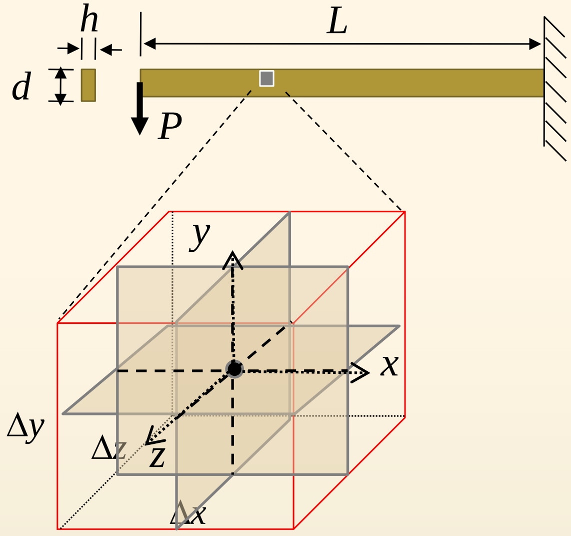
9 Stress distribution in an elastic body

6 Equations of force and moment equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Consider an infinitesimal element to construct a differential equation



$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}$$

Euler's equation $I\dot{\omega} = \sum \mathbf{M}$ $,\omega = \dfrac{d\theta}{dt}$:angular velocity of rigid body

Moment about z$_0$-axis

$$I_{z_0}\dot{\omega}_{z_0} = \sum \mathbf{M}_{z_0} = \left(\tau_{xy} - \tau_{yx}\right)\Delta x \Delta y \Delta z \quad \textbf{in equilibrium} \quad \dot{\omega} = 0$$

Equilibrium : under the action of external forces, it is at rest or moving in straight line with constant velocity

$$\therefore \tau_{xy} = \tau_{yx}$$

$$\begin{pmatrix} \tau_{xz} = \tau_{zx} \\ \tau_{yz} = \tau_{zy} \end{pmatrix}$$

C.f.) rotational equilibrium of the element*

※ Mass moment of inertia $I_{z_0} = \dfrac{m}{12}\left(dx^2 + dy^2\right) = \dfrac{\rho dxdydz}{12}\left(dx^2 + dy^2\right)$

$$I_{z_0}\dot{\omega} = \sum \mathbf{M}_z$$

$$\frac{\rho dxdydz}{12}\left(dx^2 + dy^2\right)\dot{\omega} = \left(\tau_{xy} - \tau_{yx}\right)dxdydz$$

To the center point dx → 0 , dy → 0

$$\frac{\rho}{12}\left(\cancel{dx^2 + dy^2}\right)\dot{\omega} = \tau_{xy} - \tau_{yx} \quad = 0$$

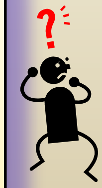
$$\boxed{\therefore \tau_{xy} = \tau_{yx} \qquad \left(\tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}\right)}$$

in equilibrium

$$X + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$Y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$Z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$



9 stress and 3 equations?

*Kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

Stress Analysis

18 Variables
- *9 Stress* $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

9 Stress distribution in an elastic body

6 Equations of force and moment equilibrium

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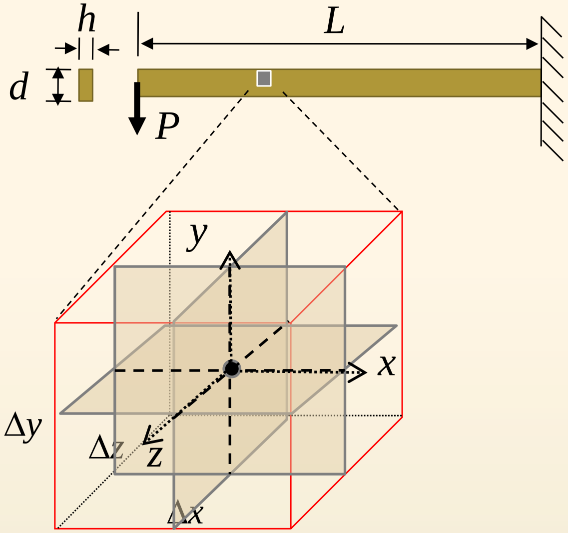
6 Relations between 6 Strain and 6 Stress

Consider an infinitesimal element to construct a differential equation



$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}$$

in equilibrium

$$X + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

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9 stress and 3 equations?

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

in equilibrium

$$X + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

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$$Z + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

6 stress and 3 equations

CAUTION So, from now on..

18 Variables, 18 Equations → **15 Variables, 15 Equations**

Stress Analysis

15 Variables

- *6 Stress* $\sigma_x, \; \sigma_y, \; \sigma_z, \; \tau_{xy}, \; \tau_{yz}, \; \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

✓**Specification of Stress at a Point**

Stress within a body is completely determined when we know the values of the six stress components at each point

in equilibrium

$$X + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$Y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0$$

$$Z + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

Review

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

- *6 Stress distribution in an elastic body*
- *3 Equations of force equilibrium*
- *6 Relations between 6 Strain and 3 Displacement*
- *6 Relations between 6 Strain and 6 Stress*

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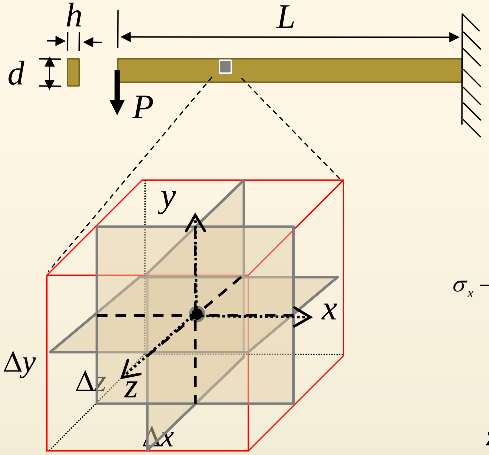
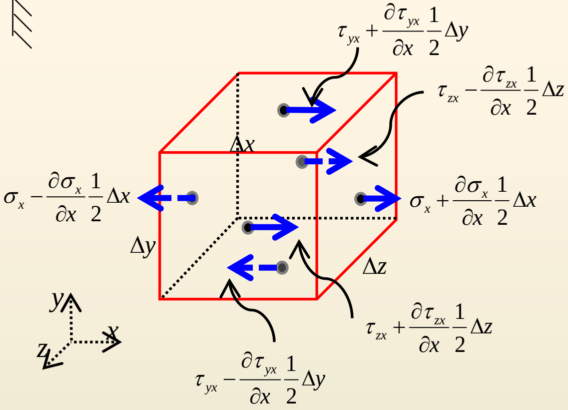
Stress within a body is completely determined when we know the values of the six stress components at each point



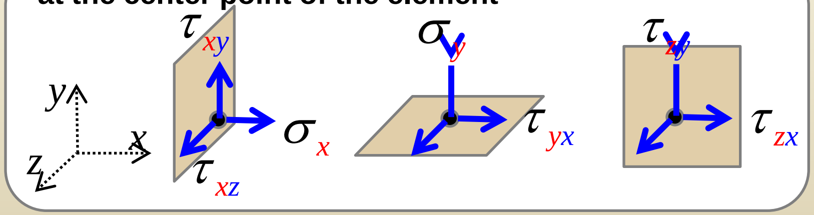


$$m\frac{d\mathbf{V}}{dt}=\sum\mathbf{F}=\mathbf{F}_{Body}+\boxed{\mathbf{F}_{Surface}}$$

$$\sum F_{Surface,x}=\left[\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z$$

$$\sum F_x=F_{Body,x}+F_{Surface,x}$$
$$=\left[X+\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right]\Delta x\Delta y\Delta z$$

at the center point of the element



Force in equilibrium

$$X+\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=0$$
$$Y+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0$$
$$Z+\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}=0$$

Moment in equilibrium

$$\tau_{xy}=\tau_{yx}$$
$$\tau_{xz}=\tau_{zx}$$
$$\tau_{yz}=\tau_{zy}$$

6 stress and
3 equations

Strain Analysis

Strain Analysis

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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6 Stress distribution in an elastic body

3 Equations of force equilibrium

➡ *6 Relations between 6 Strain and 3 Displacement*

6 Relations between 6 Strain and 6 Stress

✓**Strain Component**

A body is said to be *strained* whenever the relative positions of points in the body are altered

The coordinates of a particle of material in the body at any point

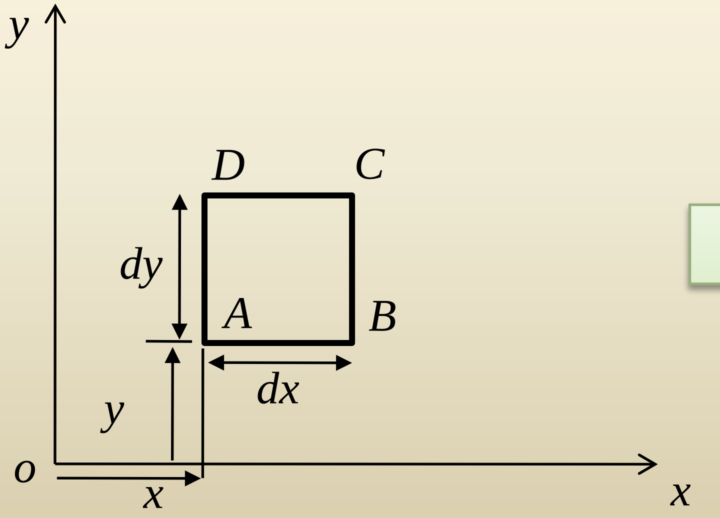
$$x, y, z$$

After strain ⇩

u, v, w **: displacement , function of x,y,z**

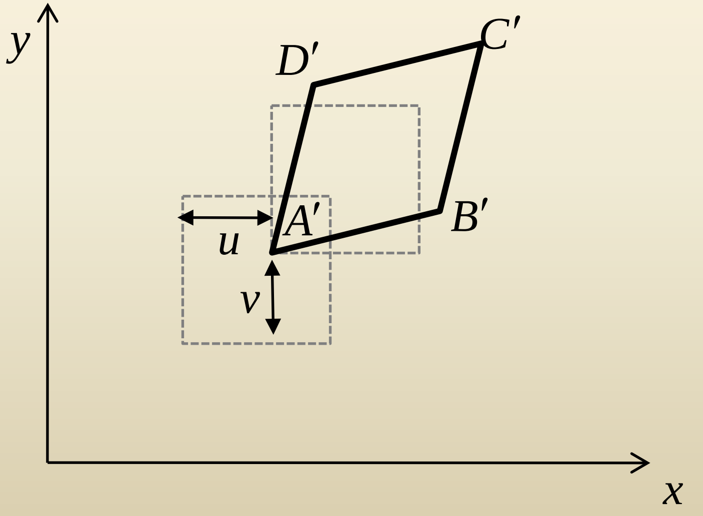
$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$

$$x + u, y + v, z + w$$



A small rectangular element ABCD

After strain



The element displaced to the position A′B′C′D′

Strain Analysis

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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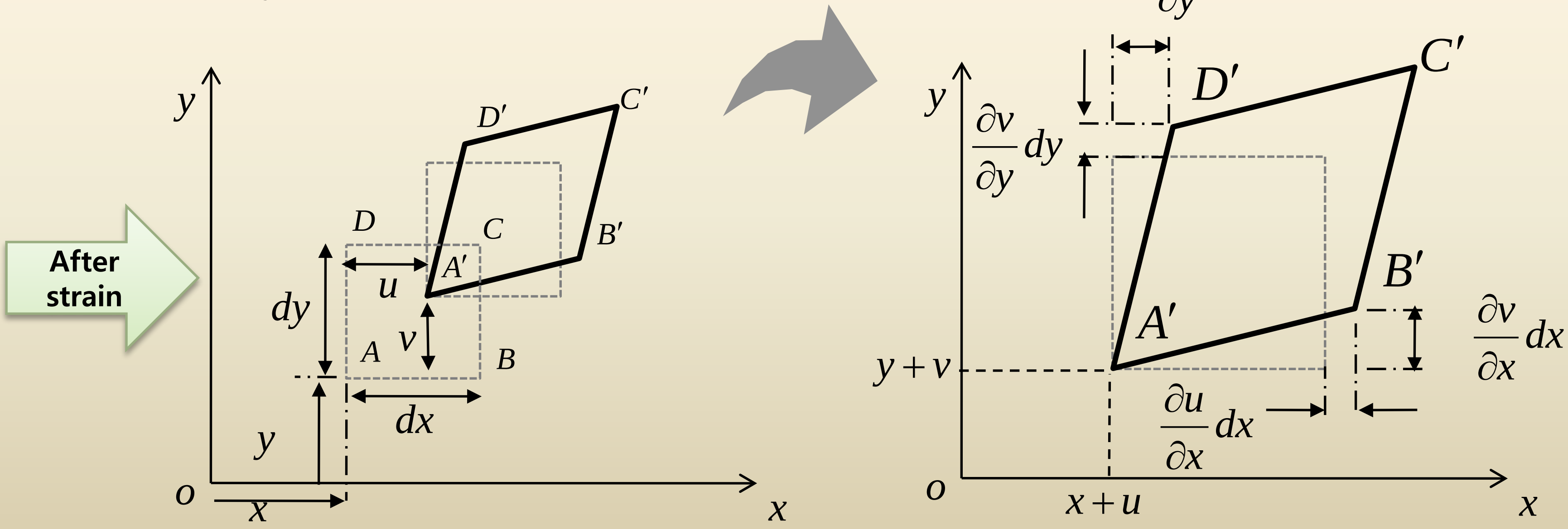
$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$

$x + u, y + v, z + w$



The element displaced to the position A′B′C′D′

Strain Analysis

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
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$$x, y, z \longrightarrow x+u, y+v, z+w$$

After strain

u, v, w **: displacement , function of x,y,z**

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$

Displaced position

$\bullet A \rightarrow A'$

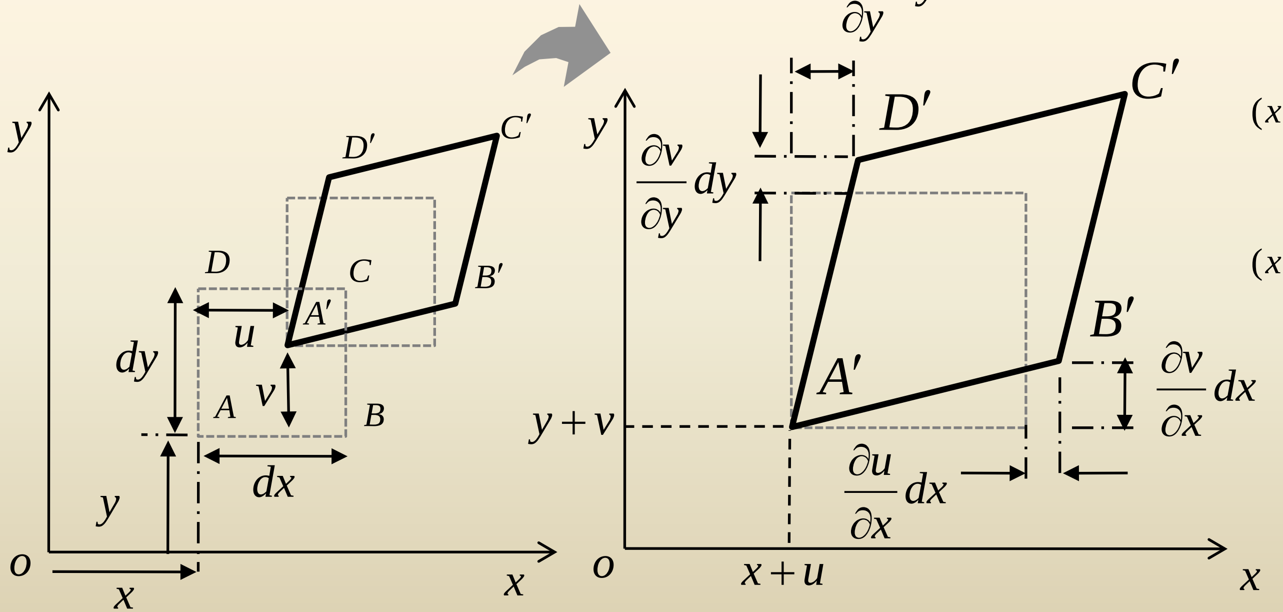
$$(x, y) \rightarrow (x+u, y+v)$$

$\bullet B \rightarrow B'$

$$(x+dx, y) \rightarrow (x+dx+u+\frac{\partial u}{\partial x}dx, y+v+\frac{\partial v}{\partial x}dy)$$

$\bullet D \rightarrow D'$

$$(x, y+dy) \rightarrow (x+u+\frac{\partial u}{\partial y}dx, y+dy+v+\frac{\partial v}{\partial y}dy)$$

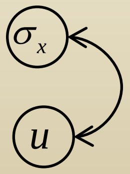


The element displaced to the position A′B′C′D′

analogue

$$\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x$$

$$u + \frac{\partial u}{\partial x}dx$$



Strain Analysis

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

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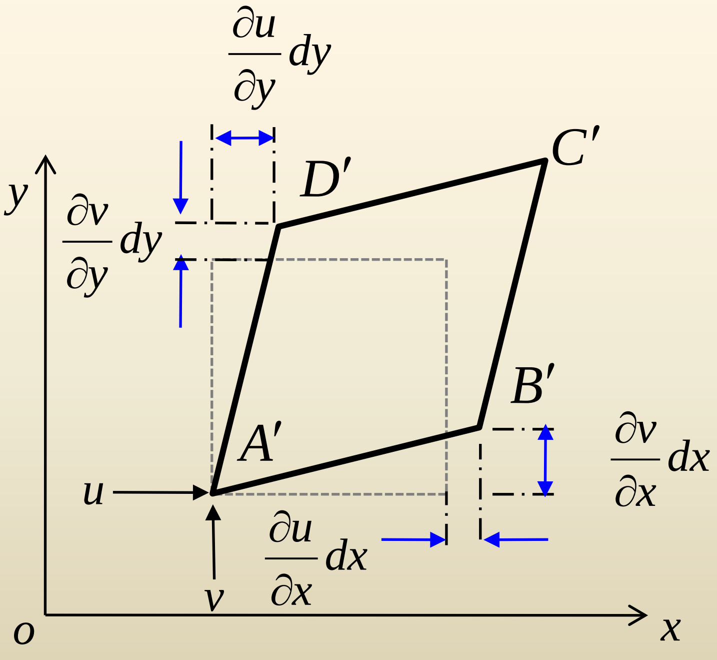
A body is said to be *strained* whenever the relative positions of points in the body are altered

$$x, y, z \rightarrow x+u, y+v, z+w$$

After strain

u, v, w **: displacement , function of x,y,z**

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$



The element displaced to the position A′B′C′D′

Displaced position

$$A \rightarrow A' \qquad B \rightarrow B'$$

$$(x, y) \rightarrow (x+u, y+v) \qquad (x+dx, y) \rightarrow (x+dx+u+\frac{\partial u}{\partial x}dx, y+v+\frac{\partial v}{\partial x}dx)$$

1) Longitudinal strain : x-direction

$$\left|\overline{AB}\right| = \sqrt{(x+dx-x)^2 + (y-y)^2}$$

$$\therefore \left|\overline{AB}\right| = dx$$

$$\left|\overline{A'B'}\right| = \sqrt{\left((x+dx+u+\frac{\partial u}{\partial x}dx)-(x+u)\right)^2 + \left((y+v+\frac{\partial v}{\partial x}dx)-(y+v)\right)^2}$$

$$\therefore \left|\overline{A'B'}\right| = \sqrt{\left(dx+\frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2}$$

Strain Analysis

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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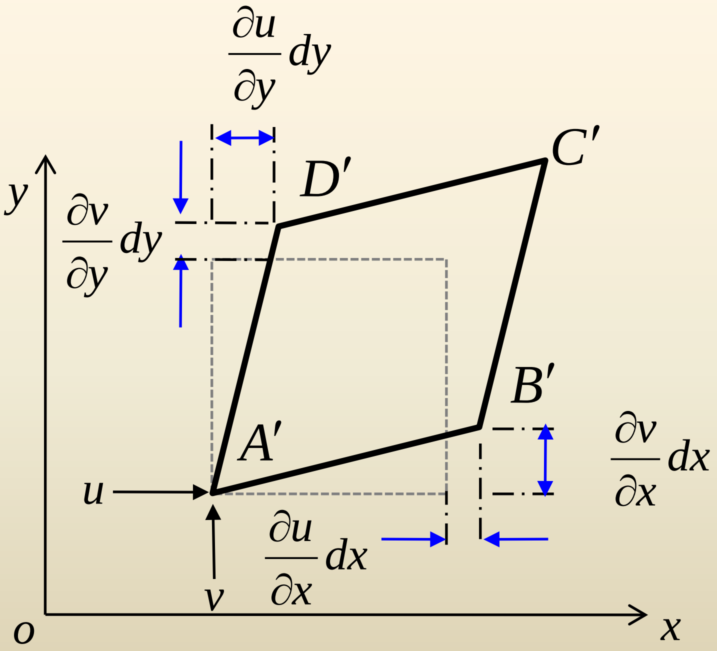
A body is said to be *strained* whenever the relative positions of points in the body are altered

$x, y, z \longrightarrow x+u, y+v, z+w$

After strain

u, v, w **: displacement , function of x,y,z**

$u = u(x,y,z), v = v(x,y,z), w = w(x,y,z)$



The element displaced to the position A'B'C'D'

Displaced position

$A \to A'$ $\qquad$ $B \to B'$

$(x,y) \to (x+u, y+v)$ $\qquad$ $(x+dx, y) \to (x+dx+u+\frac{\partial u}{\partial x}dx, y+v+\frac{\partial v}{\partial x}dx)$

1) Longitudinal strain : x-direction

$$|\overline{AB}| = dx \ , |\overline{A'B'}| = \sqrt{\left(dx+\frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2}$$

$$\boxed{\varepsilon = \frac{|\overline{A'B'}| - |\overline{AB}|}{|\overline{AB}|}} \Rightarrow (1+\varepsilon)|\overline{AB}| = |\overline{A'B'}|$$

$$(1+\varepsilon)dx = \sqrt{\left(dx+\frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2}$$

$$((1+\varepsilon)dx)^2 = \left(dx+\frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2$$

Strain Analysis

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

✓Strain Component

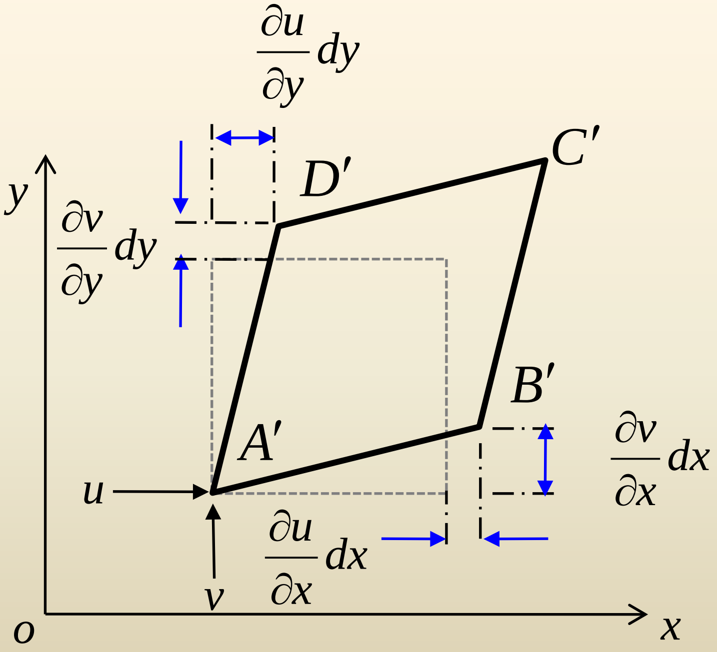
A body is said to be *strained* whenever the relative positions of points in the body are altered

$x, y, z \rightarrow x+u, y+v, z+w$

After strain

u, v, w **: displacement , function of x,y,z**

$u = u(x,y,z), v = v(x,y,z), w = w(x,y,z)$



The element displaced to the position A'B'C'D'

Displaced position

$A \rightarrow A'$ $B \rightarrow B'$

$$(x,y) \rightarrow (x+u, y+v) \quad (x+dx, y) \rightarrow (x+dx+u+\frac{\partial u}{\partial x}dx, y+v+\frac{\partial v}{\partial x}dx)$$

1) Longitudinal strain : x-direction

$$((1+\varepsilon)dx)^2 = \left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2$$

$$\left[1+2\varepsilon+\varepsilon^2\right](dx)^2 = \left[1+2\frac{\partial u}{\partial x}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right](dx)^2$$

$$2\varepsilon+\cancel{\varepsilon^2} = 2\frac{\partial u}{\partial x}+\cancel{\left(\frac{\partial u}{\partial x}\right)^2}+\cancel{\left(\frac{\partial v}{\partial x}\right)^2}$$

If we confine our attention to the case where the deformation of the body is very small

$$\therefore \varepsilon_x = \frac{\partial u}{\partial x}$$

in same way, $\varepsilon_y = \frac{\partial v}{\partial x}, \varepsilon_z = \frac{\partial w}{\partial x}$

Strain Analysis

15 Variables

- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

✓Strain Component

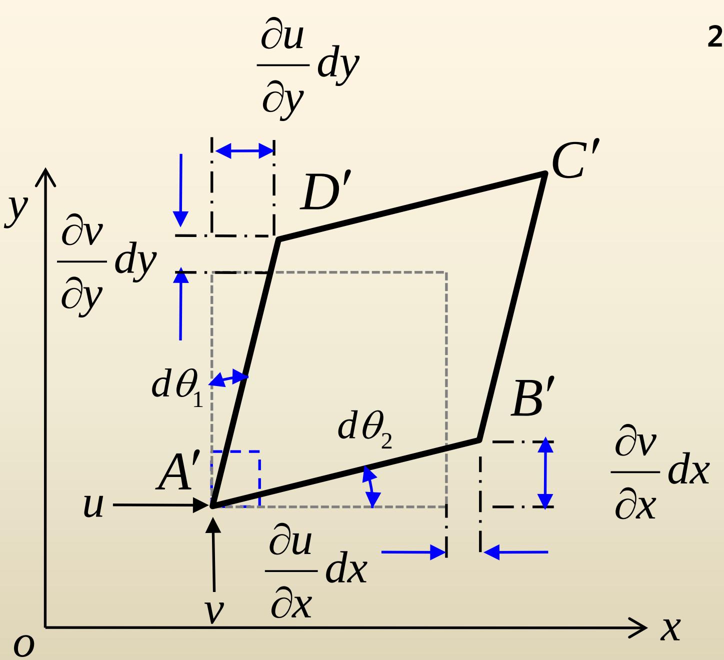
A body is said to be *strained* whenever the relative positions of points in the body are altered

$$x, y, z \longrightarrow x+u, y+v, z+w$$

After strain

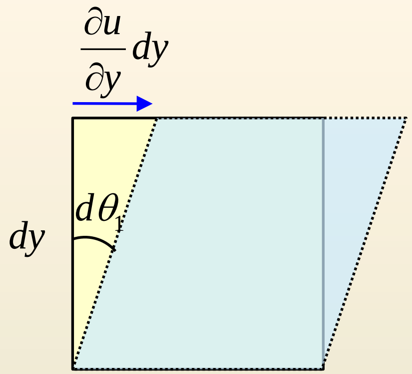
u, v, w **: displacement , function of x,y,z**

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$



The element displaced to the position A'B'C'D'

2) Shearing strain



$$d\theta_1 \approx \tan d\theta_1 = \frac{\frac{\partial u}{\partial y}dy}{dy}$$

$$\therefore d\theta_1 = \frac{\partial u}{\partial y}$$

A shearing strain γ_{xy} at a point is defined as the change in the value of the angle between the two elements originally parallel to the x and y axes at that point

---

θ_1 : rotation angle

recall, Euler's equation

$$I\dot{\omega} = \sum \mathbf{M} \quad , \omega = \frac{d\theta}{dt} \text{ :angular velocity of rigid body}$$

Strain Analysis

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

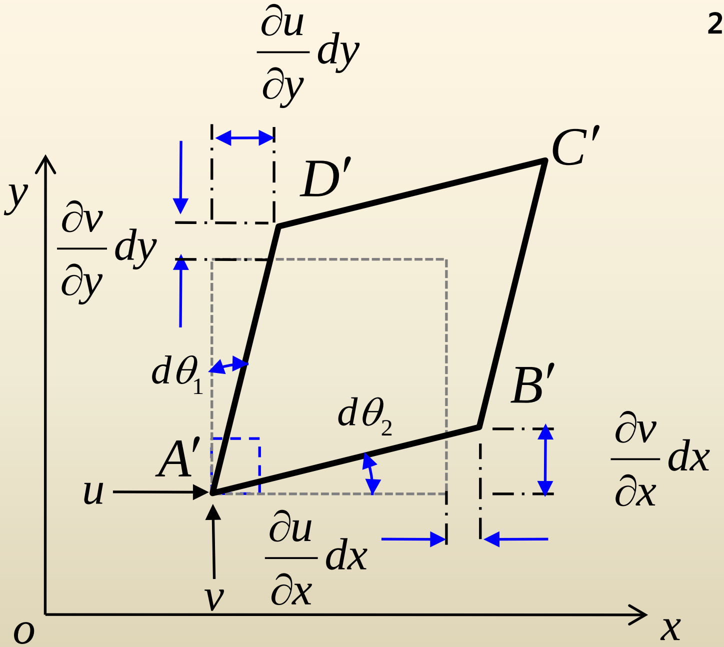
✓Strain Component

A body is said to be *strained* whenever the relative positions of points in the body are altered

$$x, y, z \xrightarrow{\text{After strain}} x+u, y+v, z+w$$

u, v, w **: displacement , function of x,y,z**

$$u = u(x,y,z), v = v(x,y,z), w = w(x,y,z)$$



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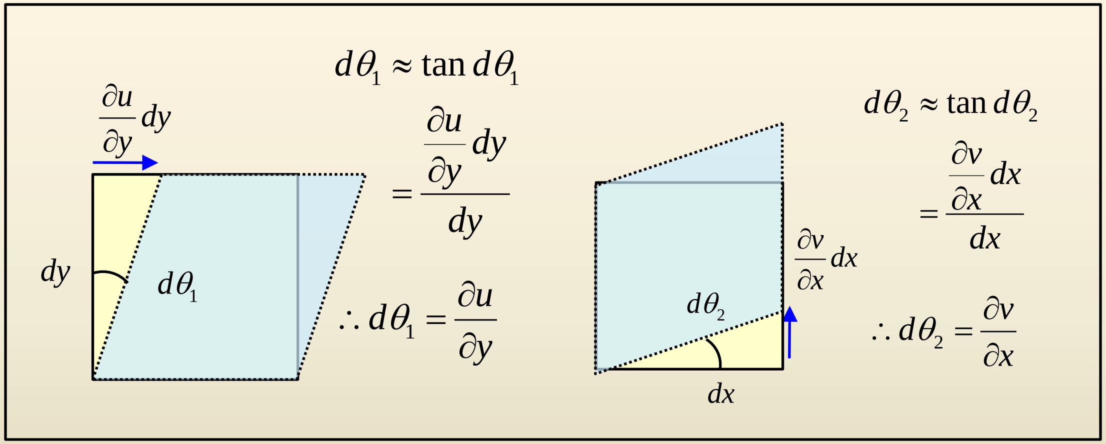
2) Shearing strain



$$d\theta_1 \approx \tan d\theta_1 = \frac{\frac{\partial u}{\partial y}dy}{dy}$$

$$\therefore d\theta_1 = \frac{\partial u}{\partial y}$$

$$d\theta_2 \approx \tan d\theta_2 = \frac{\frac{\partial v}{\partial x}dx}{dx}$$

$$\therefore d\theta_2 = \frac{\partial v}{\partial x}$$

Strain Analysis

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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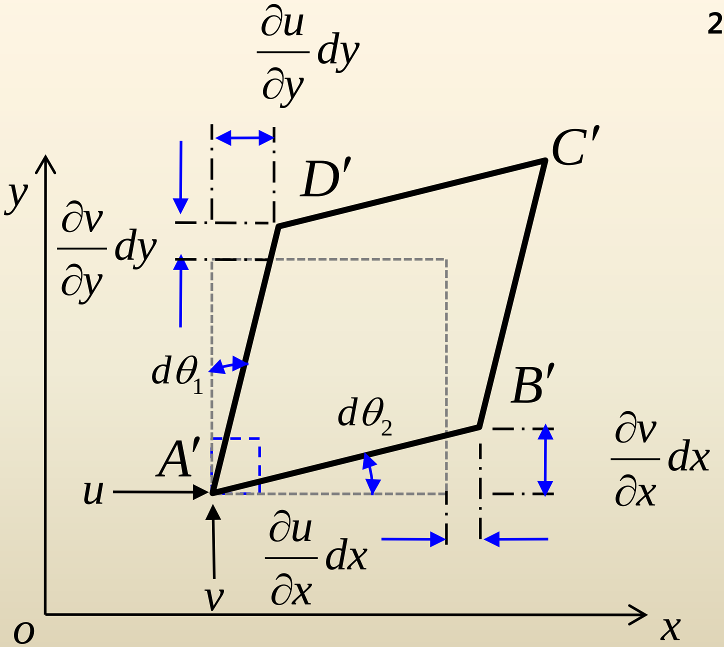
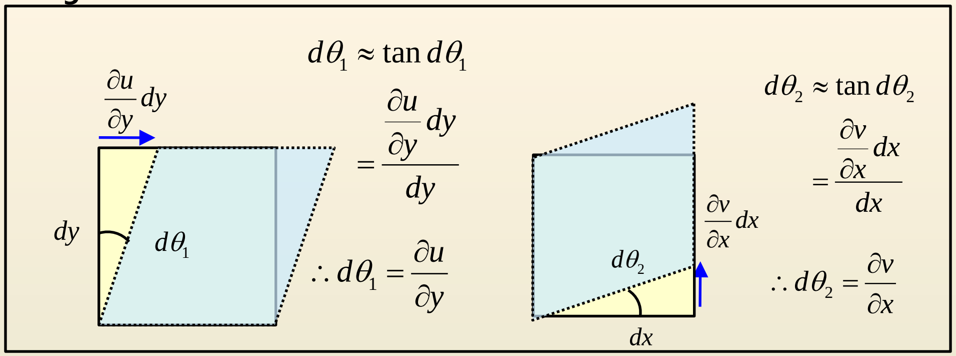
$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$



The element displaced to the position A'B'C'D'

2) Shearing strain



A shearing strain γ_{xy} at a point is defined as the change in the value of the angle between the two elements originally parallel to the x and y axes at that point

$$\gamma_{xy} = d\theta_1 + d\theta_2 \approx \tan d\theta_1 + \tan d\theta_2 = \frac{\frac{\partial u}{\partial y}dy}{dy} + \frac{\frac{\partial v}{\partial x}dx}{dx}$$

$$\therefore \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Strain Analysis

15 Variables
- 6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- 6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- 3 Displacement u, v, w

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

✓Strain Component

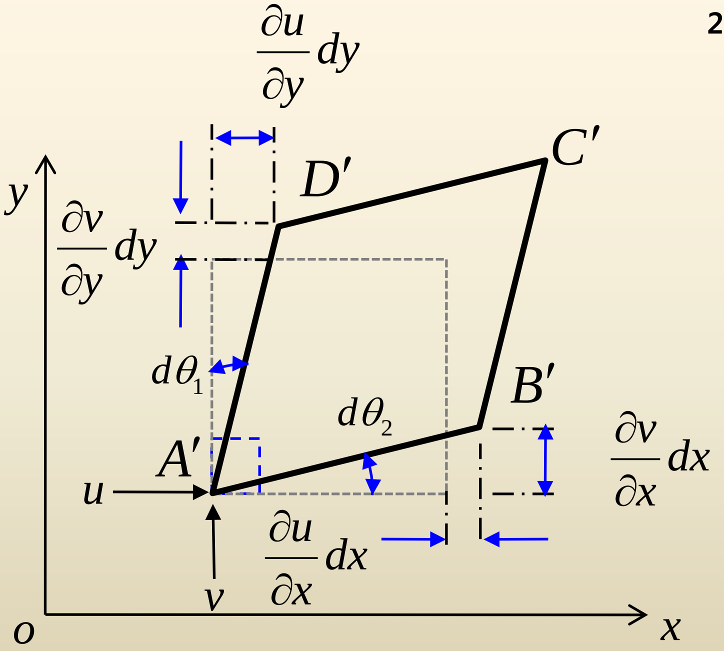
A body is said to be *strained* whenever the relative positions of points in the body are altered

$$x, y, z \rightarrow x+u, y+v, z+w$$

After strain

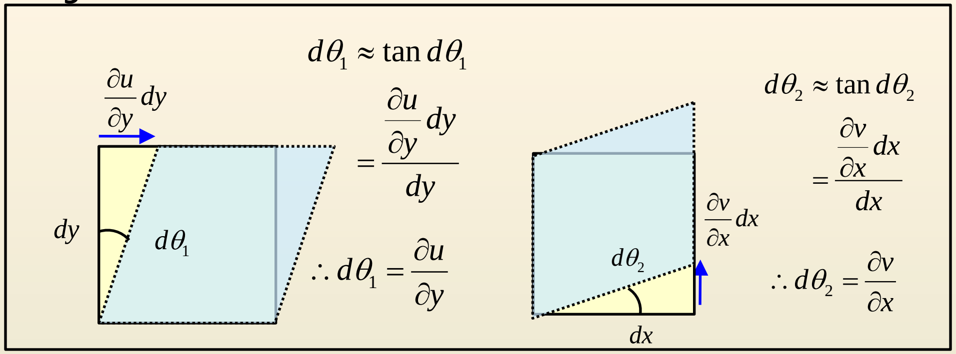
u, v, w **: displacement , function of x,y,z**

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z)$$



The element displaced to the position A'B'C'D'

2) Shearing strain



$$d\theta_1 \approx \tan d\theta_1 = \frac{\frac{\partial u}{\partial y}dy}{dy}$$

$$\therefore d\theta_1 = \frac{\partial u}{\partial y}$$

$$d\theta_2 \approx \tan d\theta_2 = \frac{\frac{\partial v}{\partial x}dx}{dx}$$

$$\therefore d\theta_2 = \frac{\partial v}{\partial x}$$

A shearing strain γ_{xy} at a point is defined as the change in the value of the angle between the two elements originally parallel to the x and y axes at that point

$$\gamma_{xy} = d\theta_1 + d\theta_2 \approx \tan d\theta_1 + \tan d\theta_2 = \frac{\frac{\partial u}{\partial y}dy}{dy} + \frac{\frac{\partial v}{\partial x}dx}{dx} \qquad \therefore \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

in same way, $\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$

Strain Analysis

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

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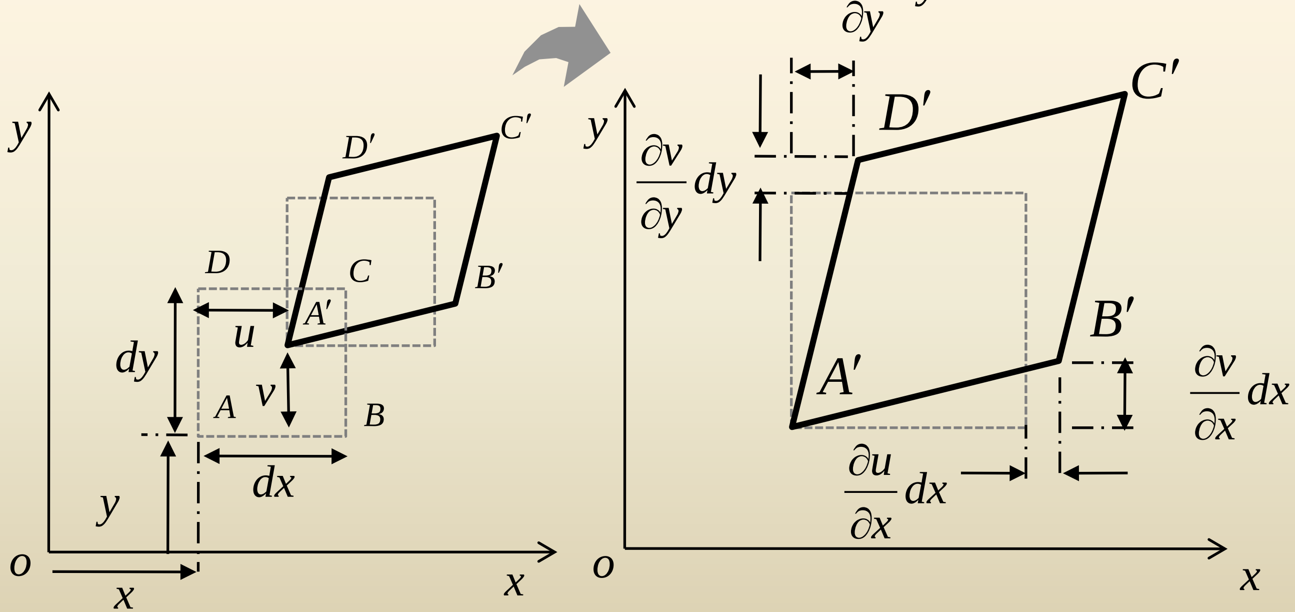

The element displaced to the position A'B'C'D'

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

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$$\varepsilon_z = \frac{\partial w}{\partial x}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

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Compatibility equations

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

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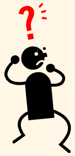

What is the compatibility equations ?

!) in case of the displacement components are not to be found, the *compatibility equations* must be satisfied

if we found displacement first, the compatibility equations are not required

For 6 relations between 6 strain and 3 displacement :

These equations maybe regarded as a system of P.D.E of $u, \ v, \ w$ **when strain components are given functions of** $x, \ y, \ z$

Given : 6 equations **find : 3 unknowns**

$$\varepsilon_x(x,y,z), \varepsilon_y(x,y,z), \varepsilon_z(x,y,z), \gamma_{xy}(x,y,z), \gamma_{yz}(x,y,z), \gamma_{zx(x,y,z)} \quad > \quad u(x,y,z), \ v(x,y,z), \ w(x,y,z)$$

There must be some conditions to be imposed on the strain components in order that these *six equations will give a set of 'single-valued continuous' solutions for the three displacement components*

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

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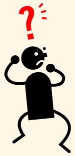

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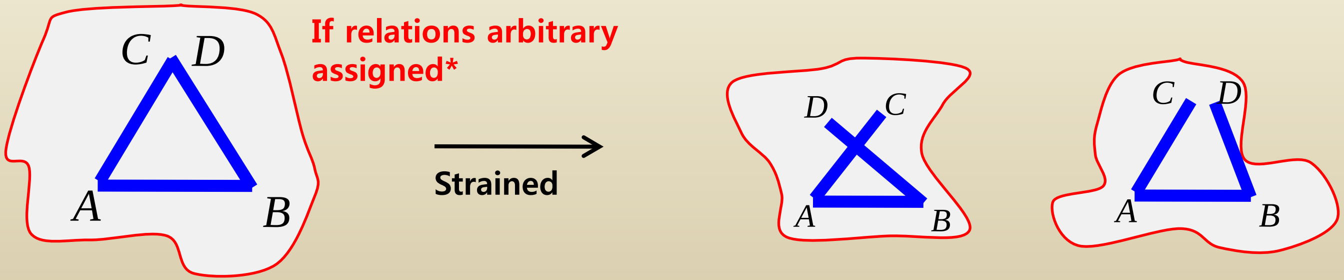

If relations arbitrary assigned*

Strained

Unstrained Specified Field

*** Fung Y.C., Foundation of Solid Mechanics, Prentich-Hall , 1968, p99-100**

Compatibility Equation

6 Relations *between* *6 Strain and 3 Displacement* *:*

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

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from shearing-strain :

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \xrightarrow{\frac{\partial^2}{\partial x \partial y}} \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \boxed{\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y}\right)} + \boxed{\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial v}{\partial x}\right)}$$

$$\boxed{\frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x}} \qquad \boxed{\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial y}}$$

u,v,w : single-valued continuous

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)$$

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

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$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial v}{\partial x}\right)$$

u,v,w : single-valued continuous

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2}\frac{\partial v}{\partial y}$$

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

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There must be some conditions to be imposed on the strain components in order that these *six equations will give a set of 'single-valued continuous' solutions for the three displacement components*

from shearing-strain :

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2}\frac{\partial v}{\partial y} \rightarrow \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

in same way,

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

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$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \xrightarrow{\frac{\partial^2}{\partial x \partial z}} \qquad \frac{\partial^2 \gamma_{xy}}{\partial x \partial z} = \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \xrightarrow{\frac{\partial^2}{\partial y \partial x}} \qquad \frac{\partial^2 \gamma_{zx}}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

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$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\xrightarrow{\dfrac{\partial^2}{\partial x\partial z}}\frac{\partial^2\gamma_{xy}}{\partial x\partial z}=\frac{\partial^2}{\partial x\partial z}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\xrightarrow{\dfrac{\partial^2}{\partial y\partial x}}\quad +\Big)\ \frac{\partial^2\gamma_{zx}}{\partial y\partial x}=\frac{\partial^2}{\partial y\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial^2\gamma_{xy}}{\partial x\partial z}+\frac{\partial^2\gamma_{zx}}{\partial y\partial x}=\frac{\partial^2}{\partial x\partial z}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial^2}{\partial y\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

Compatibility Equation

6 Relations *between* *6 Strain and 3 Displacement* *:*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

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$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} + \frac{\partial^2 \gamma_{zx}}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$= \boxed{\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u}{\partial y}\right)} + \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial w}{\partial x}\right) + \boxed{\frac{\partial^2}{\partial y \partial x}\left(\frac{\partial u}{\partial z}\right)}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\boxed{\frac{\partial^2}{\partial y \partial z}\frac{\partial u}{\partial x}} \qquad\qquad\qquad\qquad \boxed{\frac{\partial^2}{\partial y \partial z}\frac{\partial u}{\partial x}}$$

u,v,w : single-valued continuous

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Compatibility Equation

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$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

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$$\begin{array}{ccccccc} = \boxed{\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u}{\partial y}\right)} & + & \boxed{\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial v}{\partial x}\right)} & + & \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial w}{\partial x}\right) & + & \boxed{\frac{\partial^2}{\partial y \partial x}\left(\frac{\partial u}{\partial z}\right)} \\ \downarrow & & \downarrow & & & & \downarrow \\ \boxed{\frac{\partial^2}{\partial y \partial z}\frac{\partial u}{\partial x}} & & \boxed{\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial z}\right)} & & & & \boxed{\frac{\partial^2}{\partial y \partial z}\frac{\partial u}{\partial x}} \end{array}$$

u,v,w : single-valued continuous

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\right)$$

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

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$$\begin{tikzcd}
= \boxed{\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u}{\partial y}\right)} \arrow[d] & + \boxed{\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial v}{\partial x}\right)} \arrow[d] & + \boxed{\frac{\partial^2}{\partial y \partial x}\left(\frac{\partial w}{\partial x}\right)} \arrow[d] & + \boxed{\frac{\partial^2}{\partial y \partial x}\left(\frac{\partial u}{\partial z}\right)} \arrow[d] \\
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\end{tikzcd}$$

u,v,w : single-valued continuous

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)$$

Compatibility Equation

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$$= \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial u}{\partial z}\right)$$

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from shearing-strain :

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} + \frac{\partial^2 \gamma_{zx}}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial z}\frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial z}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right) + \frac{\partial^2}{\partial y \partial z}\frac{\partial u}{\partial x}$$

$$= 2\frac{\partial^2}{\partial y \partial z}\frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} + \frac{\partial^2 \gamma_{yz}}{\partial x^2}$$

⇩

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} + \frac{\partial^2 \gamma_{zx}}{\partial y \partial x} = 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} + \frac{\partial^2 \gamma_{yz}}{\partial x^2}$$

Compatibility Equation

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

What is the compatibility equations ?

