Chapter 7. In situ and induced stresses.

- Rock at depth
  - subjected to stresses (weight of overlying strata + tectonic stress)
  - disrupted when excavated. → new set of stresses
- Magnitudes and directions of in situ & induced stresses → essential for design.

1. In situ stresses

(1) Vertical stress = 2700 tonnes/m² = 27 MPa per 1000 m.

$$\sigma_v = \gamma z$$

σ_v = vertical stress
γ = unit weight of the overlying rock
z = depth below surface.

Text Fig. 7.1

(2) Horizontal stress – difficult to estimate

① $\sigma_h = k\sigma_v = k\gamma z$

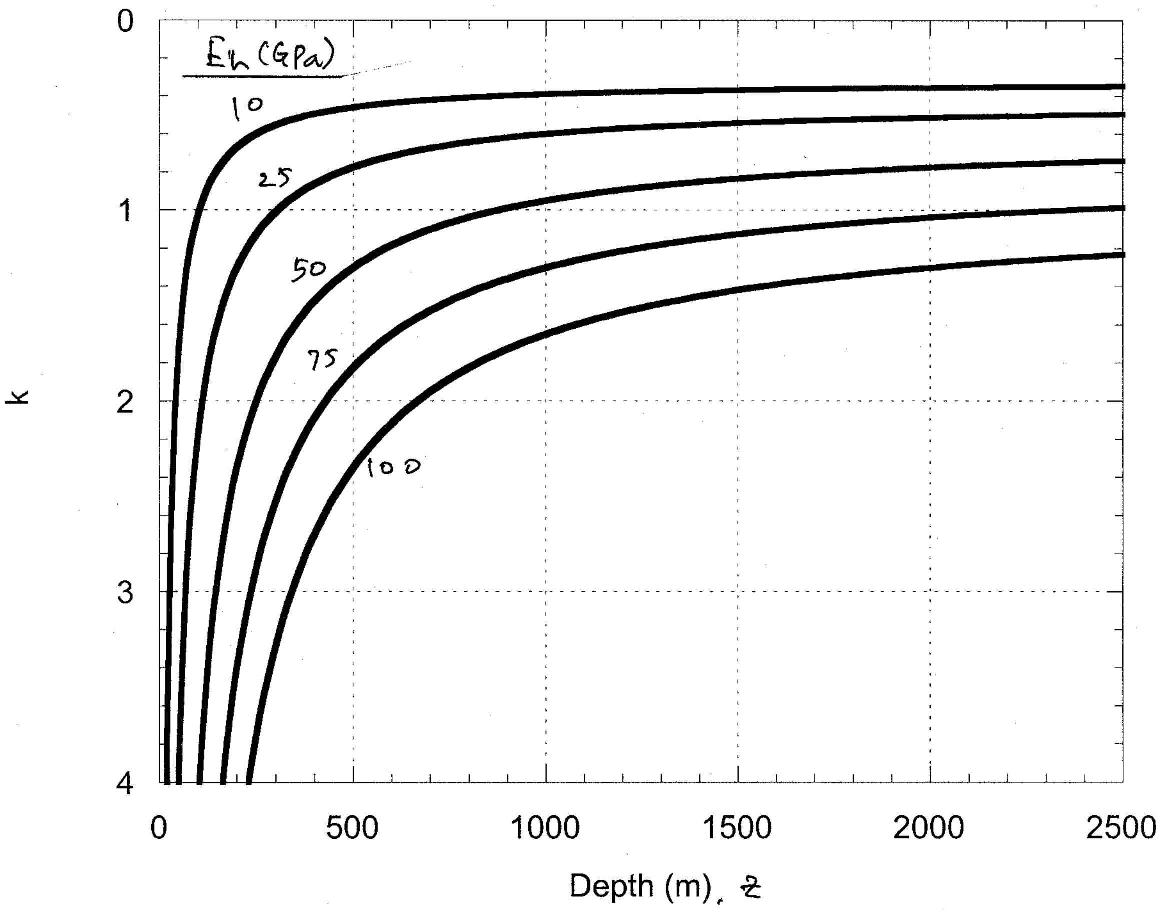
② Terzaghi & Richart (1952). $k = \frac{\nu}{1-\nu}$ (Inaccurate!)
when no lateral strain was permitted during formation of the overlying strata.

③ k { high at shallow depth / low at great depth.

④ Sheorey (1994)

$$k = 0.25 + 7E_h\left(0.001 + \frac{1}{z}\right)$$

z = depth below surface (m).
E_h = average deformation modulus (GPa) horizontal direction (∵ sedimentation).
How? Curvature of the crust and variation of elastic constants, density and thermal expansion coefficients considered.



$$k = 0.25 + 7 E_h \left(0.001 + \frac{1}{z}\right)$$

Measured σ_v > calculated γz ?
presence of high σ_h ?
$\sigma_h \neq \sigma_H$?

