Engineering Mathematics II

Prof. Dr. Yong-Su Na

(32-206, Tel. 880-7204)

Text book: Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, Wiley (2006)

Ch. 10 Vector Integral Calculus. Integral Theorems

Ch. 10 Vector Integral Calculus. Integral Theorems (벡터적분법. 적분정리)

- 적분을 곡선(선적분), 면(면적분), 고체에 대한 적분으로 확장:

  고체역학, 유체흐름, 열역학에서 공학적 기본 응용으로 활용

- 적분의 변환은 계산을 간단히 하거나, 유용한 일반적인 공식을 얻기 위해 수행

  예) 퍼텐셜 이론(Potential Theory)

- 적분변환 공식:

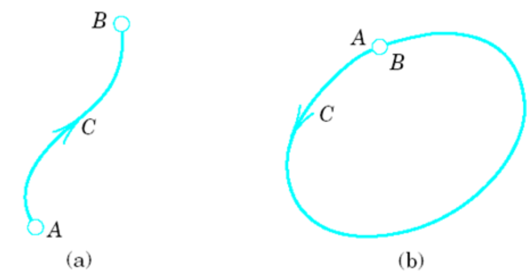
  Green의 공식, Gauss 공식, Stokes 공식

10.1 Line Integrals (선적분)

- **선적분의 개념**: 미적분학에서 공부한 정적분의 간단한 일반화
  - Line Integral(선적분) 또는 Curve Integral(곡선적분):

    Integrand(피적분함수)를 공간(혹은 평면)내의 곡선을 따라 적분.
  - Path of Integration (적분경로): 곡선 C

$$C:\mathbf{r}(t)=[x(t),\ y(t),\ z(t)]=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}\quad (a\le t\le b)$$



❖ **General Assumption:**

선적분의 모든 적분경로를 구분적으로 매끄럽다 (Piecewise Smooth).

- **Definition and Evaluation of Line Integrals**

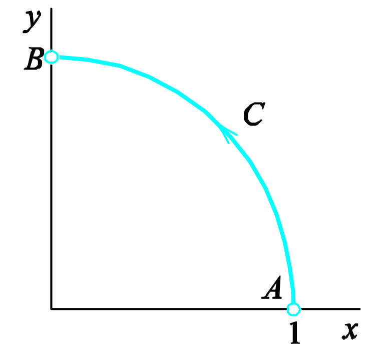
곡선 C:$\mathbf{r}(t)$에서 벡터함수$\mathbf{F}(\mathbf{r})$의 선적분 : $\int_C \mathbf{F}(\mathbf{r})\bullet d\mathbf{r}=\int_a^b \mathbf{F}(\mathbf{r}(t))\bullet \mathbf{r}'(t)dt$ $\qquad \mathbf{r}'=\dfrac{d\mathbf{r}}{dt}$

$$\int_C \mathbf{F}(\mathbf{r})\bullet d\mathbf{r}=\int_C (F_1dx+F_2dy+F_3dz)=\int_a^b (F_1x'+F_2y'+F_3z')dt$$

10.1 Line Integrals (선적분)

■ Ex.1 **Evaluation of a Line Integral in the Plane (평면에서 선적분의 계산)**

$$\mathbf{F}(\mathbf{r}) = [-y,\ -xy] = -y\mathbf{i} - xy\mathbf{j}$$



10.1 Line Integrals (선적분)

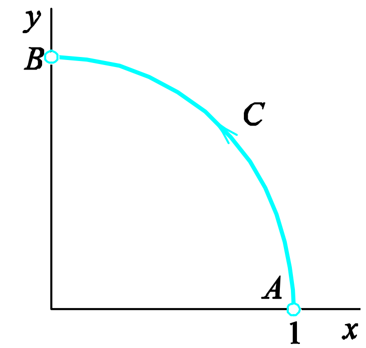
■ Ex.1 **Evaluation of a Line Integral in the Plane (평면에서 선적분의 계산)**

$$\mathbf{F}(\mathbf{r}) = [-y,\ -xy] = -y\mathbf{i} - xy\mathbf{j}$$

C를 $\mathbf{r}(t) = [\cos t,\ \sin t] = \cos t\mathbf{i} + \sin t\mathbf{j},\ (0 \le t \le \pi/2)$로 표현

$\Rightarrow\quad x(t) = \cos t,\ \ y(t) = \sin t$

$\Rightarrow\quad \mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j} = -\sin t\mathbf{i} - \cos t \sin t\mathbf{j}$



10.1 Line Integrals (선적분)

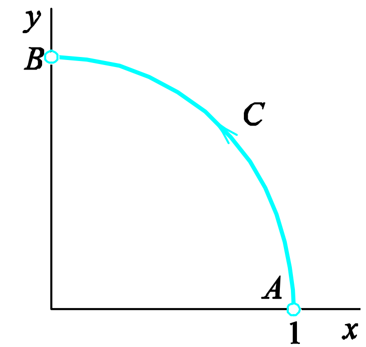
■ Ex.1 **Evaluation of a Line Integral in the Plane (평면에서 선적분의 계산)**

$$\mathbf{F}(\mathbf{r}) = [-y,\ -xy] = -y\mathbf{i} - xy\mathbf{j}$$

C를 $\mathbf{r}(t) = [\cos t,\ \sin t] = \cos t\mathbf{i} + \sin t\mathbf{j},\ (0 \le t \le \pi/2)$로 표현

$\Rightarrow\quad x(t) = \cos t,\ \ y(t) = \sin t$

$\Rightarrow\quad \mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j} = -\sin t\mathbf{i} - \cos t \sin t\mathbf{j}$



$\mathbf{r}'(t) = [-\sin t,\ \cos t] = -\sin t\mathbf{i} + \cos t\mathbf{j}$

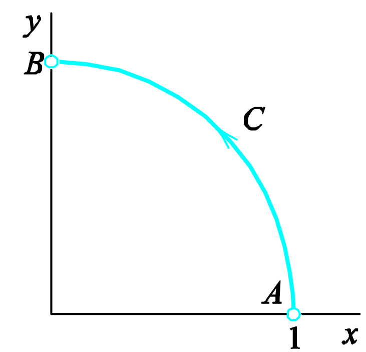
$$\Rightarrow\quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_0^{\pi/2} [-\sin t,\ -\cos t \sin t] \bullet [-\sin t,\ \cos t]\, dt$$

$$= \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\, dt = \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2t)\, dt - \int_1^0 u^2(-du) = \frac{\pi}{4} - 0 - \frac{1}{3} \approx 0.4521$$

10.1 Line Integrals (선적분)

■ Ex.1 **Evaluation of a Line Integral in the Plane (평면에서 선적분의 계산)**

$$\mathbf{F}(\mathbf{r}) = [-y,\ -xy] = -y\mathbf{i} - xy\mathbf{j}$$



C를 $\mathbf{r}(t) = [\cos t,\ \sin t] = \cos t\mathbf{i} + \sin t\mathbf{j},\ (0 \le t \le \pi/2)$로 표현