!) in case of the displacement components are not to be found, the *compatibility equations* must be satisfied

if we found displacement first, the compatibility equations are not required

There must be some conditions to be imposed on the strain components in order that these *six equations will give a set of 'single-valued continuous' solutions for the three displacement components*

from shearing-strain :

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} + \frac{\partial^2 \gamma_{zx}}{\partial y \partial x} = 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} + \frac{\partial^2 \gamma_{yz}}{\partial x^2}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$

in same way,

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right)$$

Compatibility Equation

6 Relations *between* *6 Strain and 3 Displacement* *:*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

What is the compatibility equations ?

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There must be some conditions to be imposed on the strain components in order that these *six equations will give a set of 'single-valued continuous' solutions for the three displacement components*

compatibility equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$
$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$
$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

or

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Compatibility Equation

six equations?

compatibility equations

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\xrightarrow{\frac{\partial^2}{\partial z^2}}\frac{\partial^4\varepsilon_x}{\partial z^2\partial y^2}+\frac{\partial^4\varepsilon_y}{\partial z^2\partial x^2}=\frac{\partial^4\gamma_{xy}}{\partial z^2\partial x\partial y} \quad ①$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\xrightarrow{\frac{\partial^2}{\partial x^2}}\frac{\partial^4\varepsilon_y}{\partial x^2\partial z^2}+\frac{\partial^4\varepsilon_z}{\partial x^2\partial y^2}=\frac{\partial^4\gamma_{yz}}{\partial x^2\partial y\partial z} \quad ②$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}\xrightarrow{\frac{\partial^2}{\partial y^2}}\frac{\partial^4\varepsilon_z}{\partial y^2\partial x^2}+\frac{\partial^4\varepsilon_x}{\partial y^2\partial z^2}=\frac{\partial^4\gamma_{zx}}{\partial y^2\partial z\partial x} \quad ③$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)\xrightarrow{\frac{\partial^2}{\partial y\partial z}}2\frac{\partial^4\varepsilon_x}{\partial y^2\partial z^2}=-\frac{\partial^4\gamma_{yz}}{\partial x^2\partial y\partial z}+\frac{\partial^4\gamma_{zx}}{\partial x\partial y^2\partial z}+\frac{\partial^4\gamma_{xy}}{\partial x\partial y\partial z^2} \quad Ⓐ$$

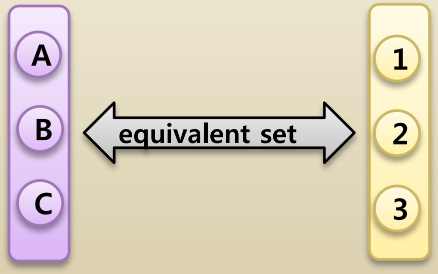
$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)\xrightarrow{\frac{\partial^2}{\partial z\partial x}}2\frac{\partial^4\varepsilon_y}{\partial z^2\partial x^2}=\frac{\partial^4\gamma_{yz}}{\partial x^2\partial y\partial z}-\frac{\partial^4\gamma_{zx}}{\partial x\partial y^2\partial z}+\frac{\partial^4\gamma_{xy}}{\partial x^2\partial y\partial z^2} \quad Ⓑ$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)\xrightarrow{\frac{\partial^2}{\partial x\partial y}}2\frac{\partial^4\varepsilon_z}{\partial x^2\partial y^2}=\frac{\partial\gamma_{yz}}{\partial x^2\partial y\partial z}+\frac{\partial\gamma_{zx}}{\partial x\partial y^2\partial z}-\frac{\partial\gamma_{xy}}{\partial x\partial y\partial z^2} \quad Ⓒ$$

Ⓐ = ① + ③ - ②

Ⓑ = ② + ① - ③

Ⓒ = ③ + ② - ①



Ⓐ Ⓑ Ⓒ ⟷ equivalent set ⟷ ① ② ③

6 Relations **between** *6 Strain and 3 Displacement* :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

this system of equations does not, in general, possess a solution for u,v and w unless the six strain components are somehow related....

the strain components must satisfy these expressions in order that solutions for the displacement components exist....

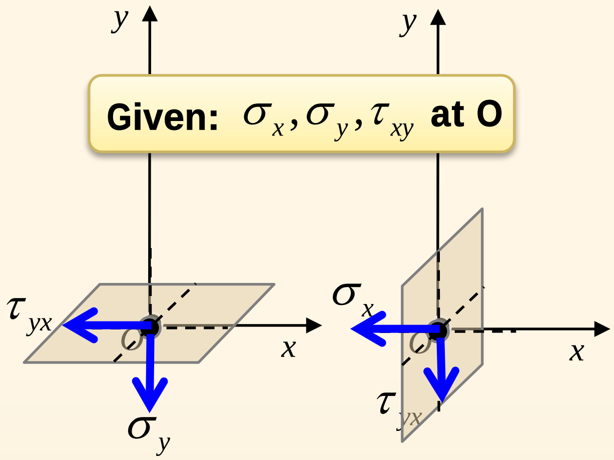
Although six compatibility equations are written, they are equivalent to three independent fourth-order equations...

It is usually more convenient, however, to use the six second-order equations rather than three fourth-order equations.*

Chou, P.C., Elasticity, Van Nostard Company, 1967, p39-41*

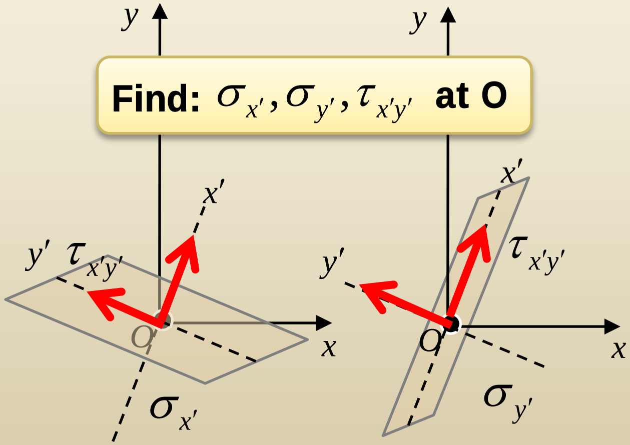
Transformation of Stress

Specification of Stress At a Point (xy axis)

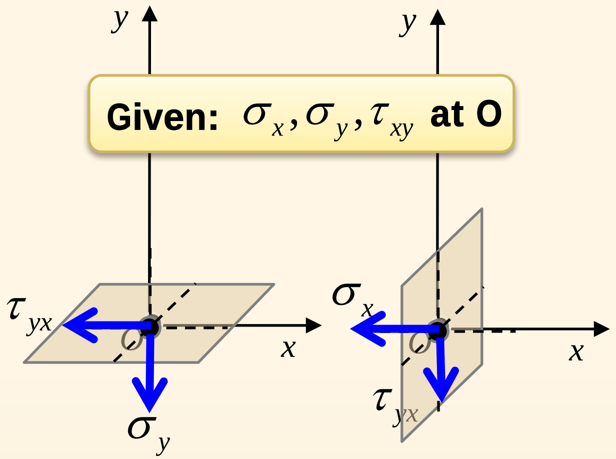

Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**

✓**Specification of Stress at a Point**

Stress within a body is completely determined when we know the values of the six stress components at each point


Find: $\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}$ **at O**

Specification of Stress At a Point (xy axis)



Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**

- ✓**Stress on BC**

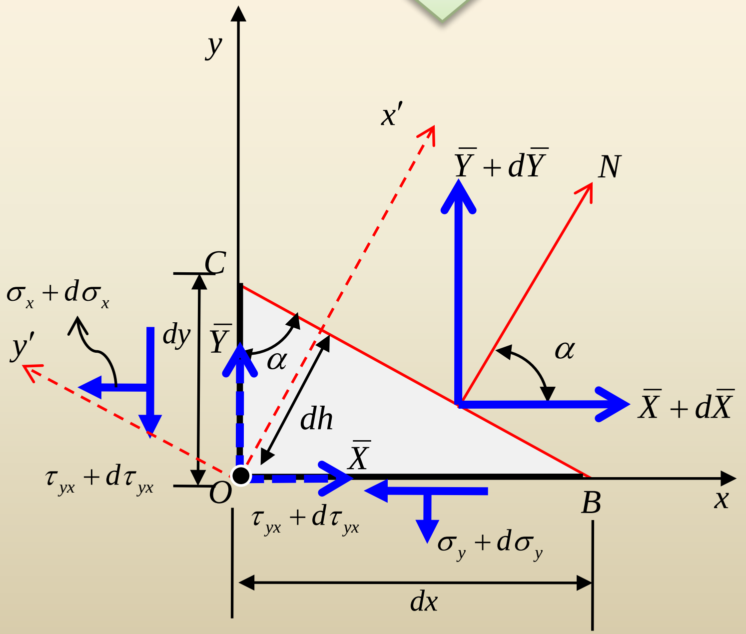
  $\bar{X}$ **: Stress along x axis** $\bar{Y}$ **: Stress along y axis**

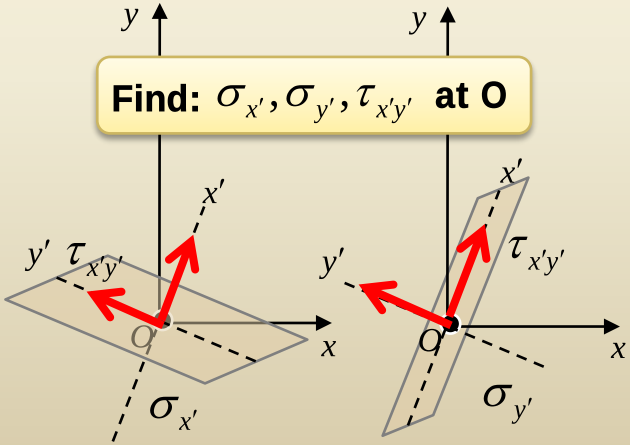
- ✓**Direction Cosine**

  $$\cos\alpha = \cos Nx = l, \quad \cos(90-\alpha) = \cos Ny = m$$

- ✓**Area on BC = A**

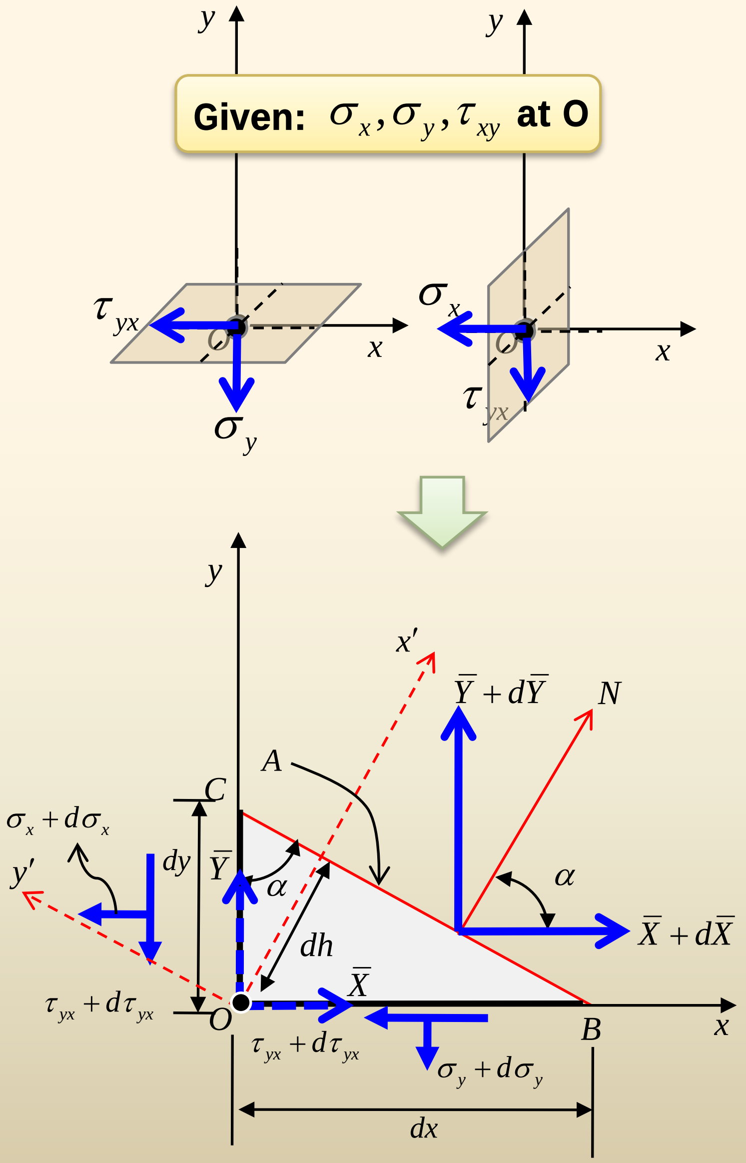
  $$Area_{OC} = Al, \quad Area_{OB} = Am,$$





Find: $\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}$ **at O**

Specification of Stress At a Point (xy axis)


Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**

✓**Stress on BC**

$\bar{X}$ **: Stress along x axis** $\bar{Y}$ **: Stress along y axis**

✓**Direction Cosine**

$$\cos\alpha = \cos Nx = l, \quad \cos(90-\alpha) = \cos Ny = m$$

✓**Area on BC = A**

$$Area_{OC} = Al, \quad Area_{OB} = Am,$$

✓**Considering the forces for OBC in equilibrium**

$$\sum F_x = \left(\bar{X} + d\bar{X}\right)A - \left(\sigma_x + d\sigma_x\right)Al - \left(\tau_{xy} + d\tau_{xy}\right)Am - X\left(\frac{1}{2}Adh\right) = 0$$

$$\sum F_y = \left(\bar{Y} + d\bar{Y}\right)A - \left(\sigma_y + d\sigma_y\right)Am - \left(\tau_{xy} + d\tau_{xy}\right)Al - Y\left(\frac{1}{2}Adh\right) = 0$$

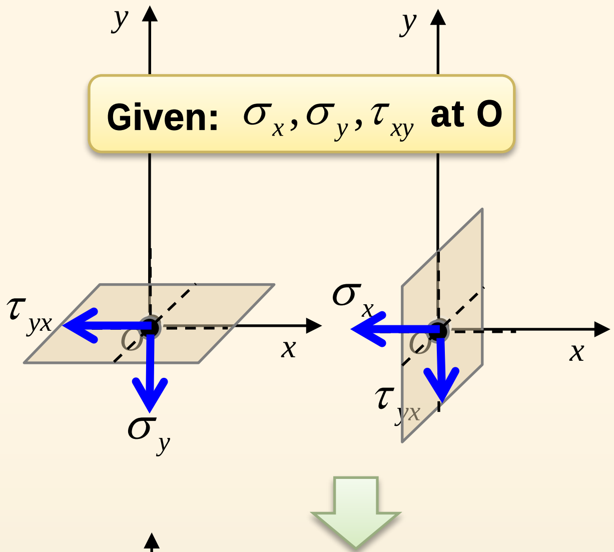
$\downarrow$ **Divided by A,** $dh \to 0$

$$\left(\bar{X} + \cancel{d\bar{X}}\right) - \left(\sigma_x + \cancel{d\sigma_x}\right)l - \left(\tau_{xy} + \cancel{d\tau_{xy}}\right)m - X\left(\cancel{\frac{1}{2}dh}\right) = 0$$

$$\left(\bar{Y} + \cancel{d\bar{Y}}\right) - \left(\sigma_y + \cancel{d\sigma_y}\right)m - \left(\tau_{xy} + \cancel{d\tau_{xy}}\right)l - Y\left(\cancel{\frac{1}{2}dh}\right) = 0$$

$$\therefore \bar{X} = l\sigma_x + m\tau_{xy}, \quad \bar{Y} = l\tau_{xy} + m\sigma_y \quad \cdots(1)$$

Specification of Stress At a Point (xy axis)



Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**

✓**Stress on BC**

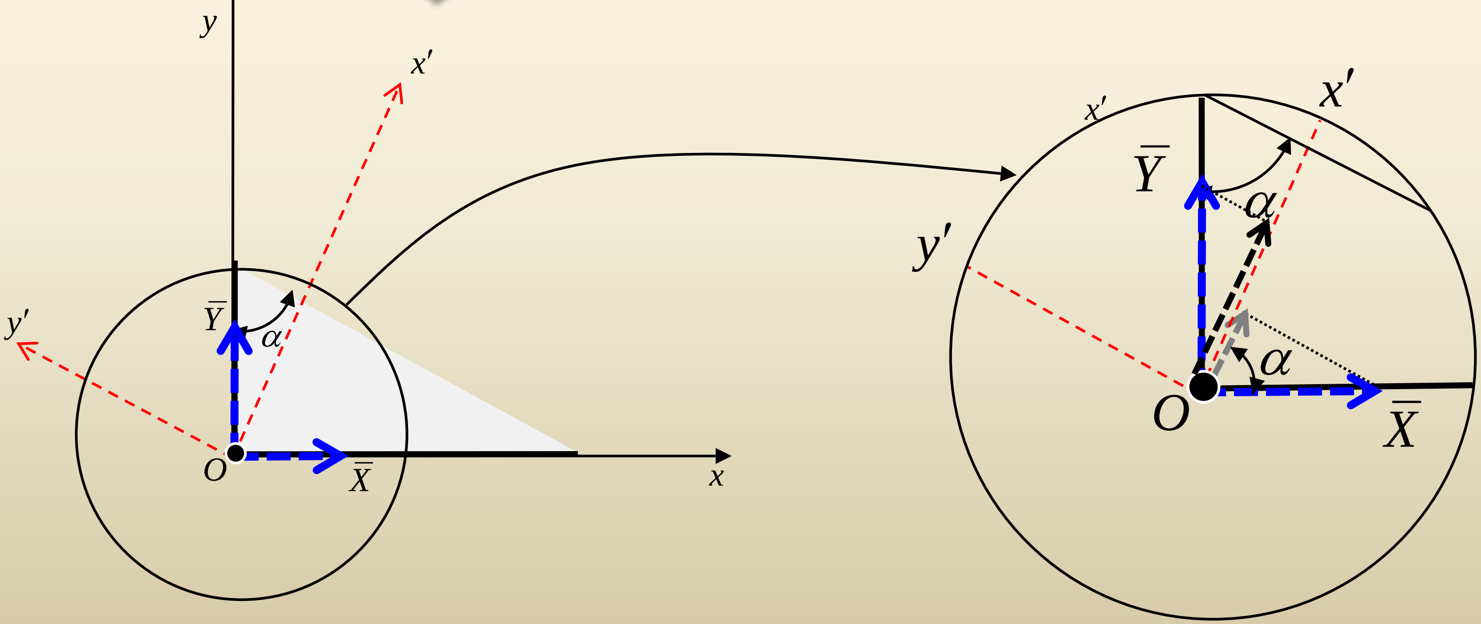
$\bar{X}$ **: Stress along x axis** $\bar{Y}$ **: Stress along y axis**

✓**Direction Cosine**

$$\cos\alpha = \cos Nx = l, \quad \cos(90-\alpha) = \cos Ny = m$$

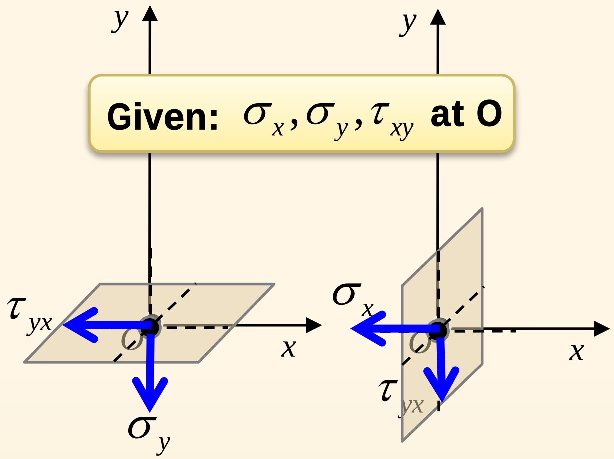
✓**From force equilibrium**

$$\bar{X} = l\sigma_x + m\tau_{xy}, \quad \bar{Y} = l\tau_{xy} + m\sigma_y \quad \cdots(1)$$



$$\sigma_{x'} = l\bar{X} + m\bar{Y}$$

Specification of Stress At a Point (xy axis)



Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**

✓**Stress on BC**

$\bar{X}$: **Stress along x axis** $\bar{Y}$: **Stress along y axis**

✓**Direction Cosine**

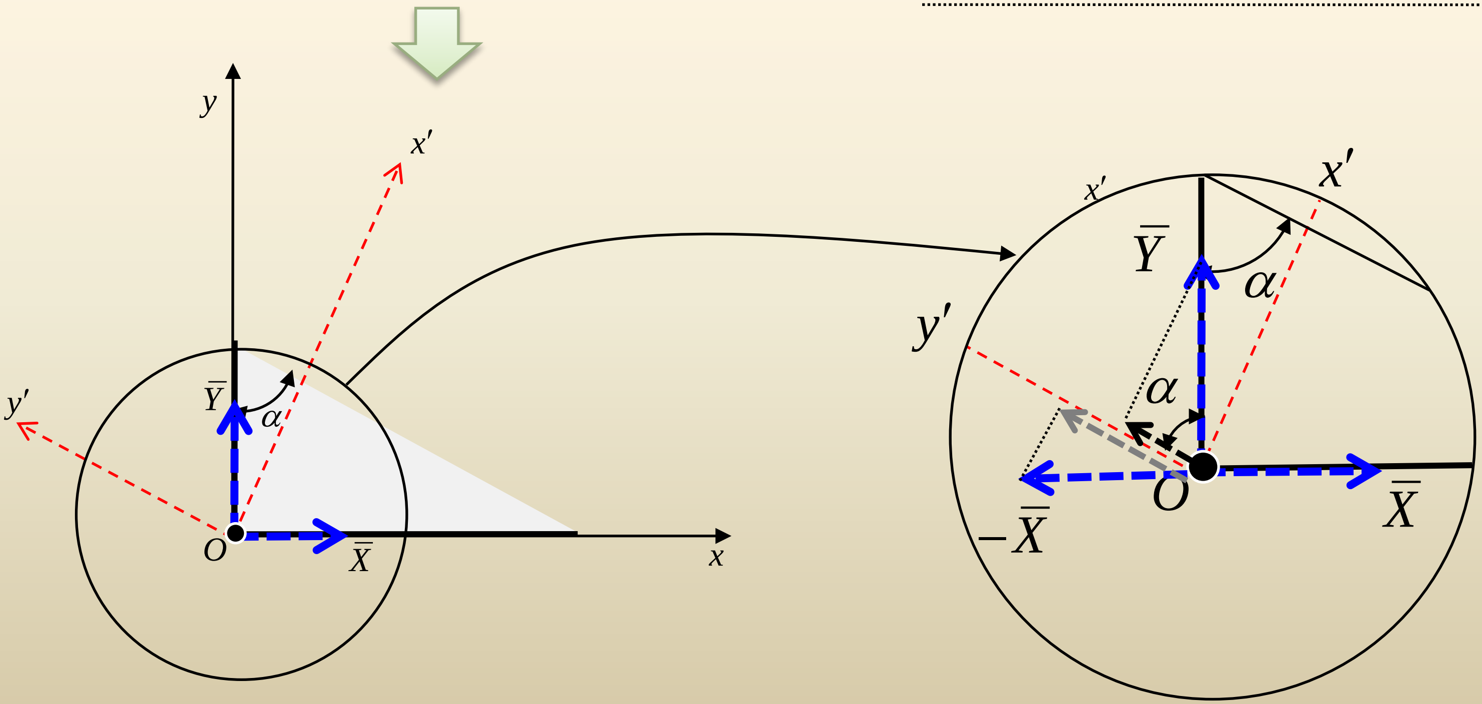
$$\cos\alpha = \cos Nx = l, \quad \cos(90-\alpha) = \cos Ny = m$$

✓**From force equilibrium**

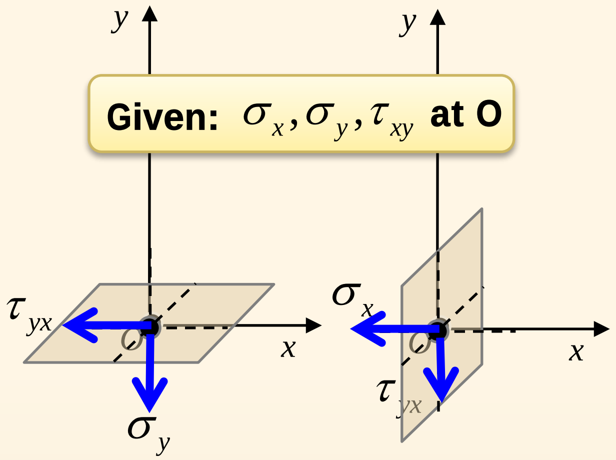
$$\bar{X} = l\sigma_x + m\tau_{xy}, \quad \bar{Y} = l\tau_{xy} + m\sigma_y \quad \cdots(1)$$

$$\sigma_{x'} = l\bar{X} + m\bar{Y}$$

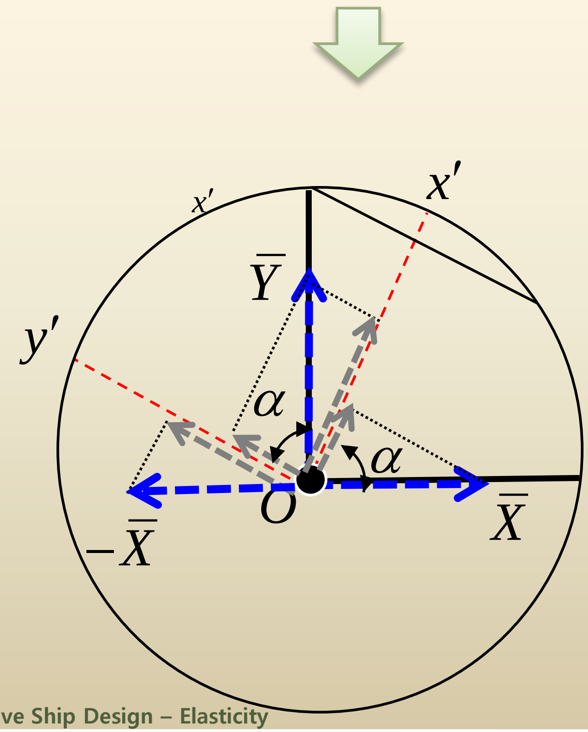
$$\tau_{x'y'} = l\bar{Y} - m\bar{X}$$



Specification of Stress At a Point (xy axis)



Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**



✓**Stress on BC**

$\bar{X}$ **: Stress along x axis** $\bar{Y}$ **: Stress along y axis**

✓**Direction Cosine**

$$\cos\alpha = \cos Nx = l, \quad \cos(90-\alpha) = \cos Ny = m$$

✓**From force equilibrium**

$$\bar{X} = l\sigma_x + m\tau_{xy}, \quad \bar{Y} = l\tau_{xy} + m\sigma_y \quad \cdots(1)$$

$$\sigma_{x'} = l\bar{X} + m\bar{Y} \qquad \tau_{x'y'} = l\bar{Y} - m\bar{X}$$

$$\sigma_{x'} = l\bar{X} + m\bar{Y}$$

Using (1)

$$= l(l\sigma_x + m\tau_{xy}) + m(l\tau_{xy} + m\sigma_y)$$

$$= l^2\sigma_x + m^2\sigma_y + 2lm\tau_{xy}$$

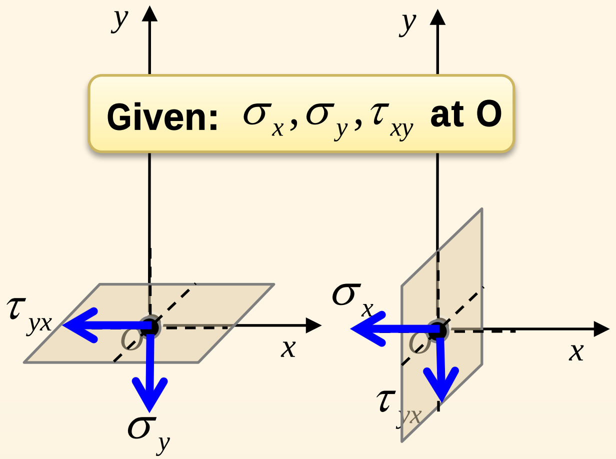
$$\tau_{x'y'} = l\bar{Y} - m\bar{X}$$

Using (1)

$$= l(l\tau_{xy} + m\sigma_y) - m(l\sigma_x + m\tau_{xy})$$

$$= \left(l^2 - m^2\right)\tau_{xy} + lm\left(\sigma_y - \sigma_x\right)$$

Specification of Stress At a Point (xy axis)


Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**

✓**Direction Cosine**

$$\cos\alpha = \cos Nx = l, \quad \cos(90-\alpha) = \cos Ny = m$$

✓**From force equilibrium**

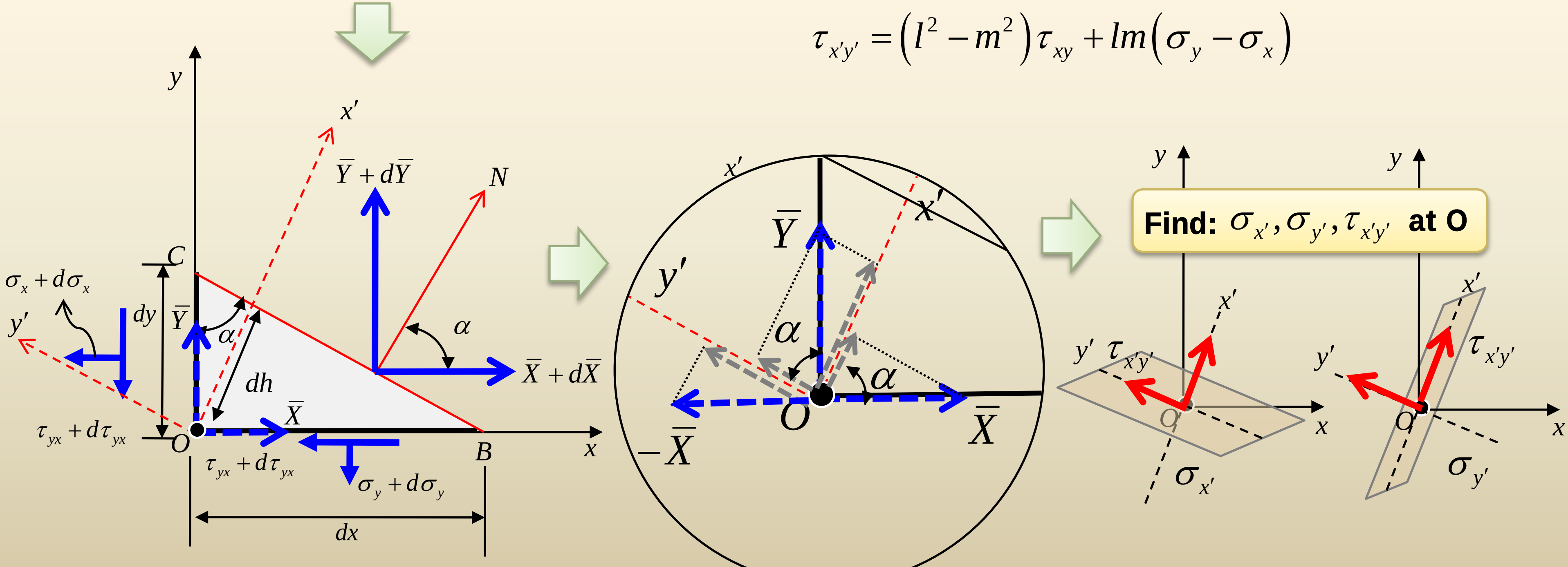
$$\bar{X} = l\sigma_x + m\tau_{xy}, \quad \bar{Y} = l\tau_{xy} + m\sigma_y \quad \cdots(1)$$

✓**From relation between** $\bar{X}, \bar{Y}$ **and** x', y' **axis**

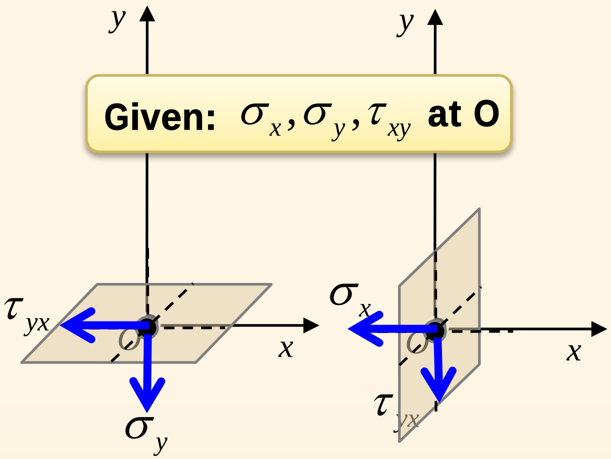
$$\sigma_{x'} = l\bar{X} + m\bar{Y}, \ \tau_{x'y'} = l\bar{Y} - m\bar{X} \quad \cdots(2)$$

$$\therefore \sigma_{x'} = l^2\sigma_x + m^2\sigma_y + 2lm\tau_{xy}$$

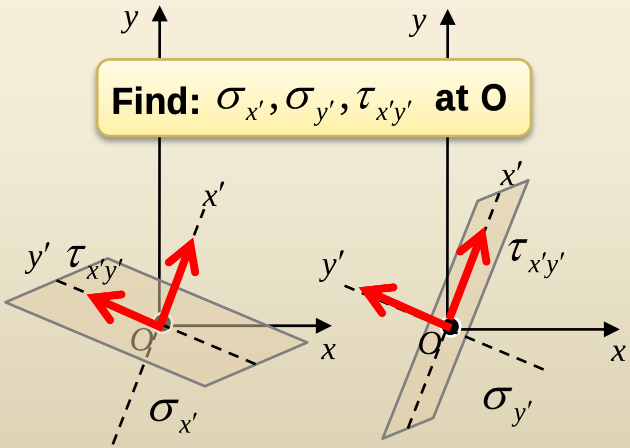
$$\tau_{x'y'} = \left(l^2 - m^2\right)\tau_{xy} + lm\left(\sigma_y - \sigma_x\right)$$


Find: $\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}$ **at O**

Specification of Stress At a Point (xy axis)



Given: $\sigma_x, \sigma_y, \tau_{xy}$ **at O**



Find: $\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}$ **at O**

✓**Direction Cosine**

$$\cos\alpha = \cos Nx = l, \quad \cos(90-\alpha) = \cos Ny = m$$

✓**From force equilibrium**

$$\bar{X} = l\sigma_x + m\tau_{xy}, \quad \bar{Y} = l\tau_{xy} + m\sigma_y \quad \cdots(1)$$

✓**From relation between** $\bar{X}, \bar{Y}$ **and** x', y' **axis**

$$\sigma_{x'} = l\bar{X} + m\bar{Y}, \quad \tau_{x'y'} = l\bar{Y} - m\bar{X} \quad \cdots(2)$$

$$\therefore \sigma_{x'} = l^2\sigma_x + m^2\sigma_y + 2lm\tau_{xy}$$

$$\tau_{x'y'} = \left(l^2 - m^2\right)\tau_{xy} + lm\left(\sigma_y - \sigma_x\right)$$

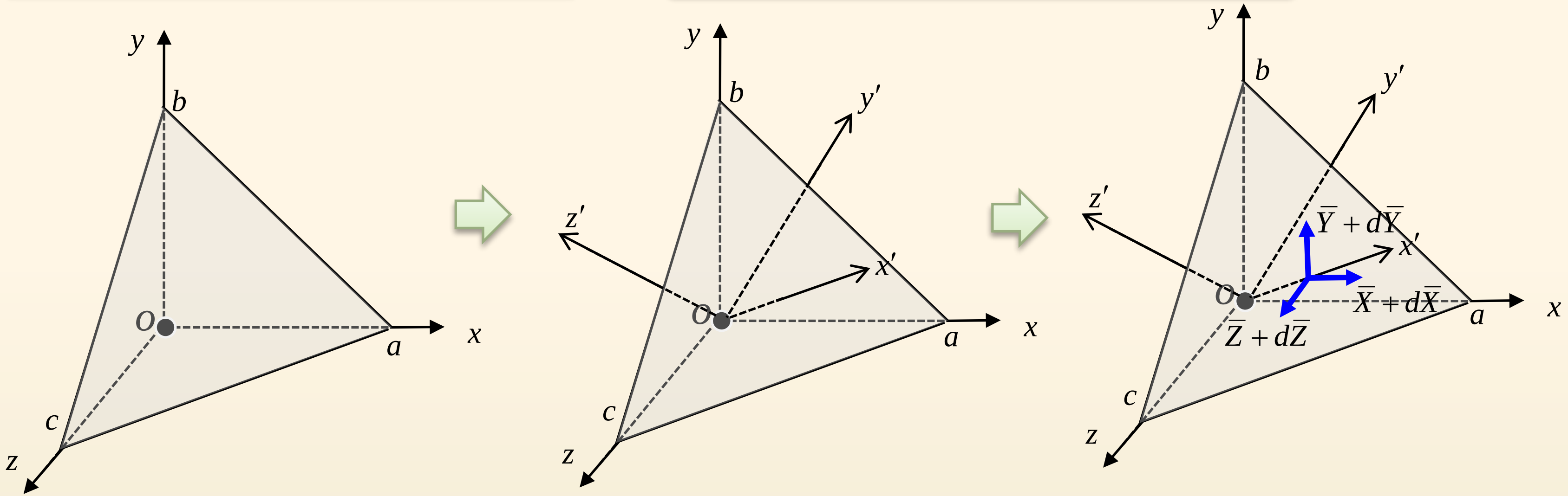
Stress at point O about x'y' frame expresses in terms of stress about xy frame at the same point

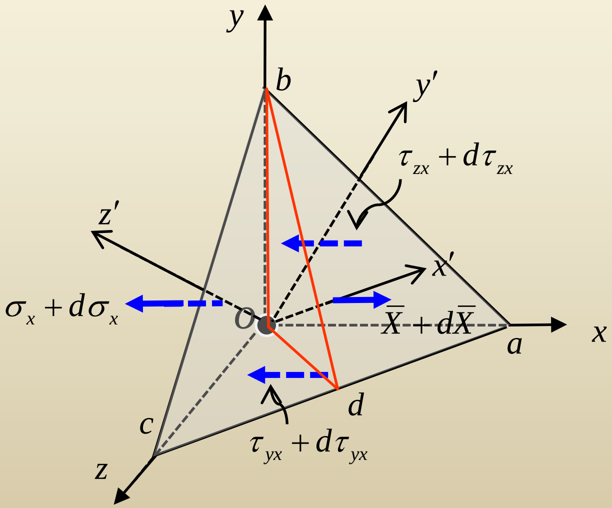
Extend to 3-Dimension

Specification of Stress At a Point (xyz axis)

Given: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **at O**

Find: $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ **at O**





✓**Body force in x-direction :** $X\left(\frac{1}{2}Adh\right)$

✓**Stress in x-direction :** $\bar{X}+d\bar{X}$ on Δabc, $\tau_{yx}+d\tau_{yx}$ on Δoca

$\sigma_x+d\sigma_x$ on Δobc, $\tau_{zx}+d\tau_{zx}$ on Δoab

✓**Area :** $\Delta abc = A$

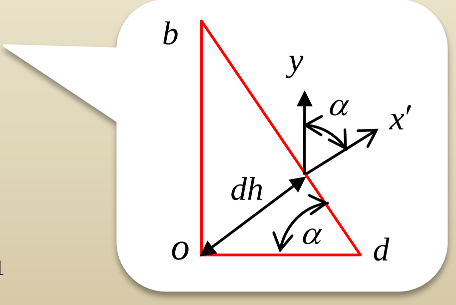
$$\cos\alpha = \cos x'y = m_1$$

$$\Delta oca = \Delta abc\cdot\cos\alpha = Am_1$$

In same way,

$$\Delta obc = \Delta abc\cdot\cos x'x = Al_1$$

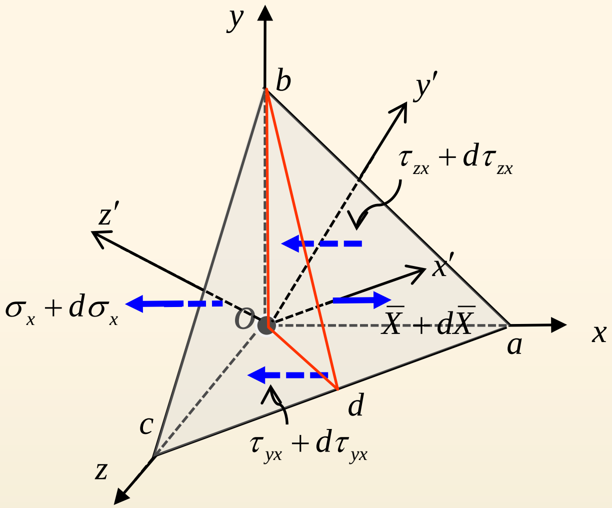
$$\Delta oab = \Delta abc\cdot\cos x'z = An_1$$



Specification of Stress At a Point (xyz axis)

Given: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **at O**

Find: $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ **at O**



✓**Body force in x-direction :** $X\left(\frac{1}{2}Adh\right)$

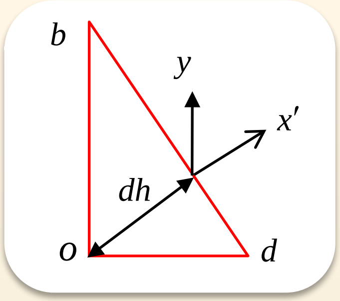
✓**Stress in x-direction :**

$\bar{X}+d\bar{X}$ on Δabc, $\quad \Delta abc = A$

$\sigma_x+d\sigma_x$ on Δobc, $\quad \Delta obc = Al_1$

$\tau_{yx}+d\tau_{yx}$ on Δoca, $\quad \Delta oca = Am_1$

$\tau_{zx}+d\tau_{zx}$ on Δoab, $\quad \Delta oab = An_1$



✓**Force Equilibrium Equation in x-direction**

$$\sum F_x = \left(\bar{X}+d\bar{X}\right)A-\left(\sigma_x+d\sigma_x\right)Al_1-\left(\tau_{yx}+d\tau_{yx}\right)Am_1-\left(\tau_{zx}+d\tau_{zx}\right)An_1-X\left(\frac{1}{2}Adh\right)=0$$

Divided by A, $dh \rightarrow 0$

$$\left(\bar{X}+\cancel{d\bar{X}}\right)-\left(\sigma_x+\cancel{d\sigma_x}\right)l_1-\left(\tau_{yx}+\cancel{d\tau_{yx}}\right)m_1-\left(\tau_{zx}+\cancel{d\tau_{zx}}\right)n_1-X\left(\frac{1}{2}\cancel{dh}\right)=0$$

$$\therefore \bar{X} = l_1\sigma_x + m_1\tau_{yx} + n_1\tau_{zx}$$

✓**In same way,**

$$\bar{Y} = l_2\tau_{yx} + m_2\sigma_x + n_2\tau_{yz}$$

$$\bar{Z} = l_3\tau_{zx} + m_3\tau_{yz} + n_3\sigma_x$$

✓**Directional Cosine**

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

ex)