Topographic + geologic features.

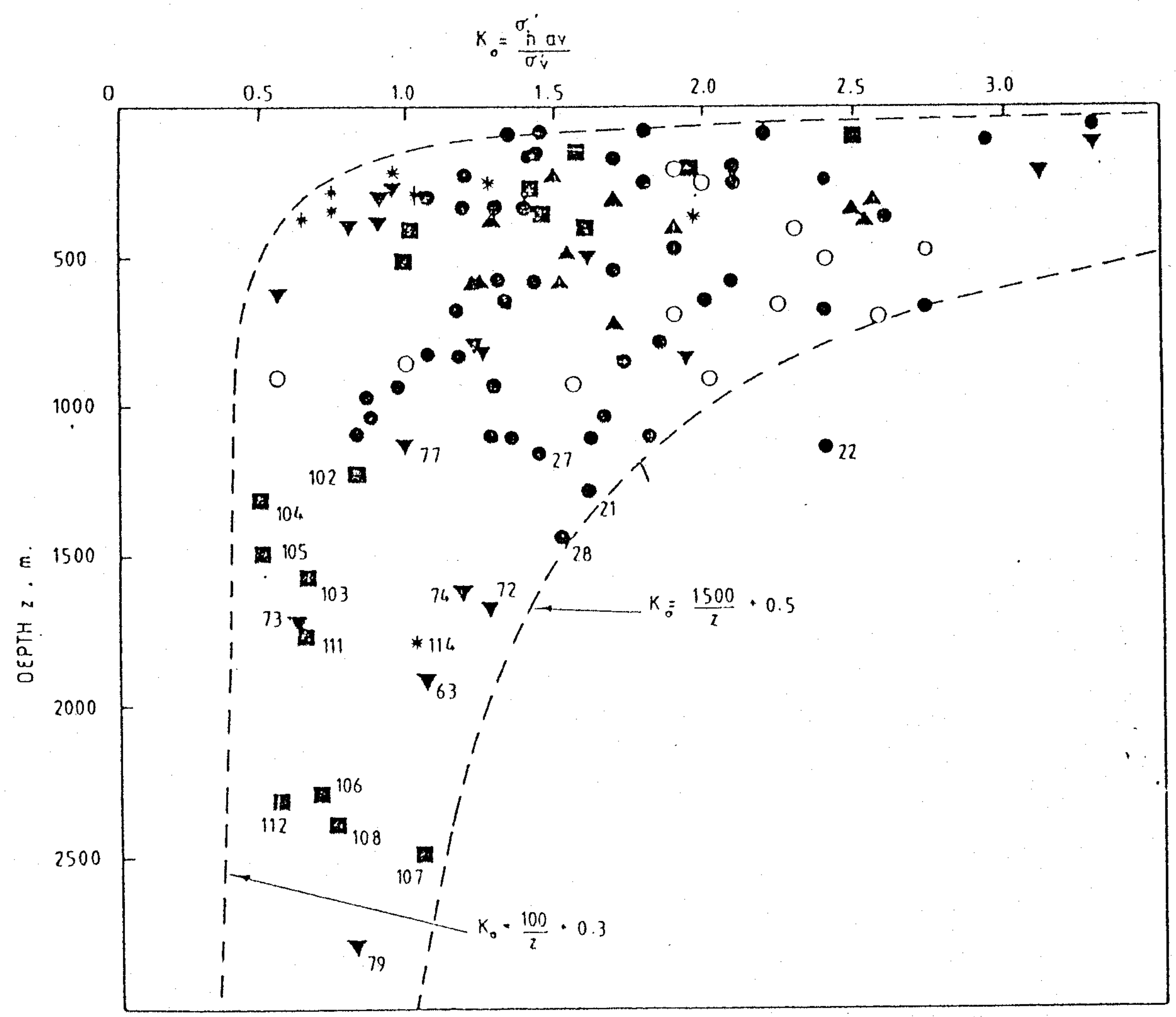
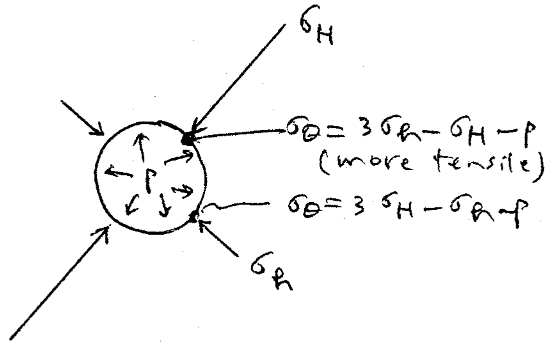