$\Rightarrow \quad x(t) = \cos t,\ \ y(t) = \sin t$

$\Rightarrow \quad \mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j} = -\sin t\mathbf{i} - \cos t \sin t\mathbf{j}$

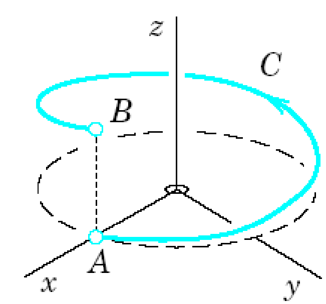
$\mathbf{r}'(t) = [-\sin t,\ \cos t] = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$$\Rightarrow \quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_0^{\pi/2} [-\sin t,\ -\cos t \sin t] \bullet [-\sin t,\ \cos t]\,dt$$

$$= \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\,dt = \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2t)\,dt - \int_1^0 u^2(-du) = \frac{\pi}{4} - 0 - \frac{1}{3} \approx 0.4521$$

■ Ex.2 **Line Integral in Space**

$$\mathbf{F}(\mathbf{r}) = [z,\ x,\ y],\quad \mathbf{r}(t) = [\cos t,\ \sin t,\ 3t]\ \ (0 \le t \le 2\pi)$$



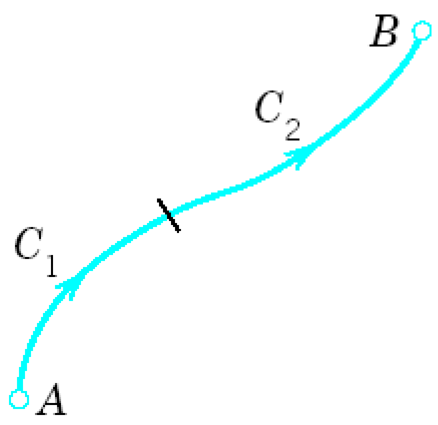
10.1 Line Integrals (선적분)

- **Simple General Properties of the Line Integral**

$$\int_C k\mathbf{F} \bullet d\mathbf{r} = k\int_C \mathbf{F} \bullet d\mathbf{r} \qquad (k\text{는 상수})$$

$$\int_C (\mathbf{F} + \mathbf{G}) \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet d\mathbf{r} + \int_C \mathbf{G} \bullet d\mathbf{r}$$

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$$



10.1 Line Integrals (선적분)

■ Ex.3 **Work Done by a Variable Force**

- 직선분 d를 따른 변위에서 일정한 힘 F에 의한 일 $W = F \bullet d$이다.
- 곡선 $C\colon \mathbf{r}(t)$를 따르는 변위에서 힘 F가 변할 때, 행해진 일 W는 C의 작은 현을 따른 변위에서 행해진 일의 합의 극한으로 정의할 수 있다.
- 선적분으로 W를 정하는 것과 같다.

10.1 Line Integrals (선적분)

■ Ex.3 **Work Done by a Variable Force**

- 직선분 d를 따른 변위에서 일정한 힘 F에 의한 일 $W = F \bullet d$이다.
- 곡선 C:$\mathbf{r}(t)$를 따르는 변위에서 힘 F가 변할 때, 행해진 일 W는 C의 작은 현을 따른 변위에서 행해진 일의 합의 극한으로 정의할 수 있다.
- 선적분으로 W를 정하는 것과 같다.

■ Ex.4 **Work Done Equals the Gain in Kinetic Energy.**

$\mathbf{F}$가 힘이면 선적분은 일이다. t를 시간이라 하면 $d\mathbf{r}/dt = \mathbf{v}$는 속도이다.

$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{v}(t)\,dt$$

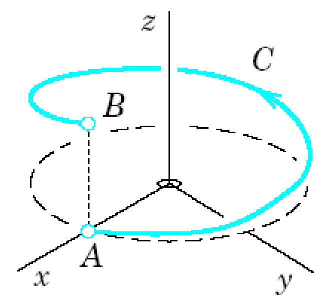
$$Newton\text{의 제2법칙} \Rightarrow \mathbf{F} = m\mathbf{r}''(t) = m\mathbf{v}'(t) \;\Rightarrow\; W = \int_a^b m\mathbf{v}'(t) \bullet \mathbf{v}(t)\,dt = \int_a^b m\left(\frac{\mathbf{v} \bullet \mathbf{v}}{2}\right)' dt = \frac{m}{2}|\mathbf{v}|^2 \Big|_{t=a}^{t=b}$$

10.1 Line Integrals (선적분)

- **Other Forms of Line Integrals**

값이 벡터인 선적분 : $\int_C \mathbf{F}(\mathbf{r})dt = \int_a^b \mathbf{F}(\mathbf{r}(t))dt = \int_a^b [F_1(\mathbf{r}(t)),\ F_2(\mathbf{r}(t)),\ F_3(\mathbf{r}(t))]dt$

- **Ex.5** Integrate $\mathbf{F}(\mathbf{r}) = [xy, yz, z]$ along the helix.



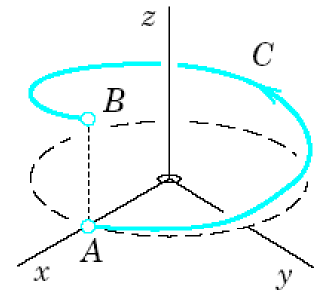
10.1 Line Integrals (선적분)

- **Other Forms of Line Integrals**

$$\text{값이 벡터인 선적분}: \int_C \mathbf{F}(\mathbf{r})dt = \int_a^b \mathbf{F}(\mathbf{r}(t))dt = \int_a^b [F_1(\mathbf{r}(t)),\ F_2(\mathbf{r}(t)),\ F_3(\mathbf{r}(t))]dt$$

■ **Ex.5** Integrate $\mathbf{F}(\mathbf{r}) = [xy,\ yz, z]$ along the helix.

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))dt = \left[-\frac{1}{2}\cos^2 t,\ 3\sin t - 3t\cos t,\ \frac{3}{2}t^2\right]\Bigg|_0^{2\pi} = [0,\ -6\pi,\ 6\pi^2]$$



- **Path Dependence**

선적분은 일반적으로 피적분함수와 경로의 끝점과 관련이 있을 뿐 아니라 적분이 취해지는 경로 그 자체와도 관련이 있다.

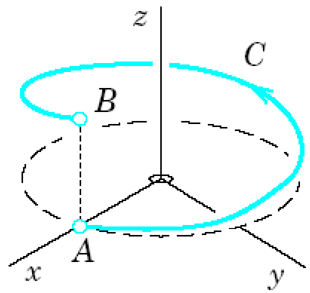
10.1 Line Integrals (선적분)

- **Other Forms of Line Integrals**

$$\text{값이 벡터인 선적분 : } \int_C \mathbf{F}(\mathbf{r})dt = \int_a^b \mathbf{F}(\mathbf{r}(t))dt = \int_a^b [F_1(\mathbf{r}(t)),\ F_2(\mathbf{r}(t)),\ F_3(\mathbf{r}(t))]dt$$

■ **Ex.5** Integrate $\mathbf{F}(\mathbf{r}) = [xy, yz, z]$ along the helix.