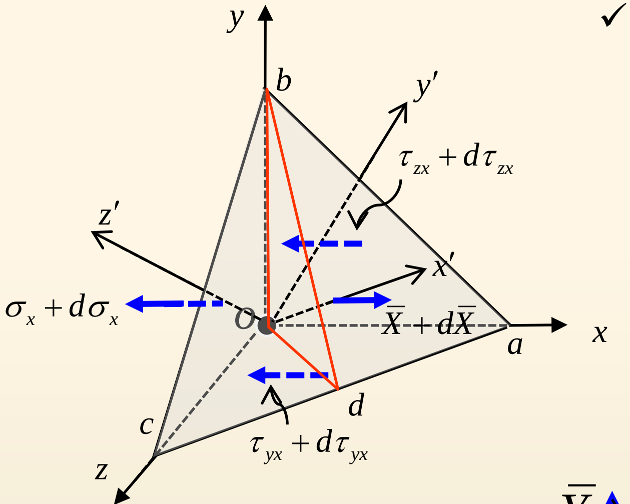
$l_1 = \cos x'x$

$m_2 = \cos y'y$

Specification of Stress At a Point (xyz axis)

Given: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **at O**

Find: $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ **at O**



✓**From force equilibrium**

$$\bar{X} = l_1\sigma_x + m_1\tau_{yx} + n_1\tau_{zx}$$
$$\bar{Y} = l_2\tau_{yx} + m_2\sigma_x + n_2\tau_{yz} \quad \cdots(1)$$
$$\bar{Z} = l_3\tau_{zx} + m_3\tau_{yz} + n_3\sigma_x$$

✓**Directional Cosine**

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

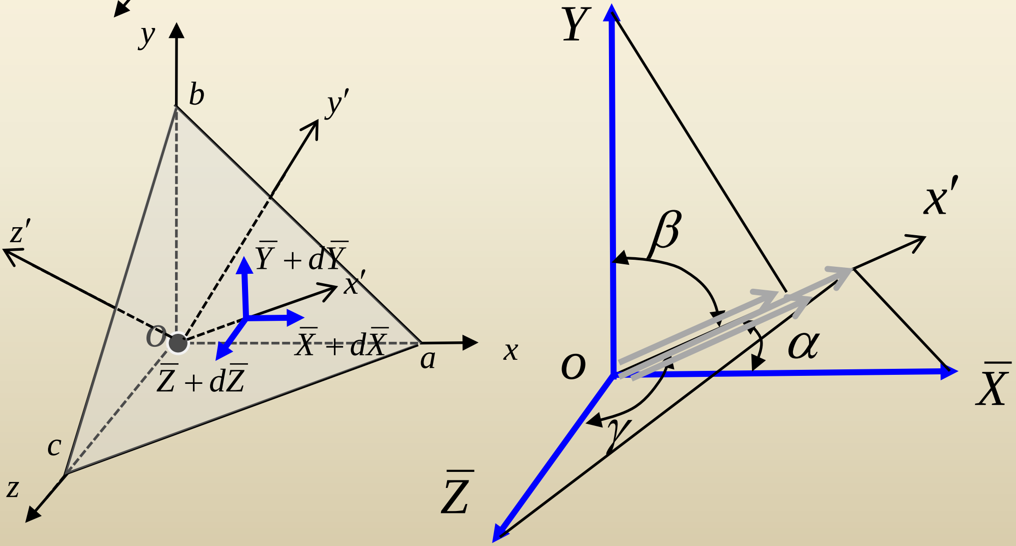
ex)

$l_1 = \cos x'x$

$m_2 = \cos y'y$

✓**From relation between** $\bar{X}, \bar{Y}, \bar{Z}$ **and** x' **axis**

$$\sigma_{x'} = \bar{X}\cos\alpha + \bar{Y}\cos\beta + \bar{Z}_1\cos\gamma$$
$$= \bar{X}\cos xx' + \bar{Y}\cos yx' + \bar{Z}_1\cos zx'$$
$$\therefore \sigma_{x'} = l_1\bar{X} + m_1\bar{Y} + n_1\bar{Z}_1$$



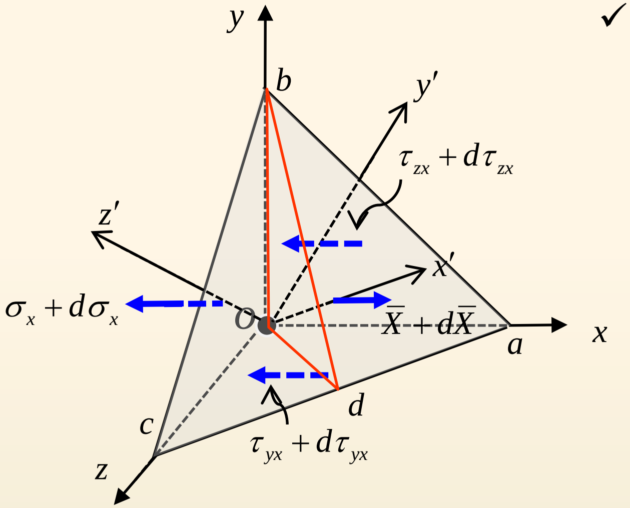
In same way,

$$\tau_{x'y'} = l_2\bar{X} + m_2\bar{Y} + n_2\bar{Z}$$

Specification of Stress At a Point (xyz axis)

Given: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **at O**

Find: $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ **at O**



✓**From force equilibrium**

$$\bar{X} = l_1\sigma_x + m_1\tau_{yx} + n_1\tau_{zx}$$
$$\bar{Y} = l_2\tau_{yx} + m_2\sigma_x + n_2\tau_{yz} \quad \cdots(1)$$
$$\bar{Z} = l_3\tau_{zx} + m_3\tau_{yz} + n_3\sigma_x$$

✓**Directional Cosine**

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

ex)

$l_1 = \cos x'x$

$m_2 = \cos y'y$

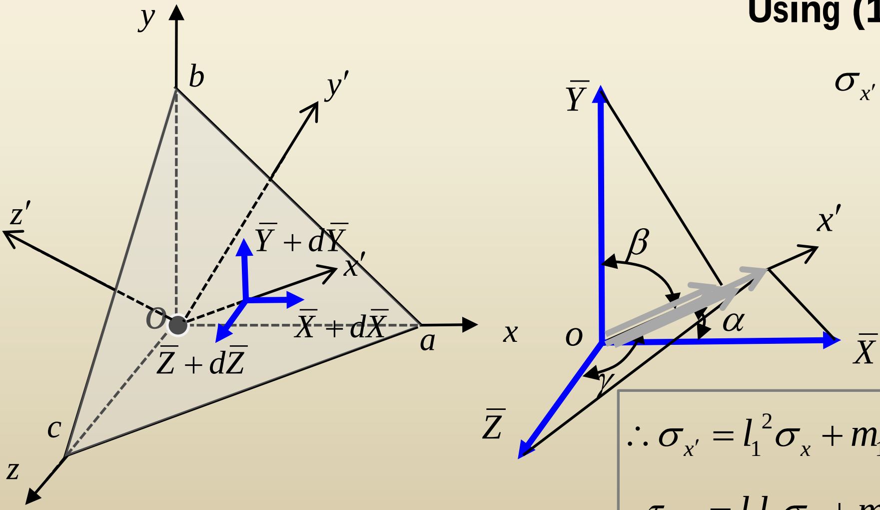
✓**From relation between** $\bar{X}, \bar{Y}, \bar{Z}$ **and** x' **axis**

$$\sigma_{x'} = l_1\bar{X} + m_1\bar{Y} + n_1\bar{Z}_1 \qquad \tau_{x'y'} = l_2\bar{X} + m_2\bar{Y} + n_2\bar{Z}$$

Using (1)

$$\sigma_{x'} = l_1(l_1\sigma_x + m_1\tau_{yx} + n_1\tau_{zx}) + m_1(l_2\tau_{yx} + m_2\sigma_x + n_2\tau_{yz}) + n_1(l_3\tau_{zx} + m_3\tau_{yz} + n_3\sigma_x)$$

$$\tau_{x'y'} = l_2\left(l_1\sigma_x + m_1\tau_{xy} + n_1\tau_{zx}\right) + m_2\left(l_1\tau_{xy} + m_1\sigma_y + n_1\tau_{yz}\right) + n_2\left(l_1\tau_{zx} + m_1\tau_{yz} + n_1\sigma_z\right)$$



$$\therefore \sigma_{x'} = l_1{}^2\sigma_x + m_1{}^2\sigma_y + n_1{}^2\sigma_z + 2l_1m_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zy}$$

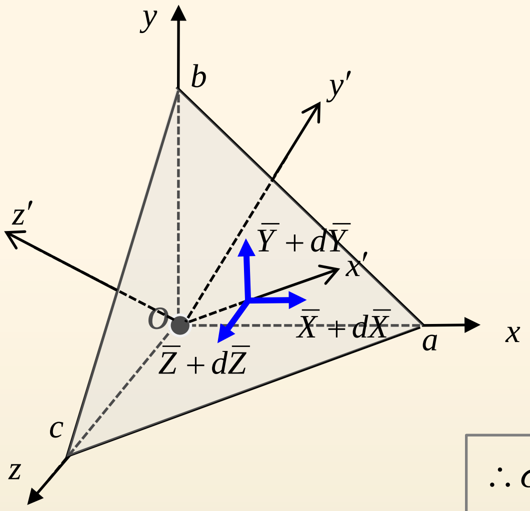
$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\sigma_z + \left(l_1m_2 + m_1l_2\right)\tau_{xy} + \left(m_1n_2 + n_1m_2\right)\tau_{yz} + \left(n_1l_2 + l_1n_2\right)\tau_{zx}$$

Stress at point O about x′y′z′ frame expresses in terms of stress about xyz frame at the same point

Specification of Stress At a Point (xyz axis)

Given: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **at O**

Find: $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ **at O**



✓**From force equilibrium**

$$\bar{X} = l_1\sigma_x + m_1\tau_{yx} + n_1\tau_{zx}$$
$$\bar{Y} = l_2\tau_{yx} + m_2\sigma_x + n_2\tau_{yz} \quad \cdots(1)$$
$$\bar{Z} = l_3\tau_{zx} + m_3\tau_{yz} + n_3\sigma_x$$

✓**Directional Cosine**

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

$ex)$

$$l_1 = \cos x'x$$
$$m_2 = \cos y'y$$

✓**From relation between** $\bar{X}, \bar{Y}, \bar{Z}$ **and** x' **axis**

$$\sigma_{x'} = l_1\bar{X} + m_1\bar{Y} + n_1\bar{Z}_1 \qquad \tau_{x'y'} = l_2\bar{X} + m_2\bar{Y} + n_2\bar{Z} \cdots(2)$$

$$\therefore \sigma_{x'} = l_1{}^2\sigma_x + m_1{}^2\sigma_y + n_1{}^2\sigma_z + 2l_1m_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zy}$$
$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\sigma_z + (l_1m_2 + m_1l_2)\tau_{xy} + (m_1n_2 + n_1m_2)\tau_{yz} + (n_1l_2 + l_1n_2)\tau_{zx}$$

Stress at point O about x′y′z′ frame expresses in terms of stress about xyz frame at the same point

In same way,

$$\sigma_y{}' = l_2{}^2\sigma_x + m_2{}^2\sigma_y + n_2{}^2\sigma_z + 2l_2m_2\tau_{xy} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{xz}$$
$$\sigma_z{}' = l_3{}^2\sigma_x + m_3{}^2\sigma_y + n_3{}^2\sigma_z + 2l_3m_3\tau_{xy} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx}$$
$$\tau_{y'z'} = l_2l_3\sigma_x + m_2m_3\sigma_y + n_2n_3\sigma_z + (l_2m_3 + m_2l_3)\tau_{xy} + (m_2n_3 + n_2m_3)\tau_{yz} + (n_2l_3 + l_2n_3)\tau_{zx}$$
$$\tau_{z'x'} = l_3l_1\sigma_x + m_3m_1\sigma_y + n_3n_1\sigma_z + (l_3m_1 + m_3l_1)\tau_{xy} + (m_3n_1 + n_3m_1)\tau_{yz} + (n_3l_1 + l_3n_1)\tau_{zx}$$

Specification of Stress At a Point (xyz axis)

Given: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **at O**

Find: $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ **at O**

Stress at point O about x′y′z′ frame expresses in terms of stress about xyz frame at the same point

✓Directional Cosine

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

$ex)$

$l_1 = \cos x'x$

$m_2 = \cos y'y$

$$\sigma_{x'} = l_1^2\sigma_x + m_1^2\sigma_y + n_1^2\sigma_z + 2l_1m_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zx}$$

$$\sigma_{y'} = l_2^2\sigma_x + m_2^2\sigma_y + n_2^2\sigma_z + 2l_2m_2\tau_{xy} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{zx}$$

$$\sigma_{z'} = l_3^2\sigma_x + m_3^2\sigma_y + n_3^2\sigma_z + 2l_3m_3\tau_{xy} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx}$$

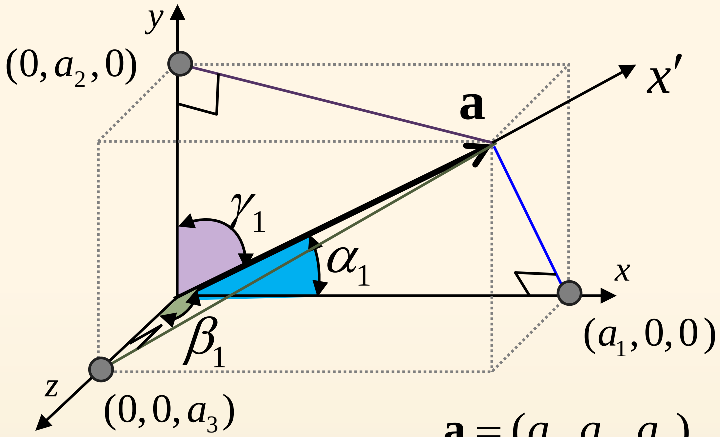
$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\sigma_z + \left(l_1m_2 + m_1l_2\right)\tau_{xy} + \left(m_1n_2 + n_1m_2\right)\tau_{yz} + \left(n_1l_2 + l_1n_2\right)\tau_{zx}$$

$$\tau_{y'z'} = l_2l_3\sigma_x + m_2m_3\sigma_y + n_2n_3\sigma_z + \left(l_2m_3 + m_2l_3\right)\tau_{xy} + \left(m_2n_3 + n_2m_3\right)\tau_{yz} + \left(n_2l_3 + l_2n_3\right)\tau_{zx}$$

$$\tau_{z'x'} = l_3l_1\sigma_x + m_3m_1\sigma_y + n_3n_1\sigma_z + \left(l_3m_1 + m_3l_1\right)\tau_{xy} + \left(m_3n_1 + n_3m_1\right)\tau_{yz} + \left(n_3l_1 + l_3n_1\right)\tau_{zx}$$

Specification of Stress At a Point (xyz axis)

Property of Directional Cosine



$$\mathbf{a}=(a_1,a_2,a_3)$$

✓**Directional Cosine**

	x	y	z
x'	$l_1=\cos\alpha_1$	$m_1=\cos\beta_1$	$n_1=\cos\gamma_1$
y'	$l_2=\cos\alpha_2$	$m_2=\cos\beta_2$	$n_2=\cos\gamma_2$
z'	$l_3=\cos\alpha_3$	$m_3=\cos\beta_3$	$n_3=\cos\gamma_3$

ex)

$$l_1=\cos x'x$$

$$m_2=\cos y'y$$

$$a_1=|\mathbf{a}|\cos\alpha_1 \quad =|\mathbf{a}|\cos x'x \quad =|\mathbf{a}|\,l_1$$

$$a_2=|\mathbf{a}|\cos\beta_1 \quad =|\mathbf{a}|\cos x'y \quad =|\mathbf{a}|\,m_1$$

$$a_3=|\mathbf{a}|\cos\gamma_1 \quad =|\mathbf{a}|\cos x'z \quad =|\mathbf{a}|\,n_1$$

⇨

$$l_1=a_1/|\mathbf{a}|$$

$$m_1=a_2/|\mathbf{a}|$$

$$n_1=a_3/|\mathbf{a}|$$

Unit tangent vector of x'

$$\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{1}{|\mathbf{a}|}(a_1,a_2,a_3)=(l_1,m_1,n_1)$$ ⇨ **Length of Unit Tangent vector :** $$\left|\frac{\mathbf{a}}{|\mathbf{a}|}\right|=1$$ ⇨ $$\therefore l_1^2+m_1^2+n_1^2=1$$

In same way,

$$l_2^2+m_2^2+n_2^2=1$$

$$l_3^2+m_3^2+n_3^2=1$$

Unit tangent vector of y' **:** $$\frac{\mathbf{b}}{|\mathbf{b}|}=\frac{1}{|\mathbf{b}|}(b_1,b_2,b_3)=(l_2,m_2,n_2)$$

Unit tangent vector of z' **:** $$\frac{\mathbf{c}}{|\mathbf{c}|}=\frac{1}{|\mathbf{c}|}(c_1,c_2,c_3)=(l_3,m_3,n_3)$$

Since x',y',z' are orthogonal

$$(l_1,m_1,n_1)\bullet(l_2,m_2,n_2)=0$$

$$(l_2,m_2,n_2)\bullet(l_3,m_3,n_3)=0$$

$$(l_3,m_3,n_3)\bullet(l_1,m_1,n_1)=0$$

⇨

$$l_1l_2+m_1m_2+n_1n_2=0$$

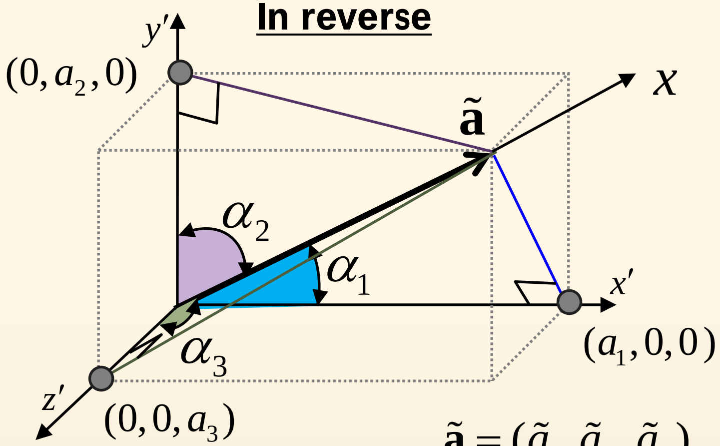
$$l_2l_3+m_2m_3+n_2n_3=0$$

$$l_3l_1+m_3m_1+n_3n_1=0$$

Specification of Stress At a Point (xyz axis)

Property of Directional Cosine

In reverse



$$\tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$$

✓**Directional Cosine**

	x	y	z
x'	$l_1 = \cos\alpha_1$	$m_1 = \cos\beta_1$	$n_1 = \cos\gamma_1$
y'	$l_2 = \cos\alpha_2$	$m_2 = \cos\beta_2$	$n_2 = \cos\gamma_2$
z'	$l_3 = \cos\alpha_3$	$m_3 = \cos\beta_3$	$n_3 = \cos\gamma_3$

ex)

$$l_1 = \cos x'x$$

$$m_2 = \cos y'y$$

$$\tilde{a}_1 = |\tilde{\mathbf{a}}|\cos\alpha_1 = |\tilde{\mathbf{a}}|\cos xx' = |\tilde{\mathbf{a}}| l_1$$

$$\tilde{a}_2 = |\tilde{\mathbf{a}}|\cos\alpha_2 = |\tilde{\mathbf{a}}|\cos xy' = |\tilde{\mathbf{a}}| l_2$$

$$\tilde{a}_3 = |\tilde{\mathbf{a}}|\cos\alpha_3 = |\tilde{\mathbf{a}}|\cos xz' = |\tilde{\mathbf{a}}| l_3$$

⇨

$$l_1 = \tilde{a}_1 / |\tilde{\mathbf{a}}|$$

$$m_1 = \tilde{a}_2 / |\tilde{\mathbf{a}}|$$

$$n_1 = \tilde{a}_3 / |\tilde{\mathbf{a}}|$$

Unit tangent vector of x

$$\frac{\tilde{\mathbf{a}}}{|\tilde{\mathbf{a}}|} = \frac{1}{|\tilde{\mathbf{a}}|}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = (l_1, l_2, l_3)$$

⇨ **Length of Unit Tangent vector :** $\left|\frac{\mathbf{a}}{|\mathbf{a}|}\right| = 1$ ⇨ $\therefore l_1^2 + l_2^2 + l_3^2 = 1$

In same way,

$$m_1^2 + m_2^2 + m_3^2 = 1$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

Unit tangent vector of y : $\frac{\tilde{\mathbf{b}}}{|\tilde{\mathbf{b}}|} = \frac{1}{|\tilde{\mathbf{b}}|}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = (m_1, m_2, m_3)$

Unit tangent vector of z : $\frac{\tilde{\mathbf{c}}}{|\tilde{\mathbf{c}}|} = \frac{1}{|\tilde{\mathbf{c}}|}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) = (n_1, n_2, n_3)$

Since x,y,z are orthogonal

$$(l_1, l_2, l_3) \bullet (m_1, m_2, m_3) = 0$$

$$(m_1, m_2, m_3) \bullet (n_1, n_2, n_3) = 0$$

$$(n_1, n_2, n_3) \bullet (l_1, l_2, l_3) = 0$$

⇨

$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$$

$$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$$

$$n_1 l_1 + n_2 l_2 + n_3 l_3 = 0$$

Stress-Strain Relation

Specification of Stress At a Point (xyz axis)

Given: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **at O**

Find: $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}, \tau_{z'x'}$ **at O**

Stress at point O about x'y'z' frame expresses in terms of stress about xyz frame at the same point

$$\sigma_{x'} = l_1{}^2\sigma_x + m_1{}^2\sigma_y + n_1{}^2\sigma_z + 2l_1m_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zx}$$

$$\sigma_{y'} = l_2{}^2\sigma_x + m_2{}^2\sigma_y + n_2{}^2\sigma_z + 2l_2m_2\tau_{xy} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{zx}$$

$$\sigma_{z'} = l_3{}^2\sigma_x + m_3{}^2\sigma_y + n_3{}^2\sigma_z + 2l_3m_3\tau_{xy} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx}$$

$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\sigma_z + \left(l_1m_2 + m_1l_2\right)\tau_{xy} + \left(m_1n_2 + n_1m_2\right)\tau_{yz} + \left(n_1l_2 + l_1n_2\right)\tau_{zx}$$

$$\tau_{y'z'} = l_2l_3\sigma_x + m_2m_3\sigma_y + n_2n_3\sigma_z + \left(l_2m_3 + m_2l_3\right)\tau_{xy} + \left(m_2n_3 + n_2m_3\right)\tau_{yz} + \left(n_2l_3 + l_2n_3\right)\tau_{zx}$$

$$\tau_{z'x'} = l_3l_1\sigma_x + m_3m_1\sigma_y + n_3n_1\sigma_z + \left(l_3m_1 + m_3l_1\right)\tau_{xy} + \left(m_3n_1 + n_3m_1\right)\tau_{yz} + \left(n_3l_1 + l_3n_1\right)\tau_{zx}$$

✓Invariant Equation

$$\begin{aligned}\sigma_{x'} + \sigma_{y'} + \sigma_{z'} = &\ l_1{}^2\sigma_x + m_1{}^2\sigma_y + n_1{}^2\sigma_z + 2l_1m_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zy}\\ &+ l_2{}^2\sigma_x + m_2{}^2\sigma_y + n_2{}^2\sigma_z + 2l_2m_2\tau_{xy} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{xz}\\ &+ l_3{}^2\sigma_x + m_3{}^2\sigma_y + n_3{}^2\sigma_z + 2l_3m_3\tau_{xy} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx}\\ &= (1)\sigma_x + (1)\sigma_y + (1)\sigma_z + 2(0)\tau_{xy} + 2(0)\tau_{yz} + 2(0)\tau_{xz}\\ &= \sigma_x + \sigma_y + \sigma_z\end{aligned}$$

The quantity $\sigma_x + \sigma_y + \sigma_z$ is invariant with respect to orthogonal transformations of coordinates

✓Directional Cosine

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

$$l_1^2 + m_1^2 + n_1^2 = 1$$
$$l_2^2 + m_2^2 + n_2^2 = 1$$
$$l_3^2 + m_3^2 + n_3^2 = 1$$
$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$l_2l_3 + m_2m_3 + n_2n_3 = 0$$
$$l_3l_1 + m_3m_1 + n_3n_1 = 0$$

$$l_1^2 + l_2^2 + l_3^2 = 1$$
$$m_1^2 + m_2^2 + m_3^2 = 1$$
$$n_1^2 + n_2^2 + n_3^2 = 1$$
$$l_1m_1 + l_2m_2 + l_3m_3 = 0$$
$$m_1n_1 + m_2n_2 + m_3n_3 = 0$$
$$n_1l_1 + n_2l_2 + n_3l_3 = 0$$

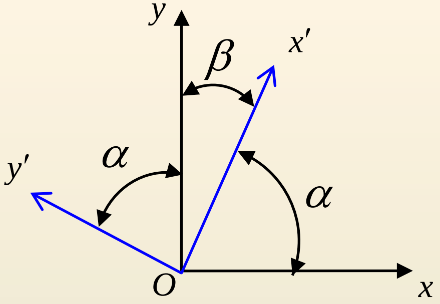
Transformation Equations for Plane Stress

✓**Plane Stress :** $\sigma_x, \sigma_y, \tau_{xy},$ **and** τ_{yx} **are independent of** z **and the other components are zero**

the body force X and Y are independent of z **and Z is zero**

$$\sigma_{x'} = l_1{}^2\sigma_x + m_1{}^2\sigma_y + n_1{}^2\cancel{\sigma_z} + 2l_1m_1\tau_{xy} + 2m_1n_1\cancel{\tau_{yz}} + 2n_1l_1\cancel{\tau_{zy}}$$

$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\cancel{\sigma_z} + \left(l_1m_2 + m_1l_2\right)\tau_{xy} + \left(m_1n_2 + n_1m_2\right)\cancel{\tau_{yz}} + \left(n_1l_2 + l_1n_2\right)\cancel{\tau_{zx}}$$



$$\cos xx' = l_1 = \cos\alpha$$

$$\cos yx' = m_1 = \cos\beta = \cos(90-\alpha) = \sin\alpha$$

$$\cos yy' = m_2 = \cos\alpha$$

$$\cos xy' = l_2 = \cos(90+\alpha) = -\sin\alpha$$

✓**Directional Cosine**

	x	y	$\cancel{z}$
x'	l_1	m_1	$\cancel{n_1}$
y'	l_2	m_2	$\cancel{n_2}$
$\cancel{z'}$	$\cancel{l_3}$	$\cancel{m_3}$	$\cancel{n_3}$

$$\sigma_{x'} = l_1{}^2\sigma_x + m_1{}^2\sigma_y + 2l_1m_1\tau_{xy} = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha$$

$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + \left(l_1m_2 + m_1l_2\right)\tau_{xy} = -\sigma_x\cos\alpha\sin\alpha + \sigma_y\sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\tau_{xy}$$

Transformation Equations for Plane Stress

✓**Plane Stress :** $\sigma_x, \sigma_y, \tau_{xy},$ **and** τ_{yx} **are independent of** z **and the other components are zero**

the body force X and Y are independent of z **and Z is zero**

$$\sigma_{x'} = l_1^{\,2}\sigma_x + m_1^{\,2}\sigma_y + 2l_1m_1\tau_{xy} = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha$$

$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + \left(l_1m_2 + m_1l_2\right)\tau_{xy} = -\sigma_x\cos\alpha\sin\alpha + \sigma_y\sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\tau_{xy}$$

$$\begin{aligned}\sigma_{x'} &= \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha\\ &= \sigma_x\frac{\left(1+\cos 2\alpha\right)}{2} + \sigma_y\frac{\left(1-\cos 2\alpha\right)}{2} + \tau_{xy}\sin 2\alpha\\ &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha\end{aligned}$$

$$\begin{aligned}\tau_{x'y'} &= \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right) + \left(\sigma_y - \sigma_x\right)\sin\alpha\cos\alpha\\ &= \tau_{xy}\left(\frac{1+\cos 2\alpha}{2} - \frac{1-\cos 2\alpha}{2}\right) - \frac{\left(\sigma_x-\sigma_y\right)}{2}\sin 2\alpha\\ &= -\frac{\left(\sigma_x-\sigma_y\right)}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha\\ &= \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha)\\ &= -1 + 2\cos^2\alpha\end{aligned}$$

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\begin{aligned}\cos 2\alpha &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha\\ &= \cos^2\alpha - \sin^2\alpha = (1-\sin^2\alpha) - \sin^2\alpha\\ &= 1 - 2\sin^2\alpha\end{aligned}$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$\sin 2\alpha = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 2\sin\alpha\cos\alpha$$

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = 2\left(\frac{\sin\alpha}{\cos\alpha}\right)\frac{1}{1-\left(\dfrac{\sin\alpha}{\cos\alpha}\right)^2} = \frac{\cos\alpha\sin\alpha}{\cos^2\alpha - \sin^2\alpha}$$

Transformation Equations for Plane Stress

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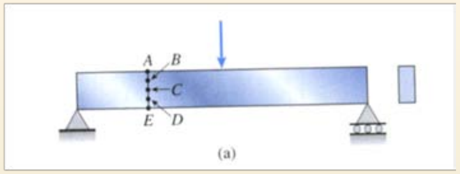
✓**Transformation Equations for Plane Stress**

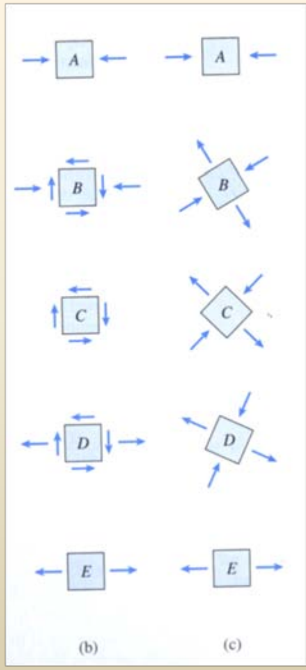
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

$$\tau_{x'y'} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

Stress at point O about x'y'z' frame expresses in terms of stress about xyz frame at the same point



Stress in beam of rectangular cross section*
(a) Simple beam with points A,B,C,D, and E
(b) Normal and shear stresses acting on stress elements at point A,B,C,D, and E
(c) Principal stress at point A,B,C,D, and E



*Gere J.M., Mechanics of materials, 6th Edition, International Student Edition, 2006, p557

Principal Direction & Principle Stress

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✓**principal stress* : the maximum and minimum normal stresses, called the *principal stresses*, can be found from the transformation equation for the normal stress**

From the differential calculus, we could obtain Min. or Max value in condition $\frac{\partial \sigma_{x'}}{\partial \alpha} = 0$

$$\frac{\partial \sigma_{x'}}{\partial \alpha} = \left(\sigma_y - \sigma_x\right) 2\cos\alpha \sin\alpha + 2\tau_{xy}\cos 2\alpha = \left(\sigma_y - \sigma_x\right) 2\cos\alpha \sin\alpha + 2\tau_{xy}(\cos^2\alpha - \sin^2\alpha)$$

$$\therefore \left(\sigma_y - \sigma_x\right) 2\cos\alpha \sin\alpha + 2\tau_{xy}(\cos^2\alpha - \sin^2\alpha) = 0$$

$$\frac{2\cos\alpha\sin\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad ⇨ \quad \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad ⇨ \quad \tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$\sigma_{x'}$ **is maximum or minimum at** $\alpha = \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$ ***principal angle***

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* James M.Gere, Mechanics of materials, 6th Edition, International Student Edition, 2006, p474-476

Principal Direction & Principle Stress

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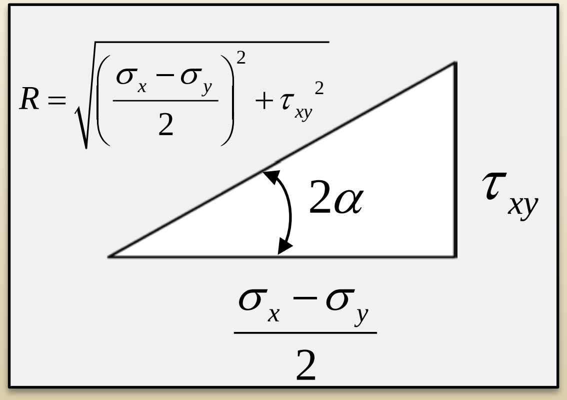

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

At principal angle $\cos 2\alpha = \dfrac{\sigma_x - \sigma_y}{2R}, \quad \sin 2\alpha = \dfrac{\tau_{xy}}{R}$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cdot\frac{\sigma_x - \sigma_y}{2R} + \tau_{xy}\cdot\frac{\tau_{xy}}{R} = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{R}\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]$$

$$= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

*** James M.Gere, Mechanics of materials, 6th Edition, International Student Edition, 2006, p474-476**

Principal Direction & Principle Stress

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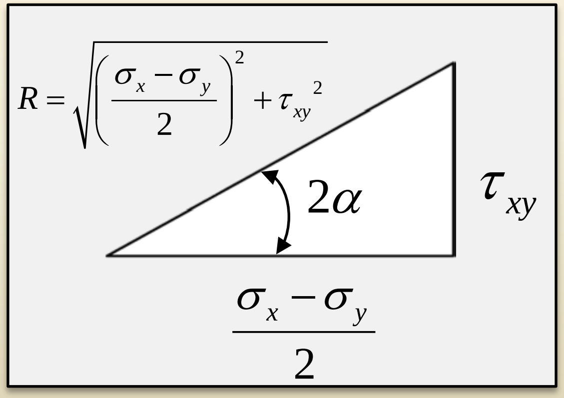
✓**Transformation Equations for Plane Stress**

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \qquad \tau_{x'y'} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

✓**principal stress* : the maximum and minimum normal stresses, called the *principal stresses*, can be found from the transformation equation for the normal stress**

From $\dfrac{\partial \sigma_{x'}}{\partial \alpha} = 0 \rightarrow \tan 2\alpha = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$ $, \sigma_{x'}$ **is maximum or minimum at** $\alpha = \dfrac{1}{2}\arctan\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$ ***Principal Angle***



$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

At principal angle $\sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$$\sigma_1 = \sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Magnitude of Principal Stress (max)

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

Recall, invariant relation

$$\sigma_2 = \sigma_x + \sigma_y - \sigma_1$$

$$\sigma_2 = \sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Magnitude of Principal Stress (min)

* James M.Gere, Mechanics of materials, 6th Edition, International Student Edition, 2006, p474-476

Principal Direction & Principle Stress

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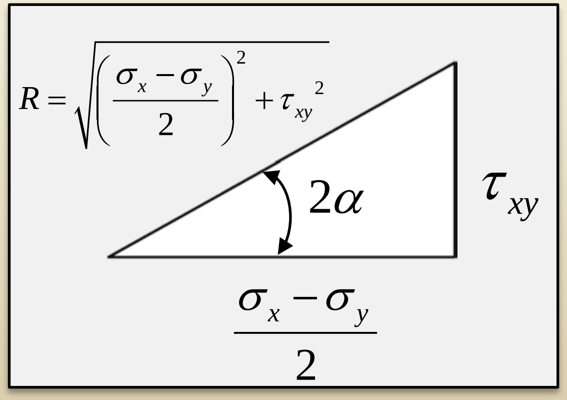
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \qquad \tau_{x'y'} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

✓**principal stress* : the maximum and minimum normal stresses, called the *<u>principal stresses</u>*, can be found from the transformation equation for the normal stress**

From the differential calculus, we could obtain Min. or Max value in condition $\dfrac{\partial \sigma_{x'}}{\partial \alpha} = 0$

$\tan 2\alpha = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$, $\sigma_{x'}$ **is maximum or minimum at** $\alpha = \dfrac{1}{2}\arctan\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$ ***<u>Principal Angle</u>***



$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

<u>At principal angle</u> $\cos 2\alpha = \dfrac{\sigma_x - \sigma_y}{2R}, \quad \sin 2\alpha = \dfrac{\tau_{xy}}{R}$

✓ **Shearing Stress in principal stress condition**

$$\tau_{x'y'} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\cdot\frac{\tau_{xy}}{R} + \tau_{xy}\cdot\frac{\sigma_x - \sigma_y}{2R} = 0$$

$\tau_{x'y'} = 0$ **in condition of principal stress**

* James M.Gere, Mechanics of materials, 6th Edition, International Student Edition, 2006, p474-476

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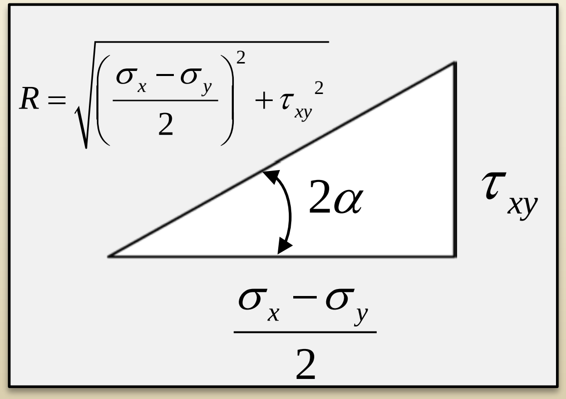

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$$\sigma_1 = \sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

Magnitude of Principal Stress(max)

$$\sigma_2 = \sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

Magnitude of Principal Stress(min)

$\tau_{x'y'} = 0$ **in condition of principal stress**

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* James M.Gere, Mechanics of materials, 6th Edition, International Student Edition, 2006, p474-476

Transformation of Strain

Specification of Strain at a Point

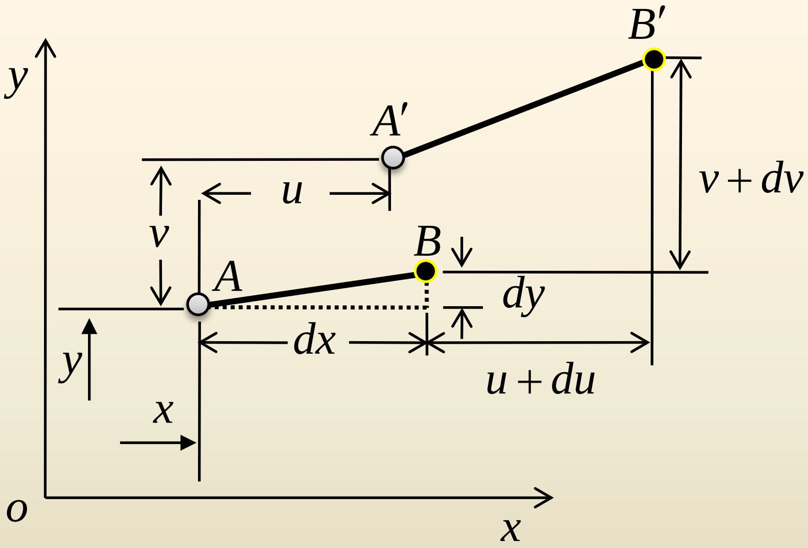
$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial x} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

✓**Specification of Strain at a Point**

Given the *three longitudinal strain components* **and the** *three shearing components at a point*, **it can be shown that** *the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated*



displacement of point A:

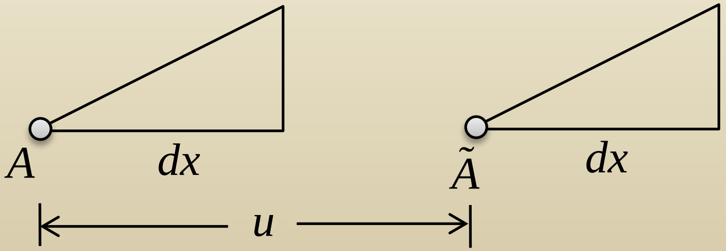
$$A=(x,y)\rightarrow A'=(x+u,\,y+v)$$

displacement of point B: $u+du,\,v+dv$

$du,\,dv\,?$

$$B=(x+dx,\,y+dy)\rightarrow B'=(x+dx+u+du,\,y+dy+v+dv)$$

1) x-direction



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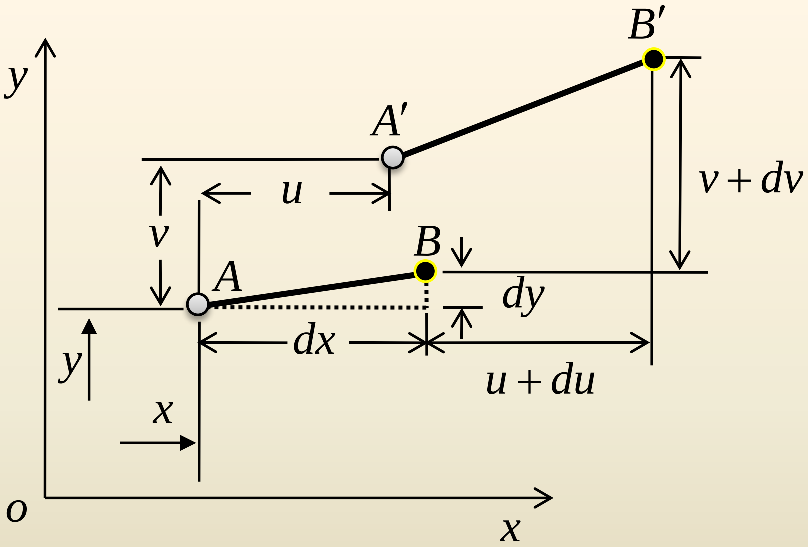


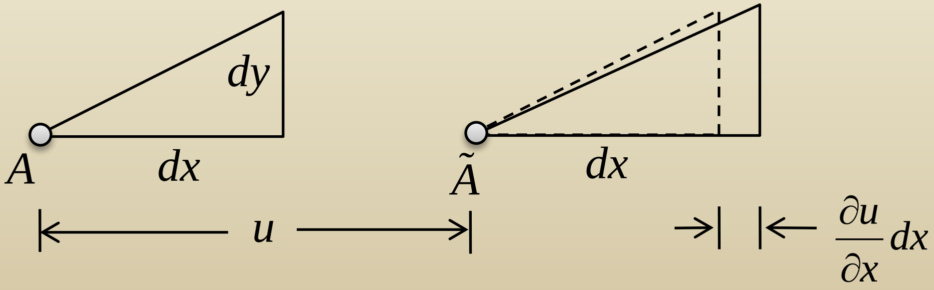
displacement of point A:

$$A = (x, y) \rightarrow A' = (x + u, y + v)$$

displacement of point B: $u + du, v + dv$ $du, dv?$

$$B = (x + dx, y + dy) \rightarrow B' = (x + dx + u + du, y + dy + v + dv)$$

1) x-direction



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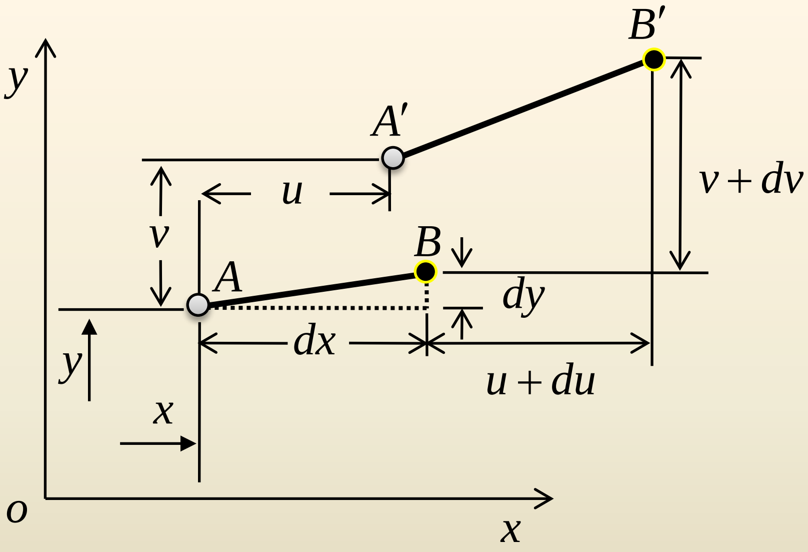

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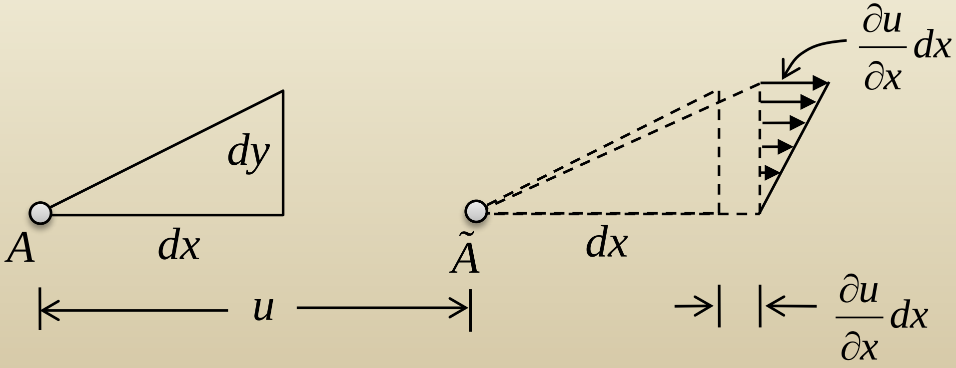