Figure 2.11 *Relation between horizontal geostatic (total) stress and depth from data collected from various sources (after Brown and Hoek 1978); see Fig. 2.10 for legend.*

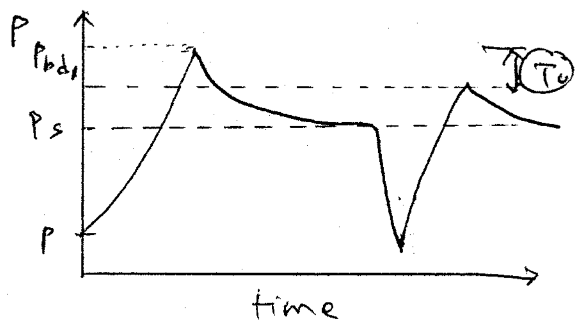
Hydraulic Fracturing.



If $\sigma_1^f = \sigma_H$, $\sigma_3^f = \sigma_h$
at $r/R = 1$ (boundary)
equations reduce to

$$\begin{cases} \sigma_r = p \\ \sigma_\theta = \sigma_H + \sigma_h + 2(\sigma_H - \sigma_h)\cos 2\theta - p \\ \tau_{r\theta} = 0 \end{cases}$$

$3\sigma_h - \sigma_H - p = -T_0 \Rightarrow \underline{P_{bd_1} = 3\sigma_h - \sigma_H + T_0}$ (Break down pressure)



Shut-in pressure $P_s = \underline{\sigma_h}, \min$

on reload $P_{bd_2} = 3\sigma_h - \sigma_H - 0$

$\therefore \quad T_0 = P_{bd_1} - P_{bd_2}$

$\sigma_h = P_s$

$\sigma_H = 3\sigma_h + T_0 - P_{bd_1} = 3P_s + T_0 - P_{bd_1}$

※ This scenario applies only for the vertical fracturing.

if internal pressure $\geq$ (vertical stress + tensile strength)
$\Rightarrow$ horizontal fracturing. $\quad (p \geq \sigma_v + T_0)$

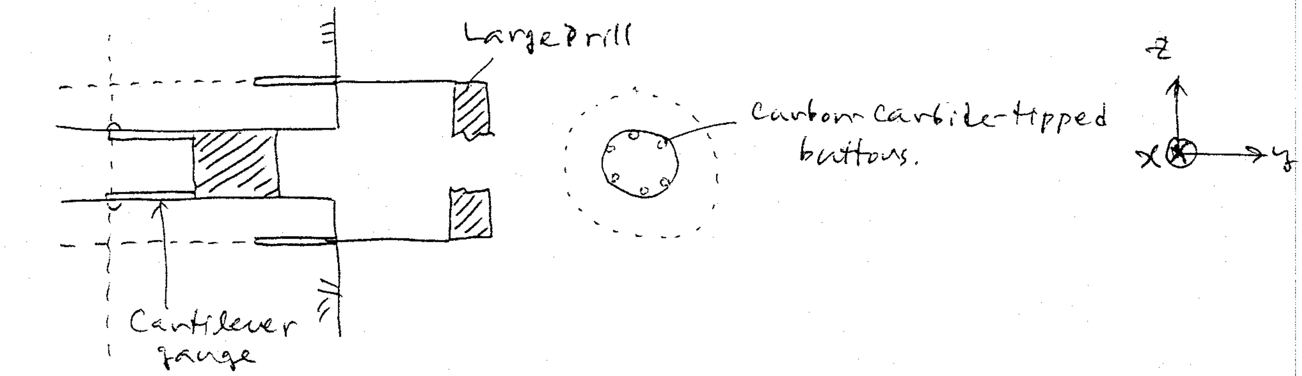
$\therefore$ To get horizontal fracture.

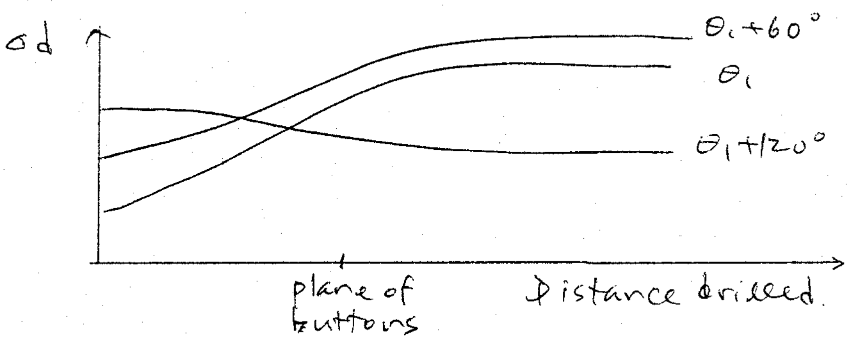
$\sigma_v + T_0 \leq 3\sigma_h - \sigma_H + T_0$