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))dt = \left[-\frac{1}{2}\cos^2 t,\ 3\sin t - 3t\cos t,\ \frac{3}{2}t^2\right]_0^{2\pi} = [0,\ -6\pi,\ 6\pi^2]$$



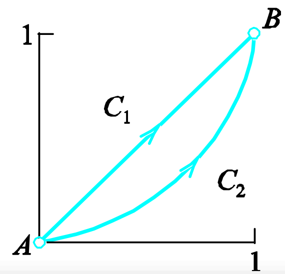
- **Path Dependence**

선적분은 일반적으로 피적분함수와 경로의 끝점과 관련이 있을 뿐 아니라 적분이 취해지는 경로 그 자체와도 관련이 있다.

■ Example: $0 \le t \le 1,\quad C_1 : \mathbf{r}_1(t) = [t, t, 0],\quad C_2 : \mathbf{r}_2(t) = [t, t^2, 0],\quad \mathbf{F} = [0, xy, 0]$

$$\mathbf{F}(\mathbf{r}_1(t)) \bullet \mathbf{r}_1'(t) = t^2 \quad \Rightarrow \quad \int_{C_1} \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 = \frac{1}{3}$$

$$\mathbf{F}(\mathbf{r}_2(t)) \bullet \mathbf{r}_2'(t) = 2t^4 \quad \Rightarrow \quad \int_{C_2} \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = \frac{2}{5}$$



10.1 Line Integrals (선적분)

PROBLEM SET 10.1

HW: 19, 20

10.2 Path Independence of Line Integrals (선적분의 경로 무관성)

- **Path Independence (경로 무관성)**

- 공간의 영역 D에서 F_1, F_2, F_3가 연속인 선적분은 만약 $\mathbf{F} = [F_1, F_2, F_3]$가 어떤 함수 f 의 기울기이면 영역에서 경로에 무관하다.

$$\mathbf{F} = \text{grad } f \quad \left(F_1 = \frac{\partial f}{\partial x},\ F_2 = \frac{\partial f}{\partial y},\ F_3 = \frac{\partial f}{\partial z} \right) \quad \Rightarrow \quad \int_A^B (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A)$$

- 영역 D의 모든 닫힌 곡선에서 선적분의 적분값이 0이면, 적분은 영역 D에서 경로무관하다.
- 미분형식 $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$가 영역 D에서 연속적인 계수함수 $F_1,\ F_2,\ F_3$를 가지고 완전하면, 선적분은 영역 D에서 경로 무관하다.

- **Exact (완전)**

영역 D의 모든 곳에서 미분가능한 함수 f가 존재하여 $\mathbf{F} \bullet d\mathbf{r} = df$ 의 관계가 성립

10.2 Path Independence of Line Integrals
(선적분의 경로 무관성)

- Ex.1 **Path Independence**

$$\int_C F \cdot dr = \int_C (2x\,dx + 2y\,dy + 4z\,dz), \quad A:(0,\ 0,\ 0) \text{ to } B:(2,\ 2,\ 2)$$

10.2 Path Independence of Line Integrals
(선적분의 경로 무관성)

■ Ex.1 **Path Independence**

$$\int_C F \cdot dr = \int_C (2xdx + 2ydy + 4zdz), \quad A:(0,\ 0,\ 0) \text{ to } B:(2,\ 2,\ 2)$$

$$\mathbf{F} = [2x,\ 2y,\ 4z] = \text{grad } f$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 2x = F_1, \ \frac{\partial f}{\partial y} = 2y = F_2, \ \frac{\partial f}{\partial z} = 4z = F_3$$

$$\Rightarrow \quad f = x^2 + y^2 + 2z^2$$

∴ 적분은 경로와 무관하다.

$$\int_C (2xdx + 2ydy + 4zdz) = f(B) - f(A) = f(2,\ 2,\ 2) - f(0,\ 0,\ 0) = 4 + 4 + 8 = 16$$

10.2 Path Independence of Line Integrals (선적분의 경로 무관성)

- **Criterion for Exactness and Path Independence (완전성과 경로 무관성에 대한 판별기준)**

선적분 $\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$에서

F_1, F_2, F_3가 영역에서 연속적이고, 연속적인 일차편미분도함수를가진다고 하자.

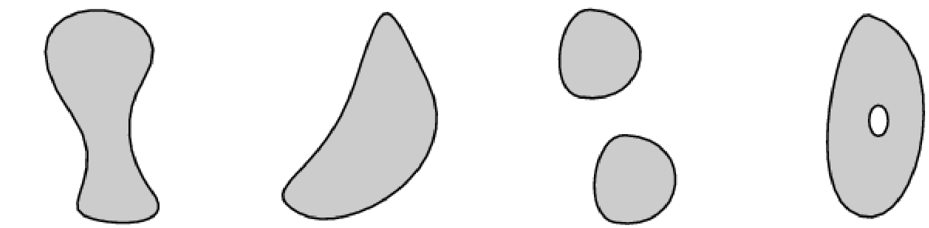
- 미분형식 $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$이 완전

$$\Rightarrow \quad \operatorname{curl} \mathbf{F} = \mathbf{0} \left(\text{즉}, \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \right) \qquad \nabla \times F = \nabla \times (\nabla f) = 0$$

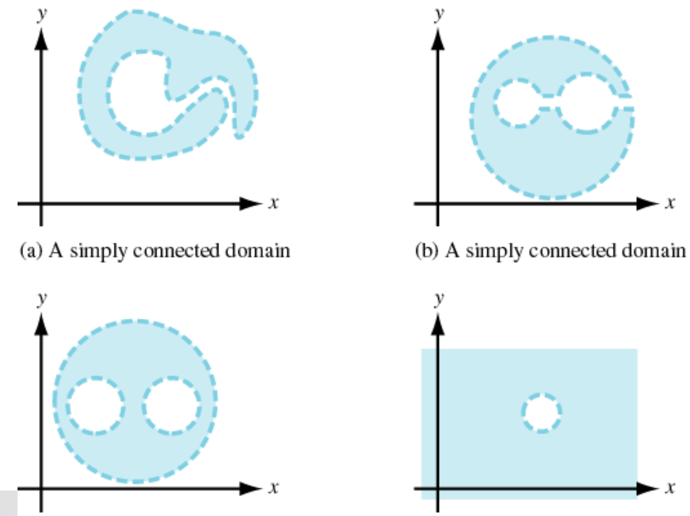
- $\operatorname{curl} \mathbf{F} = \mathbf{0}$이 성립하고 D가 단순연결 $\Rightarrow$ $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$은 완전

$\Rightarrow$ 선적분은 경로 무관하다.