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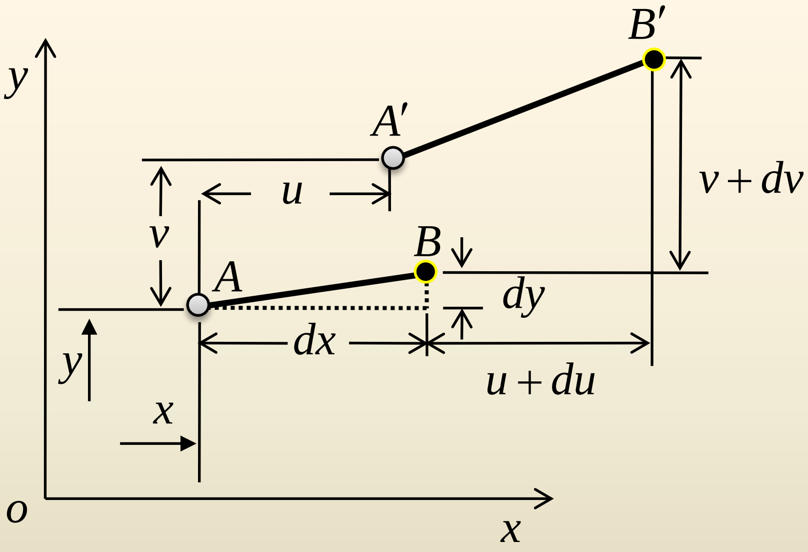

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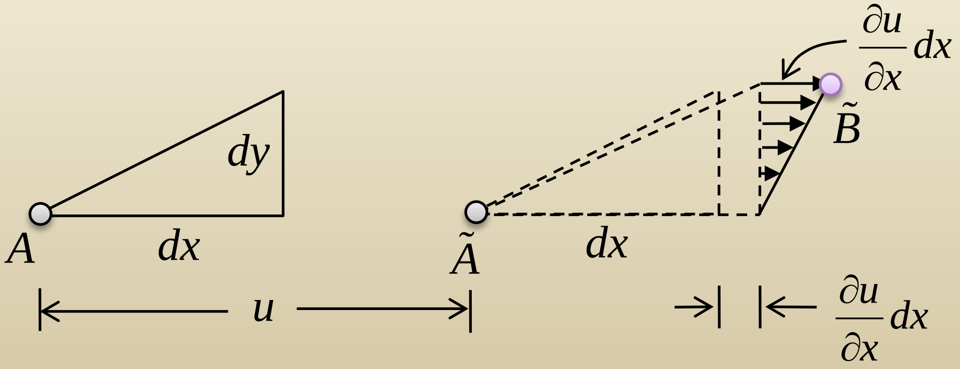
1) x-direction

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$



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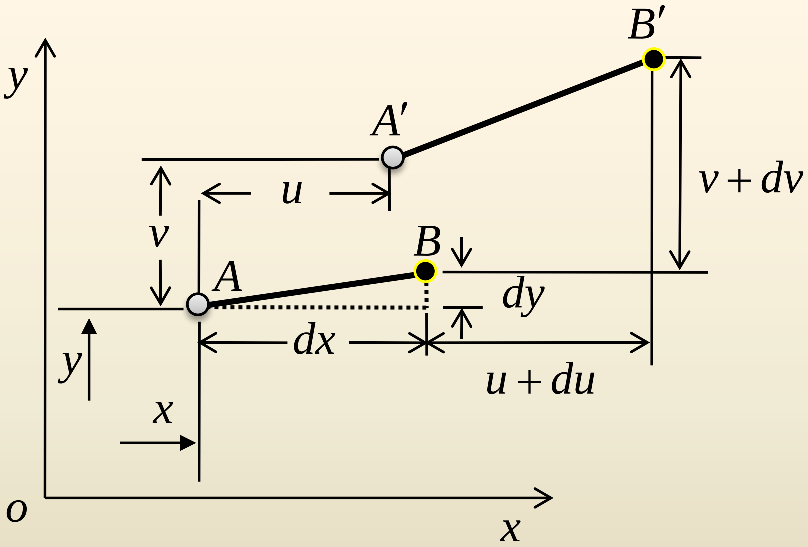

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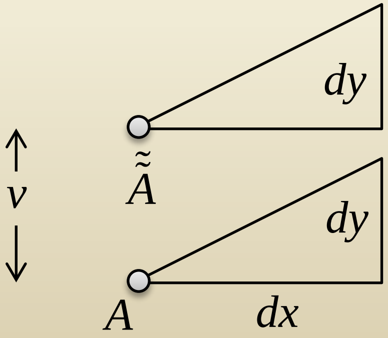
2) y-direction



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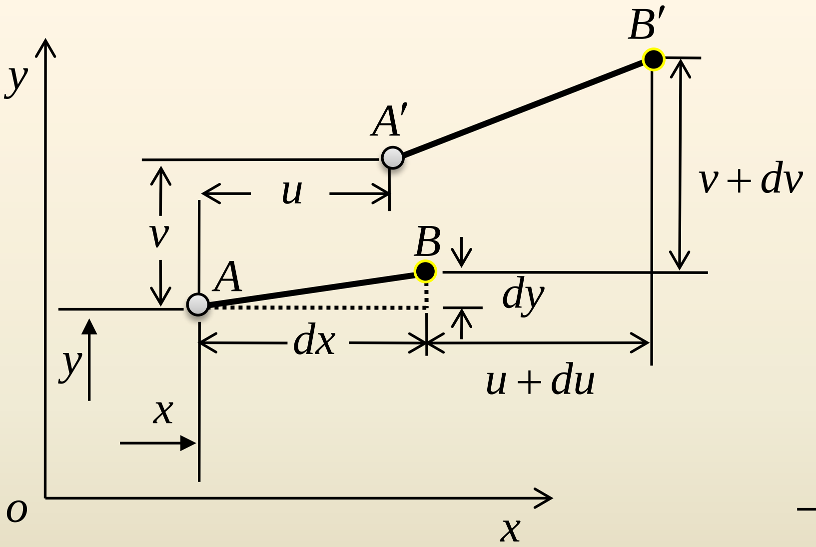

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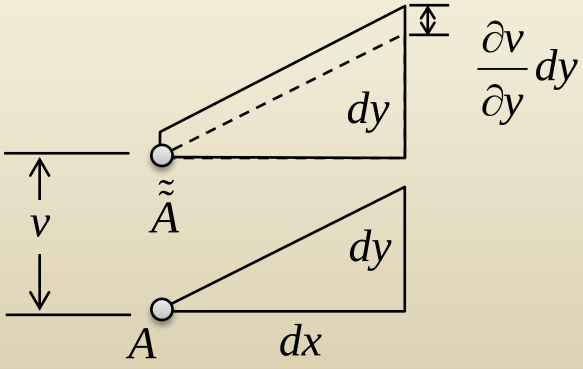
2) y-direction



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$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

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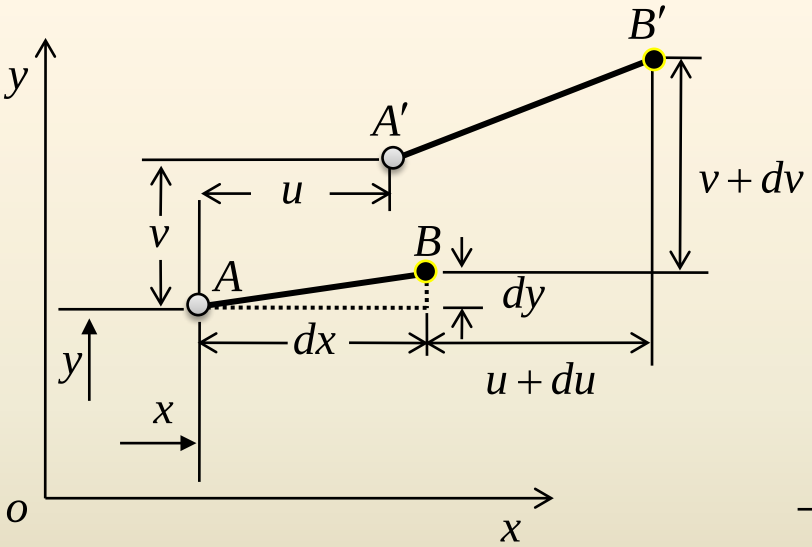
Given the *three longitudinal strain components* **and the** *three shearing components at a point*, **it can be shown that** *the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated*



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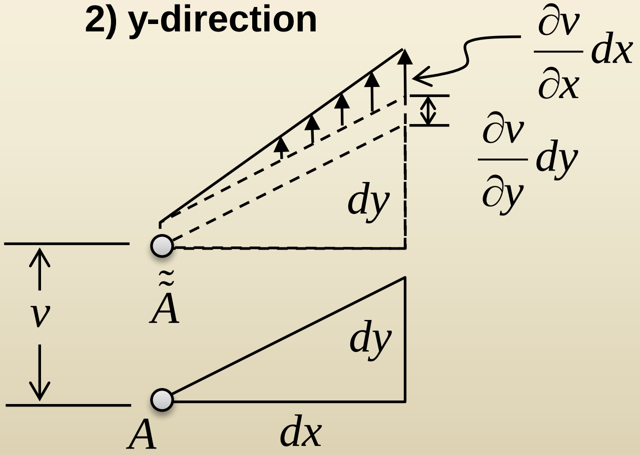

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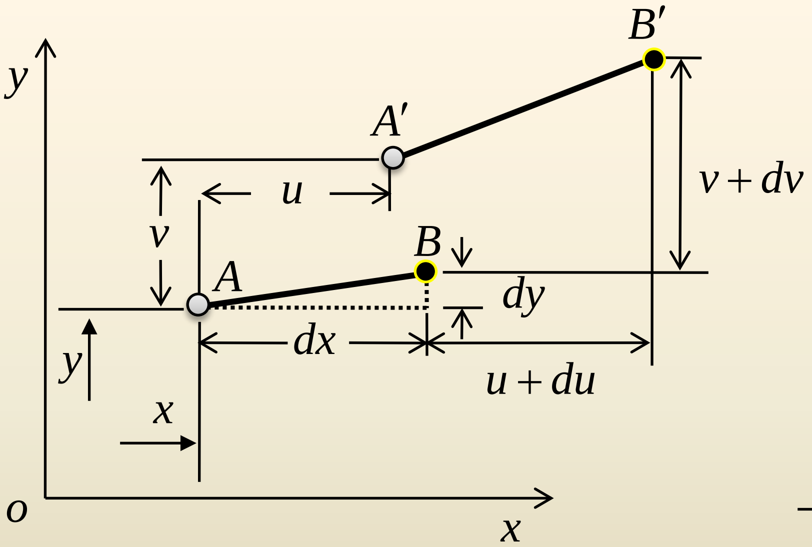

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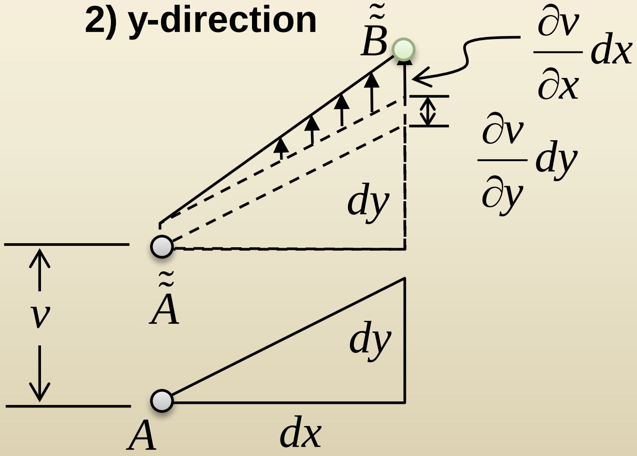

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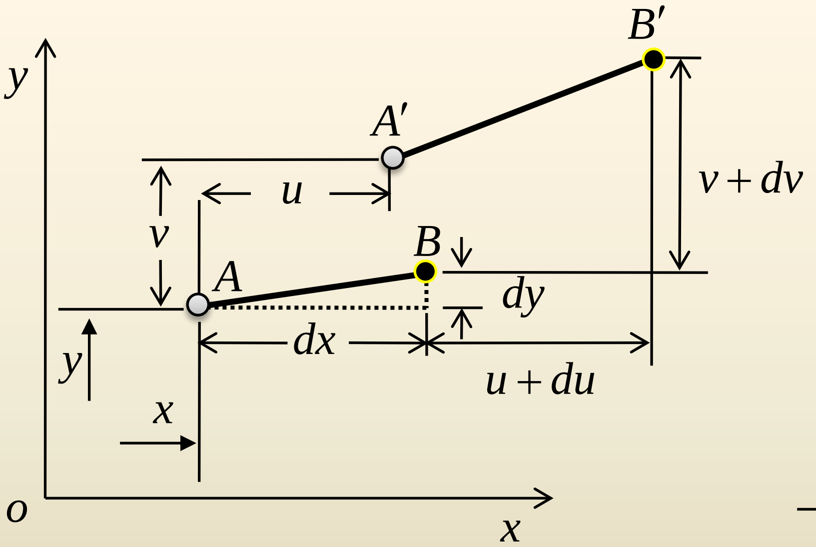

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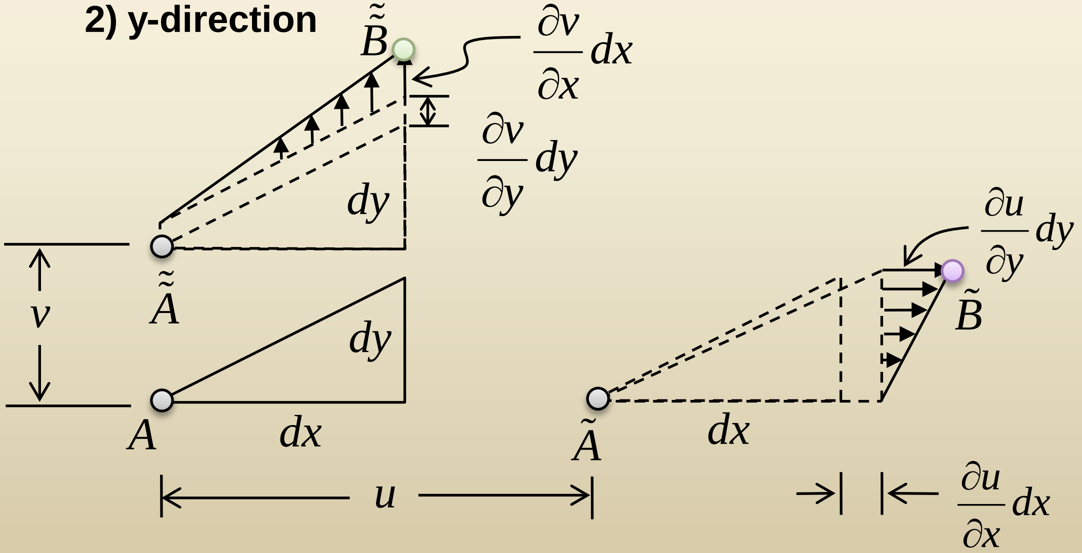

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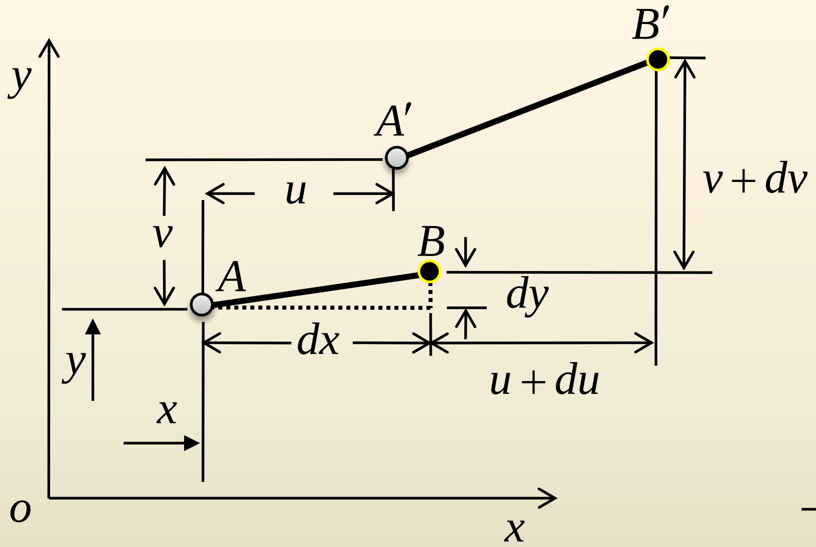

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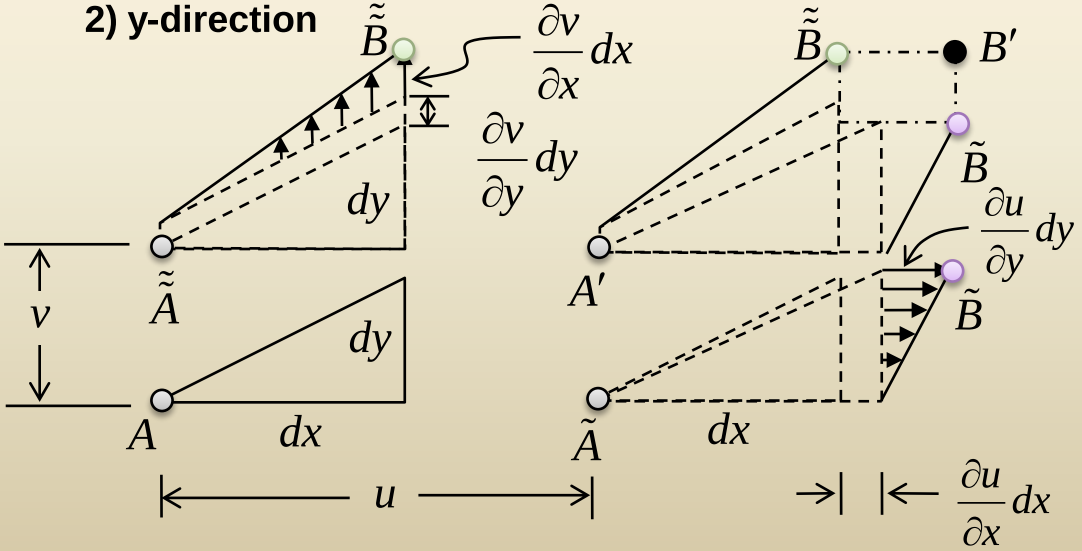

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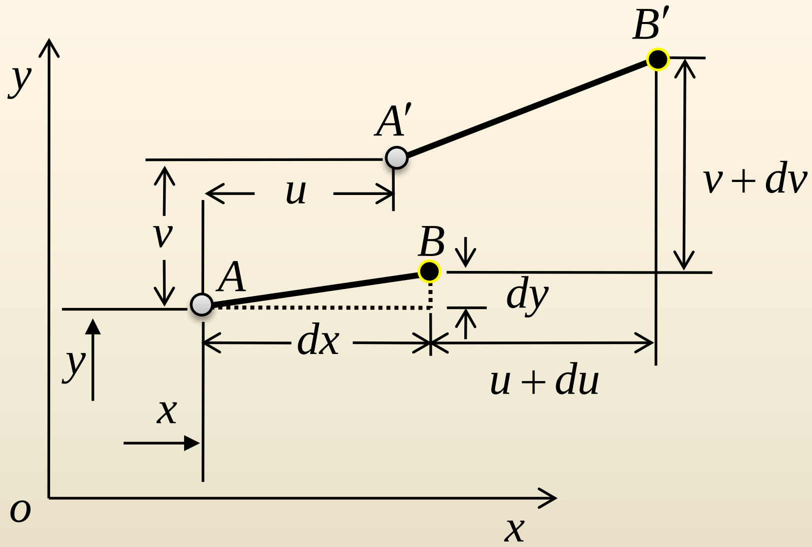

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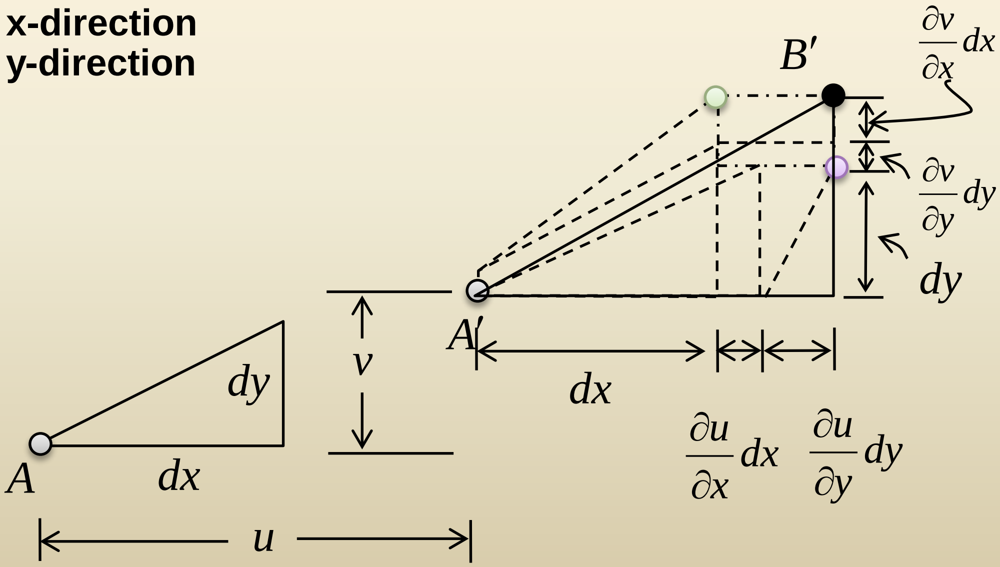

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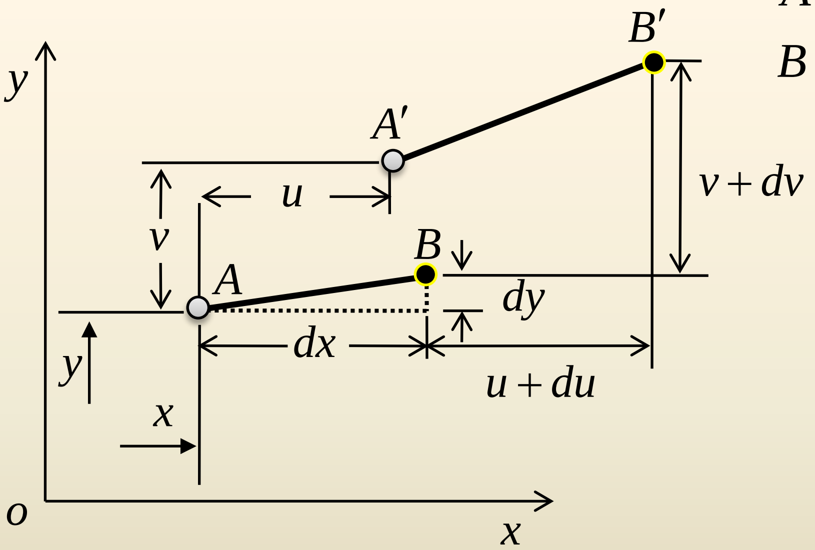

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Let ε **be the longitudinal strain in the direction AB :**

$$\varepsilon = \frac{A'B' - AB}{AB} = \frac{dL' - dL}{dL} \qquad \therefore dL' = \left(1+\varepsilon\right)dL$$

From $(dL')^2 = \left(\overline{A'B'}\right)^2$

L.H.S : $(dL')^2 = \left(1+\varepsilon\right)^2 dL^2 = \left(1+2\varepsilon+\varepsilon^2\right)dL^2$

R.H.S : $\left(\overline{A'B'}\right)^2 = \left(\overline{OB'} - \overline{OA'}\right)^2$

$$= (dx + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy)^2 + (dy + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy)^2$$

$$\therefore \left(1+2\varepsilon+\varepsilon^2\right)dL^2 = (dx + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy)^2 + (dy + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy)^2$$

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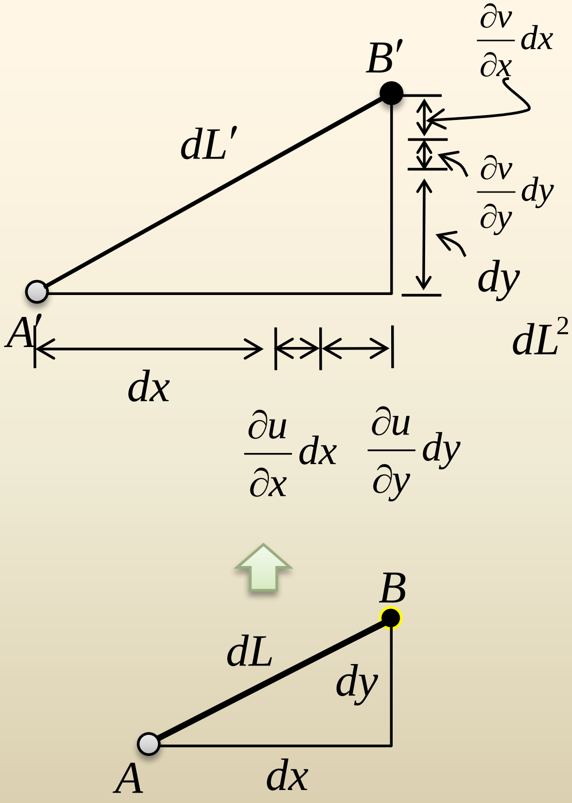

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$$dL^2 + 2\varepsilon\, dL^2 + \varepsilon^2 dL^2 = dx^2 + dy^2 + 2\frac{\partial u}{\partial x}dx^2 + 2\frac{\partial u}{\partial y}dxdy + 2\frac{\partial v}{\partial x}dxdy + 2\frac{\partial v}{\partial y}dy^2$$

$$+\left(\frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial u}{\partial y}\right)^2 dy^2 + \left(\frac{\partial v}{\partial x}\right)^2 dx^2 + \left(\frac{\partial v}{\partial y}\right)^2 dy^2$$

Since $dL^2 = dx^2 + dy^2$

$$2\varepsilon\, dL^2 + \varepsilon^2 dL^2 = 2\frac{\partial u}{\partial x}dx^2 + 2\frac{\partial u}{\partial y}dxdy + 2\frac{\partial v}{\partial x}dxdy + 2\frac{\partial v}{\partial y}dy^2$$

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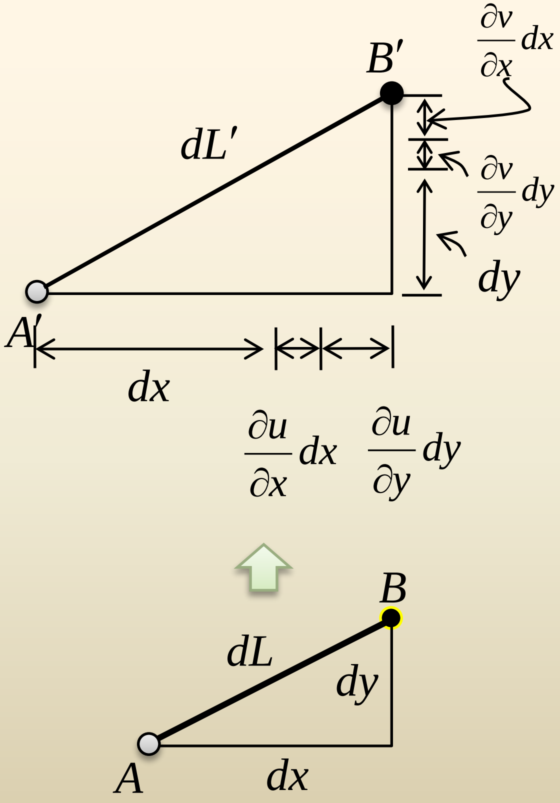

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Since $\varepsilon, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ **are infinitesimal quantities,**

$$2\varepsilon\, dL^2 = 2\frac{\partial u}{\partial x}dx^2+2\frac{\partial u}{\partial y}dxdy+2\frac{\partial v}{\partial x}dxdy+2\frac{\partial v}{\partial y}dy^2$$

$o(\varepsilon^4)$ **neglected and** $o(\varepsilon^3)$ **remained**

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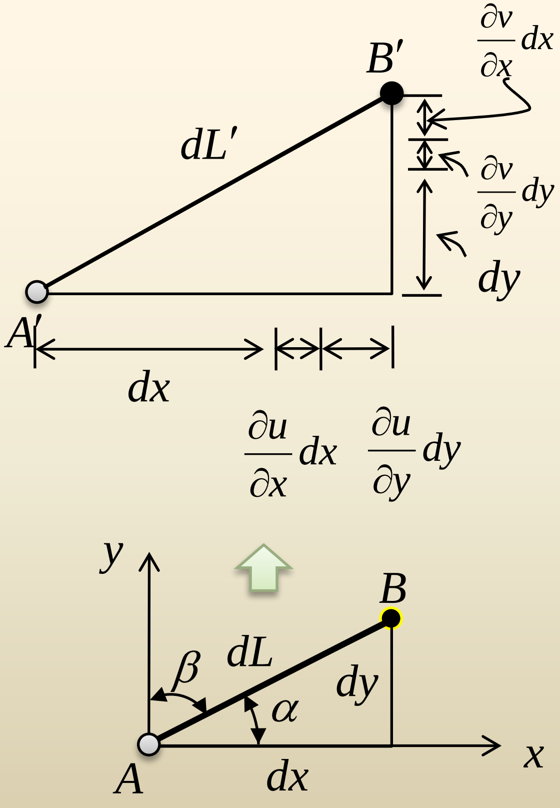

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$$2\varepsilon\, dL^2 = 2\frac{\partial u}{\partial x}dx^2 + 2\frac{\partial u}{\partial y}dxdy + 2\frac{\partial v}{\partial x}dxdy + 2\frac{\partial v}{\partial y}dy^2$$

divided by $2dL^2$

$$\varepsilon = \frac{\partial u}{\partial x}\left(\frac{dx}{dL}\right)^2 + \frac{\partial u}{\partial y}\frac{dx}{dL}\frac{dy}{dL} + \frac{\partial v}{\partial x}\frac{dx}{dL}\frac{dy}{dL} + \frac{\partial v}{\partial y}\left(\frac{dy}{dL}\right)^2$$

since $\frac{dx}{dL} = \cos\alpha, \quad \frac{dy}{dL} = \sin\alpha = \sin(90-\beta) = \cos\beta$

$$\varepsilon = \frac{\partial u}{\partial x}\cos^2\alpha + \frac{\partial u}{\partial y}\cos\alpha\sin\alpha + \frac{\partial v}{\partial x}\cos\alpha\sin\alpha + \frac{\partial v}{\partial y}\sin^2\alpha$$

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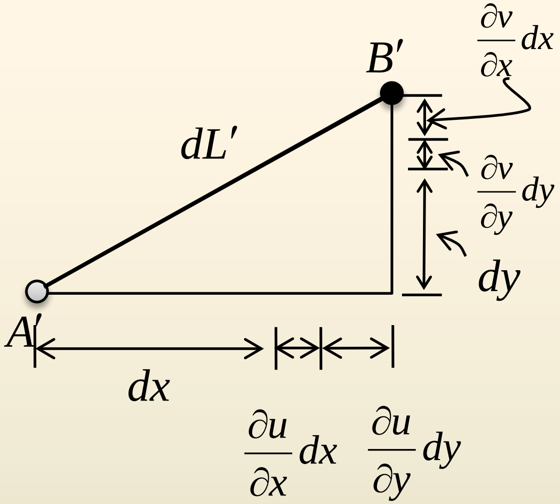

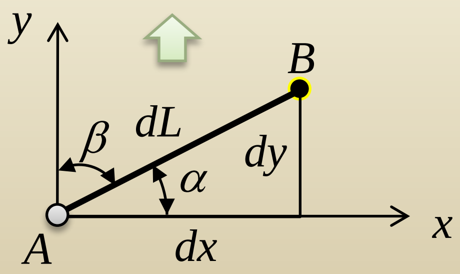

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directional cosine : $\cos\alpha = l, \ \cos\beta = m$

$$\varepsilon = \frac{\partial u}{\partial x}l^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)lm + \frac{\partial v}{\partial y}m^2$$

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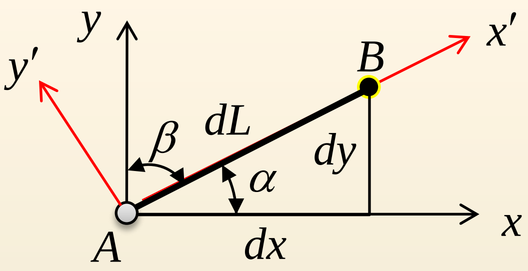

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directional cosine : $\cos\alpha = l = \cos xx', \ \cos\beta = m = \cos yx'$

$$\varepsilon = \varepsilon_x l^2 + \gamma_{xy} lm + \varepsilon_y m^2$$

If we denote by x' and y' the direction of the new coordinate axes

$$\varepsilon_{x'} = \varepsilon_x l^2 + \varepsilon_y m^2 + \gamma_{xy} lm$$

***Strain* at point O about x'y' frame expresses in terms of stress about xy frame at the same point**

c.f.)

$$\therefore \sigma_x{}' = l^2\sigma_x + m^2\sigma_y + 2lm\tau_{xy}$$

$$\tau_{x'y'} = \left(l^2 - m^2\right)\tau_{xy} + lm\left(\sigma_y - \sigma_x\right)$$

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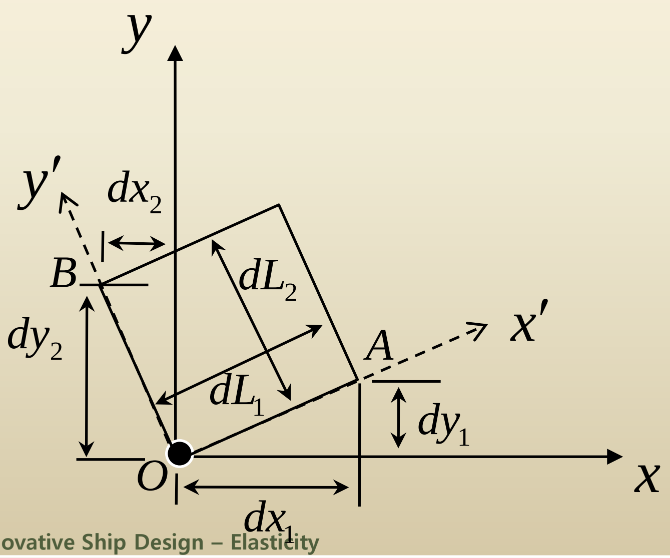

$$dL_1 = \overline{OA}$$

$$dL_2 = \overline{OB}$$

$$\cos xx' = l_1$$

$$\cos yx' = m_1$$

$$\cos xy' = l_2$$

$$\cos yy' = m_2$$

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$

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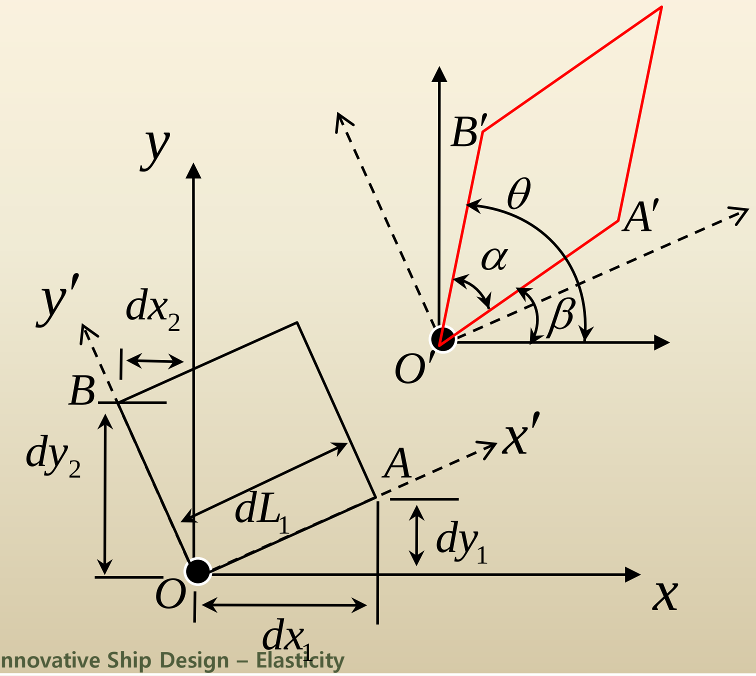
$$dL_1 = \overline{OA} \qquad \cos xx' = l_1 \qquad \cos xy' = l_2 \qquad \varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$

$$dL_2 = \overline{OB} \qquad \cos yx' = m_1 \qquad \cos yy' = m_2 \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

We find that the directional cosines l_1', m_1' **of** $O'A'$



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✓Specification of Strain at a Point

Given the *three longitudinal strain components* **and the** *three shearing components at a point*, **it can be shown that** *the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated*

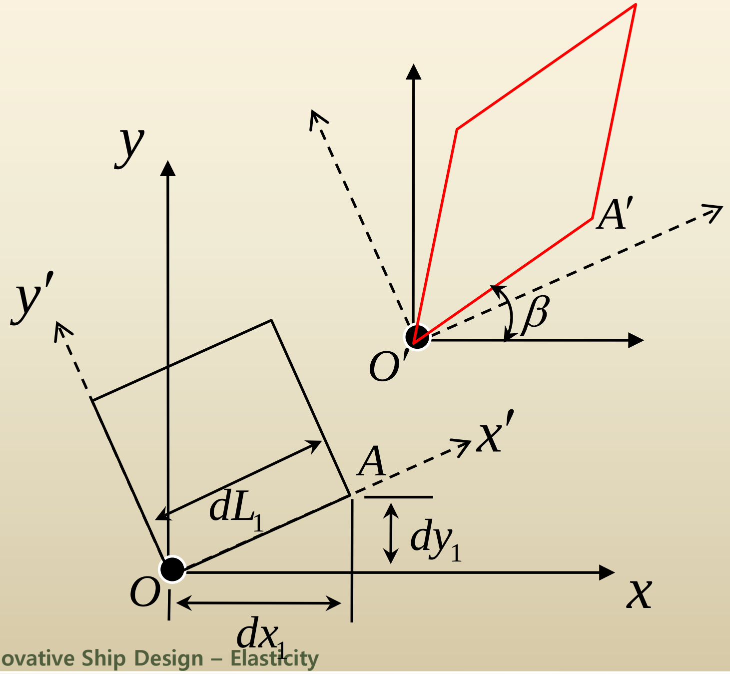
To find shearing strain when referring to these new directions, let us consider

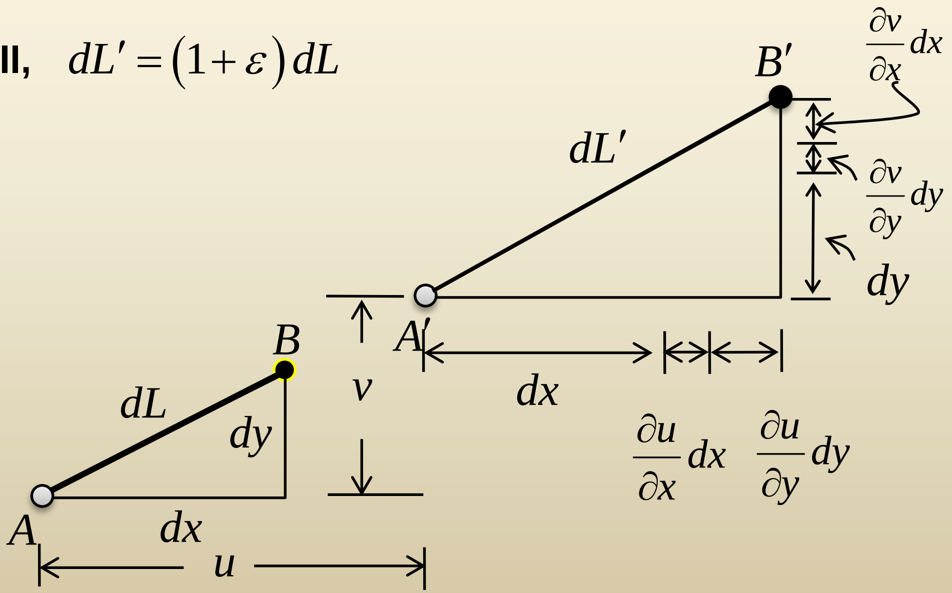
$$dL_1 = \overline{OA} \qquad \cos xx' = l_1 \qquad \cos xy' = l_2 \qquad \varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$

$$dL_2 = \overline{OB} \qquad \cos yx' = m_1 \qquad \cos yy' = m_2 \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

We find that the directional cosines l_1', m_1' **of** $O'A'$

recall, $dL' = \left(1+\varepsilon\right)dL$





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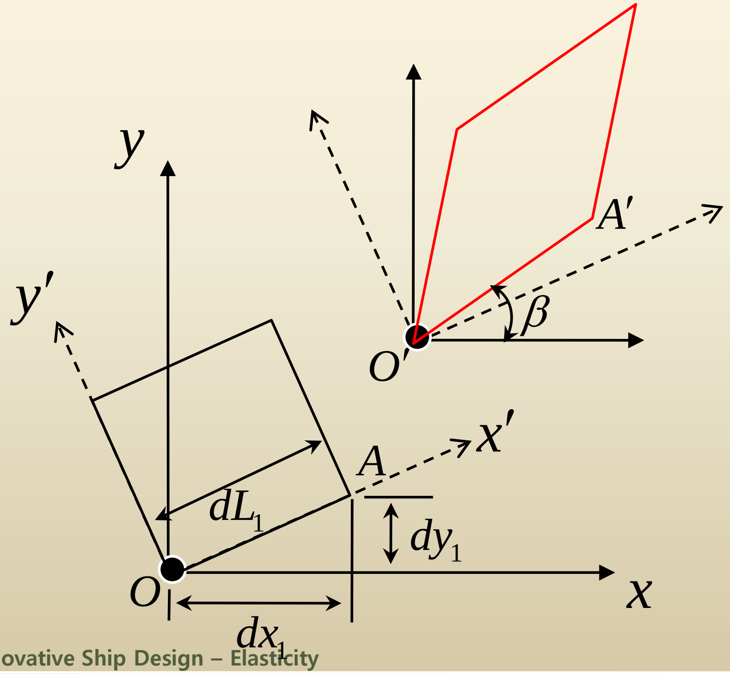
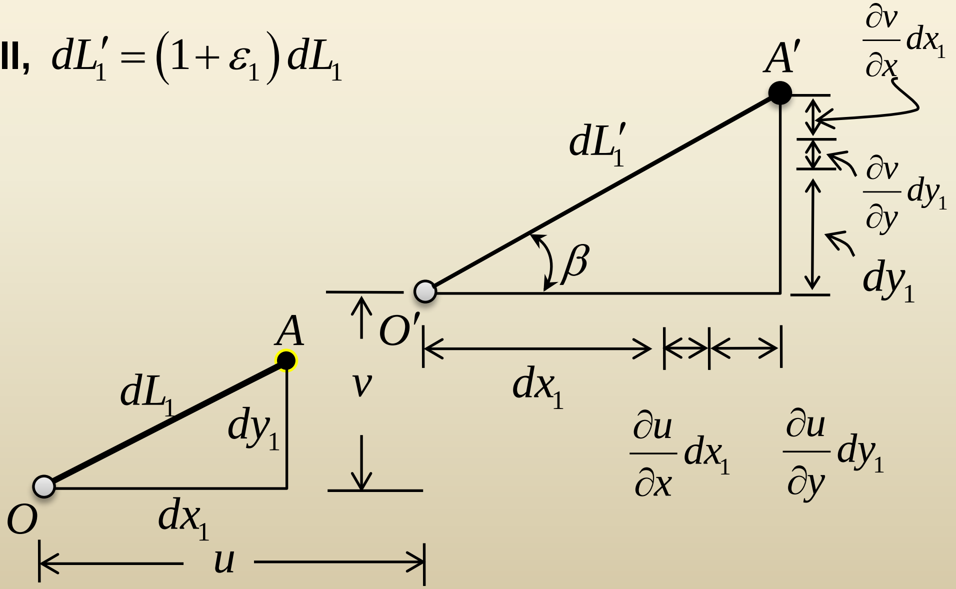


recall, $dL_1' = (1+\varepsilon_1)\,dL_1$



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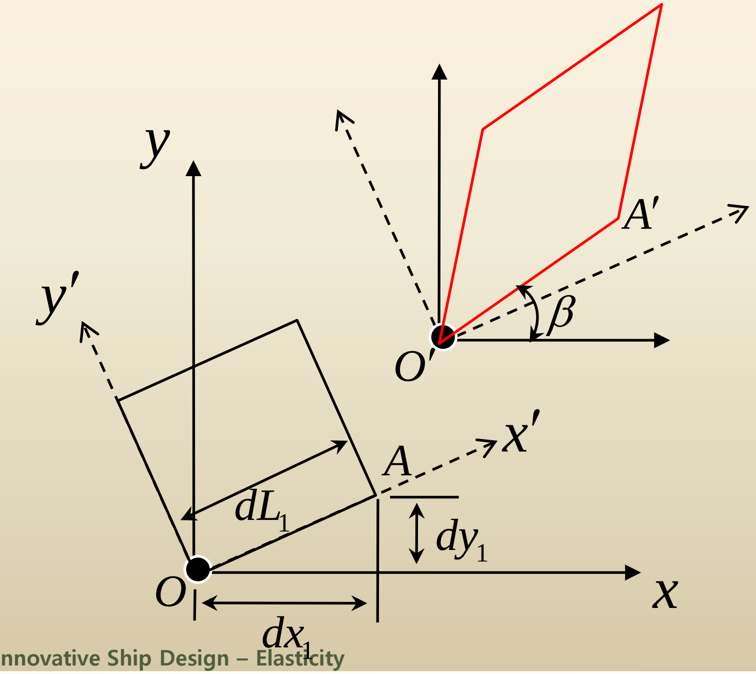