$\sigma_v \leq 3\sigma_h - \sigma_H = \left(3\frac{\sigma_h}{\sigma_H} - 1\right)\sigma_H \quad$ where $N = \frac{\sigma_h}{\sigma_H}$

$= (3N - 1)\sigma_H$

Overcoring





$\Big\{$ Knowns: $\delta d(\theta)$, $f_1 \sim f_4$, E, ν, θ,
Unknowns: $\sigma_x, \sigma_y, \sigma_z, \tau_{xz}$

$$\delta d(\theta) = \sigma_x f_1 + \sigma_y f_2 + \sigma_z f_3 + \tau_{xz} f_4$$

where
$$\begin{cases} f_1 = d\,(1 + 2\cos 2\theta)\dfrac{1-\nu^2}{E} + \dfrac{d\,\nu^2}{E} \\ f_2 = -\dfrac{d\,\nu}{E} \\ f_3 = d\,(1 - 2\cos 2\theta)\dfrac{1-\nu^2}{E} + \dfrac{d\,\nu^2}{E} \\ f_4 = d\,(4\sin 2\theta)\dfrac{1-\nu^2}{E} \end{cases}$$

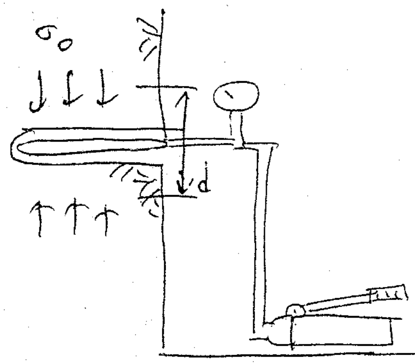
d: diameter of the borehole

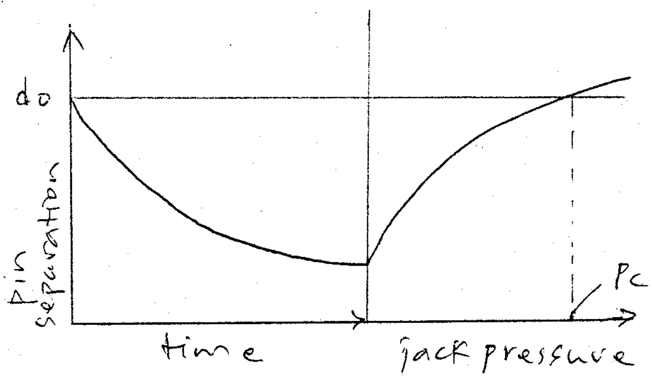
There are 3 equations.
One of the stress components is known or assumed.
i.e. $\sigma_y = 0$ (if near opening)
or $\sigma_y = \rho g z$ (if vertical hole)

$$\begin{Bmatrix} \sigma d(\theta_1) - f_2\sigma_y \\ \sigma d(\theta_1+60) - f_2\sigma_y \\ \sigma d(\theta_1+120) - f_2\sigma_y \end{Bmatrix} = \begin{pmatrix} f_{11} & f_{13} & f_{14} \\ f_{21} & f_{23} & f_{24} \\ f_{31} & f_{33} & f_{34} \end{pmatrix} \begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix}$$

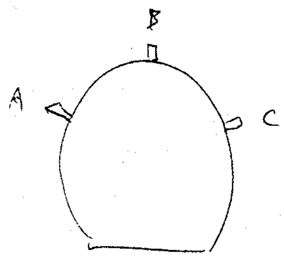
⊛ Three independent holes can solve 6 unknowns.

Flat Jack Method. (Tincelin, 1952)

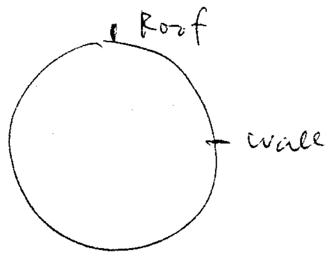




$\sigma_0 = P_c$



$$\begin{Bmatrix} \sigma_{\theta,A} \\ \sigma_{\theta,B} \\ \sigma_{\theta,C} \end{Bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$