10.2 Path Independence of Line Integrals
(선적분의 경로 무관성)

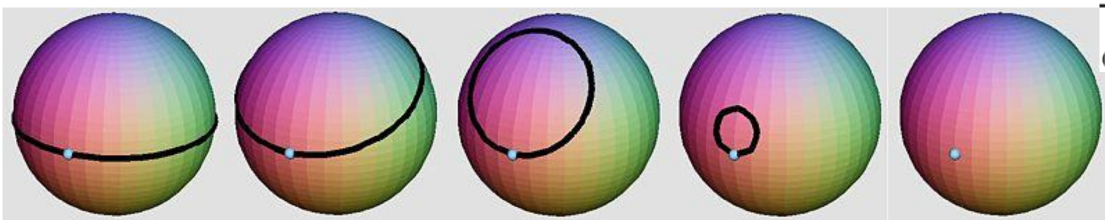


simply connected *simply connected* *not simply connected* *not simply connected*



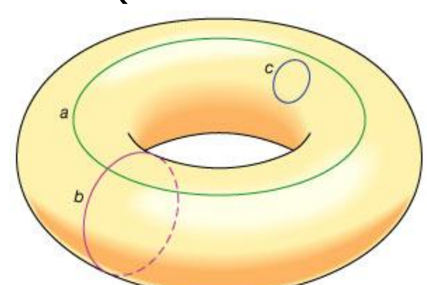
(a) A simply connected domain (b) A simply connected domain

(c) A multiply connected domain (d) A multiply connected domain



A sphere is simply connected because every loop can be contracted (on the surface) to a point.

A torus is not simply connected. Neither of the colored loops can be contracted to a point without leaving the surface.



10.2 Path Independence of Line Integrals
(선적분의 경로 무관성)

- Ex.3 **Exactness and Independence of Path. Determination of a Potential**

  Show that the differential form is exact. Find the value of I
  from A: (0, 0, 1) to B (1, π/4, 2)

$$I = \int_C \left[2xyz^2 dx + \left(x^2 z^2 + z \cos yz \right) dy + \left(2x^2 yz + y \cos yz \right) dz \right]$$

10.2 Path Independence of Line Integrals
(선적분의 경로 무관성)

- **Ex.3 Exactness and Independence of Path. Determination of a Potential**

  Show that the differential form is exact. Find the value of I
  from A: (0, 0, 1) to B (1, π/4, 2)

$$I = \int_C \left[2xyz^2 dx + \left(x^2 z^2 + z\cos yz\right)dy + \left(2x^2 yz + y\cos yz\right)dz\right]$$

완전성 :

$$\left(F_3\right)_y = 2x^2 z + \cos yz - yz\sin yz = \left(F_2\right)_z, \quad \left(F_1\right)_z = 4xyz = \left(F_3\right)_x, \quad \left(F_2\right)_x = 2xz^2 = \left(F_1\right)_y$$

f 를 구하기

$$f = \int F_2 dy = \int \left(x^2 z^2 + z\cos yz\right)dy = x^2 yz^2 + \sin yz + g(x,z)$$

$$\Rightarrow \quad f_x = 2xyz^2 + g_x = F_1 = 2xyz^2 \quad \Rightarrow \quad g_x = 0 \quad \Rightarrow \quad g = h(z)$$

$$f_z = 2x^2 yz + y\cos yz + h' = F_3 = 2x^2 yz + y\cos yz \quad \Rightarrow \quad h' = 0 \quad \Rightarrow \quad h = \text{상수}$$

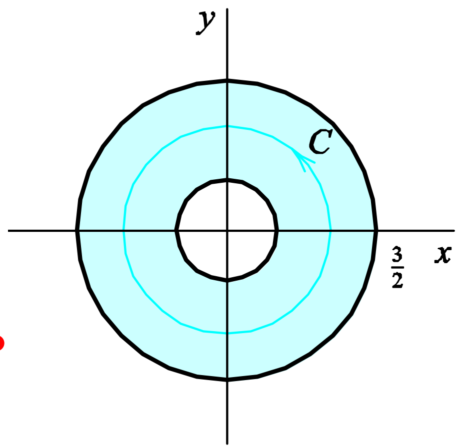
$$\therefore\ f = x^2 yz^2 + \sin yz, \quad f(B) - f(A) = 1\cdot\frac{\pi}{4}\cdot 4 + \sin\frac{\pi}{2} - 0 = \pi + 1$$

10.2 Path Independence of Line Integrals (선적분의 경로 무관성)

- Ex.4 **On the Assumption of Simple Connectedness**

$$F_1 = -\frac{y}{x^2+y^2}, \quad F_2 = \frac{x}{x^2+y^2}, \quad F_3 = 0,$$

$$I = \int_C (F_1 dx + F_2 dy) = \int_C \frac{-y dx + x dy}{x^2+y^2}$$

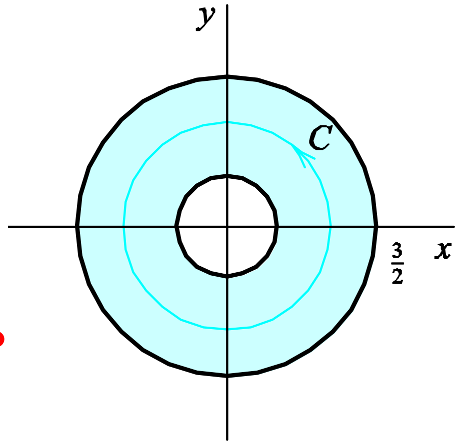


10.2 Path Independence of Line Integrals
(선적분의 경로 무관성)

■ Ex.4 **On the Assumption of Simple Connectedness**

$$F_1 = -\frac{y}{x^2+y^2}, \quad F_2 = \frac{x}{x^2+y^2}, \quad F_3 = 0,$$

$$I = \int_C (F_1 dx + F_2 dy) = \int_C \frac{-y dx + x dy}{x^2+y^2}$$



$$\frac{\partial F_2}{\partial x} = \frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial F_1}{\partial y} = -\frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$ 이므로 미분형식이 D에서 완전하다.

만약 적분 I가 D에서 경로에 무관하면 D의 임의 닫힌 곡선에서 $I=0$이다.

그러나, $x = r\cos\theta, \quad y = r\sin\theta, \quad r = 1 \quad \Rightarrow \quad dx = -\sin\theta d\theta, \quad dy = \cos\theta d\theta$

$\Rightarrow \quad -y dx + x dy = \sin^2\theta d\theta + \cos^2\theta d\theta = d\theta \quad \Rightarrow$ 반시계방향 적분 $I = \int_0^{2\pi} \frac{d\theta}{1} = 2\pi$

$\therefore$ D가 단순연결이 아니므로, D에서 I가 경로무관하다고 결론을 내릴 수 없다.

10.2 Path Independence of Line Integrals (선적분의 경로 무관성)

PROBLEM SET 10.2

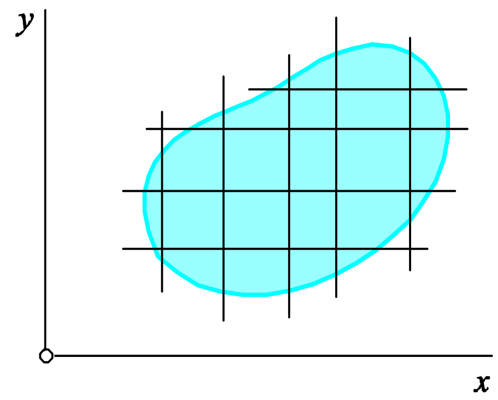
HW: 10, 16

10.3 Calculus Review: Double Integrals (이중적분)

- **Double Integrals:** 피적분함수를 평면의 닫힌 유한한 영역에서 적분
- 영역 R을 x축과 y축에 평행한 직선을 그어 분할한다.
- 각 직사각형내의 한 점 (x_k, y_k)을 택하여

  $J_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$와 같은 형태의 합을 만든다.