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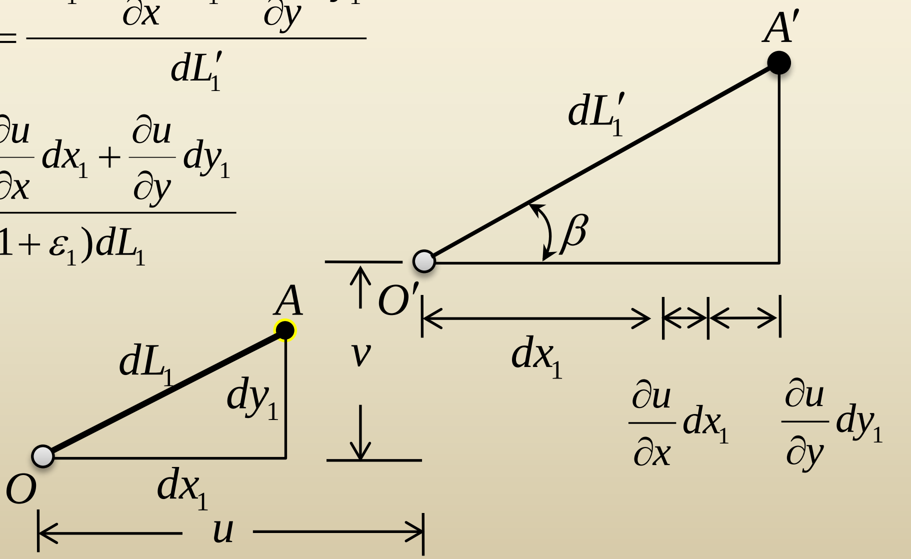
$$l_1' = \cos\beta = \frac{dx_1 + \dfrac{\partial u}{\partial x}dx_1 + \dfrac{\partial u}{\partial y}dy_1}{dL_1'}$$
$$= \frac{dx_1 + \dfrac{\partial u}{\partial x}dx_1 + \dfrac{\partial u}{\partial y}dy_1}{(1+\varepsilon_1)dL_1}$$



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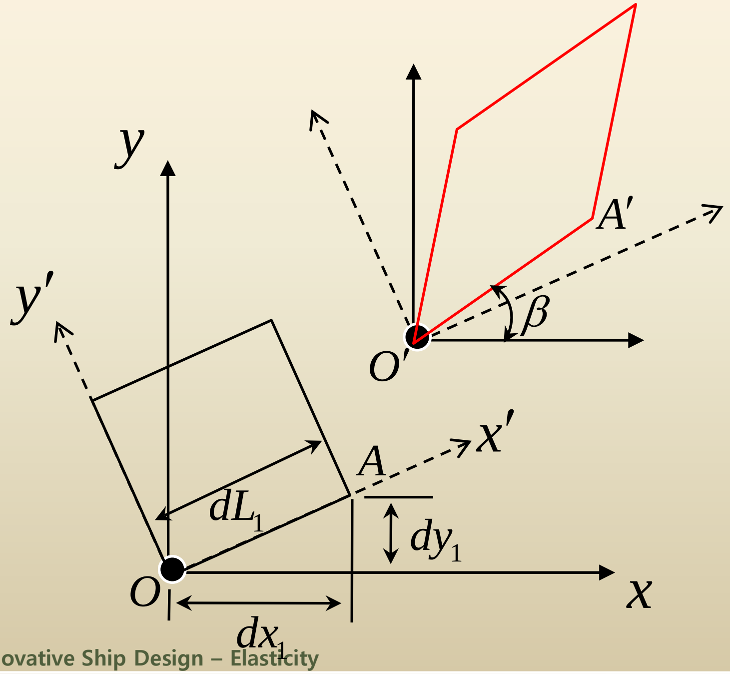

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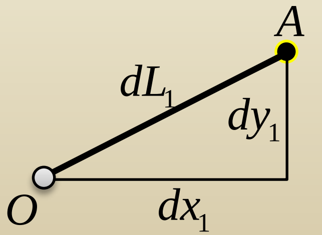
$$l_1' = \cos\beta = \frac{dx_1 + \frac{\partial u}{\partial x} dx_1 + \frac{\partial u}{\partial y} dy_1}{dL_1'}$$

$$= \frac{dx_1 + \frac{\partial u}{\partial x} dx_1 + \frac{\partial u}{\partial y} dy_1}{(1+\varepsilon_1) dL_1} = \frac{\left(1+\frac{\partial u}{\partial x}\right)}{(1+\varepsilon_1)} \cdot \frac{dx_1}{dL_1} + \frac{\frac{\partial u}{\partial y}}{(1+\varepsilon_1)} \cdot \frac{dy_1}{dL_1}$$

$$= \frac{\left(1+\frac{\partial u}{\partial x}\right)}{(1+\varepsilon_1)} \cdot \frac{dx_1}{dL_1} + \frac{\frac{\partial u}{\partial y}}{(1+\varepsilon_1)} \cdot \frac{dy_1}{dL_1}$$



$$= \frac{\left(1+\frac{\partial u}{\partial x}\right)}{(1+\varepsilon_1)} \cdot l_1 + \frac{\frac{\partial u}{\partial y}}{(1+\varepsilon_1)} \cdot m_1$$

Specification of Strain at a Point

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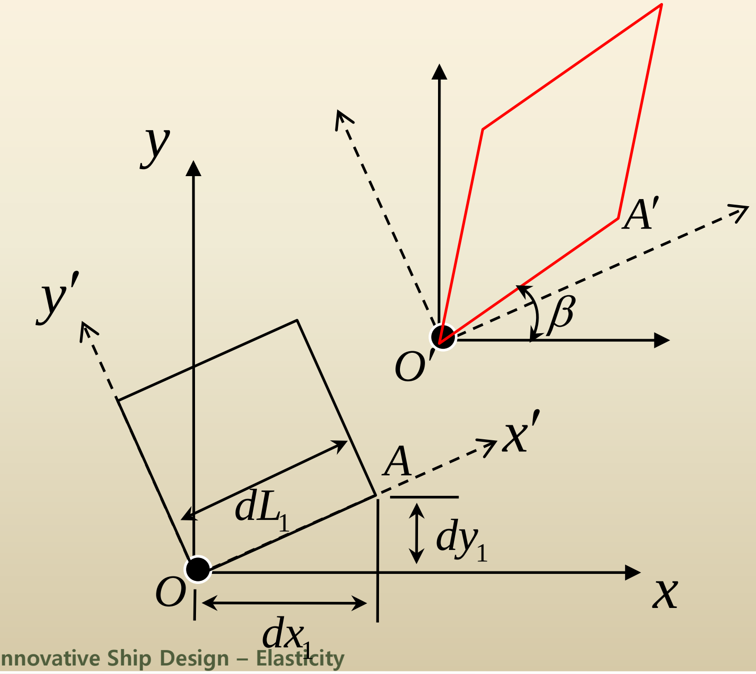
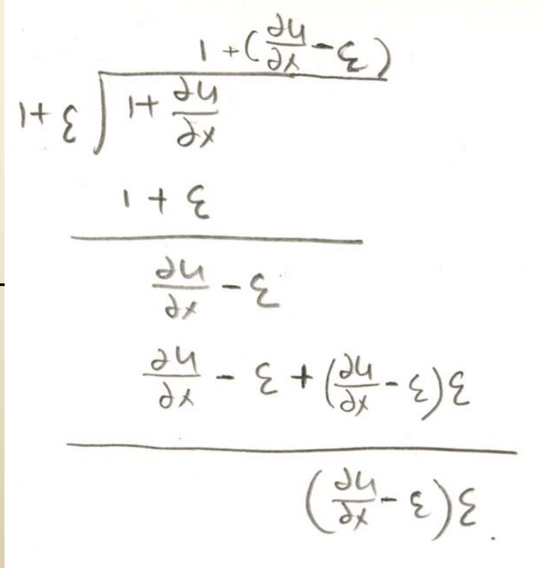


We find that the directional cosines l_1', m_1' **of** $O'A'$

$$l_1' = \boxed{\frac{\left(1 + \dfrac{\partial u}{\partial x}\right)}{(1 + \varepsilon_1)}} \cdot l_1 + \frac{\dfrac{\partial u}{\partial y}}{(1 + \varepsilon_1)} \cdot m_1$$



$$1 + \left(\frac{\partial u}{\partial x} - \varepsilon\right)$$

$$1 + \varepsilon \,\big)\, 1 + \frac{\partial u}{\partial x}$$

$$\underline{1 + \varepsilon}$$

$$\frac{\partial u}{\partial x} - \varepsilon$$

$$\underline{\frac{\partial u}{\partial x} - \varepsilon + \left(\frac{\partial u}{\partial x} - \varepsilon\right)\varepsilon}$$

$$\left(\frac{\partial u}{\partial x} - \varepsilon\right)\varepsilon.$$

$$\left(1 + \frac{\partial u}{\partial x} - \varepsilon_1\right) + \underbrace{\left(\frac{\partial u}{\partial x} - \varepsilon_1\right)\varepsilon_1}_{o(\varepsilon_1{}^2)}$$

$$\approx \left(1 + \frac{\partial u}{\partial x} - \varepsilon_1\right)$$

Specification of Strain at a Point

$$\varepsilon_x=\frac{\partial u}{\partial x}\qquad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

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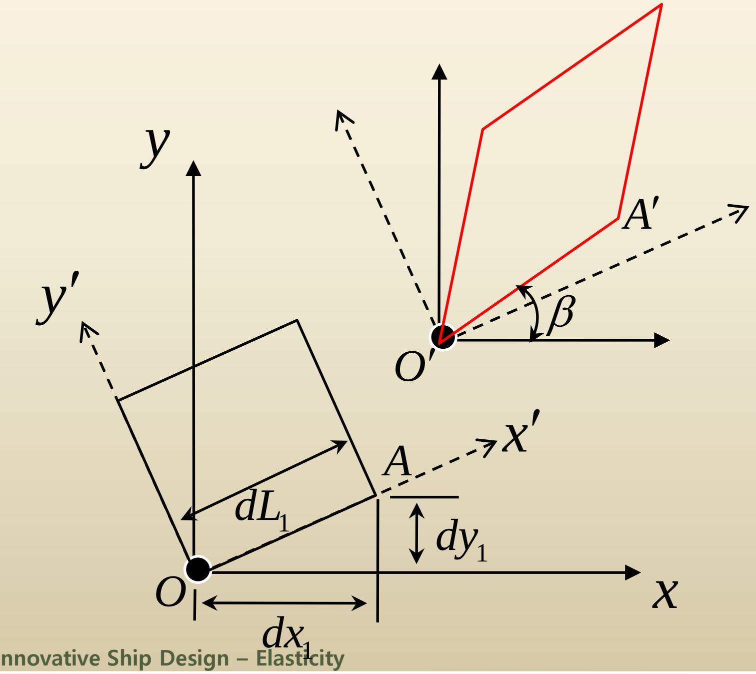

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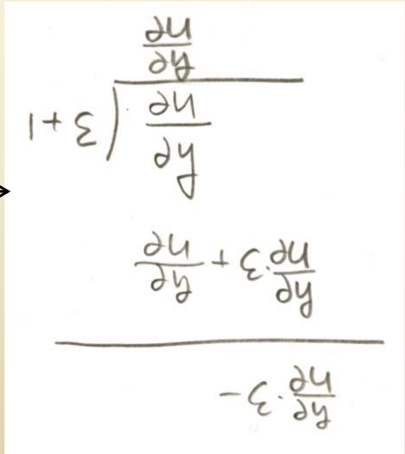
$$l_1'=\frac{\left(1+\dfrac{\partial u}{\partial x}\right)}{(1+\varepsilon_1)}\cdot l_1+\boxed{\frac{\dfrac{\partial u}{\partial y}}{(1+\varepsilon_1)}}\cdot m_1$$



$$\begin{array}{r} \frac{\partial u}{\partial y} \\ 1+\varepsilon\,\big)\overline{\;\frac{\partial u}{\partial y}\qquad\qquad} \\ \frac{\partial u}{\partial y}+\varepsilon\cdot\frac{\partial u}{\partial y} \\ \hline -\varepsilon\cdot\frac{\partial u}{\partial y} \end{array}$$

$$\begin{array}{r} \frac{\partial u}{\partial y}+\varepsilon_1\frac{\partial u}{\partial y} \\ \hline o(\varepsilon_1{}^2) \end{array}$$

$$\approx\frac{\partial u}{\partial y}$$

Specification of Strain at a Point

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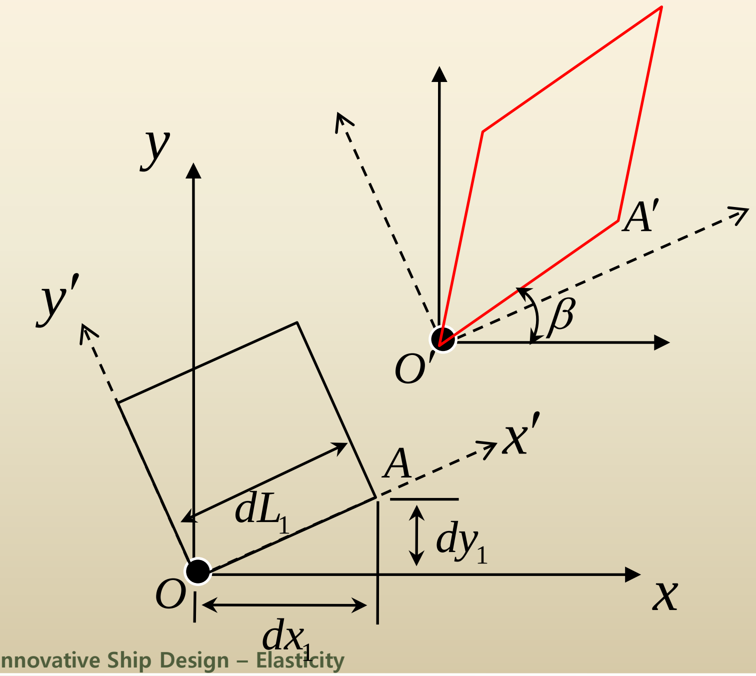

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$$l_1' = \frac{\left(1+\dfrac{\partial u}{\partial x}\right)}{(1+\varepsilon_1)} \cdot l_1 + \frac{\dfrac{\partial u}{\partial y}}{(1+\varepsilon_1)} \cdot m_1 \approx l_1\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right) + m_1\frac{\partial u}{\partial y}$$

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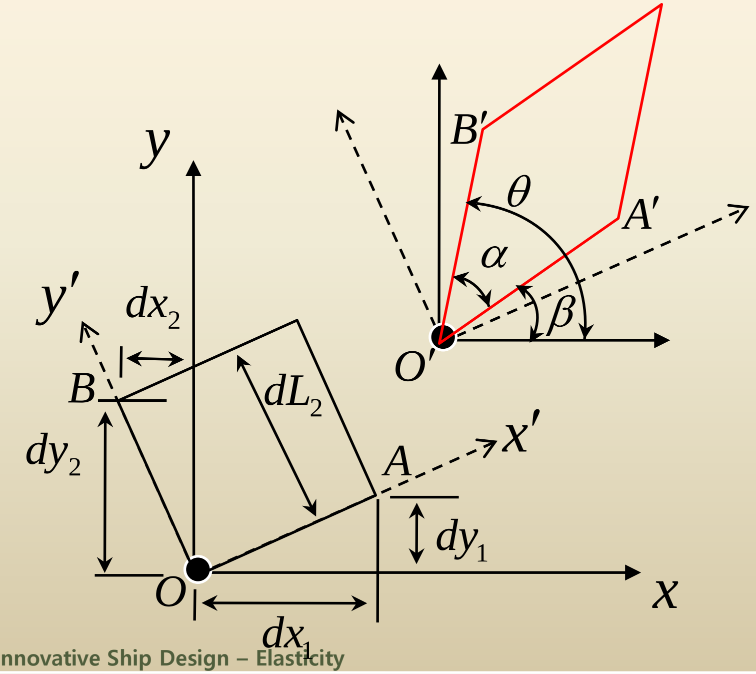

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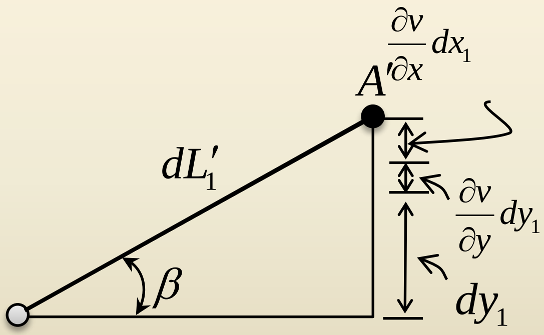
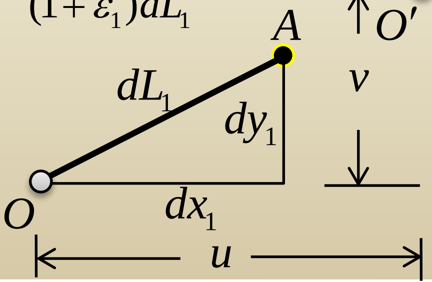
$$m_1' = \cos(90-\beta) = \sin\beta$$

$$= \frac{dy_1 + \frac{\partial v}{\partial x}dx_1 + \frac{\partial v}{\partial y}dy_1}{dL_1'}$$

$$= \frac{dy_1 + \frac{\partial v}{\partial x}dx_1 + \frac{\partial v}{\partial y}dy_1}{(1+\varepsilon_1)dL_1}$$





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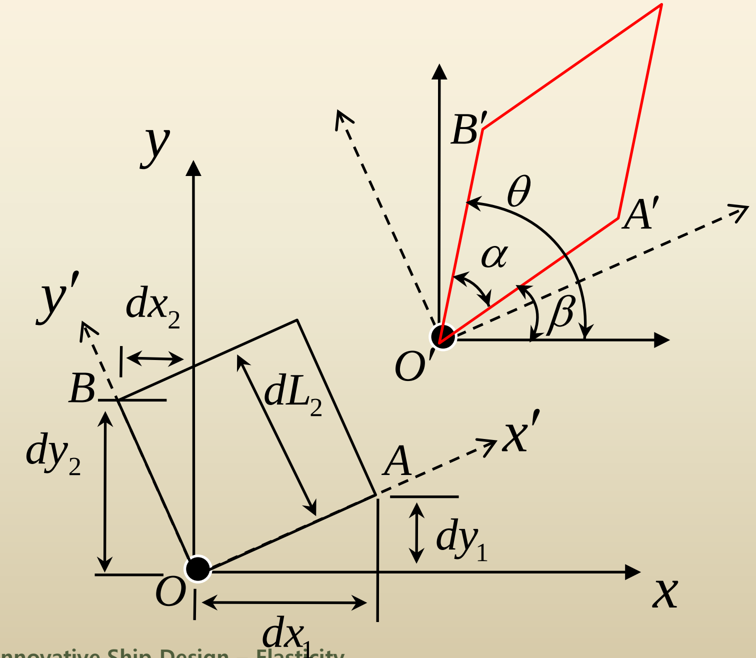
$$l_1' = l_1\left(1 - \varepsilon_1 + \frac{\partial u}{\partial x}\right) + m_1 \frac{\partial u}{\partial y}$$



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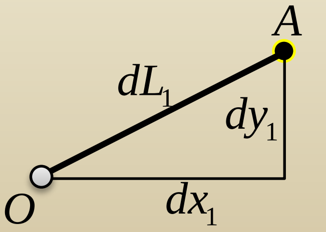
$$m_1' = \frac{dy_1 + \frac{\partial v}{\partial x} dx_1 + \frac{\partial v}{\partial y} dy_1}{(1+\varepsilon_1) dL_1} = \frac{\left(1 + \frac{\partial v}{\partial x}\right)}{(1+\varepsilon_1)} \cdot \frac{dx_1}{dL_1} + \frac{\frac{\partial v}{\partial y}}{(1+\varepsilon_1)} \cdot \frac{dy_1}{dL_1}$$

$$= \frac{\left(1 + \frac{\partial v}{\partial x}\right)}{(1+\varepsilon_1)} \cdot \frac{dx_1}{dL_1} + \frac{\frac{\partial v}{\partial y}}{(1+\varepsilon_1)} \cdot \frac{dy_1}{dL_1} = \frac{\left(1 + \frac{\partial v}{\partial x}\right)}{(1+\varepsilon_1)} \cdot l_1 + \frac{\frac{\partial v}{\partial y}}{(1+\varepsilon_1)} \cdot m_1$$

In same way with l_1'

$$m_1{}' \approx l_1 \frac{\partial v}{\partial x} + m_1 \left(1 - \varepsilon_1 + \frac{\partial v}{\partial y}\right)$$



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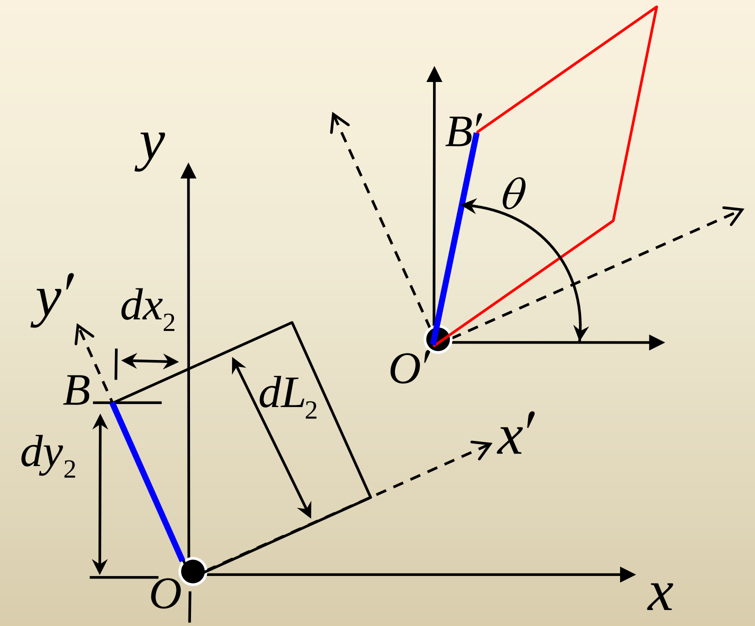
$$dL_2 = \overline{OB} \qquad \cos yx' = m_1 \qquad \cos yy' = m_2 \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB} \qquad \sin\beta = m_1' \qquad \sin\theta = m_2'$$

$$l_1' = l_1\left(1 - \varepsilon_1 + \frac{\partial u}{\partial x}\right) + m_1\frac{\partial u}{\partial y}$$

$$m_1{}' = l_1\frac{\partial v}{\partial x} + m_1\left(1 - \varepsilon_1 + \frac{\partial v}{\partial y}\right)$$



We find that the directional cosines l_2', m_2' **of** $O'B'$

For OB **and** $O'B'$ **, in same way**

$$l_2' = l_2\left(1 - \varepsilon_2 + \frac{\partial u}{\partial x}\right) + m_2\frac{\partial u}{\partial y}$$

$$m_2{}' = l_2\frac{\partial v}{\partial x} + m_2\left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)$$

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$$dL_2 = \overline{OB} \qquad \cos yx' = m_1 \qquad \cos yy' = m_2 \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB} \qquad \sin\beta = m_1' \qquad \sin\theta = m_2'$$

· α **: angle between O'A' and O'B'**

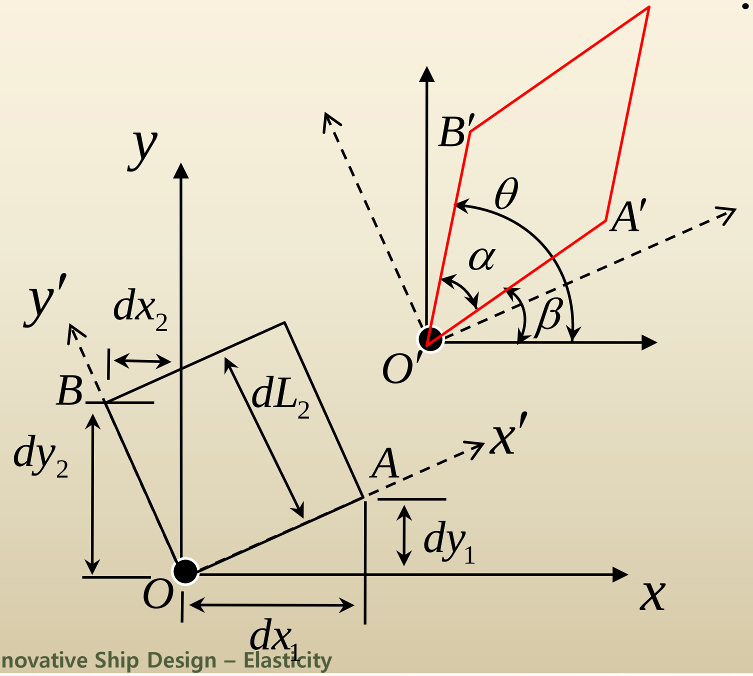
$$\cos\alpha = \cos(\theta - \beta)$$
$$= \cos\beta\cos\theta + \sin\beta\sin\theta$$
$$= l_1' l_2' + m_1' m_2'$$

$$l_1' = l_1\left(1 - \varepsilon_1 + \frac{\partial u}{\partial x}\right) + m_1\frac{\partial u}{\partial y}$$

$$m_1' = l_1\frac{\partial v}{\partial x} + m_1\left(1 - \varepsilon_1 + \frac{\partial v}{\partial y}\right)$$

$$l_2' = l_2\left(1 - \varepsilon_2 + \frac{\partial u}{\partial x}\right) + m_2\frac{\partial u}{\partial y}$$

$$m_2' = l_2\frac{\partial v}{\partial x} + m_2\left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)$$



Specification of Strain at a Point

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial x} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

✓**Specification of Strain at a Point**

Given the ***three longitudinal strain components*** **and the** ***three shearing components at a point*****, it can be shown that** ***the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated***

To find shearing strain when referring to these new directions, let us consider

$$dL_1 = \overline{OA} \qquad \cos xx' = l_1 \qquad \cos xy' = l_2 \qquad \varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \cos\beta = l_1' \qquad \cos\theta = l_2'$$

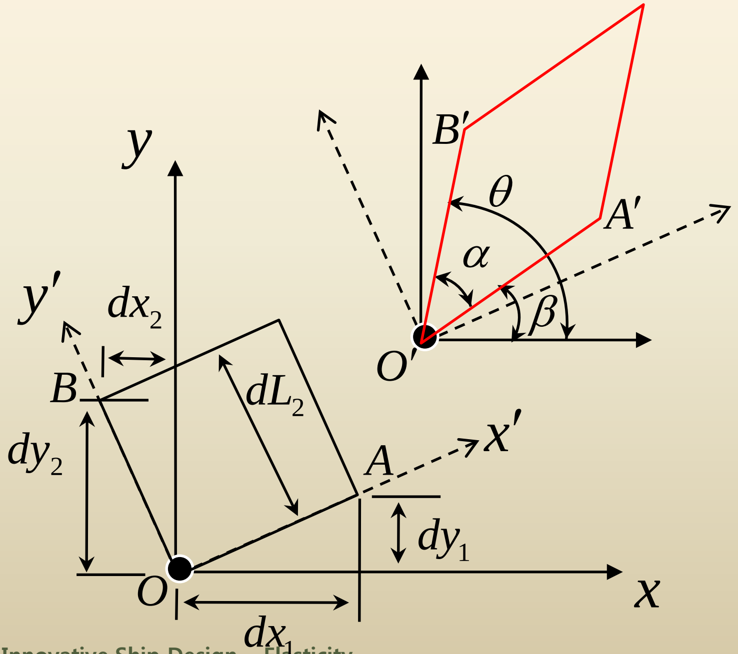
$$dL_2 = \overline{OB} \qquad \cos yx' = m_1 \qquad \cos yy' = m_2 \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB} \qquad \sin\beta = m_1' \qquad \sin\theta = m_2'$$

$$l_1' = l_1\left(1 - \varepsilon_1 + \frac{\partial u}{\partial x}\right) + m_1 \frac{\partial u}{\partial y}$$

$$m_1' = l_1 \frac{\partial v}{\partial x} + m_1\left(1 - \varepsilon_1 + \frac{\partial v}{\partial y}\right)$$

$$l_2' = l_2\left(1 - \varepsilon_2 + \frac{\partial u}{\partial x}\right) + m_2 \frac{\partial u}{\partial y}$$

$$m_2' = l_2 \frac{\partial v}{\partial x} + m_2\left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)$$



· α **: angle between O'A' and O'B'**

$$\begin{aligned}\cos\alpha &= \cos(\theta - \beta) \\ &= \cos\beta\cos\theta + \sin\beta\sin\theta \\ &= l_1' l_2' + m_1' m_2' \\ &= \left[l_1\left(1 - \varepsilon_1 + \frac{\partial u}{\partial x}\right) + m_1\frac{\partial u}{\partial y}\right]\left[l_2\left(1 - \varepsilon_2 + \frac{\partial u}{\partial x}\right) + m_2\frac{\partial u}{\partial y}\right] \\ &+ \left[l_1\frac{\partial v}{\partial x} + m_1\left(1 - \varepsilon_1 + \frac{\partial v}{\partial y}\right)\right]\left[l_2\frac{\partial v}{\partial x} + m_2\left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)\right]\end{aligned}$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$

$$\varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

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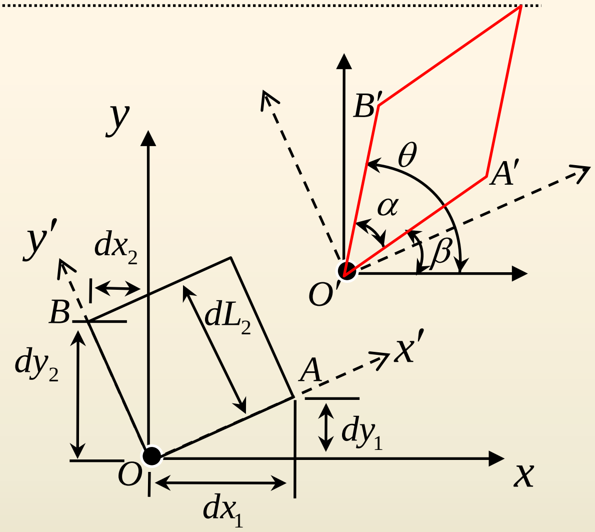
Given the ***three longitudinal strain components*** **and the** ***three shearing components at a point*****, it can be shown that** ***the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated***

- α **: angle between O′A′ and O′B′**



$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= \left[l_1\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right) + m_1\frac{\partial u}{\partial y}\right]\left[l_2\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + m_2\frac{\partial u}{\partial y}\right]$$

$$+\left[l_1\frac{\partial v}{\partial x} + m_1\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\right]\left[l_2\frac{\partial v}{\partial x} + m_2\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\right]$$

$$= l_1 l_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + m_1 l_2\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + l_1 m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + m_1 m_2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

$$+ l_1 l_2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + m_1 l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + l_1 m_2\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + m_1 m_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial x} \qquad \varepsilon_z = \frac{\partial w}{\partial x}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

✓Specification of Strain at a Point

Given the *three longitudinal strain components* **and the** *three shearing components at a point*, **it can be shown that** *the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated*

- α **: angle between O'A' and O'B'**

$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= \boxed{l_1 l_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)} + m_1 l_2 \left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + l_1 m_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + m_1 m_2 \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

$$+ l_1 l_2 \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + m_1 l_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + l_1 m_2 \left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + m_1 m_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)$$

$$l_1 l_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) = l_1 l_2 \left[\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) - \varepsilon_1\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial x}\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\right]$$

$$= l_1 l_2 \left[\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) - \varepsilon_1 + \boxed{\varepsilon_1\varepsilon_2} - \boxed{\varepsilon_1\frac{\partial u}{\partial x}} + \frac{\partial u}{\partial x} - \boxed{\frac{\partial u}{\partial x}\varepsilon_2} + \boxed{\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}}\right] \qquad \boxed{o(\varepsilon^2)}$$

$$\therefore l_1 l_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) \approx l_1 l_2 \left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial u}{\partial x}\right)$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

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$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= l_1 l_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + \boxed{m_1 l_2 \left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y}} + l_1 m_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + m_1 m_2 \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

$$+ l_1 l_2 \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + m_1 l_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + l_1 m_2 \left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + m_1 m_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)$$

$$m_1 l_2 \left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} = m_1 l_2 \left(\frac{\partial u}{\partial y} - \boxed{\varepsilon_2 \frac{\partial u}{\partial y}} + \boxed{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}\right) \qquad \boxed{o(\varepsilon^2)}$$

$$\therefore m_1 l_2 \left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} \approx m_1 l_2 \frac{\partial u}{\partial y}$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

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$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= l_1 l_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + m_1 l_2\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + \boxed{l_1 m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y}} + m_1 m_2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

$$+ l_1 l_2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + m_1 l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + l_1 m_2\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + m_1 m_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)$$

$$l_1 m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} = l_1 m_2\left(\frac{\partial u}{\partial y} - \boxed{\varepsilon_1\frac{\partial u}{\partial y}} + \boxed{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}\right) \qquad \boxed{o(\varepsilon^2)}$$

$$\therefore l_1 m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} \approx l_1 m_2\frac{\partial u}{\partial y}$$

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$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

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$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= l_1 l_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + m_1 l_2\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + l_1 m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + \boxed{m_1 m_2 \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}}$$

$$\boxed{+ l_1 l_2 \frac{\partial v}{\partial x}\frac{\partial v}{\partial x}} + m_1 l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + l_1 m_2\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + m_1 m_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)$$

$$m_1 m_2 \boxed{\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}} \qquad\qquad l_1 l_2 \boxed{\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}}$$

since $\boxed{}$ **~** $o(\varepsilon^2)$

$$\therefore m_1 m_2 \frac{\partial u}{\partial y}\frac{\partial u}{\partial y} \approx 0 \qquad\qquad \therefore l_1 l_2 \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} \approx 0$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$
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$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$
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$$\cos\alpha = l_1'l_2' + m_1'm_2'$$
$$= l_1l_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + m_1l_2\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + l_1m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + m_1m_2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$
$$+ l_1l_2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + \boxed{m_1l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x}} + l_1m_2\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + m_1m_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)$$

$$m_1l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} = m_1l_2\left(\frac{\partial v}{\partial x} - \boxed{\varepsilon_1\frac{\partial v}{\partial x}} + \boxed{\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}}\right)$$

since $\boxed{} \sim o(\varepsilon^2)$ $\quad \therefore m_1l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} \approx m_1l_2\frac{\partial v}{\partial x}$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

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$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= l_1 l_2 \left(1 - \varepsilon_1 + \frac{\partial u}{\partial x}\right)\left(1 - \varepsilon_2 + \frac{\partial u}{\partial x}\right) + m_1 l_2 \left(1 - \varepsilon_2 + \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + l_1 m_2 \left(1 - \varepsilon_1 + \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + m_1 m_2 \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

$$+ l_1 l_2 \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + m_1 l_2 \left(1 - \varepsilon_1 + \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + \boxed{l_1 m_2 \left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x}} + m_1 m_2 \left(1 - \varepsilon_1 + \frac{\partial v}{\partial y}\right)\left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)$$

$$l_1 m_2 \left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} = l_1 m_2 \left(\frac{\partial v}{\partial x} - \boxed{\varepsilon_2 \frac{\partial v}{\partial x}} + \boxed{\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}}\right)$$

since $\boxed{} \sim o(\varepsilon^2)$ $\qquad \therefore l_1 m_2 \left(1 - \varepsilon_2 + \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} \approx l_1 m_2 \frac{\partial v}{\partial x}$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

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$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= l_1 l_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + m_1 l_2 \left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + l_1 m_2 \left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + m_1 m_2 \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

$$+ l_1 l_2 \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + m_1 l_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + l_1 m_2 \left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + \boxed{m_1 m_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)}$$

$$m_1 m_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right) = m_1 m_2 \left(1-\varepsilon_2+\frac{\partial v}{\partial y} - \varepsilon_1\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial y}\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\right)$$

$$= m_1 m_2 \left(1-\varepsilon_2+\frac{\partial v}{\partial y} - \varepsilon_1 \boxed{+\varepsilon_1\varepsilon_2} \boxed{-\varepsilon_1\frac{\partial v}{\partial y}} + \frac{\partial v}{\partial y} \boxed{-\frac{\partial v}{\partial y}\varepsilon_2} + \boxed{\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}}\right)$$

since $\boxed{}$ ~ $o(\varepsilon^2)$

$$\therefore m_1 m_2 \left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right) \approx m_1 m_2 \left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial v}{\partial y}\right)$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$
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$$\cancel{o(\varepsilon^2)}$$

· α **: angle between O'A' and O'B'**

$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= l_1 l_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) + m_1 l_2\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + l_1 m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + m_1 m_2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

$$+ l_1 l_2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + m_1 l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + l_1 m_2\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} + m_1 m_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)$$

$$\approx l_1 l_2\left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial u}{\partial x}\right) + m_1 l_2\frac{\partial u}{\partial y} + l_1 m_2\frac{\partial u}{\partial y} + m_1 l_2\frac{\partial v}{\partial x} + l_1 m_2\frac{\partial v}{\partial x} + m_1 m_2\left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial v}{\partial y}\right)$$

$$= l_1 l_2\left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial u}{\partial x}\right) + m_1 l_2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + l_1 m_2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + m_1 m_2\left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial v}{\partial y}\right)$$

$$= l_1 l_2\left(1-\varepsilon_1-\varepsilon_2+2\varepsilon_x\right) + m_1 l_2\gamma_{xy} + l_1 m_2\gamma_{xy} + m_1 m_2\left(1-\varepsilon_1-\varepsilon_2+2\varepsilon_y\right)$$

$$l_1 l_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right) \approx l_1 l_2\left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial u}{\partial x}\right)$$

$$m_1 l_2\left(1-\varepsilon_2+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} \approx m_1 l_2\frac{\partial u}{\partial y}$$

$$l_1 m_2\left(1-\varepsilon_1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} \approx l_1 m_2\frac{\partial u}{\partial y}$$

$$m_1 m_2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y} \approx 0 \qquad l_1 l_2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x} \approx 0$$

$$m_1 l_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} \approx m_1 l_2\frac{\partial v}{\partial x}$$

$$l_1 m_2\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x} \approx l_1 m_2\frac{\partial v}{\partial x}$$

$$m_1 m_2\left(1-\varepsilon_1+\frac{\partial v}{\partial y}\right)\left(1-\varepsilon_2+\frac{\partial v}{\partial y}\right) = m_1 m_2\left(1-\varepsilon_1-\varepsilon_2+2\frac{\partial v}{\partial y}\right)$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$

$$\varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial x} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

✓**Specification of Strain at a Point**

Given the *three longitudinal strain components* **and the** *three shearing components at a point*, **it can be shown that** *the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated*

- α **: angle between O′A′ and O′B′**

$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= l_1 l_2 \left(1 - \varepsilon_1 - \varepsilon_2 + 2\varepsilon_x\right) + m_1 l_2 \gamma_{xy} + l_1 m_2 \gamma_{xy} + m_1 m_2 \left(1 - \varepsilon_1 - \varepsilon_2 + 2\varepsilon_y\right)$$

$$= l_1 l_2 \left(1 - \varepsilon_1 - \varepsilon_2\right) + 2 l_1 l_2 \varepsilon_x + \left(m_1 l_2 + l_1 m_2\right)\gamma_{xy} + m_1 m_2 \left(1 - \varepsilon_1 - \varepsilon_2\right) + 2 m_1 m_2 \varepsilon_y$$

$$= \left(l_1 l_2 + m_1 m_2\right)\left(1 - \varepsilon_1 - \varepsilon_2\right) + 2\left(l_1 l_2 \varepsilon_x + m_1 m_2 \varepsilon_y\right) + \left(l_1 m_2 + l_2 m_1\right)\gamma_{xy}$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial x} \qquad \varepsilon_z = \frac{\partial w}{\partial x}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

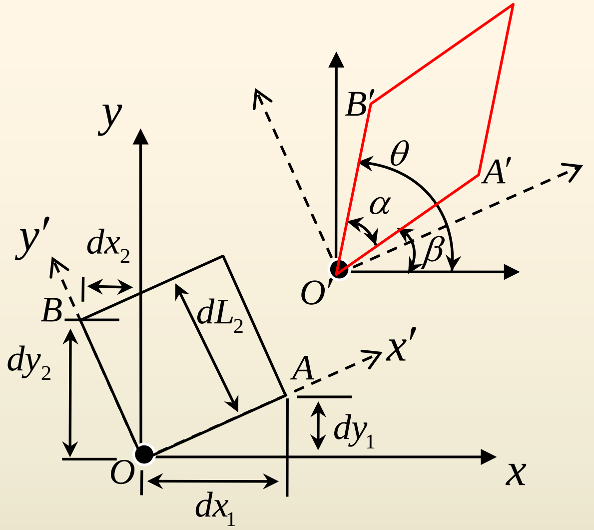
✓**Specification of Strain at a Point**

Given the ***three longitudinal strain components*** **and the** ***three shearing components at a point*****, it can be shown that** ***the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated***

- α : **angle between O′A′ and O′B′**

$$\cos\alpha = l_1' l_2' + m_1' m_2'$$

$$= (l_1 l_2 + m_1 m_2)(1 - \varepsilon_1 - \varepsilon_2) + 2(l_1 l_2 \varepsilon_x + m_1 m_2 \varepsilon_y) + (l_1 m_2 + l_2 m_1)\gamma_{xy}$$



✓**From the definition of shearing strain,** $\gamma_{x'y'} = \dfrac{\pi}{2} - \alpha$

For small shearing strain, the angle $\dfrac{\pi}{2} - \alpha$ **is small**

$$\gamma_{x'y'} = \frac{\pi}{2} - \alpha \approx \sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$$

$$\therefore \gamma_{x'y'} = (l_1 l_2 + m_1 m_2)(1 - \varepsilon_1 - \varepsilon_2) + 2(l_1 l_2 \varepsilon_x + m_1 m_2 \varepsilon_y) + (l_1 m_2 + l_2 m_1)\gamma_{xy}$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA} \qquad \varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial x} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

✓**Specification of Strain at a Point**

Given the *three longitudinal strain components* and the *three shearing components at a point*, it can be shown that *the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated*

$$\gamma_{x'y'} = (l_1 l_2 + m_1 m_2)(1 - \varepsilon_1 - \varepsilon_2) + 2(l_1 l_2 \varepsilon_x + m_1 m_2 \varepsilon_y) + (l_1 m_2 + l_2 m_1)\gamma_{xy}$$

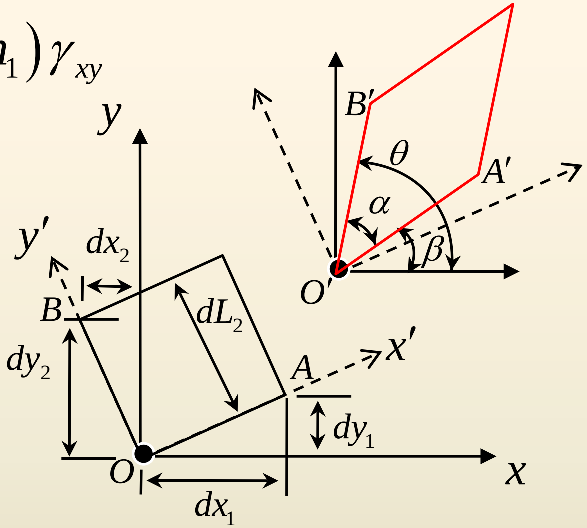
$\angle A\odot B$ **: angle between OA and OB**

$$\cos(\angle AOB) = \cos\frac{\pi}{2} = 0$$

and

$$\begin{aligned}\cos(\angle AOB) &= \cos(\angle xOB - \angle xOA) \\ &= \cos(\angle xOB)\cos(\angle xOA) + \sin(\angle xOB)\sin(\angle xOA) \\ &= \cos xy'\cos xx' + \sin xx'\sin xy' \\ &= l_1 l_2 + m_1 m_2\end{aligned}$$

$$\therefore l_1 l_2 + m_1 m_2 = 0$$



***Strain* at point O about x'y' frame expresses in terms of stress about xy frame at the same point**

$$\therefore \gamma_{x'y'} = 2(l_1 l_2 \varepsilon_x + m_1 m_2 \varepsilon_y) + (l_1 m_2 + l_2 m_1)\gamma_{xy}$$

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$
$$\varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$
$$\varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$
$$\varepsilon_z = \frac{\partial w}{\partial x} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

✓**Specification of Strain at a Point**

Given the ***three longitudinal strain components*** **and the** ***three shearing components at a point*****, it can be shown that** ***the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated***

……

In same way, for 3 dimensional frame

Given: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \lambda_{zx}$ **at O**

Find: $\varepsilon_{x'}, \varepsilon_{y'}, \varepsilon_{z'}, \gamma_{x'y'}, \gamma_{y'z'}, \gamma_{z'x'}$ **at O**

$$\varepsilon_{x'} = {l_1}^2\varepsilon_x + {m_1}^2\varepsilon_y + {n_1}^2\varepsilon_z + l_1m_1\gamma_{xy} + m_1n_1\gamma_{yz} + n_1l_1\gamma_{zx}$$
$$\varepsilon_{y'} = {l_2}^2\varepsilon_x + {m_2}^2\varepsilon_y + {n_2}^2\varepsilon_z + l_2m_2\gamma_{xy} + m_2n_2\gamma_{yz} + n_2l_2\gamma_{zx}$$
$$\varepsilon_{z'} = {l_3}^2\varepsilon_x + {m_3}^2\varepsilon_y + {n_3}^2\varepsilon_z + l_3m_3\gamma_{xy} + m_3n_3\gamma_{yz} + n_3l_3\gamma_{zx}$$
$$\gamma_{x'y'} = 2l_1l_2\varepsilon_x + 2m_1m_2\varepsilon_y + 2n_1n_2\varepsilon_z + (l_1m_2 + m_1l_2)\gamma_{xy} + (m_1n_2 + n_1m_2)\gamma_{yz} + (n_1l_2 + l_1n_2)\gamma_{zx}$$
$$\gamma_{y'z'} = 2l_2l_3\varepsilon_x + 2m_2m_3\varepsilon_y + 2n_2n_3\varepsilon_z + (l_2m_3 + m_2l_3)\gamma_{xy} + (m_2n_3 + n_2m_3)\gamma_{yz} + (n_2l_3 + l_2n_3)\gamma_{zx}$$
$$\gamma_{z'x'} = 2l_3l_1\varepsilon_x + 2m_3m_1\varepsilon_y + 2n_3n_1\varepsilon_z + (l_3m_1 + m_3l_1)\gamma_{xy} + (m_3n_1 + n_3m_1)\gamma_{yz} + (n_3l_1 + l_3n_1)\gamma_{zx}$$