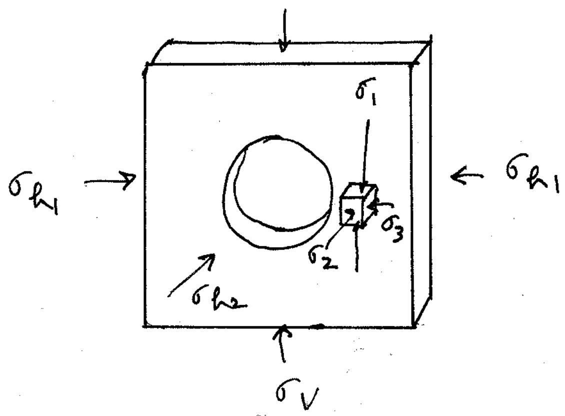
$$\begin{Bmatrix} \sigma_{\theta,W} \\ \sigma_{\theta,R} \end{Bmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \begin{Bmatrix} \sigma_h \\ \sigma_v \end{Bmatrix}$$

$$\begin{cases} \sigma_h = \frac{1}{8}\sigma_{\theta,W} + \frac{3}{8}\sigma_{\theta,R} \\ \sigma_v = \frac{3}{8}\sigma_{\theta,W} + \frac{1}{8}\sigma_{\theta,R} \end{cases}$$

7.2.1 The world stress map.

7.2.2 Developing a stress measuring programme.

2. Analysis of induced stresses.



$\sigma_V > \sigma_{h2} > \sigma_{h1}$

$\sigma_1, \sigma_2, \sigma_3$ uniform?

Yes, before excavation

↓

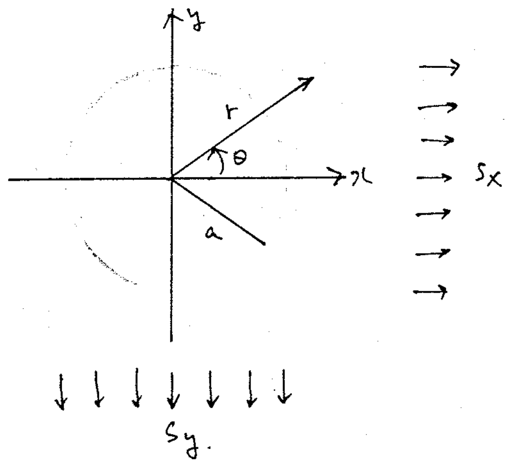
Re-distributed after excavation.

↓

How?

Infinite plate with a circular hole

(1) Configuration



Infinite plate
Thickness t
Circular hole of radius a
Applied far field stress S_x, S_y

(2) Boundary conditions.

$$(\sigma_r)_{r=\infty} = \frac{1}{2}(S_x+S_y) + \frac{1}{2}(S_x-S_y)\cos 2\theta \qquad (4.7.1)$$

$$(\tau_{r\theta})_{r=\infty} = -\frac{1}{2}(S_x-S_y)\sin 2\theta.$$

At $r=a$, $\quad (\sigma_r)_{r=a} = (\tau_{r\theta})_{r=a} = 0. \qquad (4.7.2)$

(3) stress function assumed.

$$\Phi = A\log r + Br^2 + (Cr^2 + Dr^4 + Er^{-2} + F)\cos 2\theta \qquad (4.7.3)$$

stress components

$$\begin{cases} \sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = \frac{A}{r^2} + 2B + (-2C - 6Er^{-4} - 4Fr^{-2})\cos 2\theta \\ \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} = -\frac{A}{r^2} + 2B + (2C + 12Dr^2 + 6Er^{-4})\cos 2\theta \\ \tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} = (2C + 6Dr^2 - 6Er^{-4} - 2Fr^{-2})\sin 2\theta \end{cases} \qquad (4.7.4)$$

When $r=\infty$.

$$\begin{cases} 2B = \frac{1}{2}(S_x+S_y) \\ -2C = \frac{1}{2}(S_x-S_y) \end{cases}$$

$$(\sigma_r)_{r=\infty} = 2B - 2C\cos 2\theta = \frac{1}{2}(S_x+S_y) + \frac{1}{2}(S_x-S_y)\cos 2\theta.$$

$$(\tau_{r\theta})_{r=\infty} = [2C + 6D(\infty)^2]\sin 2\theta = -\frac{1}{2}(S_x-S_y)\sin 2\theta. \quad D=0 \qquad (4.7.5)$$

When $r=a$.

$$(\sigma_r)_{r=a} = \frac{A}{a^2} + 2B + (-2C - 6Ea^{-4} - 4Fa^{-2})\cos 2\theta = 0 \quad \begin{cases} \frac{A}{a^2} + 2B = 0 \\ -2C - 6Ea^{-4} - 4Fa^{-2} = 0 \end{cases}$$

$$(\tau_{r\theta})_{r=a} = (2C + 6Da^2 - 6Ea^{-4} - 2Fa^{-2})\sin 2\theta.$$

$$-2C - 6Ea^{-4} - 2Fa^{-2} = 0$$

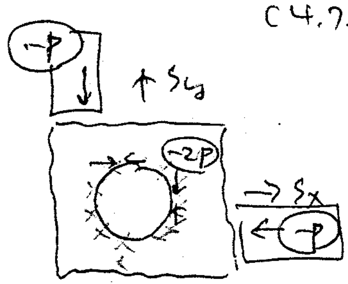
Solving Eq (4.7.5).