  * ΔA_k는 k번째 직사각형의 면적



- $f(x, y)$가 R에서 연속이고 R이 유한개의 매끄러운 곡선을 경계로 한다고 가정

  ⇒ 수열 J_n이 수렴, 극한을 영역 R에서의 $f(x, y)$의 이중적분(Double Integral)이라한다.

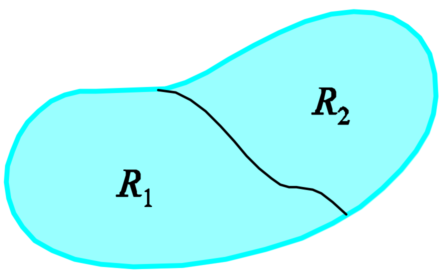
10.3 Calculus Review: Double Integrals (이중적분)

- **이중적분의 성질**

* $\iint_R kf\,dxdy = k\iint_R f\,dxdy$

* $\iint_R (f+g)dxdy = \iint_R f\,dxdy + \iint_R g\,dxdy$

* $\iint_R f\,dxdy = \iint_{R_1} f\,dxdy + \iint_{R_2} f\,dxdy$



* 이중적분에대한 **평균값 정리**(Mean Value Theorem)

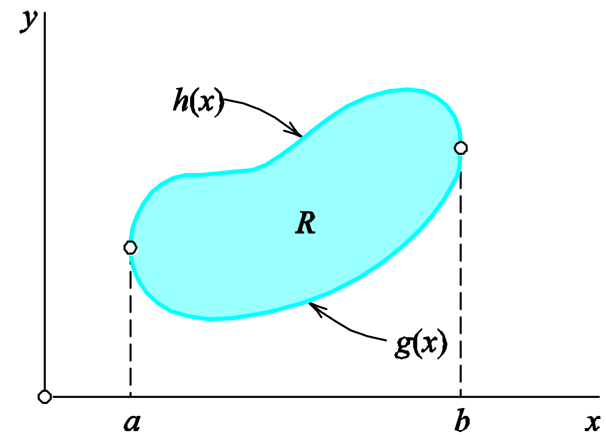
R이 단순연결되었으면

$\iint_R f(x,y)dxdy = f(x_0, y_0)A$을 만족하는 점 (x_0, y_0)가 적어도하나 R에 존재한다.

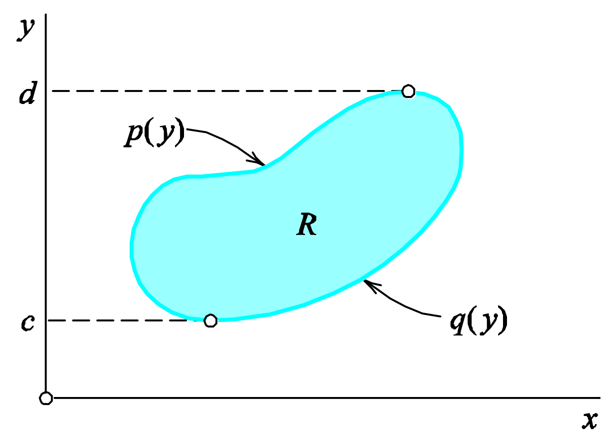
A는 R의 면적이다.

10.3 Calculus Review: Double Integrals (이중적분)

- **Evaluation of Double Integrals by Two Successive Integrations (연속적인 두 적분에 의한 이중적분의 계산)**



- $$\iint_R f(x,y)\,dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx$$

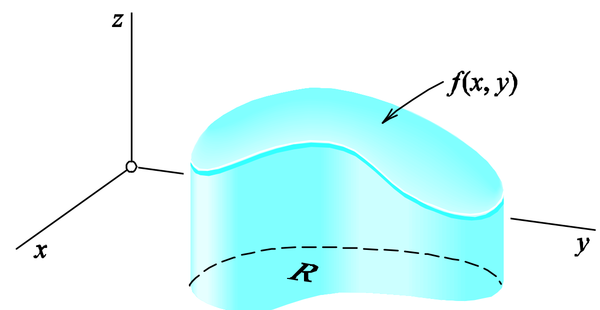


- $$\iint_R f(x,y)\,dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx \right] dy$$

10.3 Calculus Review: Double Integrals (이중적분)

- **Applications of Double Integrals**
  - 영역 R의 면적 : $A = \iint_R dxdy$
  - $z = f(x,y)\ (>0)$ 아래와 xy 평면 영역 R 위로 이루어지는 체적

    : $V = \iint_R f(x,y)dxdy$



  - $f(x,y)$: xy 평면에서 질량분포의 밀도(=단위 면적당 질량)

    $*$ R에서 전체 질량 : $M = \iint_R f(x,y)dxdy$

    $*$ R에서 질량의 무게중심(Center of Gravity) : $\bar{x} = \frac{1}{M}\iint_R xf(x,y)dxdy,\ \ \bar{y} = \frac{1}{M}\iint_R yf(x,y)dxdy$

    $*$ R에서 질량의 관성모멘트(Moments of Inertia) : $I_x = \iint_R y^2 f(x,y)dxdy,\ I_y = \iint_R x^2 f(x,y)dxdy$

    $*$ R에서 질량의 극관성모멘트(Polar Moments of Inertia) : $I_0 = I_x + I_y = \iint_R (x^2 + y^2) f(x,y)dxdy$

10.3 Calculus Review: Double Integrals (이중적분)

- **Change of Variables in Double Integrals (이중적분에서 변수변환). Jacobian**

- 이중적분에서 $x,\ y$에서 $u,\ v$로의 변수 변환공식

$$: \iint_R f(x,y)\,dxdy = \iint_{R^*} f\big(x(u,v), y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$\text{Jacobian}:\ J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- 극좌표계 r과 θ로서, $x = r\cos\theta,\ y = r\sin\theta$

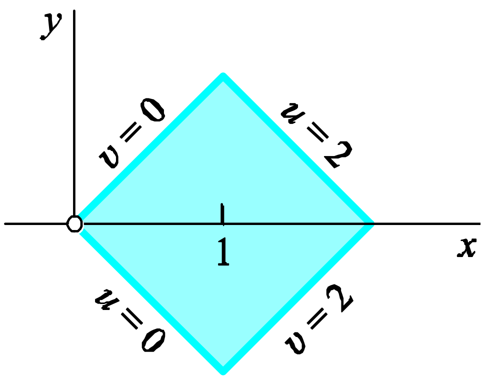
$$\Rightarrow\quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\Rightarrow\quad \iint_R f(x,y)\,dxdy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

10.3 Calculus Review: Double Integrals (이중적분)

- Ex. 1 Evaluate the following double integral over the square *R*.