✓**Directional Cosine**

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Strain at point O about x′y′z′ frame expresses in terms of strain about xyz frame at the same point

Specification of Strain at a Point

$$\varepsilon_1 = \varepsilon_{x'} = \varepsilon_{OA}$$

$$\varepsilon_2 = \varepsilon_{y'} = \varepsilon_{OB}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial x} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

✓**Specification of Strain at a Point**

Given the *three longitudinal strain components* and the *three shearing components at a point*, it can be shown that *the elongation in any direction and the distortion of the angle between any two perpendicular directions can be calculated*

$$\varepsilon_{x'} = l_1{}^2\varepsilon_x + m_1{}^2\varepsilon_y + n_1{}^2\varepsilon_z + l_1m_1\gamma_{xy} + m_1n_1\gamma_{yz} + n_1l_1\gamma_{zx}$$

$$\varepsilon_{y'} = l_2{}^2\varepsilon_x + m_2{}^2\varepsilon_y + n_2{}^2\varepsilon_z + l_2m_2\gamma_{xy} + m_2n_2\gamma_{yz} + n_2l_2\gamma_{zx}$$

$$\varepsilon_{z'} = l_3{}^2\varepsilon_x + m_3{}^2\varepsilon_y + n_3{}^2\varepsilon_z + l_3m_3\gamma_{xy} + m_3n_3\gamma_{yz} + n_3l_3\gamma_{zx}$$

Strain **at point O about x'y' frame expresses in terms of strain about xy frame at the same point**

$$\begin{aligned}\varepsilon_{x'} + \varepsilon_{y'} + \varepsilon_{z'} = {} & l_1{}^2\varepsilon_x + m_1{}^2\varepsilon_y + n_1{}^2\varepsilon_z + l_1m_1\gamma_{xy} + m_1n_1\gamma_{yz} + n_1l_1\gamma_{zx} \\ & + l_2{}^2\varepsilon_x + m_2{}^2\varepsilon_y + n_2{}^2\varepsilon_z + l_2m_2\gamma_{xy} + m_2n_2\gamma_{yz} + n_2l_2\gamma_{zx} \\ & + l_3{}^2\varepsilon_x + m_3{}^2\varepsilon_y + n_3{}^2\varepsilon_z + l_3m_3\gamma_{xy} + m_3n_3\gamma_{yz} + n_3l_3\gamma_{zx} \\ = {} & (1)\varepsilon_x + (1)\varepsilon_y + (1)\varepsilon_z + (0)\gamma_{xy} + (0)\gamma_{yz} + (0)\gamma_{zy} \\ = {} & \varepsilon_x + \varepsilon_y + \varepsilon_z\end{aligned}$$

The quantity $\varepsilon_x + \varepsilon_y + \varepsilon_z$ is invariant with respect to orthogonal transformations of coordinates

✓**Directional Cosine**

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

$$l_1^2 + l_2^2 + l_3^2 = 1$$

$$m_1^2 + m_2^2 + m_3^2 = 1$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

$$l_1m_1 + l_2m_2 + l_3m_3 = 0$$

$$m_1n_1 + m_2n_2 + m_3n_3 = 0$$

$$n_1l_1 + n_2l_2 + n_3l_3 = 0$$

Specification of Strain at a Point

✓**Plane Strain :** $\varepsilon_x, \varepsilon_y, \gamma_{xy},$ **and** γ_{yx} **are independent of** z **and the other components are zero**

the body force X and Y are independent of z **and Z is zero**

$$\varepsilon_{x'} = l_1^{\,2}\varepsilon_x + m_1^{\,2}\varepsilon_y + n_1^{\,2}\varepsilon_z + l_1 m_1\gamma_{xy} + m_1 n_1\gamma_{yz} + n_1 l_1\gamma_{zx}$$

$$\varepsilon_{y'} = l_2^{\,2}\varepsilon_x + m_2^{\,2}\varepsilon_y + n_2^{\,2}\varepsilon_z + l_2 m_2\gamma_{xy} + m_2 n_2\gamma_{yz} + n_2 l_2\gamma_{zx}$$

$$\varepsilon_{z'} = l_3^{\,2}\varepsilon_x + m_3^{\,2}\varepsilon_y + n_3^{\,2}\varepsilon_z + l_3 m_3\gamma_{xy} + m_3 n_3\gamma_{yz} + n_3 l_3\gamma_{zx}$$

$$\gamma_{x'y'} = 2l_1 l_2\varepsilon_x + 2m_1 m_2\varepsilon_y + 2n_1 n_2\varepsilon_z + \left(l_1 m_2 + m_1 l_2\right)\gamma_{xy} + \left(m_1 n_2 + n_1 m_2\right)\gamma_{yz} + \left(n_1 l_2 + l_1 n_2\right)\gamma_{zx}$$

$$\gamma_{y'z'} = 2l_2 l_3\varepsilon_x + 2m_2 m_3\varepsilon_y + 2n_2 n_3\varepsilon_z + \left(l_2 m_3 + m_2 l_3\right)\gamma_{xy} + \left(m_2 n_3 + n_2 m_3\right)\gamma_{yz} + \left(n_2 l_3 + l_2 n_3\right)\gamma_{zx}$$

$$\gamma_{z'x'} = 2l_3 l_1\varepsilon_x + 2m_3 m_1\varepsilon_y + 2n_3 n_1\varepsilon_z + \left(l_3 m_1 + m_3 l_1\right)\gamma_{xy} + \left(m_3 n_1 + n_3 m_1\right)\gamma_{yz} + \left(n_3 l_1 + l_3 n_1\right)\gamma_{zx}$$

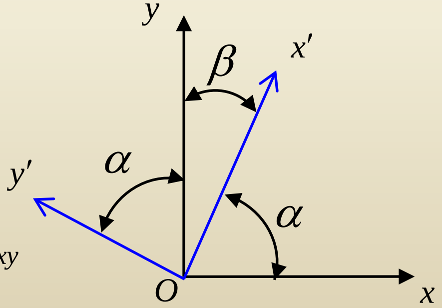
⇩

$$\varepsilon_{x'} = l_1^{\,2}\varepsilon_x + m_1^{\,2}\varepsilon_y + l_1 m_1\gamma_{xy}$$

$$\varepsilon_{y'} = l_2^{\,2}\varepsilon_x + m_2^{\,2}\varepsilon_y + l_2 m_2\gamma_{xy}$$

$$\gamma_{x'y'} = 2l_1 l_2\varepsilon_x + 2m_1 m_2\varepsilon_y + \left(l_1 m_2 + m_1 l_2\right)\gamma_{xy}$$

✓**Directional Cosine**



$$\cos xx' = l_1 = \cos\alpha$$

$$\cos yx' = m_1 = \cos\beta = \cos(90-\alpha) = \sin\alpha$$

$$\cos yy' = m_2 = \cos\alpha$$

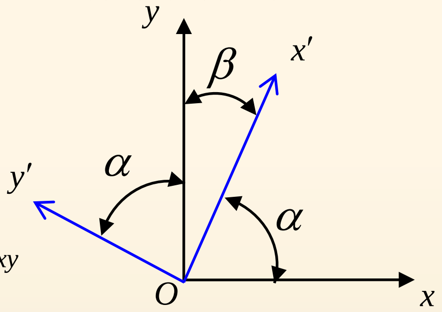
$$\cos xy' = l_2 = \cos(90+\alpha) = -\sin\alpha$$

Specification of Strain at a Point

✓**Plane Strain :** $\varepsilon_x, \varepsilon_y, \gamma_{xy},$ **and** γ_{yx} **are independent of** z **and the other components are zero**

the body force X and Y are independent of z **and Z is zero**

✓**Directional Cosine**



$$\varepsilon_{x'} = l_1{}^2\varepsilon_x + m_1{}^2\varepsilon_y + l_1 m_1 \gamma_{xy}$$

$$\varepsilon_{y'} = l_2{}^2\varepsilon_x + m_2{}^2\varepsilon_y + l_2 m_2 \gamma_{xy}$$

$$\gamma_{x'y'} = 2l_1 l_2 \varepsilon_x + 2m_1 m_2 \varepsilon_y + \left(l_1 m_2 + m_1 l_2\right)\gamma_{xy}$$

$$\cos xx' = l_1 = \cos\alpha$$

$$\cos yx' = m_1 = \cos\beta = \cos(90-\alpha) = \sin\alpha$$

$$\cos yy' = m_2 = \cos\alpha$$

$$\cos xy' = l_2 = \cos(90+\alpha) = -\sin\alpha$$

⇩

$$\varepsilon_{x'} = l_1{}^2\varepsilon_x + m_1{}^2\varepsilon_y + l_1 m_1 \gamma_{xy} = \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha$$

$$\gamma_{x'y'} = 2l_1 l_2 \varepsilon_x + 2m_1 m_2 \varepsilon_y + \left(l_1 m_2 + m_1 l_2\right)\gamma_{xy} = -2\varepsilon_x \cos\alpha\sin\alpha + 2\varepsilon_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\gamma_{xy}$$

Specification of Strain at a Point

✓**Plane Strain :** $\varepsilon_x, \varepsilon_y, \gamma_{xy},$ **and** γ_{yx} **are independent of** z **and the other components are zero**

the body force X and Y are independent of z **and Z is zero**

$$\begin{cases} \varepsilon_{x'} = \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha \\ \gamma_{x'y'} = -2\varepsilon_x \cos\alpha\sin\alpha + 2\varepsilon_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\gamma_{xy} \end{cases}$$

let $2\bar{\gamma} = \gamma$ **,**

$$\gamma_{x'y'} = -2\varepsilon_x \cos\alpha\sin\alpha + 2\varepsilon_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\gamma_{xy}$$

$$2\bar{\gamma}_{x'y'} = -2\varepsilon_x \cos\alpha\sin\alpha + 2\varepsilon_y \sin\alpha\cos\alpha + 2(\cos^2\alpha - \sin^2\alpha)\bar{\gamma}_{xy}$$

$$\bar{\gamma}_{x'y'} = -\varepsilon_x \cos\alpha\sin\alpha + \varepsilon_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\bar{\gamma}_{xy}$$

$$\therefore \begin{cases} \varepsilon_{x'} = \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha \\ \bar{\gamma}_{x'y'} = -\varepsilon_x \cos\alpha\sin\alpha + \varepsilon_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\bar{\gamma}_{xy} \end{cases}$$

Recall, principal stress)

$$\sigma_{x'} = l_1^2\sigma_x + m_1^2\sigma_y + 2l_1 m_1 \tau_{xy}$$
$$= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha$$

$$\tau_{x'y'} = l_1 l_2 \sigma_x + m_1 m_2 \sigma_y + \left(l_1 m_2 + m_1 l_2\right)\tau_{xy}$$
$$= -\sigma_x \cos\alpha\sin\alpha + \sigma_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\tau_{xy}$$

Min. or Max value in condition $\dfrac{\partial\sigma_{x'}}{\partial\alpha} = 0$

At principal angle $\alpha = \dfrac{1}{2}\arctan\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$

Principal Stress

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$\tau_{x'y'} = 0$ **in condition of principal stress**

Specification of Strain at a Point

✓**Plane Strain :** $\varepsilon_x, \varepsilon_y, \gamma_{xy}$, and γ_{yx} are independent of z and the other components are zero

the body force X and Y are independent of z and Z is zero

$$\begin{cases} \varepsilon_{x'} = \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha \\ \gamma_{x'y'} = -2\varepsilon_x \cos\alpha\sin\alpha + 2\varepsilon_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\gamma_{xy} \end{cases}$$

let $2\bar{\gamma} = \gamma$,

$$\varepsilon_{x'} = \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha$$

$$\bar{\gamma}_{x'y'} = -\varepsilon_x \cos\alpha\sin\alpha + \varepsilon_y \sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\bar{\gamma}_{xy}$$

same in format

Min. or Max value in condition $\dfrac{\partial \varepsilon_{x'}}{\partial \alpha} = 0$

At principal angle $\alpha = \dfrac{1}{2}\arctan\left(\dfrac{2\bar{\gamma}_{xy}}{\varepsilon_x - \varepsilon_y}\right)$

Principal Strain

$$\varepsilon_{\max,\min} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \bar{\gamma}_{xy}^{\,2}}$$ where $2\bar{\gamma} = \gamma$,

$\bar{\gamma}_{x'y'} = 0$ in condition of principal strain

Recall, principal stress)

$$\sigma_{x'} = l_1^2\sigma_x + m_1^2\sigma_y + 2l_1m_1\tau_{xy}$$
$$= \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha$$

$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + (l_1m_2 + m_1l_2)\tau_{xy}$$
$$= -\sigma_x\cos\alpha\sin\alpha + \sigma_y\sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\tau_{xy}$$

Min. or Max value in condition $\dfrac{\partial \sigma_{x'}}{\partial \alpha} = 0$

At principal angle $\alpha = \dfrac{1}{2}\arctan\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$

Principal Stress

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

$\tau_{x'y'} = 0$ in condition of principal stress

Generalized Hooke’s Law

Generalized Hooke's Law

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

✓Idealization of material

Assume that the material is *perfectly elastic*, i.e., we shall limit our attention to the behavior of the material before the elastic limit is reached

✓Material Assumption

Continuous : Material having the nature of a structureless mass

Homogeneous : In case the elastic properties are the same throughout the body, i.e., are independent of the location

Isotropic : In case the elastic properties of the body are the same in all directions about any given point

✓Generalized Hooke's Law

- **Hooke's Law : "Extension is proportional to force"** $\sigma = E\varepsilon$
- **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

$$
\begin{aligned}
\sigma_x &= c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\
\sigma_y &= c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\
\sigma_z &= c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\
\tau_{xy} &= c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\
\tau_{yz} &= c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\
\tau_{zx} &= c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx}
\end{aligned}
$$

36 constants?

the principal directions and isotropy

the principal strains and principal stress occur in the same directions1)

reduced to only *two independent constants*

$$
\begin{cases}
\sigma_1 = \lambda e + 2\mu\varepsilon_1 \\
\sigma_2 = \lambda e + 2\mu\varepsilon_2 \\
\sigma_3 = \lambda e + 2\mu\varepsilon_3
\end{cases}
$$

μ, λ : Lame Elastic constant

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G$$

ν : Poisson's Ratio
G : Shear Modulus
E : Young's Modulus

The relation between the principal stress components $\sigma_1,\ \sigma_2,\ \sigma_3$ **and the principal strain components** $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$

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1) Gere, J.M., Mechanics of Materials, Sixth Edition, Thomson, 2006, p517

Generalized Hooke's Law

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Isotropic **: In case the elastic properties of the body are the same in all directions about any given point**

● **Generalized Hooke's Law** : ***"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

the translation of *stress* between principal direction 1,2,3 and arbitrary x,y,z

$$\sigma_x = l_1^{\,2}\sigma_1 + m_1^{\,2}\sigma_2 + n_1^{\,2}\sigma_3 \qquad \tau_{xy} = l_1 l_2\sigma_1 + m_1 m_2\sigma_2 + n_1 n_2\sigma_3$$

$$\sigma_x = \lambda e + 2\mu(l_1^{\,2}\varepsilon_1 + m_1^{\,2}\varepsilon_2 + n_1^{\,2}\varepsilon_3)$$

$$\tau_{xy} = 2\mu(l_1 l_2\varepsilon_1 + m_1 m_2\varepsilon_2 + n_1 n_2\varepsilon_3)$$

where $, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\sigma_x = \lambda e + 2\mu\varepsilon_x$$

$$\tau_{xy} = \mu\gamma_{xy}$$

where $, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

the translation of *strain* between principal direction 1,2,3 and arbitrary x,y,z

$$\varepsilon_x = l_1^{\,2}\varepsilon_1 + m_1^{\,2}\varepsilon_2 + n_1^{\,2}\varepsilon_3 \qquad \gamma_{xy} = 2l_1 l_2\varepsilon_1 + 2m_1 m_2\varepsilon_2 + 2n_1 n_2\varepsilon_3$$

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law

the principal directions and isotropy

$$\sigma_1 = \lambda e + 2\mu\varepsilon_1$$

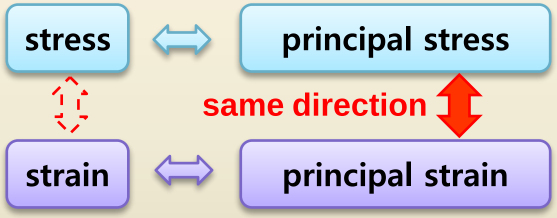
$$\sigma_2 = \lambda e + 2\mu\varepsilon_2$$

$$\sigma_3 = \lambda e + 2\mu\varepsilon_3$$

$, \lambda, \mu$: constant of Lamė

$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

relation for arbitrary system of cartesian coordinate axes?



✓Directional Cosine

$l_1^2 + l_2^2 + l_3^2 = 1$

$m_1^2 + m_2^2 + m_3^2 = 1$

$n_1^2 + n_2^2 + n_3^2 = 1$

$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$

$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$

$n_1 l_1 + n_2 l_2 + n_3 l_3 = 0$

arbitrary system of coordinates

principal directions

	1	2	3
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

$l_1^2 + l_2^2 + l_3^2 = 1$
$m_1^2 + m_2^2 + m_3^2 = 1$
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Generalized Hooke's Law

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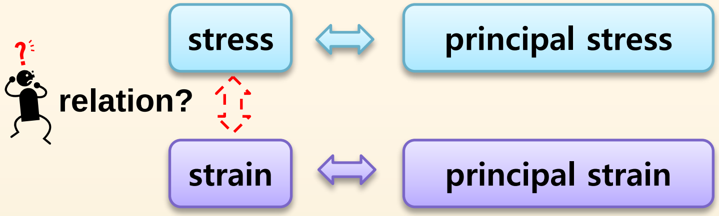
- **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

We shall show that *the directions of the principal stresses* coincides with *the directions of the principal strains*



stress ⇔ **principal stress**

strain ⇔ **principal strain**

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$$\sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx}$$
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$$\tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx}$$

Let 1,2,3 be the principal direction of *strain*

$$\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3 + c_{44}\cancel{\gamma_{12}} + c_{45}\cancel{\gamma_{23}} + c_{46}\cancel{\gamma_{31}}$$

the shearing-strain components referring to these directions are therefore zero

$$\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

recall, Principal Strain

$$\varepsilon_{\max,\min} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \bar{\gamma}_{xy}{}^2}$$

$\bar{\gamma}_{x'y'} = 0$ **in condition of principal strain**

where $2\bar{\gamma} = \gamma$

Generalized Hooke’s Law

15 Variables

6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement u, v, w

6 Stress distribution in an elastic body

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6 Relations between 6 Strain and 3 Displacement

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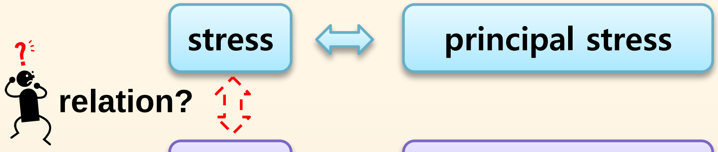

stress ⇔ principal stress

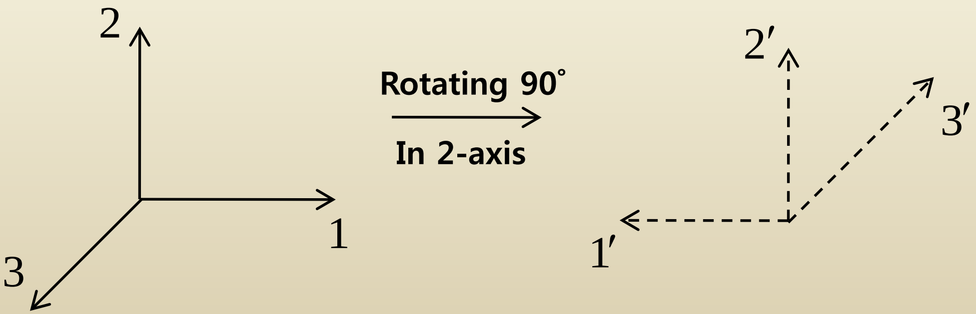
relation?

strain ⇔ principal strain

Let 1,2,3 be the principal direction of *strain*

$$\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

Introduce new frame system 1’,2’,3’



Generalized Hooke's Law

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
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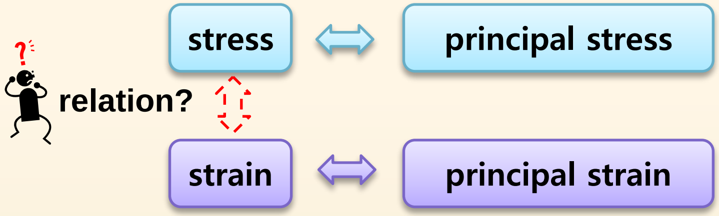

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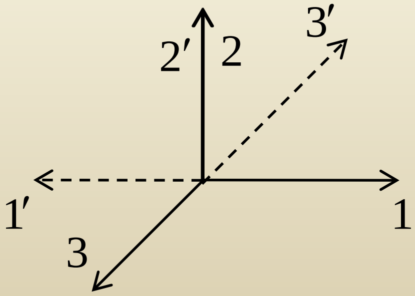
Introduce new frame system 1',2',3'



✓Directional Cosine

	1	2	3
1'	l_1	m_1	n_1
2'	l_2	m_2	n_2
3'	l_3	m_3	n_3

$$l_1 = n_3 = \cos 180° = -1, \quad m_2 = \cos 0° = 1$$

$$l_2 = l_3 = m_1 = m_3 = n_1 = n_2 = \cos 90° = 0$$

Generalized Hooke's Law

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
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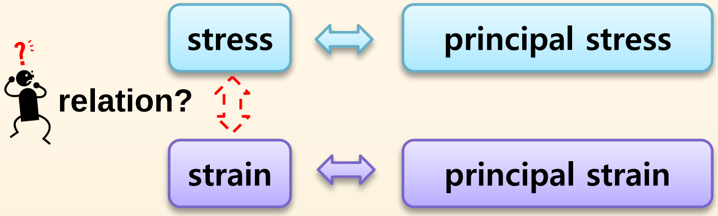


stress ⇔ **principal stress**

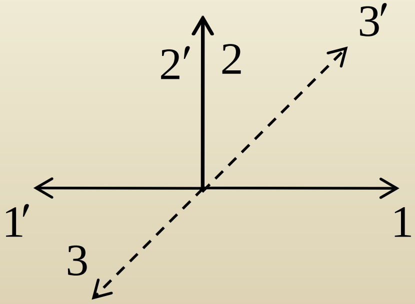
relation?

strain ⇔ **principal strain**

Let 1,2,3 be the principal direction of *strain*

$$\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

For *isotropic* materials the elastic constant such as c_{41}, c_{42}, and c_{43} should not be changed with the direction



Since 1'2'3 coordinates axes are also in the principal direction of strain

$$\tau_{1'2'} = c_{41}\varepsilon_{1'} + c_{42}\varepsilon_{2'} + c_{43}\varepsilon_{3'}$$

✓**Directional Cosine**

	1	2	3
1′	l_1	m_1	n_1
2′	l_2	m_2	n_2
3′	l_3	m_3	n_3

$$l_1 = n_3 = \cos 180° = -1, \quad m_2 = \cos 0° = 1$$

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6 Stress distribution in an elastic body

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stress ⟺ **principal stress**

relation?

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Let 1,2,3 be the principal direction of *strain*

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isotropic material
→ same c_{41}, c_{42}, c_{43}

✓**Directional Cosine**

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$1'$	$l_1 = -1$	$m_1 = 0$	$n_1 = 0$
$2'$	$l_2 = 0$	$m_2 = 1$	$n_2 = 0$
$3'$	$l_3 = 0$	$m_3 = 0$	$n_3 -1$

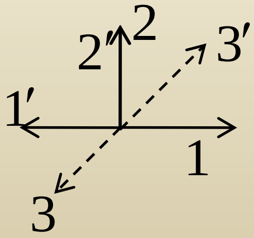
From the Transformation Equations for Stress

$$\tau_{x'y'} = l_1 l_2 \sigma_x + m_1 m_2 \sigma_y + n_1 n_2 \sigma_z + (l_1 m_2 + m_1 l_2)\tau_{xy} + (m_1 n_2 + n_1 m_2)\tau_{yz} + (n_1 l_2 + l_1 n_2)\tau_{zx}$$

$$\tau_{1'2'} = l_1 l_2 \sigma_1 + m_1 m_2 \sigma_2 + n_1 n_2 \sigma_3 + (l_1 m_2 + m_1 l_2)\tau_{12} + (m_1 n_2 + n_1 m_2)\tau_{23} + (n_1 l_2 + l_1 n_2)\tau_{31}$$

$$\tau_{1'2'} = (-1)\cdot 0\cdot\sigma_1 + 0\cdot 1\cdot\sigma_2 + 0\cdot 0\cdot\sigma_3 + ((-1)\cdot 1 + 0\cdot 0)\tau_{12} + (0\cdot 0 + 0\cdot 1)\tau_{23} + (0\cdot 0 + (-1)\cdot 0)\tau_{31}$$

$$\tau_{1'2'} = -\tau_{12}$$



Generalized Hooke’s Law

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx}$
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6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

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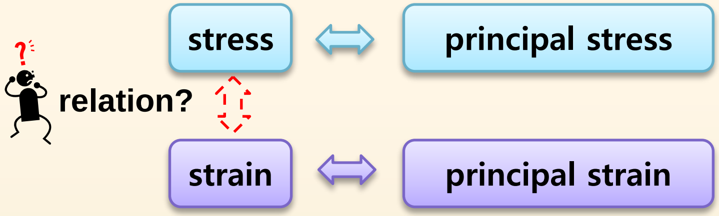


stress ⟺ **principal stress**

relation?

strain ⟺ **principal strain**

Let 1,2,3 be the principal direction of *strain*

$$\left.\begin{aligned}\tau_{12} &= c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3 \\ \tau_{1'2'} &= c_{41}\varepsilon_{1'} + c_{42}\varepsilon_{2'} + c_{43}\varepsilon_{3'}\end{aligned}\right)$$

isotropic material
→ same c_{41}, c_{42}, c_{43}

✓Directional Cosine

	1	2	3
$1'$	$l_1 = -1$	$m_1 = 0$	$n_1 = 0$
$2'$	$l_2 = 0$	$m_2 = 1$	$n_2 = 0$
$3'$	$l_3 = 0$	$m_3 = 0$	$n_3 -1$

From the Transformation Equations for Stress

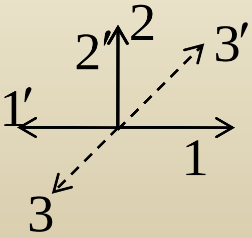
$$\tau_{1'2'} = -\tau_{12}$$

From the Transformation Equations for Strain

$$\varepsilon_{x'} = l_1{}^2\varepsilon_x + m_1{}^2\varepsilon_y + n_1{}^2\varepsilon_z + l_1m_1\gamma_{xy} + m_1n_1\gamma_{yz} + n_1l_1\gamma_{zx}$$

$$\varepsilon_{y'} = l_2{}^2\varepsilon_x + m_2{}^2\varepsilon_y + n_2{}^2\varepsilon_z + l_2m_2\gamma_{xy} + m_2n_2\gamma_{yz} + n_2l_2\gamma_{zx}$$

$$\varepsilon_{z'} = l_3{}^2\varepsilon_x + m_3{}^2\varepsilon_y + n_3{}^2\varepsilon_z + l_3m_3\gamma_{xy} + m_3n_3\gamma_{yz} + n_3l_3\gamma_{zx}$$



Generalized Hooke's Law

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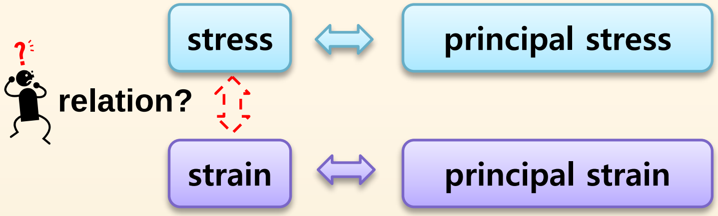

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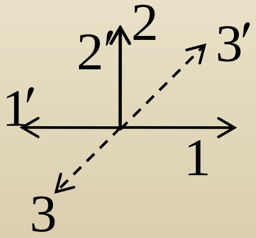
$$\tau_{1'2'} = -\tau_{12}$$

From the Transformation Equations for Strain

$$\varepsilon_{1'} = (-1)^2 \cdot \varepsilon_1 + 0^2 \cdot \varepsilon_2 + 0^2 \cdot \varepsilon_3 + (-1)\cdot 0 \cdot \gamma_{12} + 0 \cdot 0 \cdot \gamma_{23} + 0 \cdot (-1) \cdot \gamma_{31}$$

$$\varepsilon_{2'} = 0^2 \cdot \varepsilon_1 + 1^2 \cdot \varepsilon_2 + 0^2 \cdot \varepsilon_3 + 0 \cdot 1 \cdot \gamma_{12} + 1 \cdot 0 \cdot \gamma_{23} + 0 \cdot 0 \cdot \gamma_{31}$$

$$\varepsilon_{3'} = 0^2 \cdot \varepsilon_1 + 0^2 \cdot \varepsilon_2 + (-1)^2 \cdot \varepsilon_3 + 0 \cdot 0 \cdot \gamma_{12} + 0 \cdot (-1) \cdot \gamma_{23} + (-1) \cdot 0 \cdot \gamma_{31}$$



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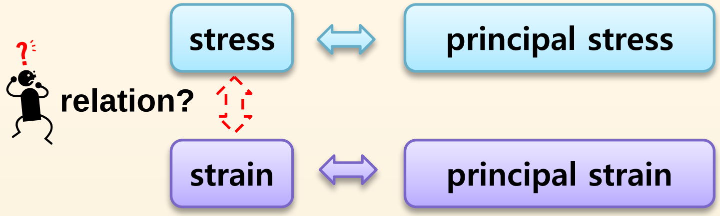
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$$\tau_{1'2'} = -\tau_{12}$$

From the Transformation Equations for Strain

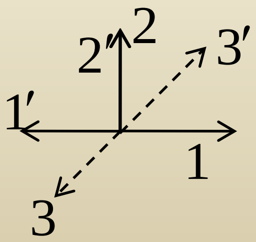
$$\varepsilon_{1'} = (-1)^2 \cdot \varepsilon_1 = \varepsilon_1$$

$$\varepsilon_{2'} = 1^2 \cdot \varepsilon_2 = \varepsilon_2$$

$$\varepsilon_{3'} = (-1)^2 \cdot \varepsilon_3 = \varepsilon_3$$

✓Directional Cosine

	1	2	3
$1'$	$l_1 = -1$	$m_1 = 0$	$n_1 = 0$
$2'$	$l_2 = 0$	$m_2 = 1$	$n_2 = 0$
$3'$	$l_3 = 0$	$m_3 = 0$	$n_3 - 1$



Generalized Hooke's Law

15 Variables

- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

6 Stress distribution in an elastic body

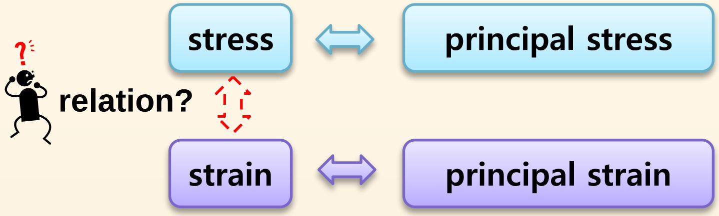
3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Isotropic **: In case the elastic properties of the body are the same in all directions about any given point**

- **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***



We shall show that *the directions of the principal stresses* coincides with *the directions of the principal strains*

Let 1,2,3 be the principal direction of *strain*

$$\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

$$\tau_{1'2'} = c_{41}\varepsilon_{1'} + c_{42}\varepsilon_{2'} + c_{43}\varepsilon_{3'}$$

isotropic material
→ same c_{41}, c_{42}, c_{43}

$$\tau_{1'2'} = -\tau_{12}$$

From the Transformation Equations for Stress

$$\varepsilon_{1'} = (-1)^2 \cdot \varepsilon_1 = \varepsilon_1$$

$$\varepsilon_{2'} = 1^2 \cdot \varepsilon_2 = \varepsilon_2$$

$$\varepsilon_{3'} = (-1)^2 \cdot \varepsilon_3 = \varepsilon_3$$

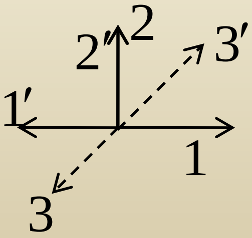
From the Transformation Equations for Strain

$$\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

$$-\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

✓**Directional Cosine**

	1	2	3
$1'$	$l_1 = -1$	$m_1 = 0$	$n_1 = 0$
$2'$	$l_2 = 0$	$m_2 = 1$	$n_2 = 0$
$3'$	$l_3 = 0$	$m_3 = 0$	$n_3 -1$



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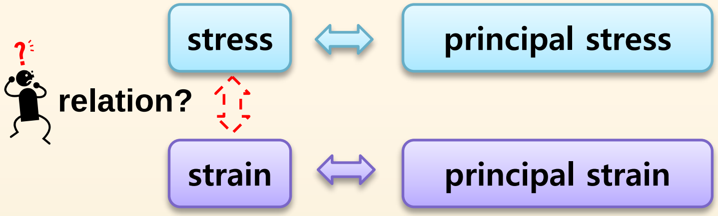

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$$\begin{aligned} \tau_{12} &= c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3 \\ \tau_{1'2'} &= c_{41}\varepsilon_{1'} + c_{42}\varepsilon_{2'} + c_{43}\varepsilon_{3'} \end{aligned}$$

isotropic material
→ same c_{41}, c_{42}, c_{43}

← the Transformation Equations for Stress / Strain

$$\begin{aligned} \tau_{12} &= c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3 \\ -\tau_{12} &= c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3 \end{aligned}$$

$$\therefore \tau_{12} = -\tau_{12}$$

$$\tau_{12} = 0$$

In same way, $\tau_{23} = 0 \, , \tau_{31} = 0$

✓Directional Cosine

	1	2	3
$1'$	$l_1 = -1$	$m_1 = 0$	$n_1 = 0$
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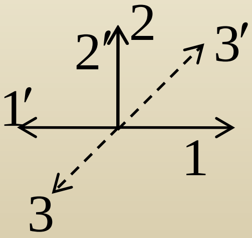

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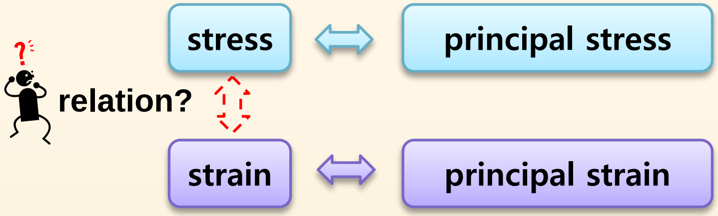
We shall show that *the directions of the principal stresses* coincides with *the directions of the principal strains*



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isotropic material

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← the Transformation Equations for Stress / Strain

$$\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

$$-\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

$$\therefore \tau_{12} = -\tau_{12} \qquad \underline{\tau_{12} = 0}$$

In same way, $\underline{\tau_{23} = 0\ , \tau_{31} = 0}$

The shearing *stress components* referred to 1,2,3 coordinate system are zero

In other word

1,2,3 are also the principal direction of *stress*

✓Directional Cosine

	1	2	3
$1'$	$l_1 = -1$	$m_1 = 0$	$n_1 = 0$
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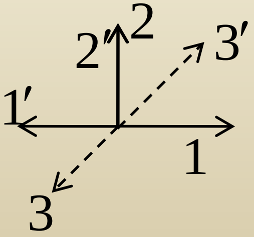

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Isotropic **: In case the elastic properties of the body are the same in all directions about any given point**

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stress ⇔ **principal stress**

relation? **same direction**

strain ⇔ **principal strain**

Let 1,2,3 be the principal direction of *strain*

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$$-\tau_{12} = c_{41}\varepsilon_1 + c_{42}\varepsilon_2 + c_{43}\varepsilon_3$$

$$\therefore \tau_{12} = -\tau_{12} \quad \Rightarrow \quad \underline{\tau_{12} = 0}$$

In same way, $\underline{\tau_{23} = 0\ , \tau_{31} = 0}$

The shearing *stress components* referred to 1,2,3 coordinate system are zero

In other word

1,2,3 are also the principal direction of *stress*

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	1	2	3
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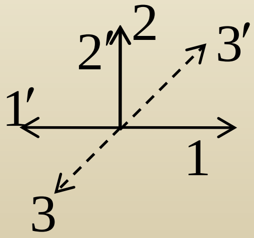

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- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
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- **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

Referring to *the principal directions*

$$\begin{cases} \sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3 \\ \sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3 \\ \sigma_3 = c_{31}\varepsilon_1 + c_{32}\varepsilon_2 + c_{33}\varepsilon_3 \end{cases}$$

c_{ij} **: the elastic property of the material relating the *stress* in the i direction to the *strain* in the j direction**

Generalized Hooke's Law

$$\begin{cases} \sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx} \end{cases}$$

36 constants? ⇩ **isotropic**

reduced to only *two independent constants*

To show this, we should introduce *principal stress* and *principal strain*

stress	⇔	**principal stress**
relation?		**same direction**
strain	⇔	**principal strain**

Generalized Hooke's Law

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
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c_{ij} **: the elastic property of the material relating the *stress* in the i direction to the *strain* in the j direction**

From the consideration of *isotropy*,

the effect of an extension ε_1 **on** σ_1

same ||

the effect of ε_2 **on** σ_2

same ||

the effect of ε_3 **on** σ_3

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law

$$\begin{cases} \sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx} \end{cases}$$

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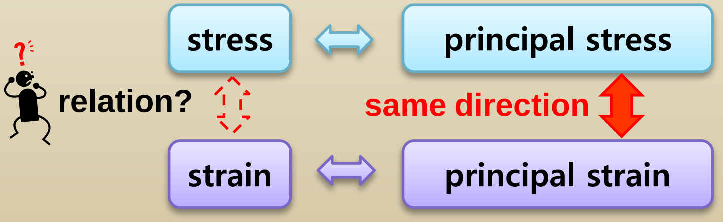

Generalized Hooke's Law

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- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

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- **Generalized Hooke's Law : "*each of the six stress components may be expresses as a linear function of the six components of strain, and conversely*" (six 'stress-strain' equations)**

Referring to *the principal directions*

$$\begin{cases} \sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3 \\ \sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3 \\ \sigma_3 = c_{31}\varepsilon_1 + c_{32}\varepsilon_2 + c_{33}\varepsilon_3 \end{cases}$$

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36 constants? **isotropic**

reduced to only *two independent constants*

To show this, we should introduce *principal stress* and *principal strain*

From the consideration of *isotropy*,

the effect of an extension ε_1 **on** σ_1

same ||

the effect of ε_2 **on** σ_2

same ||

the effect of ε_3 **on** σ_3

means, $c_{11} = c_{22} = c_{33}$

relation?

stress ⟺ **principal stress**

same direction

strain ⟺ **principal strain**

Generalized Hooke's Law

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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6 Stress distribution in an elastic body

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6 Relations between 6 Strain and 6 Stress

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isotropy

$$c_{11} = c_{22} = c_{33}$$

c_{ij} **: the elastic property of the material relating the *stress* in the i direction to the *strain* in the j direction**

Generalized Hooke's Law

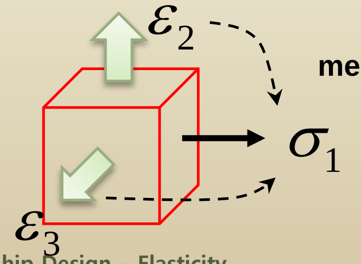
$$\begin{cases} \sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx} \end{cases}$$

36 constants? **isotropic**

reduced to only *two independent constants*

From the consideration of *isotropy*,

the effect on σ_1 of ε_2 and ε_3 must be indistinguishable

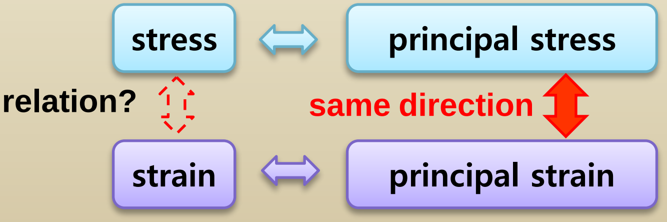


means equal effect

$$\therefore c_{12} = c_{13}$$

similarly, $c_{21} = c_{23}\,,\ c_{31} = c_{32}$

To show this, we should introduce *principal stress* and *principal strain*



relation?

stress ⇔ **principal stress**

same direction

strain ⇔ **principal strain**

Generalized Hooke's Law

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

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Referring to *the principal directions*

$$\begin{cases}\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3 \\ \sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3 \\ \sigma_3 = c_{31}\varepsilon_1 + c_{32}\varepsilon_2 + c_{33}\varepsilon_3\end{cases}$$

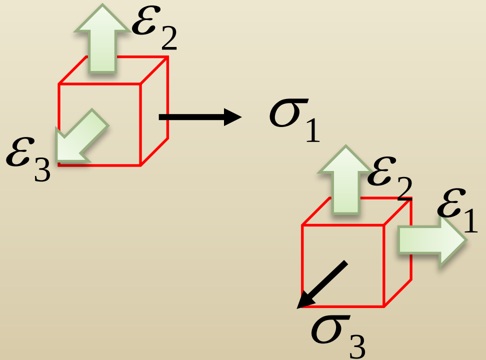
isotropy

$$c_{11} = c_{22} = c_{33}$$

$$c_{12} = c_{13},\ c_{21} = c_{23},\ c_{31} = c_{32}$$

c_{ij} **: the elastic property of the material relating the *stress* in the i direction to the *strain* in the j direction**

From the consideration of *isotropy*,



moreover,

the effect of ε_2 **or** ε_3 **on** σ_1

must be identical to

the effect of ε_1 **or** ε_3 **on** σ_2

$$\therefore c_{12} = c_{13} = c_{21} = c_{23} = c_{31} = c_{32}$$

Generalized Hooke's Law

$$\begin{cases}\sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx}\end{cases}$$

36 constants? **isotropic**

reduced to only *two independent constants*

To show this, we should introduce *principal stress* and *principal strain*

stress ⇔ **principal stress**

relation? **same direction**

strain ⇔ **principal strain**

Generalized Hooke's Law

15 Variables

- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

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---

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law

$$\begin{cases} \sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx} \end{cases}$$

36 constants? **isotropic**

reduced to only *two independent constants*

Referring to *the principal directions*

$$\begin{cases} \sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3 \\ \sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3 \\ \sigma_3 = c_{31}\varepsilon_1 + c_{32}\varepsilon_2 + c_{33}\varepsilon_3 \end{cases}$$

From the consideration of *isotropy*,

$$c_{11} = c_{22} = c_{33}$$

$$c_{12} = c_{13} = c_{21} = c_{23} = c_{31} = c_{32}$$

$$\begin{cases} \sigma_1 = \enclose{circle}{c_{11}}\varepsilon_1 + \boxed{c_{12}}\varepsilon_2 + \boxed{c_{13}}\varepsilon_3 \\ \sigma_2 = \boxed{c_{21}}\varepsilon_1 + \enclose{circle}{c_{22}}\varepsilon_2 + \boxed{c_{23}}\varepsilon_3 \\ \sigma_3 = \boxed{c_{31}}\varepsilon_1 + \boxed{c_{32}}\varepsilon_2 + \enclose{circle}{c_{33}}\varepsilon_3 \end{cases}$$

For the principal axes, we need to be concerned only with two elastic constants

$$\begin{cases} \sigma_1 = a\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3) \\ \sigma_2 = a\varepsilon_2 + b(\varepsilon_3 + \varepsilon_1) \\ \sigma_3 = a\varepsilon_3 + b(\varepsilon_1 + \varepsilon_2) \end{cases}$$

$\enclose{circle}{a}$ $\boxed{b}$

To show this, we should introduce *principal stress* and *principal strain*

relation?

stress ⇔ **principal stress**

same direction

strain ⇔ **principal strain**

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- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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● **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

---

isotropy

$$\begin{cases}\sigma_1 = a\varepsilon_1 + b(\varepsilon_2+\varepsilon_3)\\ \sigma_2 = a\varepsilon_2 + b(\varepsilon_3+\varepsilon_1)\\ \sigma_3 = a\varepsilon_3 + b(\varepsilon_1+\varepsilon_2)\end{cases}$$

←

the principal directions

$$\begin{cases}\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3\\ \sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3\\ \sigma_3 = c_{31}\varepsilon_1 + c_{32}\varepsilon_2 + c_{33}\varepsilon_3\end{cases}$$

Introduce two constant $\lambda = b\ ,\ \mu = \frac{1}{2}(a-b)$

$$b = \lambda$$
$$a = 2\mu + b = 2\mu + \lambda$$

$$\begin{cases}\sigma_1 = (2\mu+\lambda)\varepsilon_1 + \lambda(\varepsilon_2+\varepsilon_3) = 2\mu\varepsilon_1 + \lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3)\\ \sigma_2 = (2\mu+\lambda)\varepsilon_2 + \lambda(\varepsilon_3+\varepsilon_1) = 2\mu\varepsilon_2 + \lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3)\\ \sigma_3 = (2\mu+\lambda)\varepsilon_3 + \lambda(\varepsilon_1+\varepsilon_2) = 2\mu\varepsilon_3 + \lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3)\end{cases}$$

$$\therefore \begin{cases}\sigma_1 = \lambda e + 2\mu\varepsilon_1\\ \sigma_2 = \lambda e + 2\mu\varepsilon_2\\ \sigma_3 = \lambda e + 2\mu\varepsilon_3\end{cases}$$

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

➡ *6 Relations between 6 Strain and 6 Stress*

Generalized Hooke's Law

$$\begin{cases}\sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx}\\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx}\\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx}\\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx}\\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx}\\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx}\end{cases}$$

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To show this, we should introduce *principal stress* and *principal strain*

stress ⇔ **principal stress**

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Generalized Hooke's Law

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
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---

isotropy

$$\begin{cases}\sigma_1 = a\varepsilon_1 + b(\varepsilon_2+\varepsilon_3)\\ \sigma_2 = a\varepsilon_2 + b(\varepsilon_3+\varepsilon_1)\\ \sigma_3 = a\varepsilon_3 + b(\varepsilon_1+\varepsilon_2)\end{cases} \leftarrow$$

the principal directions

$$\begin{cases}\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3\\ \sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3\\ \sigma_3 = c_{31}\varepsilon_1 + c_{32}\varepsilon_2 + c_{33}\varepsilon_3\end{cases} \leftarrow$$

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$$e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\therefore \begin{cases}\sigma_1 = \lambda e + 2\mu\varepsilon_1\\ \sigma_2 = \lambda e + 2\mu\varepsilon_2\\ \sigma_3 = \lambda e + 2\mu\varepsilon_3\end{cases} ,\ \lambda, \mu : \text{constant of Lam}\dot{e}$$

6 Stress distribution in an elastic body

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6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law

$$\begin{cases}\sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx}\\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx}\\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx}\\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx}\\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx}\\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx}\end{cases}$$

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➡ *6 Relations between 6 Strain and 6 Stress*

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---

the principal directions(1,2,3) and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

$$,\ \lambda, \mu : \text{constant of Lamė} \qquad e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

Generalized Hooke's Law

$$\begin{cases} \sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{14}\gamma_{xy} + c_{15}\gamma_{yz} + c_{16}\gamma_{zx} \\ \sigma_y = c_{21}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{24}\gamma_{xy} + c_{25}\gamma_{yz} + c_{26}\gamma_{zx} \\ \sigma_z = c_{31}\varepsilon_x + c_{32}\varepsilon_y + c_{33}\varepsilon_z + c_{34}\gamma_{xy} + c_{35}\gamma_{yz} + c_{36}\gamma_{zx} \\ \tau_{xy} = c_{41}\varepsilon_x + c_{42}\varepsilon_y + c_{43}\varepsilon_z + c_{44}\gamma_{xy} + c_{45}\gamma_{yz} + c_{46}\gamma_{zx} \\ \tau_{yz} = c_{51}\varepsilon_x + c_{52}\varepsilon_y + c_{53}\varepsilon_z + c_{54}\gamma_{xy} + c_{55}\gamma_{yz} + c_{56}\gamma_{zx} \\ \tau_{zx} = c_{61}\varepsilon_x + c_{62}\varepsilon_y + c_{63}\varepsilon_z + c_{64}\gamma_{xy} + c_{65}\gamma_{yz} + c_{66}\gamma_{zx} \end{cases}$$

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same direction ↕

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Stress-Strain Relation

Stress-Strain Relation

15 Variables

6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$

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Isotropic : In case the elastic properties of the body are the same in all directions about any given point

- **Generalized Hooke's Law** : "*each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)*

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

the principal directions and isotropy

$, \lambda, \mu : \text{constant of Lamė}$

$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

The relation between the principal stress components $\sigma_1,\ \sigma_2,\ \sigma_3$ and the principal strain components $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$

The stress-strain relations referring an arbitrary system of cartesian coordinate axes?