$$A = -\frac{a^2}{2}(S_x + S_y)$$
$$2B = \frac{1}{2}(S_x + S_y)$$
$$2C = -\frac{1}{2}(S_x - S_y) \qquad (4.7.6)$$
$$D = 0$$
$$E = -\frac{1}{4}(S_x - S_y)a^4$$
$$F = \frac{1}{2}(S_x - S_y)a^2$$

$$\boxed{\begin{cases} \sigma_r = \frac{1}{2}(S_x+S_y)\left(1-\frac{a^2}{r^2}\right) + \frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta \\ \sigma_\theta = \frac{1}{2}(S_x+S_y)\left(1+\frac{a^2}{r^2}\right) - \frac{1}{2}(S_x-S_y)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} = -\frac{1}{2}(S_x-S_y)\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta \end{cases}} \qquad ☆ \quad (4.7.7)$$

When $S_x = S_y = -p$ (hydrostatic pressure).

$$\begin{cases} \sigma_r = -p\left(1-\frac{a^2}{r^2}\right) \\ \sigma_\theta = -p\left(1+\frac{a^2}{r^2}\right) \\ \tau_{r\theta} = 0 \end{cases} \qquad (4.7.8)$$



At $r=a$ $\quad \sigma_r = 0, \ \sigma_\theta = -2p, \ \tau_{r\theta} = 0$

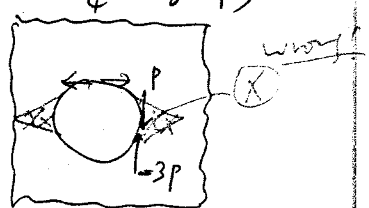
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{0-(-2p)}{2} = p$$
$(45°)$

45° shear
failure mode

When $S_x = 0$.

$$\begin{cases} \sigma_r = \frac{S_y}{2}\left(1-\frac{a^2}{r^2}\right) - \frac{S_y}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta \\ \sigma_\theta = \frac{S_y}{2}\left(1+\frac{a^2}{r^2}\right) + \frac{S_y}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} = \frac{S_y}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta \end{cases} \qquad (4.7.9)$$

$\downarrow (S_y = -p)$



At $r=a$. $\quad \sigma_r = 0, \ \sigma_\theta = \begin{cases} -3p & (\theta = 0) \\ p & (\theta = \frac{\pi}{2}) \end{cases}$
$(S_y = -p)$

$$\tau_{max} = \frac{0-(-3p)}{2} = 1.5p \ (45°)$$

(4) Displacements.

$$\begin{aligned}\epsilon_r &= \frac{\partial u}{\partial r} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)\\ \epsilon_\theta &= \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)\\ \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{2(1+\nu)}{E}\tau_{r\theta}\end{aligned} \qquad (4.7.10)$$

Eq (4.7.7) $\rightarrow$ Eq (4.7.10).

$$\left\{\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{E}\left[\left(\frac{\sigma_x+\sigma_y}{2}\right)\left(1-\frac{a^2}{r^2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta\right]\\ &\quad -\frac{\nu}{E}\left[\left(\frac{\sigma_x+\sigma_y}{2}\right)\left(1+\frac{a^2}{r^2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]\\ \frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta} &= \frac{1}{E}\left[\left(\frac{\sigma_x+\sigma_y}{2}\right)\left(1+\frac{a^2}{r^2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]\\ &\quad -\frac{\nu}{E}\left[\left(\frac{\sigma_x+\sigma_y}{2}\right)\left(1-\frac{a^2}{r^2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta\right]\\ \frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r} &= -\frac{2(1+\nu)}{E}\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta\end{aligned}\right. \qquad (4.7.11)$$

By integrating (From the 1st eq)

$$\begin{aligned}u &= \frac{1}{E}\left[\left(\frac{\sigma_x+\sigma_y}{2}\right)\left(r+\frac{a^2}{r}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r-\frac{a^4}{r^3}+\frac{4a^2}{r}\right)\cos 2\theta\right]\\ &\quad -\frac{\nu}{E}\left[\left(\frac{\sigma_x+\sigma_y}{2}\right)\left(r-\frac{a^2}{r}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r-\frac{a^4}{r^3}\right)\cos 2\theta\right] + g_1(\theta)\end{aligned} \qquad (4.7.12)$$

(Plugging into the 2nd Eq.)

$$\frac{\partial v}{\partial \theta} = \frac{1}{E}\left[-2\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r+\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\cos 2\theta\right] - \frac{\nu}{E}\left[2\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r-\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\cos 2\theta\right] - g_1(\theta)$$