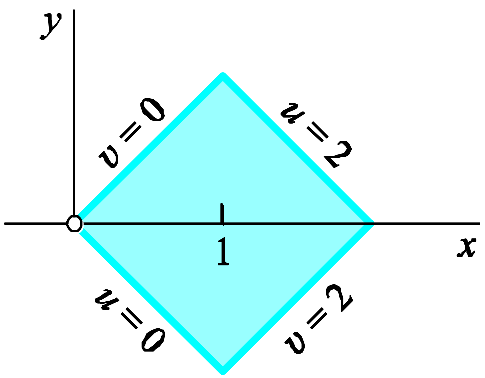
$$\iint_R \left(x^2 + y^2\right) dx\,dy$$



10.3 Calculus Review: Double Integrals (이중적분)

- Ex. 1 Evaluate the following double integral over the square *R*.

$$\iint_R \left(x^2 + y^2\right) dxdy$$



R의 모양으로부터 변환

$$: \; x + y = u, \;\; x - y = v \quad \Rightarrow \quad x = \frac{1}{2}(u + v), \;\; y = \frac{1}{2}(u - v)$$

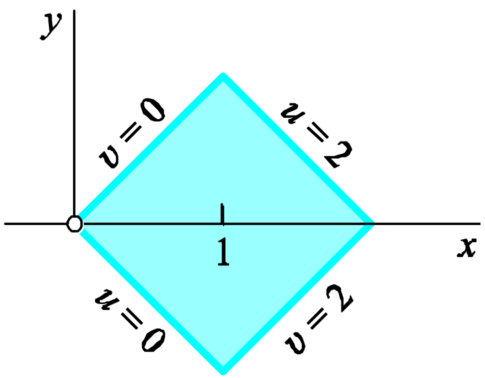
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

10.3 Calculus Review: Double Integrals (이중적분)

- Ex. 1 Evaluate the following double integral over the square *R*.

$$\iint_R \left(x^2 + y^2\right) dxdy$$



R의 모양으로부터 변환

$$: x + y = u,\ x - y = v \quad \Rightarrow \quad x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

R은 정사각형 $0 \le u \le 2,\ \ 0 \le v \le 2$

$$\iint_R \left(x^2 + y^2\right) dxdy = \int_0^2\int_0^2 \frac{1}{2}\left(u^2 + v^2\right)\frac{1}{2}\,dudv = \frac{8}{3}$$

10.3 Calculus Review: Double Integrals (이중적분)

PROBLEM SET 10.3

HW: 9, 12

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

- **Green's Theorem in the Plane**
  (Transformation between Double Integrals and Line Integrals)

R : xy 평면에서의 닫힌 유계영역

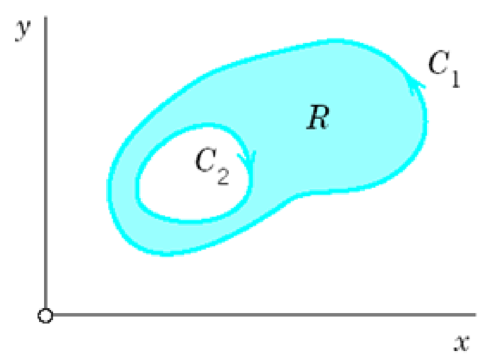
C : 유한개의 매끄러운 곡선으로 영역 R의 경계

$F_1(x, y),\ F_2(x, y)$: R을 포함하는 어떤 영역의 모든점에서 연속이고

연속인 편도함수 $\dfrac{\partial F_1}{\partial y}, \dfrac{\partial F_2}{\partial x}$를 갖는 함수

$$\Rightarrow \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

* 적분의 방향 : C를 따라 진행할 때 R이 좌측에 있는 방향



- $\mathbf{F} = [F_1, F_2] = F_1 \mathbf{i} + F_2 \mathbf{j} \quad \Rightarrow \quad \iint_R (\text{curl}\,\mathbf{F}) \bullet \mathbf{k}\, dxdy = \oint_C \mathbf{F} \bullet d\mathbf{r}$

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

■ Ex. 1 **Verification of Green's Theorem in the Plane**

$$F_1 = y^2 - 7y, \quad F_2 = 2xy + 2x,$$

$$C: \ x^2 + y^2 = 1$$

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

■ Ex. 1 **Verification of Green's Theorem in the Plane**

$$F_1 = y^2 - 7y, \quad F_2 = 2xy + 2x,$$

$$C:\; x^2 + y^2 = 1$$

원판 R의 면적 : π $\Rightarrow$ $\displaystyle\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_R [(2y+2)-(2y-7)]dxdy = 9\iint_R dxdy = 9\pi$

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

■ Ex. 1 **Verification of Green's Theorem in the Plane**

$$F_1 = y^2 - 7y, \quad F_2 = 2xy + 2x,$$

$$C:\ x^2 + y^2 = 1$$

원판 R의 면적 : $\pi \quad \Rightarrow \quad \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_R [(2y+2)-(2y-7)]dxdy = 9 \iint_R dxdy = 9\pi$

C : 반시계방향 $\quad \Rightarrow \quad \mathbf{r}(t) = [\cos t\ ,\ \sin t], \quad \mathbf{r}'(t) = [-\sin t\ ,\ \cos t]$

$$\Rightarrow F_1 = y^2 - 7y = \sin^2 t - 7\sin t, \quad F_2 = 2xy + 2x = 2\cos t \sin t + 2\cos t$$

$$\Rightarrow \oint_C (F_1 x' + F_2 y')dt = \int_0^{2\pi} [(\sin^2 t - 7\sin t)(-\sin t) + 2(\cos t \sin t + \cos t)(\cos t)]dt$$

$$= \int_0^{2\pi} (-\sin^3 t + 7\sin^2 t + 2\cos^2 t \sin t + 2\cos^2 t)dt$$

$$= 0 + 7\pi - 0 + 2\pi = 9\pi$$

10.4 Green’s Theorem in the Plane (평면에서의 Green의 정리)

- **Some Applications of Green’s Theorem**

- Ex. 2 **Area of a Plane Region as a Line Integral Over the Boundary**

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

- **Some Applications of Green's Theorem**

■ Ex. 2 **Area of a Plane Region as a Line Integral Over the Boundary**

$$F_1 = 0, \quad F_2 = x \quad \Rightarrow \quad \iint_R dxdy = \oint_C xdy$$

$$F_1 = -y, \quad F_2 = 0 \quad \Rightarrow \quad \iint_R dxdy = -\oint_C ydx$$

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

- **Some Applications of Green's Theorem**

■ Ex. 2 **Area of a Plane Region as a Line Integral Over the Boundary**

$$F_1 = 0, \quad F_2 = x \quad \Rightarrow \quad \iint_R dxdy = \oint_C xdy$$

$$F_1 = -y, \quad F_2 = 0 \quad \Rightarrow \quad \iint_R dxdy = -\oint_C ydx$$

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

■ Ex. 3 **Area of a Plane Region in a Polar Coordinates**

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

- **Some Applications of Green's Theorem**

■ Ex. 2 **Area of a Plane Region as a Line Integral Over the Boundary**

$$F_1 = 0,\quad F_2 = x \quad \Rightarrow \quad \iint_R dxdy = \oint_C xdy$$

$$F_1 = -y,\quad F_2 = 0 \quad \Rightarrow \quad \iint_R dxdy = -\oint_C ydx$$

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

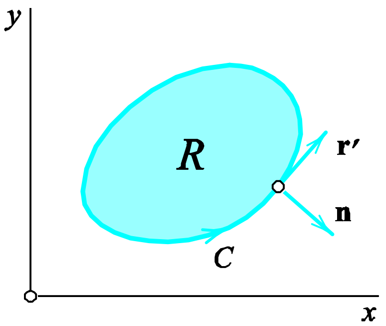
■ Ex. 3 **Area of a Plane Region in a Polar Coordinates**