By using the translation of stress and the translation of strain

✓Directional Cosine

	1	2	3
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

arbitrary system of coordinates

principal directions

6 Stress distribution in an elastic body

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Generalized Hooke's Law

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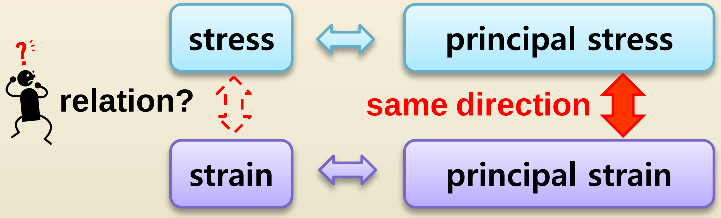

Stress-Strain Relation

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
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- **Generalized Hooke's Law** : ***"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

the transformation of *stress* between arbitrary x,y,z and arbitrary x',y',z' system

$$\sigma_{x'} = l_1^{\,2}\sigma_x + m_1^{\,2}\sigma_y + n_1^{\,2}\sigma_z + 2l_1m_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zx} \quad ①$$

$$\sigma_{y'} = l_2^{\,2}\sigma_x + m_2^{\,2}\sigma_y + n_2^{\,2}\sigma_z + 2l_2m_2\tau_{xy} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{zx} \quad ②$$

$$\sigma_{z'} = l_3^{\,2}\sigma_x + m_3^{\,2}\sigma_y + n_3^{\,2}\sigma_z + 2l_3m_3\tau_{xy} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx} \quad ③$$

$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\sigma_z + \left(l_1m_2 + m_1l_2\right)\tau_{xy} + \left(m_1n_2 + n_1m_2\right)\tau_{yz} + \left(n_1l_2 + l_1n_2\right)\tau_{zx} \quad ④$$

$$\tau_{y'z'} = l_2l_3\sigma_x + m_2m_3\sigma_y + n_2n_3\sigma_z + \left(l_2m_3 + m_2l_3\right)\tau_{xy} + \left(m_2n_3 + n_2m_3\right)\tau_{yz} + \left(n_2l_3 + l_2n_3\right)\tau_{zx} \quad ⑤$$

$$\tau_{z'x'} = l_3l_1\sigma_x + m_3m_1\sigma_y + n_3n_1\sigma_z + \left(l_3m_1 + m_3l_1\right)\tau_{xy} + \left(m_3n_1 + n_3m_1\right)\tau_{yz} + \left(n_3l_1 + l_3n_1\right)\tau_{zx} \quad ⑥$$

the transformation of *stress* between principal direction 1,2,3 and arbitrary x,y,z system

$x, y, z \rightarrow 1, 2, 3$ **principal direction**

$x', y', z' \rightarrow x, y, z$ **arbitrary system**

From ①
$$\sigma_x = l_1^{\,2}\sigma_1 + m_1^{\,2}\sigma_2 + n_1^{\,2}\sigma_3 + 2l_1m_1\cancel{\tau_{12}} + 2m_1n_1\cancel{\tau_{23}} + 2n_1l_1\cancel{\tau_{31}}$$

$$\therefore \sigma_x = l_1^{\,2}\sigma_1 + m_1^{\,2}\sigma_2 + n_1^{\,2}\sigma_3$$

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

$, \lambda, \mu$: constant of Lamė

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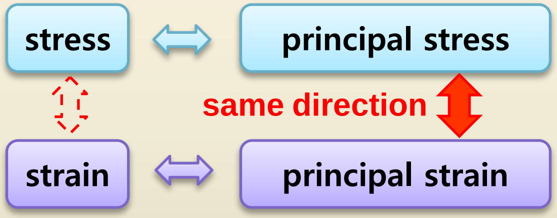
relation for arbitrary system of cartesian coordinate axes?



✓**Directional Cosine**

principal directions

arbitrary system of coordinates

	1	2	3
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$$\tau_{z'x'} = l_3l_1\sigma_x + m_3m_1\sigma_y + n_3n_1\sigma_z + (l_3m_1 + m_3l_1)\tau_{xy} + (m_3n_1 + n_3m_1)\tau_{yz} + (n_3l_1 + l_3n_1)\tau_{zx} \quad ⑥$$

the transformation of *stress* between principal direction 1,2,3 and arbitrary x,y,z system

$x, y, z \rightarrow 1, 2, 3$ **principal direction**

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From ②

$$\sigma_y = l_2^2\sigma_1 + m_2^2\sigma_2 + n_2^2\sigma_3 + 2l_2m_2\cancel{\tau_{12}} + 2m_2n_2\cancel{\tau_{23}} + 2n_2l_2\cancel{\tau_{31}}$$

$$\therefore \sigma_x = l_2^2\sigma_1 + m_2^2\sigma_2 + n_2^2\sigma_3$$

Generalized Hooke's Law

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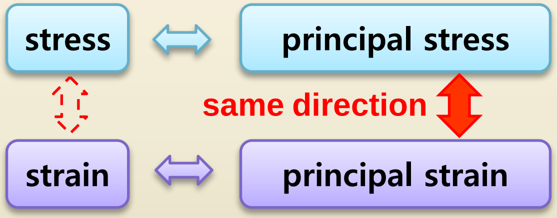

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- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$$\sigma_{y'} = l_2^2\sigma_x + m_2^2\sigma_y + n_2^2\sigma_z + 2l_2m_2\tau_{xy} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{zx} \quad ②$$

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From ③

$$\sigma_z = l_3^2\sigma_1 + m_3^2\sigma_2 + n_3^2\sigma_3 + 2l_3m_3\cancel{\tau_{12}} + 2m_3n_3\cancel{\tau_{23}} + 2n_3l_3\cancel{\tau_{31}}$$

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$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

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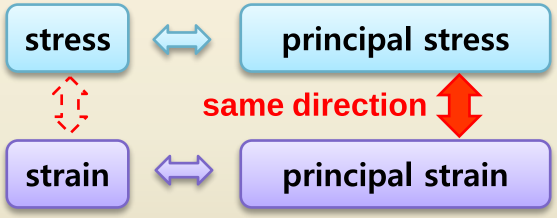
relation for arbitrary system of cartesian coordinate axes?



stress ⇔ **principal stress**

same direction

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✓Directional Cosine

principal directions

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Stress-Strain Relation

15 Variables
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$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

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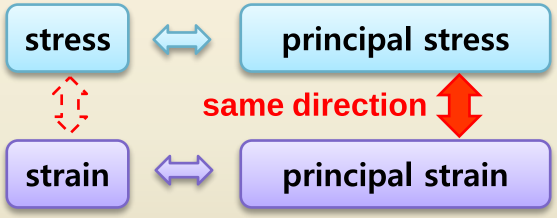

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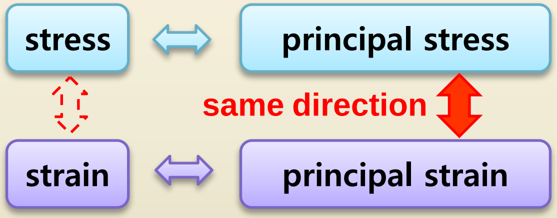

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Stress-Strain Relation

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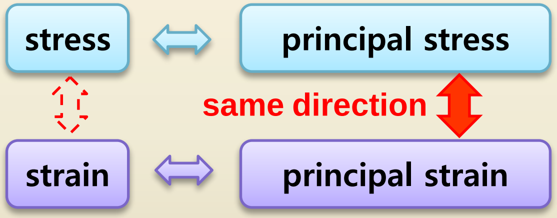

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Stress-Strain Relation

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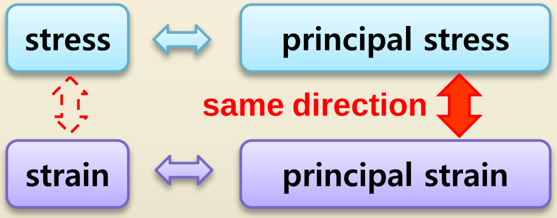

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$$\gamma_{x'y'} = 2l_1l_2\varepsilon_x + 2m_1m_2\varepsilon_y + 2n_1n_2\varepsilon_z + (l_1m_2 + m_1l_2)\gamma_{xy} + (m_1n_2 + n_1m_2)\gamma_{yz} + (n_1l_2 + l_1n_2)\gamma_{zx} \quad ④$$

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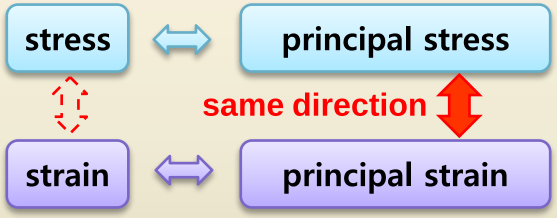
Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases} \quad \begin{array}{l} ,\lambda, \mu : \text{constant of Lamė} \\ ,e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \end{array}$$

relation for arbitrary system of cartesian coordinate axes?



✓**Directional Cosine**

principal directions

arbitrary system of coordinates

	1	2	3
x	l_1	m_1	n_1
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Stress-Strain Relation

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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the translation of *stress* between principal direction 1,2,3 and arbitrary x,y,z

$$\sigma_x = l_1^2\sigma_1 + m_1^2\sigma_2 + n_1^2\sigma_3 \qquad \tau_{xy} = l_1 l_2\sigma_1 + m_1 m_2\sigma_2 + n_1 n_2\sigma_3$$

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the translation of *strain* between principal direction 1,2,3 and arbitrary x,y,z

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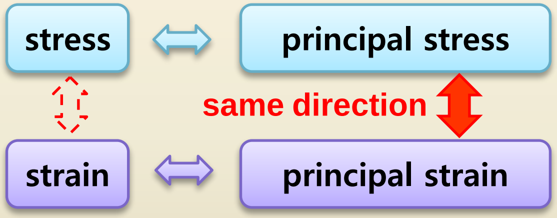
relation for arbitrary system of cartesian coordinate axes?



stress ⟺ **principal stress**

same direction

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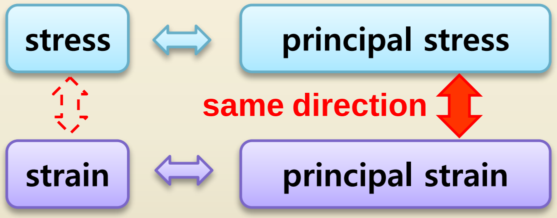

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arbitrary system of coordinates

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the translation of *strain* between principal direction 1,2,3 and arbitrary x,y,z

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where $,e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

from **"the quantity** $\varepsilon_x + \varepsilon_y + \varepsilon_z$ **is invariant with respect to orthogonal transformations of coordinates"**

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\therefore e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

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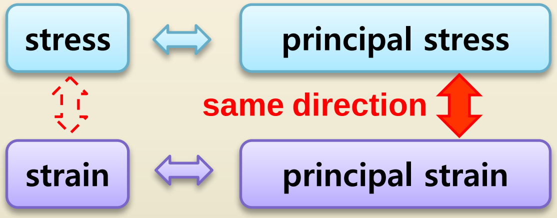

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arbitrary system of coordinates

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Stress-Strain Relation

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relation for arbitrary system of cartesian coordinate axes?

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arbitrary system of coordinates

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the translation of *strain* between principal direction 1,2,3 and arbitrary x,y,z

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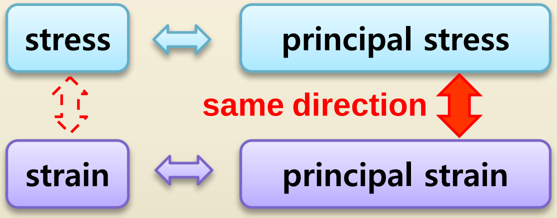

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in same way

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properties of directional cosine

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Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

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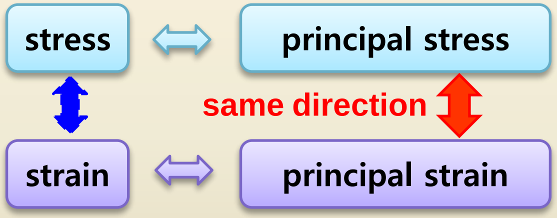

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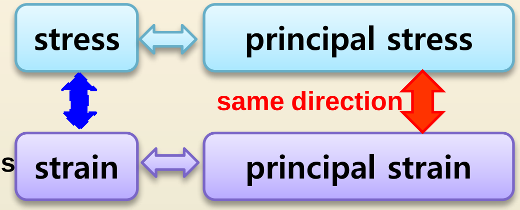
relation for arbitrary system of cartesian coordinate axes



Stress-Strain Relation

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Isotropic : In case the elastic properties of the body are the same in all directions about any given point

● **Generalized Hooke's Law** : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)*

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \qquad \tau_{xy} = \mu\gamma_{xy}$$

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Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

$, \lambda, \mu : \text{constant of Lamė}$

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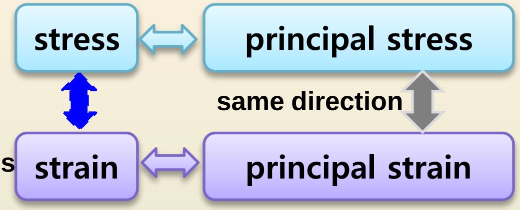
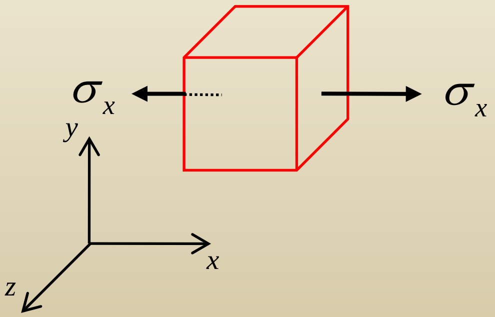


relation for arbitrary system of cartesian coordinate axes

Generalized Hooke's Law in terms of 'engineering elastic constant'

✓ ***Modulus of Elasticity 'E'*** **: the ratio of stress to strain**

in case of being submitted to normal stress σ_x but with no other stresses on the six faces



$$\varepsilon_x = E\sigma_x$$

Stress-Strain Relation

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
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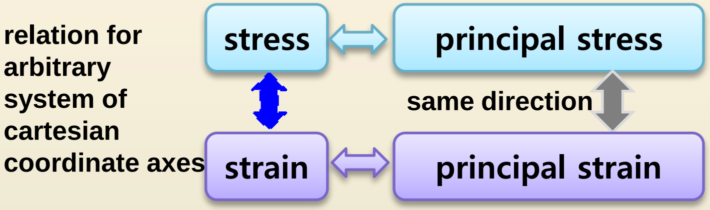


relation for arbitrary system of cartesian coordinate axes

stress ⇔ **principal stress**

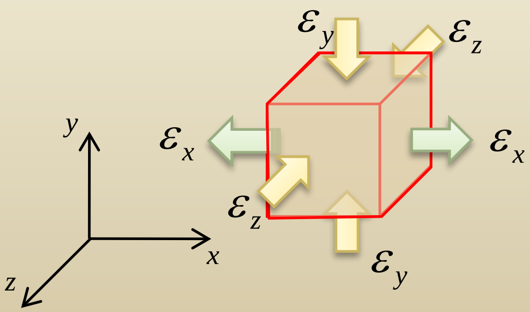
same direction

strain ⇔ **principal strain**

Generalized Hooke's Law in terms of 'engineering elastic constant'

✓***Poisson's ratio 'v'***

From experimental result, it is observed that extension of the element in x direction is accompanied by lateral contractions in the y and z direction,



$$\varepsilon_y = \varepsilon_z = -v\varepsilon_x = -v\frac{\sigma_x}{E}$$

✓***Modulus of Elasticity 'E'***

$$\varepsilon_x = \frac{1}{E}\sigma_x$$

Stress-Strain Relation

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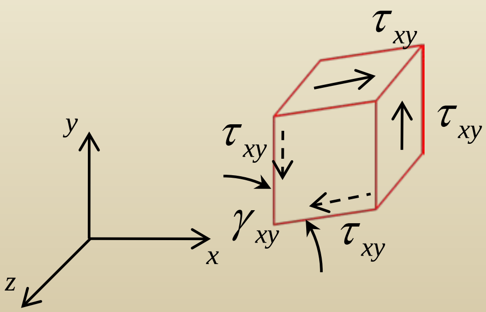
Generalized Hooke's Law in terms of 'engineering elastic constant'

✓***Modulus of elasticity in shear 'G'***

The ratio between the shearing-stress component and its corresponding shearing-strain component



$$\tau_{xy} = G\gamma_{xy}$$

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

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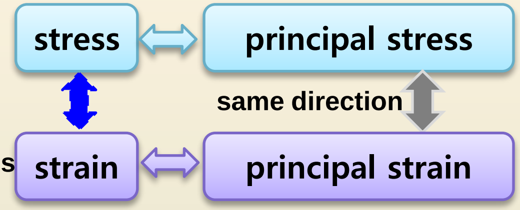

relation for arbitrary system of cartesian coordinate axes

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the principal directions and isotropy

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solving for $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

$$\sigma_x = \lambda\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z + 2\mu\varepsilon_x \cdots (1.a)$$

$$\sigma_y = \lambda\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z + 2\mu\varepsilon_y \cdots (1.b)$$

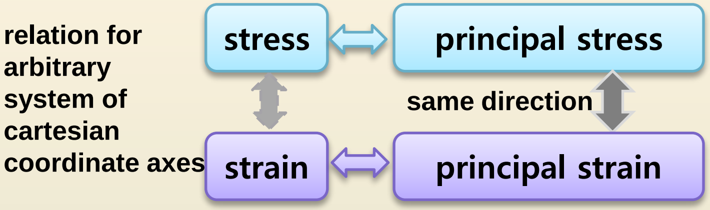
$$\sigma_z = \lambda\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z + 2\mu\varepsilon_z \cdots (1.c)$$

$$(1.a) - (1.b) : \sigma_x - \sigma_y = 2\mu\varepsilon_x - 2\mu\varepsilon_y \cdots (2.a)$$

$$(1.a) - (1.c) : \sigma_x - \sigma_z = 2\mu\varepsilon_x - 2\mu\varepsilon_z \cdots (2.b)$$

$$(2.a) : \varepsilon_y = \varepsilon_x + \frac{1}{2\mu}(\sigma_y - \sigma_x)$$

$$(2.b) : \varepsilon_z = \varepsilon_x + \frac{1}{2\mu}(\sigma_z - \sigma_x)$$


relation for arbitrary system of cartesian coordinate axes

● **engineering elastic constant**

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6 Stress distribution in an elastic body

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6 Relations between 6 Strain and 3 Displacement

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solving for $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

$$\sigma_x = \lambda\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z + 2\mu\varepsilon_x \cdots (1.a)$$

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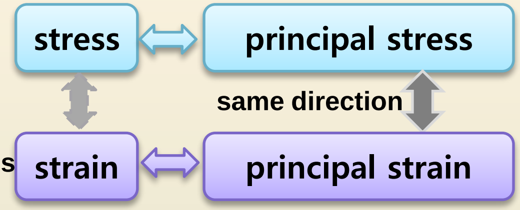
$$(1.a)-(1.b): \varepsilon_y = \varepsilon_x + \frac{1}{2\mu}(\sigma_y - \sigma_x) \cdots (2.a)$$

$$(1.a)-(1.c): \varepsilon_z = \varepsilon_x + \frac{1}{2\mu}(\sigma_z - \sigma_x) \cdots (2.b)$$

$$(2.a),(2.b) \rightarrow (1.a):$$

$$\sigma_x = \lambda\varepsilon_x + \lambda(\varepsilon_x + \frac{1}{2\mu}(\sigma_y - \sigma_x)) + \lambda(\varepsilon_x + \frac{1}{2\mu}(\sigma_z - \sigma_x)) + 2\mu\varepsilon_x$$



relation for arbitrary system of cartesian coordinate axes

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Stress-Strain Relation

15 Variables

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Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

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solving for $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

$$\sigma_x = \lambda\varepsilon_x + \lambda(\varepsilon_x + \frac{1}{2\mu}(\sigma_y - \sigma_x)) + \lambda(\varepsilon_x + \frac{1}{2\mu}(\sigma_z - \sigma_x)) + 2\mu\varepsilon_x$$

$$\sigma_x = 3\lambda\varepsilon_x + \frac{\lambda}{2\mu}(\sigma_y - \sigma_x) + \frac{\lambda}{2\mu}(\sigma_z - \sigma_x) + 2\mu\varepsilon_x$$

$$(3\lambda + 2\mu)\varepsilon_x = \sigma_x - \frac{\lambda}{2\mu}(\sigma_y - \sigma_x) - \frac{\lambda}{2\mu}(\sigma_z - \sigma_x)$$

$$(3\lambda + 2\mu)\varepsilon_x = (1 + \frac{2\lambda}{2\mu})\sigma_x - \frac{\lambda}{2\mu}(\sigma_y + \sigma_z)$$

$$\therefore \varepsilon_x = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma_x - \frac{\lambda}{2\mu(3\lambda + 2\mu)}(\sigma_y + \sigma_z)$$

Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

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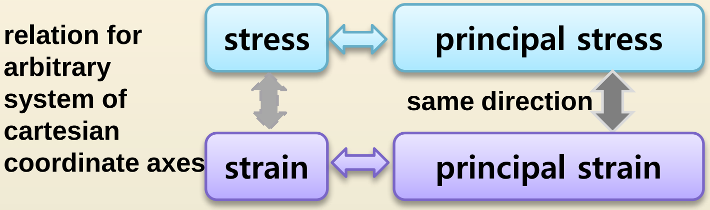

relation for arbitrary system of cartesian coordinate axes

stress ⟷ **principal stress**

same direction

strain ⟷ **principal strain**

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Stress-Strain Relation

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Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

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solving for $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

$$\therefore \varepsilon_x = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_x - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_y+\sigma_z)$$

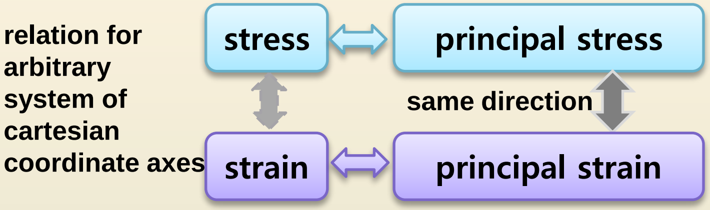
in same way

$$\varepsilon_y = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_y - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_z+\sigma_x)$$

$$\varepsilon_z = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_z - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_x+\sigma_y)$$

and

$$\gamma_{xy} = \frac{1}{\mu}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{\mu}\tau_{yz}, \quad \gamma_{zx} = \frac{1}{\mu}\tau_{zx}$$



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By comparison with engineering elastic constant

$$\varepsilon_x = \boxed{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}}\sigma_x - \boxed{\frac{\lambda}{2\mu(3\lambda+2\mu)}}(\sigma_y+\sigma_z)$$
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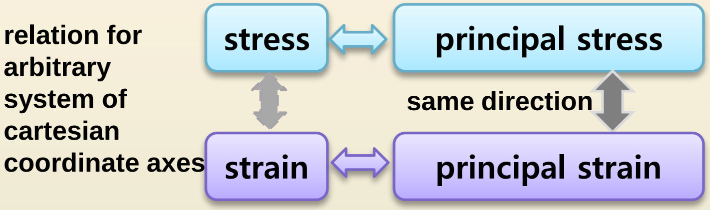
$$\gamma_{xy} = \boxed{\frac{1}{\mu}}\tau_{xy}, \quad \gamma_{yz} = \boxed{\frac{1}{\mu}}\tau_{yz}, \quad \gamma_{zx} = \boxed{\frac{1}{\mu}}\tau_{zx}$$



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$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

By comparison with engineering elastic constant

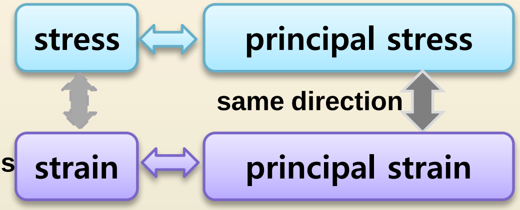
$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

$$\nu = \frac{\lambda}{2\mu(3\lambda+2\mu)} \cdot \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$
$$= \frac{\lambda}{2(\lambda+\mu)}$$

$$G = \mu$$

$$\varepsilon_x = \boxed{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}}\sigma_x - \boxed{\frac{\lambda}{2\mu(3\lambda+2\mu)}}(\sigma_y + \sigma_z)$$
$$\varepsilon_y = \boxed{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}}\sigma_y - \boxed{\frac{\lambda}{2\mu(3\lambda+2\mu)}}(\sigma_z + \sigma_x)$$
$$\varepsilon_z = \boxed{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}}\sigma_z - \boxed{\frac{\lambda}{2\mu(3\lambda+2\mu)}}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \boxed{\frac{1}{\mu}}\tau_{xy}, \quad \gamma_{yz} = \boxed{\frac{1}{\mu}}\tau_{yz}, \quad \gamma_{zx} = \boxed{\frac{1}{\mu}}\tau_{zx}$$



relation for arbitrary system of cartesian coordinate axes

stress

principal stress

same direction

strain

principal strain

- **engineering elastic constant**

✓***Modulus of Elasticity 'E'***

$$\varepsilon_x = \boxed{\frac{1}{E}}\sigma_x$$

✓***Poisson's ratio 'ν'***

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

✓***Modulus of elasticity in shear 'G'***

$$\tau_{xy} = \boxed{G}\gamma_{xy}$$

Stress-Strain Relation

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx}$
- *3 Displacement* u,v,w

Isotropic : In case the elastic properties of the body are the same in all directions about any given point

- **Generalized Hooke's Law** : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)*

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

$$\sigma_x=\lambda e+2\mu\varepsilon_x \qquad \tau_{xy}=\mu\gamma_{xy}$$
$$\sigma_y=\lambda e+2\mu\varepsilon_y \qquad \tau_{yz}=\mu\gamma_{yz}$$
$$\sigma_z=\lambda e+2\mu\varepsilon_z \qquad \tau_{zx}=\mu\gamma_{zx}$$

where

$$,e=\varepsilon_x+\varepsilon_y+\varepsilon_z$$

Generalized Hooke's Law

the principal directions and isotropy

$$\sigma_1=\lambda e+2\mu\varepsilon_1$$
$$\sigma_2=\lambda e+2\mu\varepsilon_2$$
$$\sigma_3=\lambda e+2\mu\varepsilon_3$$

$,\lambda,\mu$: constant of Lamė

$,e=\varepsilon_1+\varepsilon_2+\varepsilon_3$

By comparison with engineering elastic constant

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\ \nu=\frac{\lambda}{2(\lambda+\mu)},\ G=\mu$$

$$\varepsilon_x=\boxed{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}}\sigma_x-\boxed{\frac{\lambda}{2\mu(3\lambda+2\mu)}}(\sigma_y+\sigma_z)$$
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$$\gamma_{xy}=\boxed{\frac{1}{\mu}}\tau_{xy},\quad \gamma_{yz}=\boxed{\frac{1}{\mu}}\tau_{yz},\quad \gamma_{zx}=\boxed{\frac{1}{\mu}}\tau_{zx}$$

three independent constant?

Recall, 'only two elastic constants for isotropic material'

from $\nu=\frac{\lambda}{2(\lambda+\mu)}$ ⇨ $\lambda=2(\lambda+\mu)\nu$

$$(1-2\nu)\lambda=\mu\nu$$

$$\lambda=\frac{\mu\nu}{(1-2\nu)}$$

relation for arbitrary system of cartesian coordinate axes

stress ⇔ **principal stress**

same direction

strain ⇔ **principal strain**

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$$\varepsilon_y=\varepsilon_z=-\nu\varepsilon_x=-\boxed{\nu\frac{\sigma_x}{E}}$$

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Stress-Strain Relation

15 Variables
$$\left\{\begin{array}{ll} \text{6 Stress} & \sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx} \\ \text{6 Strain} & \varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx} \\ \text{3 Displacement} & u,v,w \end{array}\right.$$

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$$\left\{\begin{array}{l}\sigma_1=\lambda e+2\mu\varepsilon_1\\ \sigma_2=\lambda e+2\mu\varepsilon_2\\ \sigma_3=\lambda e+2\mu\varepsilon_3\end{array}\right.$$

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By comparison with engineering elastic constant

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\ \nu=\frac{\lambda}{2(\lambda+\mu)},\ G=\mu$$

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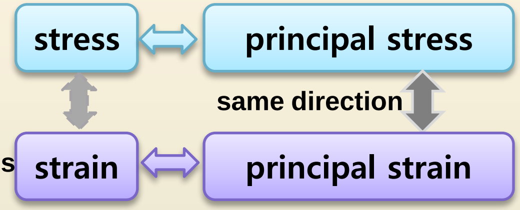
Recall, 'only two elastic constants for isotropic material'

from $\nu=\dfrac{\lambda}{2(\lambda+\mu)}$ ⇨ $\lambda=2(\lambda+\mu)\nu$ ⇩ $(1-2\nu)\lambda=2\mu\nu$ ⇩ $\lambda=\dfrac{2\mu\nu}{(1-2\nu)}$

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\leftarrow\lambda=\frac{2\mu\nu}{(1-2\nu)}$$



relation for arbitrary system of cartesian coordinate axes

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- *3 Displacement* u, v, w

6 Stress *distribution in an elastic body*

3 Equations *of force equilibrium*

6 Relations *between* *6 Strain and 3 Displacement*

6 Relations *between* *6 Strain and 6 Stress*

Isotropic **: In case the elastic properties of the body are the same in all directions about any given point**

- **Generalized Hooke's Law** : ***"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

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Generalized Hooke's Law

the principal directions and isotropy

$$\sigma_1 = \lambda e + 2\mu\varepsilon_1$$
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$, \lambda, \mu$: constant of Lamė

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By comparison with engineering elastic constant

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\ \nu = \frac{\lambda}{2(\lambda+\mu)},\ G = \mu$$

$$\varepsilon_x = \boxed{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}}\sigma_x - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_y+\sigma_z)$$
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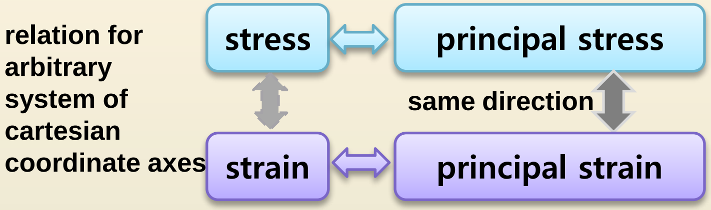
three independent constant?

Recall, 'only two elastic constants for isotropic material'

$$\lambda = \frac{2\mu\nu}{(1-2\nu)}$$

$\downarrow$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\mu\left[3\dfrac{2\mu\nu}{(1-2\nu)}+2\mu\right]}{\dfrac{2\mu\nu}{(1-2\nu)}+\mu} = \frac{\mu[6\mu\nu+2\mu(1-2\nu)]}{2\mu\nu+\mu(1-2\nu)} = \frac{\mu[2\mu\nu+2\mu]}{\mu} = 2(\nu+1)\mu$$



relation for arbitrary system of cartesian coordinate axes

stress ⇔ **principal stress**

strain ⇔ **principal strain**

same direction

- **engineering elastic constant**

✓***Modulus of Elasticity 'E'***

$$\varepsilon_x = \boxed{\frac{1}{E}}\sigma_x$$

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$$\tau_{xy} = \boxed{G}\gamma_{xy}$$

Stress-Strain Relation

15 Variables

- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

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By comparison with engineering elastic constant

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}, \quad G = \mu$$

$$\varepsilon_x = \boxed{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}}\sigma_x - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_y+\sigma_z)$$
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$$\gamma_{xy} = \boxed{\frac{1}{\mu}}\tau_{xy}, \quad \gamma_{yz} = \boxed{\frac{1}{\mu}}\tau_{yz}, \quad \gamma_{zx} = \boxed{\frac{1}{\mu}}\tau_{zx}$$

three independent constant?

Recall, 'only two elastic constants for isotropic material'

$$\lambda = \frac{2\mu\nu}{(1-2\nu)}$$
$$\downarrow$$
$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = 2(\nu+1)\mu$$

$$E = 2(\nu+1)\mu$$

$$\therefore \mu = \frac{E}{2(\nu+1)}$$

$$\therefore G = \frac{E}{2(\nu+1)}$$

Generalized Hooke's Law

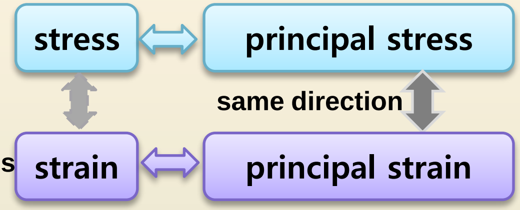
the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

$, \lambda, \mu$: constant of Lamé

$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

relation for arbitrary system of cartesian coordinate axes



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Stress-Strain Relation

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Isotropic : In case the elastic properties of the body are the same in all directions about any given point

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6 Stress distribution in an elastic body

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Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

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Generalized Hooke's Law

the principal directions and isotropy

$$\sigma_1 = \lambda e + 2\mu\varepsilon_1$$
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$, \lambda, \mu$: constant of Lamė

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relation for arbitrary system of cartesian coordinate axes

stress ⇔ **principal stress**

same direction

strain ⇔ **principal strain**

Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$
$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$
$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$
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- **engineering elastic constant**
  - ✓***Modulus of Elasticity 'E'***

$$\varepsilon_x = \boxed{\frac{1}{E}}\sigma_x$$

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$$G = \frac{E}{2(\nu+1)} \qquad \tau_{xy} = \boxed{G}\gamma_{xy}$$

Stress-Strain Relation

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
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Isotropic **: In case the elastic properties of the body are the same in all directions about any given point**

● **Generalized Hooke's Law** : ***"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

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Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

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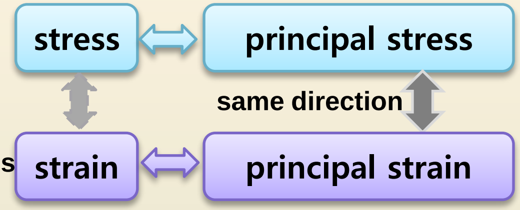
relation for arbitrary system of cartesian coordinate axes



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Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \cdots (1)$$
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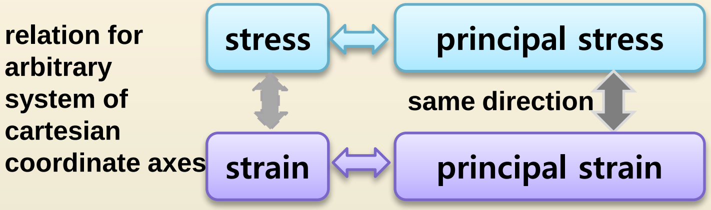
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \cdots (3)$$

$(1)-(2):$

$$E\varepsilon_x - E\varepsilon_y = \sigma_x - \sigma_y - \nu\sigma_y + \nu\sigma_x$$
$$E\varepsilon_x - E\varepsilon_y = (1+\nu)\sigma_x - (1+\nu)\sigma_y$$
$$(1+\nu)\sigma_y = (1+\nu)\sigma_x - E\varepsilon_x + E\varepsilon_y$$
$$\sigma_y = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_y \cdots (4)$$



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same direction

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$$\sigma_z = \lambda e + 2\mu\varepsilon_z \qquad \tau_{zx} = \mu\gamma_{zx}$$

where

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

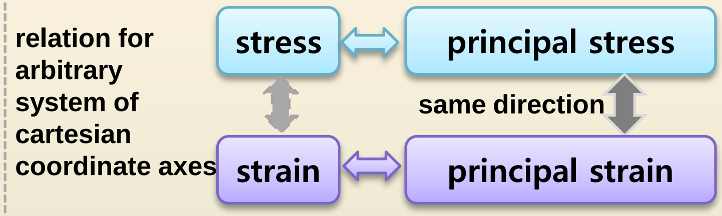
$$, \lambda, \mu : \text{constant of Lamė}$$
$$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \cdots (1)$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \cdots (2)$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \cdots (3)$$

$$\sigma_y = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_y \cdots (4)$$

$$(1)-(3):$$
$$E\varepsilon_x - E\varepsilon_z = \sigma_x - \sigma_z - \nu\sigma_z + \nu\sigma_x$$
$$E\varepsilon_x - E\varepsilon_z = (1+\nu)\sigma_x - (1+\nu)\sigma_z$$
$$(1+\nu)\sigma_z = (1+\nu)\sigma_x - E\varepsilon_x + E\varepsilon_z$$
$$\sigma_z = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_z \cdots (5)$$



relation for arbitrary system of cartesian coordinate axes

stress ⇔ **principal stress**

same direction

strain ⇔ **principal strain**

- **engineering elastic constant**
  - ✓***Modulus of Elasticity 'E'***
  - ✓***Poisson's ratio 'ν'***
  - ✓***Modulus of elasticity in shear 'G'***

Stress-Strain Relation

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

Isotropic : In case the elastic properties of the body are the same in all directions about any given point

- **Generalized Hooke's Law** : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)*

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \qquad \tau_{xy} = \mu\gamma_{xy}$$
$$\sigma_y = \lambda e + 2\mu\varepsilon_y \qquad \tau_{yz} = \mu\gamma_{yz}$$
$$\sigma_z = \lambda e + 2\mu\varepsilon_z \qquad \tau_{zx} = \mu\gamma_{zx}$$

where

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

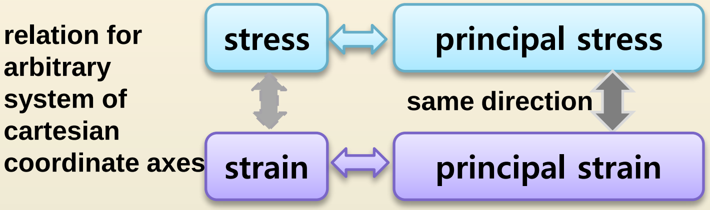
Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

$, \lambda, \mu$: constant of Lamė

$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$



relation for arbitrary system of cartesian coordinate axes

- **engineering elastic constant**
  - ✓ ***Modulus of Elasticity 'E'***
  - ✓ ***Poisson's ratio 'ν'***
  - ✓ ***Modulus of elasticity in shear 'G'***

Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \cdots (1) \qquad \sigma_y = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_y \cdots (4)$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \cdots (2) \qquad \sigma_z = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_z \cdots (5)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \cdots (3)$$

$(4),(5) \rightarrow (1):$

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_y + \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_z)]$$

$$\varepsilon_x = \frac{1}{E}[(1-2\nu)\sigma_x + \frac{2\nu E}{(1+\nu)}\varepsilon_x - \frac{\nu E}{(1+\nu)}\varepsilon_y - \frac{\nu E}{(1+\nu)}\varepsilon_z] \Rightarrow (1-2\nu)\sigma_x = E\varepsilon_x - \frac{2\nu E}{(1+\nu)}\varepsilon_x + \frac{\nu E}{(1+\nu)}\varepsilon_y + \frac{\nu E}{(1+\nu)}\varepsilon_z$$

Stress-Strain Relation

15 Variables
- 6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- 6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- 3 Displacement u, v, w

Isotropic : In case the elastic properties of the body are the same in all directions about any given point

- **Generalized Hooke's Law** : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)*

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \qquad \tau_{xy} = \mu\gamma_{xy}$$
$$\sigma_y = \lambda e + 2\mu\varepsilon_y \qquad \tau_{yz} = \mu\gamma_{yz}$$
$$\sigma_z = \lambda e + 2\mu\varepsilon_z \qquad \tau_{zx} = \mu\gamma_{zx}$$

where

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

$, \lambda, \mu$: constant of Lamé

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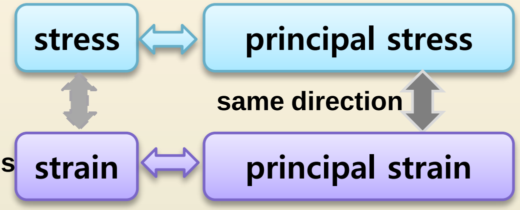

relation for arbitrary system of cartesian coordinate axes

- **engineering elastic constant**
  - ✓***Modulus of Elasticity 'E'***
  - ✓***Poisson's ratio 'ν'***
  - ✓***Modulus of elasticity in shear 'G'***

Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]\cdots(1)$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]\cdots(2)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]\cdots(3)$$

$$\sigma_y = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_y \cdots(4)$$

$$\sigma_z = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_z \cdots(5)$$

$$(4),(5) \rightarrow (1):$$

$$(1-2\nu)\sigma_x = E\varepsilon_x - \frac{2\nu E}{(1+\nu)}\varepsilon_x + \frac{\nu E}{(1+\nu)}\varepsilon_y + \frac{\nu E}{(1+\nu)}\varepsilon_z$$

$$(1-2\nu)\sigma_x = E\varepsilon_x - \frac{3\nu E}{(1+\nu)}\varepsilon_x + \frac{\nu E}{(1+\nu)}\varepsilon_x + \frac{\nu E}{(1+\nu)}\varepsilon_y + \frac{\nu E}{(1+\nu)}\varepsilon_z \Rightarrow (1-2\nu)\sigma_x = \frac{(1-2\nu)E}{(1+\nu)}\varepsilon_x + \frac{\nu E}{(1+\nu)}e \quad , e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Stress-Strain Relation

15 Variables

$$\begin{cases} \text{6 Stress} & \sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx} \\ \text{6 Strain} & \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \\ \text{3 Displacement} & u, v, w \end{cases}$$

Isotropic : In case the elastic properties of the body are the same in all directions about any given point