By integration.

$$\begin{aligned}v &= \frac{1}{E}\left[-\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r+\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right] - \frac{\nu}{E}\left[\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r-\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right]\\ &\quad -\int g_1(\theta)\,d\theta + g_2(r)\end{aligned} \qquad (4.7.13)$$

In order to meet the 3rd Eq.

$$\frac{\partial u}{\partial \theta} = \frac{1}{E}\left[-2\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r+\frac{4a^2}{r}-\frac{a^4}{r^3}\right)\sin 2\theta\right] + \frac{\nu}{E}\left[2\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(r-\frac{a^4}{r^3}\right)\sin 2\theta\right] + \frac{dg_1(\theta)}{d\theta} \qquad (4.7.14)$$

$$\frac{\partial v}{\partial r} = \frac{1}{E}\left[-\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(1-\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right] - \frac{\nu}{E}\left[\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right] + \frac{dg_2(r)}{dr} \qquad (4.7.15)$$

The 3rd Eq becomes.

$$\left[\frac{dg_1(\theta)}{d\theta}+\int g_1(\theta)\,d\theta\right]+\left[r\frac{dg_2(r)}{dr}-g_2(r)\right]=0 \qquad (4.7.16)$$

$$r\frac{dg_2(r)}{dr}-g_2(r)=k,\quad \frac{dg_1(\theta)}{d\theta}+\int g_1(\theta)\,d\theta=-k \qquad (4.7.17)$$

$$\Rightarrow \begin{cases} g_2(r)=Cr-k \\ g_1(\theta)=A\sin\theta+B\cos\theta \end{cases} \qquad (4.7.18)$$

Eq (4.7.18) → Eq (4.7.12) & Eq (4.7.13). (4.7-19)

Boundary conditions

$$v_{\theta=0}=0=A+Cr$$

$$v_{\theta=\pi/2}=0=-B+Cr$$

$$\Rightarrow A=B=C=0.$$

$$\therefore u=\frac{1}{E}\left[\left(\frac{S_x+S_y}{2}\right)\left(r+\frac{a^2}{r}\right)+\left(\frac{S_x-S_y}{2}\right)\left(r-\frac{a^4}{r^3}+\frac{4a^2}{r}\right)\cos 2\theta\right]$$
$$-\frac{\nu}{E}\left[\left(\frac{S_x+S_y}{2}\right)\left(r-\frac{a^2}{r}\right)-\left(\frac{S_x-S_y}{2}\right)\left(r-\frac{a^4}{r^3}\right)\cos 2\theta\right] \qquad (4.7.20)$$

$$v=\frac{1}{E}\left[-\left(\frac{S_x-S_y}{2}\right)\left(r+\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right]-\frac{\nu}{E}\left[\left(\frac{S_x-S_y}{2}\right)\left(r-\frac{2a^2}{r}+\frac{a^4}{r^3}\right)\sin 2\theta\right]$$

when $r=a$,

$$u=\frac{1}{E}\left[(S_x+S_y)a+2(S_x-S_y)a\cdot\cos 2\theta\right]$$
$$v=\frac{-1}{E}\left[2(S_x-S_y)a\cdot\sin 2\theta\right] \qquad (4.7.21)$$

when $S_x=S_y=-p$

$$u=-\frac{2pa}{E} \quad \text{and} \quad v=0 \qquad (4.7.22)$$

Homework → Find u, v for plane strain condition.
Compare this with the results for plane stress condition.

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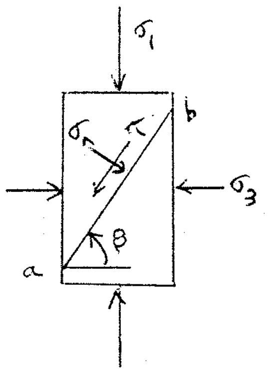
3. Strength criteria.

$$\begin{cases} \sigma_1 = f(\sigma_2, \sigma_3) \ \text{ or } \ \sigma_1 = f(\sigma_3) \\ \tau = f(\sigma_n) \end{cases}$$

◎ $\begin{cases} \text{peak strength criterion} \\ \text{residual strength criterion} \\ \text{yield criterion} \end{cases}$

◎ effective stress → total stress in ming & rock tunneling

(1) Coulomb's shear strength criterion.