$$x = r\cos\theta,\quad y = r\sin\theta \quad \Rightarrow \quad dx = \cos\theta dr - r\sin\theta d\theta,\quad dy = \sin\theta dr + r\cos\theta d\theta$$

$$\Rightarrow \quad A = \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\oint_C [(r\cos\theta)(\sin\theta dr + r\cos\theta d\theta) - (r\sin\theta)(\cos\theta dr - r\sin\theta d\theta)] = \frac{1}{2}\oint_C r^2 d\theta$$

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

■ Ex. 4 **Transformation of a Double Integral of the Laplacian of a Function into a Line Integral of Its Normal Derivative**



$w(x, y)$: 연속적이고 1차와 2차의 연속적인 도함수를 가지는 함수

$$F_1 = -\frac{\partial w}{\partial y}, \quad F_2 = \frac{\partial w}{\partial x}$$

$\Rightarrow$

$$* \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w \quad \Rightarrow \quad \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_R \nabla^2 w dxdy$$

$$* \quad \oint_C (F_1 dx + F_2 dy) = \oint_C \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds} \right) ds = \oint_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds} \right) ds = \oint_C \frac{\partial w}{\partial n} ds$$

$$\Leftarrow \quad \frac{\partial w}{\partial x}\frac{dy}{ds} - \frac{\partial w}{\partial y}\frac{dx}{ds} = \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right] \left[\frac{dy}{ds}, -\frac{dx}{ds} \right] = (\operatorname{grad} w) \bullet \mathbf{n} = \frac{\partial w}{\partial n}$$

$$\therefore \quad \iint_R \nabla^2 w dxdy = \oint_C \frac{\partial w}{\partial n} ds$$

10.4 Green's Theorem in the Plane (평면에서의 Green의 정리)

PROBLEM SET 10.4

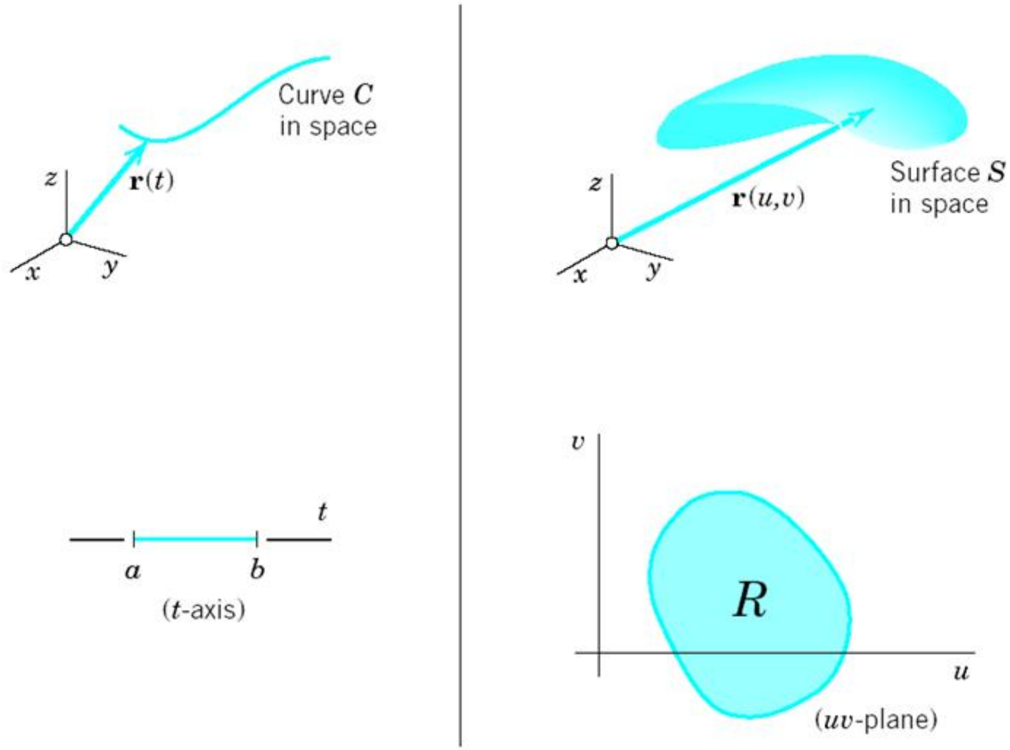
HW: 5, 20

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

- 곡면의 표현식 : $z = f(x,y)$ 또는 $g(x,y,z) = 0$

- 곡면 S의 매개변수 표현식

$$\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$$
$$= x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$
$$(u,v) \in R$$

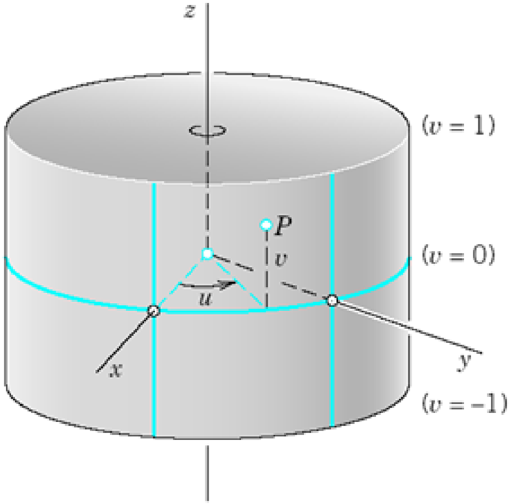


(A) Curve (B) Surface

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

■ Ex. 1 **Parametric Representation of a Cylinder**

$$x^2 + y^2 = a^2, \quad -1 \le z \le 1$$

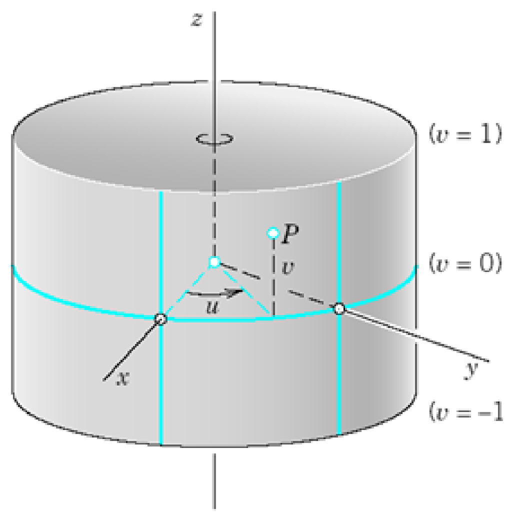


10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

■ Ex. 1 **Parametric Representation of a Cylinder**

$$x^2 + y^2 = a^2, \quad -1 \le z \le 1$$

매개변수표현식 : $\mathbf{r}(u,v) = [a\cos u,\ a\sin u,\ v] = a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k}$