- **Generalized Hooke's Law** : ***"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \qquad \tau_{xy} = \mu\gamma_{xy}$$
$$\sigma_y = \lambda e + 2\mu\varepsilon_y \qquad \tau_{yz} = \mu\gamma_{yz}$$
$$\sigma_z = \lambda e + 2\mu\varepsilon_z \qquad \tau_{zx} = \mu\gamma_{zx}$$

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Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

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Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \cdots (1)$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \cdots (2)$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \cdots (3)$$

$$\sigma_y = \sigma_x - \frac{E}{(1+\nu)}\varepsilon_x + \frac{E}{(1+\nu)}\varepsilon_y \cdots (4)$$
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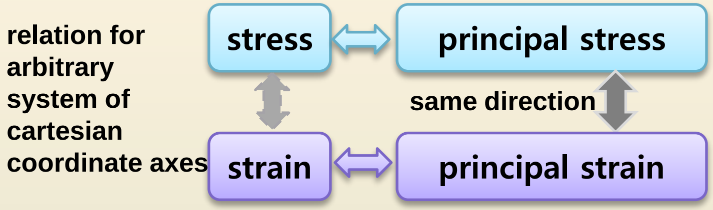

relation for arbitrary system of cartesian coordinate axes

stress ⇔ **principal stress**

same direction

strain ⇔ **principal strain**

- **engineering elastic constant**
  - ✓***Modulus of Elasticity 'E'***
  - ✓***Poisson's ratio 'v'***
  - ✓***Modulus of elasticity in shear 'G'***

$$(4),(5) \rightarrow (1):$$
$$(1-2\nu)\sigma_x = \frac{(1-2\nu)E}{(1+\nu)}\varepsilon_x + \frac{\nu E}{(1+\nu)}e \quad ⇨ \quad \therefore \sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$

in same way $\sigma_y = \dfrac{\nu E}{(1+\nu)(1-2\nu)}e + \dfrac{E}{(1+\nu)}\varepsilon_y\ ,\quad \sigma_z = \dfrac{\nu E}{(1+\nu)(1-2\nu)}e + \dfrac{E}{(1+\nu)}\varepsilon_z \qquad , e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

Stress-Strain Relation

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

6 Stress *distribution in an elastic body*

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

Isotropic **: In case the elastic properties of the body are the same in all directions about any given point**

- **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

Generalized Hooke's Law for isotropic materials referring to any arbitrary cartesian coordinates x,y,z

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \qquad \tau_{xy} = \mu\gamma_{xy}$$
$$\sigma_y = \lambda e + 2\mu\varepsilon_y \qquad \tau_{yz} = \mu\gamma_{yz}$$
$$\sigma_z = \lambda e + 2\mu\varepsilon_z \qquad \tau_{zx} = \mu\gamma_{zx}$$

where

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Generalized Hooke's Law

the principal directions and isotropy

$$\begin{cases} \sigma_1 = \lambda e + 2\mu\varepsilon_1 \\ \sigma_2 = \lambda e + 2\mu\varepsilon_2 \\ \sigma_3 = \lambda e + 2\mu\varepsilon_3 \end{cases}$$

$, \lambda, \mu$: constant of Lamė

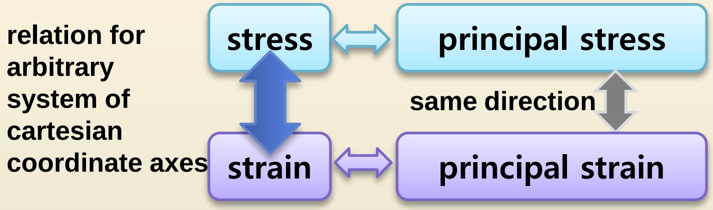
$, e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$
$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$
$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$
$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

or

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$
$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y$$
$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z$$
$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$
$$\tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx} \qquad , e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$



relation for arbitrary system of cartesian coordinate axes

stress ⇔ **principal stress**

strain ⇔ **principal strain**

same direction

- **engineering elastic constant**

✓***Modulus of Elasticity 'E'***

$$\varepsilon_x = \boxed{\frac{1}{E}}\sigma_x$$

✓***Poisson's ratio 'ν'***

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

✓***Modulus of elasticity in shear 'G'***

$$G = \frac{E}{2(\nu+1)} \qquad \tau_{xy} = \boxed{G}\gamma_{xy}$$

Stress-Strain Relation

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

6 Stress distribution in an elastic body

3 Equations of force equilibrium

6 Relations between 6 Strain and 3 Displacement

6 Relations between 6 Strain and 6 Stress

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- **Generalized Hooke's Law : *"each of the six stress components may be expresses as a linear function of the six components of strain, and conversely" (six 'stress-strain' equations)***

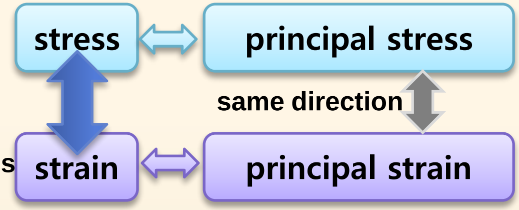
Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$
$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$
$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$
$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

or

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$
$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y$$
$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z$$
$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$
$$\tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx} \quad , e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

relation for arbitrary system of cartesian coordinate axes



- **engineering elastic constant**
  - ✓***Modulus of Elasticity 'E'***

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  - ✓***Modulus of elasticity in shear 'G'***

$$G = \frac{E}{2(\nu+1)} \qquad \tau_{xy} = \boxed{G}\gamma_{xy}$$

Three equations with three displacement components

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the three displacement components

$u,\ v,\ w$ **+ three equations of displacement**

Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

or

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z$$

$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx} \quad , e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad \gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

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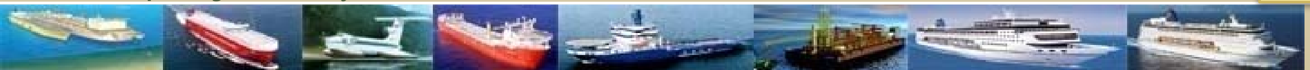

Problem in Elasticity

15 Variables

6 Stress $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the three displacement components

$u,\ v,\ w$ **+ three equations of displacement**

Generalized Hooke's Law in terms of 'engineering elastic constant'

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

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or

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$

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$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z$$

$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx} \quad , e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations between 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Problem in Elasticity

15 Variables

6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the three displacement components

$u,\ v,\ w$ **+ three equations of displacement**

①

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \Rightarrow \sigma_x = \lambda e + 2G\varepsilon_x \text{ ①}' ,\ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

②

$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy} \Rightarrow \tau_{xy} = G\gamma_{xy}$$

②′

$$\tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx} \Rightarrow \tau_{zx} = G\gamma_{zx} \quad ,G = \frac{E}{2(\nu+1)}$$

① ② → ③

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

$$\Downarrow$$

$$\frac{\partial}{\partial x}(\lambda e + 2G\varepsilon_x) + \frac{\partial}{\partial y}(G\gamma_{xy}) + \frac{\partial}{\partial z}(G\gamma_{zx}) + X = 0$$

Given : 3 Equations of force equilibrium :

③ $$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations between 6 Strain and 6 Stress

① $$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

② $$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx}$$

$$,e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

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The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the three displacement components

$u,\ v,\ w$ **+ three equations of displacement**

① ② → ③

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(\nu+1)}$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial}{\partial x}(\lambda e + 2G\varepsilon_x) + \frac{\partial}{\partial y}(G\gamma_{xy}) + \frac{\partial}{\partial z}(G\gamma_{zx}) + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + 2G\frac{\partial \varepsilon_x}{\partial x} + G\frac{\partial \gamma_{xy}}{\partial y} + G\frac{\partial \gamma_{zx}}{\partial z} + X = 0 \quad ③'$$

④ → ③'

$$\lambda\frac{\partial e}{\partial x} + 2G\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + G\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + G\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + 2G\frac{\partial^2 u}{\partial x^2} + G\frac{\partial^2 u}{\partial y^2} + G\frac{\partial^2 v}{\partial y \partial x} + G\frac{\partial^2 w}{\partial z \partial x} + G\frac{\partial^2 u}{\partial z^2} + X = 0$$

Given : 3 Equations of force equilibrium :

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$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

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①
$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z$$

②
$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

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$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

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$u,\ v,\ w$ **+ three equations of displacement**

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(\nu+1)}$$

from ① ② ③ ④

$$\lambda\frac{\partial e}{\partial x} + \boxed{2G\frac{\partial^2 u}{\partial x^2}} + \boxed{G\frac{\partial^2 u}{\partial y^2}} + G\frac{\partial^2 v}{\partial y \partial x} + G\frac{\partial^2 w}{\partial z \partial x} + \boxed{G\frac{\partial^2 u}{\partial z^2}} + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + \boxed{G\frac{\partial^2 u}{\partial x^2} + G\frac{\partial^2 u}{\partial y^2} + G\frac{\partial^2 u}{\partial z^2}} + \boxed{G\frac{\partial^2 u}{\partial x^2} + G\frac{\partial^2 v}{\partial y \partial x} + G\frac{\partial^2 w}{\partial z \partial x}} + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + \boxed{G\nabla^2 u} + \boxed{G\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\right)} + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + G\nabla^2 u + G\frac{\partial e}{\partial x} + X = 0 \quad , e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}$$

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6 Relations between 6 Strain and 6 Stress

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$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)}\varepsilon_x$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)}\varepsilon_y$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)}\varepsilon_z$$

②

$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

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from ① ② ③ ④

$$\lambda\frac{\partial e}{\partial x} + G\nabla^2 u + G\frac{\partial e}{\partial x} + X = 0$$

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0$$

in same way

$$(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 v + Y = 0$$

$$(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 w + Z = 0$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(\nu+1)}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}$$

Given : 3 Equations of force equilibrium :

③

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

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6 Relations between 6 Strain and 3 Displacement :

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6 Relations between 6 Strain and 6 Stress

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$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$

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$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

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The object : <u>find the stress distribution</u> in an elastic body or <u>find the strain</u> at any point due to <u>given body forces</u> and <u>given conditions at the boundary</u> of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the <u>three displacement components</u>

⇩

$u,\ v,\ w$ **+ three equations of displacement**

the three equations of equilibrium, expressed in terms of displacements are

$$(\lambda+G)\frac{\partial e}{\partial x}+G\nabla^2 u+X=0$$

$$(\lambda+G)\frac{\partial e}{\partial y}+G\nabla^2 v+Y=0$$

$$(\lambda+G)\frac{\partial e}{\partial z}+G\nabla^2 w+Z=0$$

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G=\frac{E}{2(\nu+1)}$$

$$e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}$$

These equations together with the <u>boundary conditions</u> completely define the three displacement components $u,\ v,\ w$

Given : <u>3 Equations</u> <u>of force equilibrium</u> :

$$\sum F_x=\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+X=0$$

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<u>6 Relations</u> between <u>6 Strain and 3 Displacement</u> :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

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<u>6 Relations</u> between 6 Strain and 6 Stress

$$\sigma_x=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$

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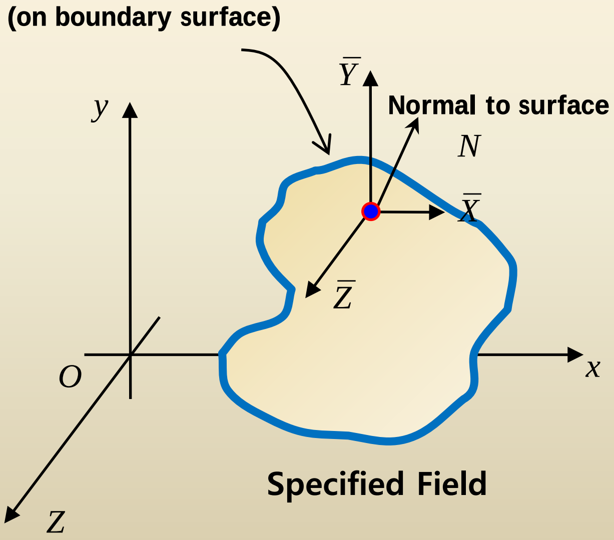
✓Boundary Conditions in terms of Given Surface Force

$$\bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$

$$\bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$$

$$\bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$$



Specified Field

directional cosine

	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
N	l	m	n

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

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6 Relations between 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

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②

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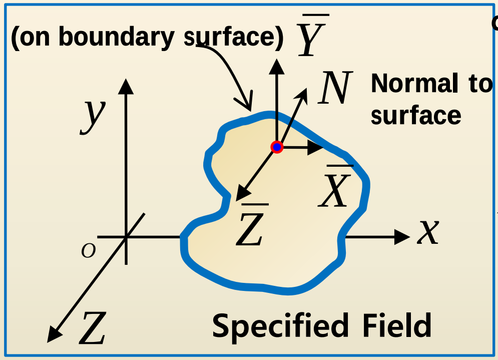

directional cosine

	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
N	l	m	n

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(\nu+1)}$$

from ①

$$\bar{X} = l(\lambda e + 2G\varepsilon_x) + mG\gamma_{xy} + nG\gamma_{zx}$$

by using ②

$$\bar{X} = l\left(\lambda e + 2G\frac{\partial u}{\partial x}\right) + mG\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + nG\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

①

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

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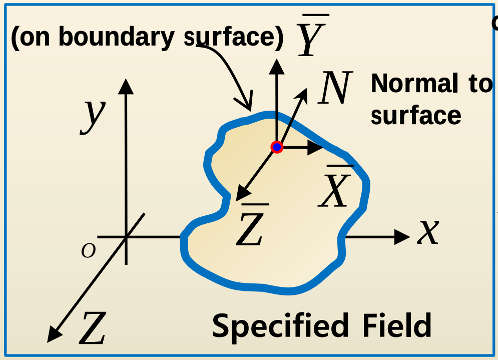

directional cosine

	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
N	l	m	n

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(\nu+1)}$$

from ① **by using** ②

$$\bar{X} = l\left(\lambda e + 2G\frac{\partial u}{\partial x}\right) + mG\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + nG\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$= l\lambda e + lG\frac{\partial u}{\partial x} + mG\frac{\partial u}{\partial y} + mG\frac{\partial v}{\partial x} + nG\frac{\partial w}{\partial x} + nG\frac{\partial u}{\partial z}$$

$$= l\lambda e + lG\frac{\partial u}{\partial x} + mG\frac{\partial u}{\partial y} + n \quad \frac{\partial u}{\partial z} GlG\frac{\partial u}{\partial x} + mG\frac{\partial v}{\partial x} + n \quad \frac{\partial w}{\partial x}$$

Given : 3 Equations of force equilibrium :

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$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

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②

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

6 Relations between 6 Strain and 6 Stress

①

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the three displacement components

⇩

$u,\ v,\ w$ **+ three equations of displacement**

These equations together with the boundary conditions completely define the three displacement components $u,\ v,\ w$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

②

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

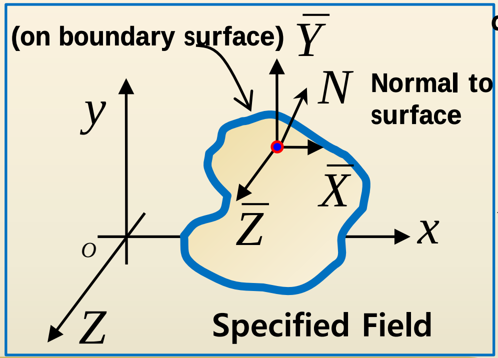
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

✓Boundary Conditions in terms of Given Surface Force

$$\bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$

$$\bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$$

$$\bar{Z} = l\tau_{zx} + m\tau_{yz} n\sigma_z$$



directional cosine

	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
N	l	m	n

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(\nu+1)}$$

from ① **by using** ②

in same way

$$\bar{X} = \lambda e l + G\left(l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z}\right) + G\left(l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x} + n\frac{\partial w}{\partial x}\right)$$

$$\bar{Y} = \lambda e m + G\left(l\frac{\partial v}{\partial x} + m\frac{\partial v}{\partial y} + n\frac{\partial v}{\partial z}\right) + G\left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y} + n\frac{\partial w}{\partial y}\right)$$

$$\bar{Z} = \lambda e n + G\left(l\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} + n\frac{\partial w}{\partial z}\right) + G\left(l\frac{\partial u}{\partial z} + m\frac{\partial v}{\partial z} + n\frac{\partial w}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

①

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)} \gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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The object : <u>find the stress distribution</u> in an elastic body or <u>find the strain</u> at any point due to <u>given body forces</u> and <u>given conditions at the boundary</u> of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the <u>three displacement components</u>

⇩

$u,\ v,\ w$ **+ three equations of displacement**

Newton's second law in equilibrium

$$m\cancel{\frac{d\mathbf{V}}{dt}} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},\ G = \frac{E}{2(\nu+1)},\ e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}$$

⇩

the three equations of equilibrium, expressed in terms of displacements are ⇨

$$(\lambda+G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0$$

$$(\lambda+G)\frac{\partial e}{\partial y} + G\nabla^2 v + Y = 0$$

$$(\lambda+G)\frac{\partial e}{\partial z} + G\nabla^2 w + Z = 0$$

Given : <u>3 Equations</u> <u>of force equilibrium</u> :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

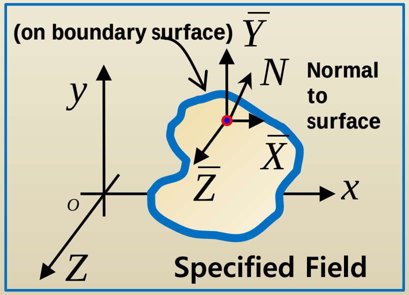
$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

<u>6 Relations</u> between <u>6 Strain and 3 Displacement</u> :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

These equations together with the <u>boundary conditions</u> completely define the three displacement components $u,\ v,\ w$



directional cosine

	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
N	l	m	n

$$\bar{X} = \lambda e l + G\left(l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z}\right) + G\left(l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x} + n\frac{\partial w}{\partial x}\right)$$

$$\bar{Y} = \lambda e m + G\left(l\frac{\partial v}{\partial x} + m\frac{\partial v}{\partial y} + n\frac{\partial v}{\partial z}\right) + G\left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y} + n\frac{\partial w}{\partial y}\right)$$

$$\bar{Z} = \lambda e n + G\left(l\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} + n\frac{\partial w}{\partial z}\right) + G\left(l\frac{\partial u}{\partial z} + m\frac{\partial v}{\partial z} + n\frac{\partial w}{\partial z}\right)$$

<u>6 Relations</u> between 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Six equations with six stress components

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

①
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\xrightarrow{\quad}$$
$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

(1′)
$$\varepsilon_y = \frac{1}{E}[(1+\nu)\sigma_y - \nu\Theta]$$
$$\varepsilon_z = \frac{1}{E}[(1+\nu)\sigma_z - \nu\Theta]$$

②
$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

③
$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

(1′) ② ⟶ ③

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{E}\left[(1+\nu)\sigma_y - \nu\Theta\right]\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\left[(1+\nu)\sigma_z - \nu\Theta\right]\right) = \frac{\partial^2}{\partial y \partial z}\left(\frac{2(1+\nu)}{E}\tau_{xy}\right)$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$
$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$
③
$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$
$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
①
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$
②
$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$
$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

①′ ② ⟶ ③

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{E}\left[(1+\nu)\sigma_y - \nu\Theta\right]\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\left[(1+\nu)\sigma_z - \nu\Theta\right]\right) = \frac{\partial^2}{\partial y \partial z}\left(\frac{2(1+\nu)}{E}\tau_{xy}\right)$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{\cancel{E}}\left[\boxed{(1+\nu)\sigma_y} - \boxed{\nu\Theta}\right]\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{\cancel{E}}\left[\boxed{(1+\nu)\sigma_z} - \boxed{\nu\Theta}\right]\right) = \frac{\partial^2}{\partial y \partial z}\left(\frac{2(1+\nu)}{\cancel{E}}\tau_{xy}\right)$$

$$(1+\nu)\left(\boxed{\frac{\partial^2\sigma_y}{\partial z^2}} + \boxed{\frac{\partial^2\sigma_z}{\partial y^2}}\right) - \nu\left(\boxed{\frac{\partial^2\Theta}{\partial z^2}} + \boxed{\frac{\partial^2\Theta}{\partial y^2}}\right) = (1+\nu)\cdot 2\frac{\partial^2\tau_{yz}}{\partial y \partial z}$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

③

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx}$
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$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ① ② ③

$$\Theta=\sigma_x+\sigma_y+\sigma_z$$

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}\right)-\nu\left(\frac{\partial^2\Theta}{\partial z^2}+\frac{\partial^2\Theta}{\partial y^2}\right)=(1+\nu)\cdot 2\frac{\partial^2\tau_{yz}}{\partial y\partial z}$$

from ④

$$\frac{\partial\tau_{zy}}{\partial z}=-\frac{\partial\tau_{xy}}{\partial x}-\frac{\partial\sigma_y}{\partial y}-Y\ \xrightarrow{\frac{\partial}{\partial y}}\ \frac{\partial^2\tau_{yz}}{\partial y\partial z}=-\frac{\partial^2\tau_{xz}}{\partial x\partial y}-\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial Y}{\partial y}$$

$$\frac{\partial\tau_{yz}}{\partial y}=-\frac{\partial\tau_{xz}}{\partial x}-\frac{\partial\sigma_z}{\partial z}-Z\ \xrightarrow{\frac{\partial}{\partial z}}\ \frac{\partial^2\tau_{zy}}{\partial y\partial z}=-\frac{\partial^2\tau_{xy}}{\partial x\partial z}-\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial Z}{\partial z}$$

Given : 3 Equations of force equilibrium :

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

④

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

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6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

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$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)]$$

①

$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)]$$

$$\varepsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]$$

$$\gamma_{xy}=\frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz}=\frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx}=\frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ① ② ③

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}\right)-\nu\left(\frac{\partial^2\Theta}{\partial z^2}+\frac{\partial^2\Theta}{\partial y^2}\right)=(1+\nu)\cdot 2\frac{\partial^2\tau_{yz}}{\partial y\partial z}$$

from ④

$$\frac{\partial\tau_{zy}}{\partial z}=-\frac{\partial\tau_{xy}}{\partial x}-\frac{\partial\sigma_y}{\partial y}-Y \xrightarrow{\frac{\partial}{\partial y}} \frac{\partial^2\tau_{yz}}{\partial y\partial z}=-\frac{\partial^2\tau_{xz}}{\partial x\partial y}-\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial Y}{\partial y}$$

$$\frac{\partial\tau_{yz}}{\partial y}=-\frac{\partial\tau_{xz}}{\partial x}-\frac{\partial\sigma_z}{\partial z}-Z \xrightarrow{\frac{\partial}{\partial z}} +\ \frac{\partial^2\tau_{zy}}{\partial y\partial z}=-\frac{\partial^2\tau_{xy}}{\partial x\partial z}-\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial Z}{\partial z}$$

$$2\frac{\partial^2\tau_{yz}}{\partial y\partial z}=-\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial}{\partial x}\left(\frac{\partial\tau_{xz}}{\partial z}+\frac{\partial\tau_{xy}}{\partial y}\right)-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}$$

Given : 3 Equations of force equilibrium :

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

④

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

$$\sum F_z=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)]$$

①

$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)]$$

$$\varepsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]$$

$$\gamma_{xy}=\frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz}=\frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx}=\frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ① ② ③

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}\right)-\nu\left(\frac{\partial^2\Theta}{\partial z^2}+\frac{\partial^2\Theta}{\partial y^2}\right)=(1+\nu)\cdot 2\frac{\partial^2\tau_{yz}}{\partial y\partial z}$$

from ④

$$\frac{\partial\tau_{zy}}{\partial z}=-\frac{\partial\tau_{xy}}{\partial x}-\frac{\partial\sigma_y}{\partial y}-Y \xrightarrow{\frac{\partial}{\partial y}} \frac{\partial^2\tau_{yz}}{\partial y\partial z}=-\frac{\partial^2\tau_{xz}}{\partial x\partial y}-\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial Y}{\partial y}$$

$$\frac{\partial\tau_{yz}}{\partial y}=-\frac{\partial\tau_{xz}}{\partial x}-\frac{\partial\sigma_z}{\partial z}-Z \xrightarrow{\frac{\partial}{\partial z}} +\ \frac{\partial^2\tau_{zy}}{\partial y\partial z}=-\frac{\partial^2\tau_{xy}}{\partial x\partial z}-\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial Z}{\partial z}$$

$$2\frac{\partial^2\tau_{yz}}{\partial y\partial z}=-\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial}{\partial x}\left(\frac{\partial\tau_{xz}}{\partial z}+\frac{\partial\tau_{xy}}{\partial y}\right)-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}$$

$$\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=-\frac{\partial\sigma_x}{\partial x}-X$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

$$\sum F_z=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)]$$

①

$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)]$$

$$\varepsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]$$

$$\gamma_{xy}=\frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz}=\frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx}=\frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ① ② ③

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2\Theta}{\partial z^2} + \frac{\partial^2\Theta}{\partial y^2}\right) = (1+\nu)\cdot 2\frac{\partial^2\tau_{yz}}{\partial y\partial z}$$

from ④

$$2\frac{\partial^2\tau_{yz}}{\partial y\partial z} = -\frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial^2\sigma_z}{\partial z^2} - \frac{\partial}{\partial x}\left(\frac{\partial\tau_{xz}}{\partial z} + \frac{\partial\tau_{xy}}{\partial y}\right) - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

$$\frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = -\frac{\partial\sigma_x}{\partial x} - X$$

$$2\frac{\partial^2\tau_{yz}}{\partial y\partial z} = -\frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial^2\sigma_z}{\partial z^2} - \frac{\partial}{\partial x}\left(-\frac{\partial\sigma_x}{\partial x} - X\right) - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

$$2\frac{\partial^2\tau_{yz}}{\partial y\partial z} = \frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial^2\sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} = \frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body,
→we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ① ② ③

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2}\right) = (1+\nu)\cdot 2\frac{\partial^2 \tau_{yz}}{\partial y \partial z}$$

from ④

$$2\frac{\partial^2 \tau_{yz}}{\partial y \partial z} = \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

③

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body,
→we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

from ④

$$\boxed{2\frac{\partial^2 \tau_{yz}}{\partial y \partial z}} = \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

from ① ② ③

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2}\right) = (1+\nu)\cdot\boxed{2\frac{\partial^2 \tau_{yz}}{\partial y \partial z}}$$

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2}\right)$$

$$= (1+\nu)\left[\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

③

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ① ② ③ ④

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}\right)-\nu\left(\frac{\partial^2\Theta}{\partial z^2}+\frac{\partial^2\Theta}{\partial y^2}\right)=(1+\nu)\left[\frac{\partial^2\sigma_x}{\partial x^2}-\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial^2\sigma_z}{\partial z^2}+\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right]$$

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}-\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)-\nu\boxed{\left(\frac{\partial^2\Theta}{\partial z^2}+\frac{\partial^2\Theta}{\partial y^2}\right)}=(1+\nu)\left[\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right]$$

$$\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial z^2}+\frac{\partial^2\Theta}{\partial y^2}-\frac{\partial^2\Theta}{\partial x^2}\right)$$

$$\left(\nabla^2\Theta-\frac{\partial^2\Theta}{\partial x^2}\right),\ \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}$$

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}-\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)-\nu\left(\nabla^2\Theta-\frac{\partial^2\Theta}{\partial x^2}\right)=(1+\nu)\left[\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right]$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

$$\sum F_z=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)]$$

①

$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)]$$

$$\varepsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]$$

$$\gamma_{xy}=\frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz}=\frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx}=\frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

from ① ② ③ ④

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

$$(1+\nu)\boxed{\left(\frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)} - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$\Downarrow$$

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)$$

$$\Downarrow$$

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \boxed{\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial x^2} + \frac{\partial^2\sigma_z}{\partial x^2}}\ \boxed{-\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial x^2} - \frac{\partial^2\sigma_z}{\partial x^2}} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)$$

$$\Downarrow$$

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2}\boxed{+\frac{\partial^2\Theta}{\partial x^2}}\ \boxed{-\frac{\partial^2\Theta}{\partial x^2}} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)$$

$$\Downarrow$$

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} = \frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

from ① ② ③ ④

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

⇩

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial x^2} \boxed{+\frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial y^2}} - \frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)$$

⇩

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial x^2} \boxed{+\frac{\partial^2\Theta}{\partial y^2}} - \frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)$$

⇩

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial y^2} \boxed{+\frac{\partial^2\sigma_x}{\partial z^2} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2}} - \frac{\partial^2\sigma_x}{\partial z^2}\right)$$

⇩

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial y^2} \boxed{+\frac{\partial^2\Theta}{\partial z^2}} - \frac{\partial^2\sigma_x}{\partial z^2}\right)$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

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6 Relations between 6 Strain and 3 Displacement :

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compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

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$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

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→we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ① ② ③ ④

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

$$(1+\nu)\boxed{\left(\frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_z}{\partial z^2}\right)} - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$\left(-\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2} - \frac{\partial^2\sigma_x}{\partial z^2}\right)$$

$$\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2} - \frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_x}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial z^2} - \frac{\partial^2\Theta}{\partial x^2}\right)$$

$$\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2}\right)$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right]$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

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6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

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compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

③

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} = \frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

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$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

from ① ② ③ ④

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

in same way

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_y - \frac{\partial^2\Theta}{\partial y^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial y^2}\right) = (1+\nu)\left[-\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_z - \frac{\partial^2\Theta}{\partial z^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial z^2}\right) = (1+\nu)\left[-\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right]$$

Given : 3 Equations of force equilibrium :

④

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

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$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

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compatibility equations *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

③

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$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

①

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

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②

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

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Problem in Elasticity

15 Variables

6 Stress $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

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$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right] \quad ①$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_y - \frac{\partial^2\Theta}{\partial y^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial y^2}\right) = (1+\nu)\left[-\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right] \quad ②$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_z - \frac{\partial^2\Theta}{\partial z^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial z^2}\right) = (1+\nu)\left[-\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right] \quad ③$$

①+②+③:

$$(1+\nu)\left(3\nabla^2\Theta - \nabla^2\Theta - \nabla^2\Theta\right) - \nu\left(3\nabla^2\Theta - \nabla^2\Theta\right) = (1+\nu)\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right]$$

⇩

$$(1-\nu)\nabla^2\Theta = -(1+\nu)\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right]$$

$$\therefore \nabla^2\Theta = -\frac{(1+\nu)}{(1-\nu)}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right] \quad ④$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad \gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

④→①:

$$\nabla^2\Theta = -\frac{(1+\nu)}{(1-\nu)}\cdot\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right] \quad ④$$

$$(1+\nu)\left(\nabla^2\Theta-\nabla^2\sigma_x-\frac{\partial^2\Theta}{\partial x^2}\right)-\nu\left(\nabla^2\Theta-\frac{\partial^2\Theta}{\partial x^2}\right)=(1+\nu)\left[\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right] \quad ①$$

$$(1+\nu)\left(-\frac{(1+\nu)}{(1-\nu)}\cdot\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right]-\nabla^2\sigma_x-\frac{\partial^2\Theta}{\partial x^2}\right)-\nu\left(-\frac{(1+\nu)}{(1-\nu)}\cdot\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right]-\frac{\partial^2\Theta}{\partial x^2}\right)=(1+\nu)\left[\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right]$$

⇩

$$\left(-(1+\nu)\cdot\frac{(1+\nu)}{(1-\nu)}\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right]-(1+\nu)\nabla^2\sigma_x-(1+\nu)\frac{\partial^2\Theta}{\partial x^2}\right)+\left(\nu\frac{(1+\nu)}{(1-\nu)}\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right]+\nu\frac{\partial^2\Theta}{\partial x^2}\right)=(1+\nu)\left[\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right]$$

⇩

$$\left(-(1+\nu)+\nu\right)\cdot\frac{(1+\nu)}{(1-\nu)}\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right]-(1+\nu)\nabla^2\sigma_x-\left((1+\nu)-\nu\right)\frac{\partial^2\Theta}{\partial x^2}=(1+\nu)\left[\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right]$$

⇩

$$-\frac{(1+\nu)}{(1-\nu)}\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right]-(1+\nu)\nabla^2\sigma_x-\frac{\partial^2\Theta}{\partial x^2}=(1+\nu)\left[\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right]$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

④→①:

$$\nabla^2\Theta = -\frac{(1+\nu)}{(1-\nu)}\cdot\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right] \quad ④$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right] \quad ①$$

$$-\frac{(1+\nu)}{(1-\nu)}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right] - (1+\nu)\nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2} = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$\Downarrow$$

$$(1+\nu)\nabla^2\sigma_x + \frac{\partial^2\Theta}{\partial x^2} = -(1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right] - \frac{(1+\nu)}{(1-\nu)}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right]$$

$$\Downarrow$$

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right] - \frac{1}{1-\nu}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right]$$

$$\Downarrow$$

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -\left(1 + \frac{1}{1-\nu}\right)\frac{\partial X}{\partial x} + \left(1 - \frac{1}{1-\nu}\right)\frac{\partial Y}{\partial y} + \left(1 - \frac{1}{1-\nu}\right)\frac{\partial Z}{\partial z} = -\left(2 + \frac{\nu}{1-\nu}\right)\frac{\partial X}{\partial x} + \left(-\frac{\nu}{1-\nu}\right)\frac{\partial Y}{\partial y} + \left(-\frac{\nu}{1-\nu}\right)\frac{\partial Z}{\partial z}$$

$$\Downarrow$$

$$\therefore \nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ + six equations of stress

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

④→①:

$$\nabla^2\Theta = -\frac{(1+\nu)}{(1-\nu)} \cdot \left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right] \quad ④$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right] \quad ①$$

$$\therefore \nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x}$$

in same way

$$\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial Y}{\partial y}$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial Z}{\partial z}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

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$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

① $$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

② $$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

③ $$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

① ② ⟶ ③

$$2\frac{\partial^2}{\partial y \partial z}\left(\frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]\right) = \frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x}\left(\frac{2(\nu+1)}{E}\tau_{yz}\right) + \frac{\partial}{\partial y}\left(\frac{2(\nu+1)}{E}\tau_{zx}\right) + \frac{\partial}{\partial z}\left(\frac{2(\nu+1)}{E}\tau_{xy}\right)\right)$$

⇩

$$\frac{2}{E}\left(\frac{\partial^2 \sigma_x}{\partial y \partial z} - \nu\frac{\partial^2 \sigma_y}{\partial y \partial z} - \nu\frac{\partial^2 \sigma_z}{\partial y \partial z}\right) = \frac{2(\nu+1)}{E}\left(-\frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial z}\right)$$

⇩

$$\left(\frac{\partial^2 \sigma_x}{\partial y \partial z} - \nu\frac{\partial^2 \sigma_y}{\partial y \partial z} - \nu\frac{\partial^2 \sigma_z}{\partial y \partial z}\right) = (\nu+1)\left(-\frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial z}\right)$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equ ③ ns *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

① $$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

② $$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

① ② → ③

$$\left(\frac{\partial^2 \sigma_x}{\partial y \partial z} - \nu \frac{\partial^2 \sigma_y}{\partial y \partial z} - \nu \frac{\partial^2 \sigma_z}{\partial y \partial z}\right) = (\nu+1)\left(-\frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial z}\right)$$

$$\Bigg(\Bigg(\boxed{\frac{\partial^2 \sigma_x}{\partial y \partial z} + \frac{\partial^2 \sigma_y}{\partial y \partial z} + \frac{\partial^2 \sigma_z}{\partial y \partial z}} - \nu \frac{\partial^2 \sigma_y}{\partial y \partial z} - \nu \frac{\partial^2 \sigma_z}{\partial y \partial z} - \frac{\partial^2 \sigma_y}{\partial y \partial z} - \frac{\partial^2 \sigma_z}{\partial y \partial z}\Bigg) = (\nu+1)\left(-\frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial z}\right)$$

$$\Bigg(\Bigg(\boxed{\frac{\partial^2 \Theta}{\partial y \partial z}} - (1+\nu)\frac{\partial^2 \sigma_y}{\partial y \partial z} - (1+\nu)\frac{\partial^2 \sigma_z}{\partial y \partial z}\Bigg) = (\nu+1)\left(-\frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial z}\right)$$

$$\frac{\partial^2 \Theta}{\partial y \partial z} = (\nu+1)\left(-\frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial z} + \frac{\partial^2 \sigma_y}{\partial y \partial z} + \frac{\partial^2 \sigma_z}{\partial y \partial z}\right)$$

$$\frac{1}{\nu+1}\frac{\partial^2 \Theta}{\partial y \partial z} = -\frac{\partial^2 \tau_{yz}}{\partial x^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial z} + \frac{\partial^2 \sigma_y}{\partial y \partial z} + \frac{\partial^2 \sigma_z}{\partial y \partial z}$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

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compatibility equ ③ **ns** *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

①

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

②

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
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$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

$$① ② \longrightarrow ③$$

$$\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = -\frac{\partial^2\tau_{yz}}{\partial x^2} + \frac{\partial^2\tau_{zx}}{\partial x\partial y} + \frac{\partial^2\tau_{xy}}{\partial x\partial z} + \boxed{\frac{\partial^2\sigma_y}{\partial y\partial z}} + \boxed{\frac{\partial^2\sigma_z}{\partial y\partial z}}$$

from ④

$$\frac{\partial\sigma_y}{\partial y} = -\frac{\partial\tau_{xy}}{\partial x} - \frac{\partial\tau_{zy}}{\partial z} - Y \xrightarrow{\frac{\partial}{\partial z}} \boxed{\frac{\partial^2\sigma_y}{\partial z\partial y}} = -\frac{\partial^2\tau_{xy}}{\partial z\partial x} - \frac{\partial^2\tau_{zy}}{\partial z^2} - \frac{\partial Y}{\partial z}$$

$$\frac{\partial\sigma_z}{\partial z} = -\frac{\partial\tau_{xz}}{\partial x} - \frac{\partial\tau_{yz}}{\partial y} - Z \xrightarrow{\frac{\partial}{\partial y}} \boxed{\frac{\partial^2\sigma_z}{\partial y\partial z}} = -\frac{\partial^2\tau_{xz}}{\partial y\partial x} - \frac{\partial^2\tau_{yz}}{\partial y^2} - \frac{\partial Z}{\partial y}$$

$$\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = -\frac{\partial^2\tau_{yz}}{\partial x^2} + \frac{\partial^2\tau_{zx}}{\partial x\partial y} + \frac{\partial^2\tau_{xy}}{\partial x\partial z} \boxed{-\frac{\partial^2\tau_{xy}}{\partial z\partial x} - \frac{\partial^2\tau_{zy}}{\partial z^2} - \frac{\partial Y}{\partial z}} \boxed{-\frac{\partial^2\tau_{xz}}{\partial y\partial x} - \frac{\partial^2\tau_{yz}}{\partial y^2} - \frac{\partial Z}{\partial y}}$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

④

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equ③**ns** *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y} \qquad 2\frac{\partial^2\varepsilon_x}{\partial y\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\partial z} \qquad 2\frac{\partial^2\varepsilon_y}{\partial z\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} = \frac{\partial^2\gamma_{zx}}{\partial z\partial x} \qquad 2\frac{\partial^2\varepsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

①

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

②

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

from ①②③④

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = -\frac{\partial^2\tau_{yz}}{\partial x^2} + \frac{\partial^2\tau_{zx}}{\partial x\partial y} + \frac{\partial^2\tau_{xy}}{\partial x\partial z} - \frac{\partial^2\tau_{xy}}{\partial z\partial x} - \frac{\partial^2\tau_{zy}}{\partial z^2} - \frac{\partial Y}{\partial z} - \frac{\partial^2\tau_{xz}}{\partial y\partial x} - \frac{\partial^2\tau_{yz}}{\partial y^2} - \frac{\partial Z}{\partial y}$$

$$\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = \boxed{-\frac{\partial^2\tau_{yz}}{\partial x^2} - \frac{\partial^2\tau_{zy}}{\partial z^2} - \frac{\partial^2\tau_{yz}}{\partial y^2}} + \frac{\partial^2\tau_{zx}}{\partial x\partial y} + \frac{\partial^2\tau_{xy}}{\partial x\partial z} - \frac{\partial^2\tau_{xy}}{\partial z\partial x} - \frac{\partial Y}{\partial z} - \frac{\partial^2\tau_{xz}}{\partial y\partial x} - \frac{\partial Z}{\partial y}$$

$$\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = \boxed{-\nabla^2\tau_{yz}} + \cancel{\frac{\partial^2\tau_{zx}}{\partial x\partial y}} + \cancel{\frac{\partial^2\tau_{xy}}{\partial x\partial z}} - \cancel{\frac{\partial^2\tau_{xy}}{\partial z\partial x}} - \frac{\partial Y}{\partial z} - \cancel{\frac{\partial^2\tau_{xz}}{\partial y\partial x}} - \frac{\partial Z}{\partial y}$$

$$\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = -\nabla^2\tau_{yz} - \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

④

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equ③**ns** *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} = \frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

①

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

②

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- 6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- 6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- 3 Displacement u, v, w

The object : <u>find the stress distribution</u> in an elastic body or <u>find the strain</u> at any point due to <u>given body forces</u> and <u>given conditions at the boundary</u> of the body

2) If we are interested in finding <u>only the stress components</u> in a body,
→we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\Theta = \sigma_x + \sigma_y + \sigma_z$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$$

from ① ② ③ ④

$$\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = -\nabla^2\tau_{yz} - \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$$

in same way

$$\nabla^2\tau_{yz} + \frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = -\left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\right)$$

$$\nabla^2\tau_{zx} + \frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x} = -\left(\frac{\partial X}{\partial z} + \frac{\partial Z}{\partial x}\right)$$

$$\nabla^2\tau_{xy} + \frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y} = -\left(\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\right)$$

Given : <u>3 Equations</u> <u>of force equilibrium</u> :

$$\sum F_x = \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0$$

④

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0$$

<u>6 Relations</u> between <u>6 Strain and 3 Displacement</u> :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equ③ns *<u>3 independent Equations</u>*

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} = \frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

<u>6 Relations</u> between 6 Strain and 6 Stress

①

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

②

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

6 Stress $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$

6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Displacement u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body,
→we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

!) Since the displacement components are not to be found in this case, the *compatibility equations* must be satisfied

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x}$$

$$\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) - 2\frac{\partial Y}{\partial y}$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) - 2\frac{\partial Z}{\partial z}$$

$$\nabla^2\tau_{xy} + \frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y} = -\left(\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\right)$$

$$\nabla^2\tau_{yz} + \frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z} = -\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)$$

$$\nabla^2\tau_{zx} + \frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x} = -\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}$$

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables

- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

Newton's second law in equilibrium

$$\cancel{m\frac{d\mathbf{V}}{dt}} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial X}{\partial x}+\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Y}{\partial y}+\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Z}{\partial z}+\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=0$$

$$\left(\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\right)+\nabla^2\tau_{xy}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y}=0$$

$$\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)+\nabla^2\tau_{yz}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z}=0$$

$$\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)+\nabla^2\tau_{zx}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x}=0$$

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0$$

$$\sum F_y = \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0$$

$$\sum F_z = \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad 2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\qquad 2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}\qquad 2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)]\qquad \gamma_{xy}=\frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)]\qquad \gamma_{yz}=\frac{2(\nu+1)}{E}\tau_{yz}$$

$$\varepsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]\qquad \gamma_{zx}=\frac{2(\nu+1)}{E}\tau_{zx}$$

Problem in Elasticity

15 Variables
- *6 Stress* $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$
- *6 Strain* $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- *3 Displacement* u, v, w

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the three displacement components

⇩

$u,\ v,\ w$ **+ three equations of displacement**

Given : 3 Equations of force equilibrium :

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

6 Relations between 6 Strain and 3 Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

compatibility equations *3 independent Equations*

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

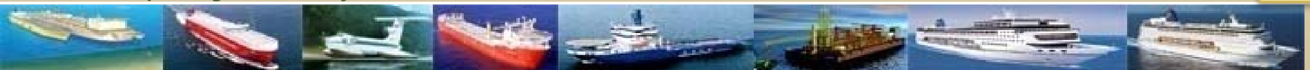
$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

6 Relations between 6 Strain and 6 Stress

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad \gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$



Problem in Elasticity

$\varepsilon_x = \frac{1}{E}\sigma_x$ $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$ $\tau_{xy} = G\gamma_{xy}$

$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ $G = \frac{E}{2(\nu+1)}$

The object : find the stress distribution in an elastic body or find the strain at any point due to given body forces and given conditions at the boundary of the body

1) The original 15 equations can also be reduced to 3 equations in terms of the three displacement components

$u,\ v,\ w$ **+ three equations of displacement**

✓3 Equations of force equilibrium

✓6 Relations between 6 Strain and 3 Displacement

✓6 Relations between 6 Strain and 6 Stress

$$(\lambda+G)\frac{\partial e}{\partial x}+G\nabla^2 u+X=0$$

$$(\lambda+G)\frac{\partial e}{\partial y}+G\nabla^2 v+Y=0$$

$$(\lambda+G)\frac{\partial e}{\partial z}+G\nabla^2 w+Z=0$$

2) If we are interested in finding only the stress components in a body, →we may reduce the system of equations to six equations with six unknown stress components

$\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ **+ six equations of stress**

✓3 Equations of force equilibrium

✓3 compatibility equations

✓6 Relations between 6 Strain and 6 Stress

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2+\frac{\partial X}{\partial x}\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Y}{\partial y}+\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Z}{\partial z}+\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=0$$

$$,\left(\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\right)+\nabla^2\tau_{xy}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y}=0$$

$$,\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)+\nabla^2\tau_{yz}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z}=0$$

$$,\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)+\nabla^2\tau_{zx}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x}=0$$