shear strength

$$s = c + \sigma_n \tan\phi \quad \cdots\cdots\cdots (3.1)$$

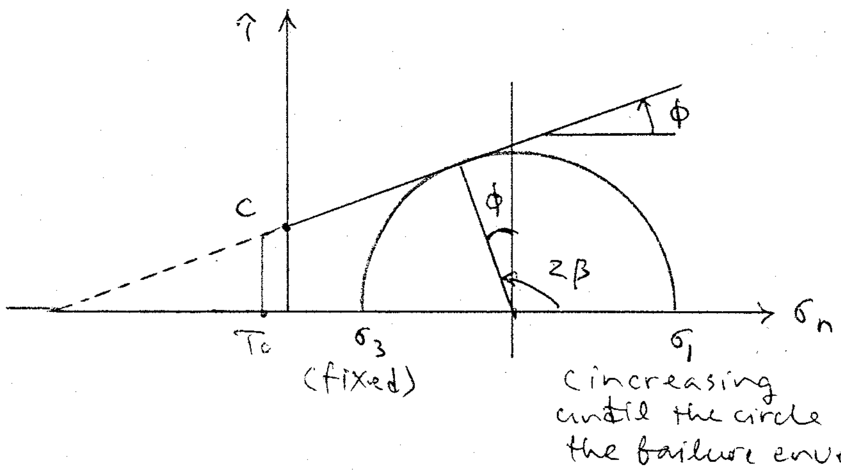
$\begin{cases} c = \text{cohesion} \\ \phi = \text{angle of internal friction} \end{cases}$

$$\begin{cases} \sigma_n = \frac{1}{2}(\sigma_1 + \sigma_3) + \frac{1}{2}(\sigma_1 - \sigma_3)\cos 2\beta \\ \tau = \frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\beta \end{cases}$$

$$\frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\beta = c + \left\{\frac{1}{2}(\sigma_1 + \sigma_3) + \frac{1}{2}(\sigma_1 - \sigma_3)\cos 2\beta\right\}\tan\phi$$

Solving for σ_1

$$\sigma_1 = \frac{2c + \sigma_3\{\sin 2\beta + \tan\phi\,(1 - \cos 2\beta)\}}{\sin 2\beta - \tan\phi\,(1 + \cos 2\beta)} \quad \cdots (3.2)$$



By geometry

$$2\beta = 90° + \phi \ \Rightarrow \ \beta = 45° + \frac{\phi}{2}$$

$$\begin{cases} \sin 2\beta = \cos\phi \\ \cos 2\beta = -\sin\phi \end{cases}$$

Eq (3.2) becomes

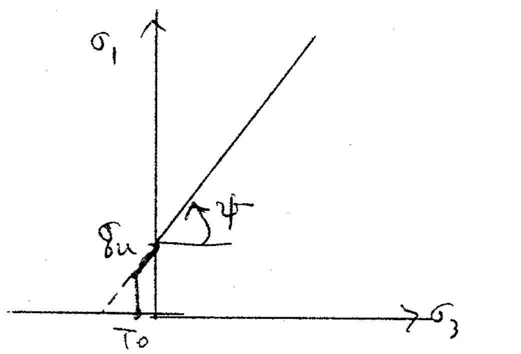
$$\sigma_1 = \frac{2c + \sigma_3[\cos\phi + \tan\phi(1+\sin\phi)]}{\cos\phi - \tan\phi(1-\sin\phi)}$$

$$= \frac{2c\cos\phi + \sigma_3[\cos^2\phi + \sin\phi(1+\sin\phi)]}{\cos^2\phi - \sin\phi(1-\sin\phi)}$$

$$= \frac{2c\cos\phi + \sigma_3[1+\sin\phi]}{1-\sin\phi}$$

$$\therefore \sigma_1 = \frac{2c\cos\phi}{1-\sin\phi} + \frac{1+\sin\phi}{1-\sin\phi}\sigma_3 \qquad (3.3)$$

Plotting $\sigma_1 - \sigma_3$ relation



$\sigma_1 = q_u + \sigma_3 \tan\psi$

$$\therefore q_u = \frac{2c\cos\phi}{1-\sin\phi}$$

$$= 2c\tan(45+\frac{\phi}{2})$$

$$= 2c\tan\beta$$

$$\tan\psi = \frac{1+\sin\phi}{1-\sin\phi}$$

$$= \tan^2(45+\frac{\phi}{2})$$

$$= \tan^2\beta$$

$$\boxed{\sigma_1 = q_u + \sigma_3\tan^2(45+\frac{\phi}{2})}$$

Mohr-Coulomb Failure Criterion.

Putting $\sigma_1 = 0$ in Eq(3.3)

$$\sigma_3 = -\frac{2c\cos\phi}{1+\sin\phi}$$

$$\therefore \sigma_T = \frac{2c\cos\phi}{1+\sin\phi} \quad \text{(Too big!)}$$

⇒ Use tension cutoff To

Limitations

① Major shear fracture at peak strength → not realistic

② Direction of shear failure → not realistic

③ Peak strength envolope → not realistic
linear

OK for residual strength condition → shear strength of discontinuities.