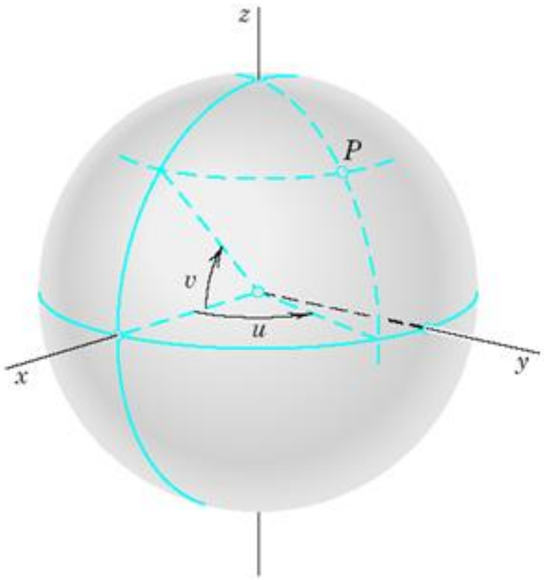


* 매개변수 u와 v는 uv평면의 직사각형 $(R : 0 \le u \le 2\pi,\ -1 \le v \le 1)$ 에서 변함
* $\mathbf{r}$의 성분 : $x = a\cos u,\ y = a\sin u,\ z = v$
* 곡선 $v =$ 상수 : 평행한 원들

  곡선 $u =$ 상수 : 수직인 직선들

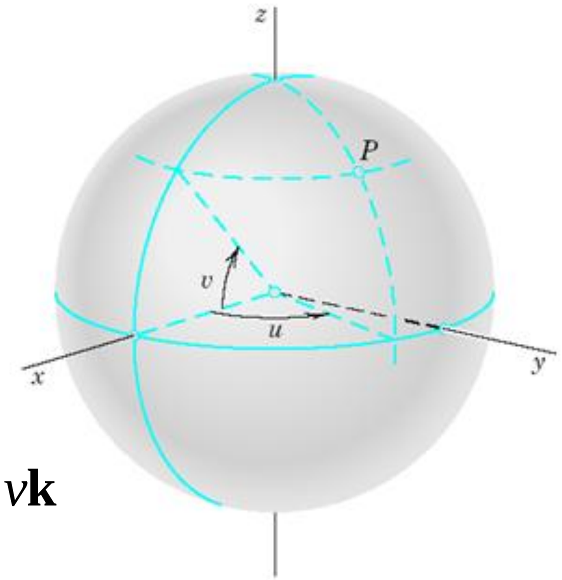
10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

- Ex. 2 **Parametric Representation of a Sphere**



10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

■ Ex. 2 **Parametric Representation of a Sphere**



* 구 $x^2 + y^2 + z^2 = a^2$의 매개변수표현식

$$: \ \mathbf{r}(u,v) = a\cos v\cos u\,\mathbf{i} + a\cos v\sin u\,\mathbf{j} + a\sin v\,\mathbf{k}$$

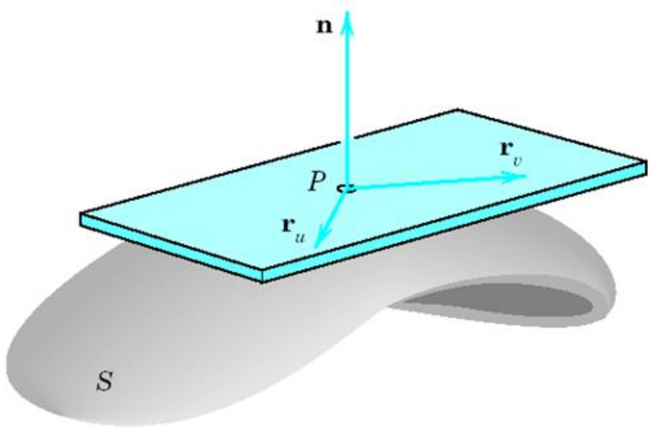
$$R: 0 \le u \le 2\pi, \quad -\frac{\pi}{2} \le v \le \frac{\pi}{2}$$

* 다른 매개변수표현식 : $\mathbf{r}(u,v) = a\cos u\sin v\,\mathbf{i} + a\sin u\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$

$$R: 0 \le u \le 2\pi, \quad 0 \le v \le \pi$$

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

- **Tangent Plane and Surface Normal (접평면과 곡면법선)**
- **Tangent Plane:** 곡면의 한 점을 통과하는 모든 곡선의 접선벡터들이 형성하는 곡면
- **Normal Vector (법선벡터):** 접평면에 수직인 벡터



$*$ 곡면 $S\ :\ \mathbf{r}=\mathbf{r}(u,v)$

$*$ S 상의 곡선 $C:\ \tilde{\mathbf{r}}(t)=\mathbf{r}(u(t),v(t))$

$*$ S 상에서 C의 접선벡터 : $\tilde{\mathbf{r}}'(t)=\dfrac{d\tilde{\mathbf{r}}}{dt}=\dfrac{d\mathbf{r}}{du}u'+\dfrac{d\mathbf{r}}{dv}v'$

P에서 편도함수 $\mathbf{r}_u$와 $\mathbf{r}_v$는 P에서 S에 접하게 됨

$\Rightarrow$ $\mathbf{r}_u$와 $\mathbf{r}_v$는 P에서 S의접평면생성 $\Rightarrow$ P에서 S의 법선벡터 : $\mathbf{N}=\mathbf{r}_u\times\mathbf{r}_v\neq\mathbf{0}$

$*$ 법선벡터의단위벡터 : $\mathbf{n}=\dfrac{1}{|\mathbf{N}|}\mathbf{N}=\dfrac{1}{|\mathbf{r}_u\times\mathbf{r}_v|}\mathbf{r}_u\times\mathbf{r}_v$

$*$ $S\ :\ g(x,y,z)=0 \quad\Rightarrow\quad \mathbf{n}=\dfrac{1}{|\operatorname{grad} g|}\operatorname{grad} g$

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

● **Tangent Plane and Surface Normal (접평면과 곡면법선)**

곡면 S는 $\mathbf{r}_u$와 $\mathbf{r}_v$에 의해 생성되는 유일한 접평면을 가지며,

S의 점들에서 방향이 연속적인 유일한 법선벡터를 가진다.

■ **Ex. 4 Unit Normal Vector of a Sphere (구의 단위법선벡터)**

$$g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$$

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

- **Tangent Plane and Surface Normal (접평면과 곡면법선)**

곡면 S는 $\mathbf{r}_u$와 $\mathbf{r}_v$에 의해 생성되는 유일한 접평면을 가지며,

S의 점들에서 방향이 연속적인 유일한 법선벡터를 가진다.

■ Ex. 4 Unit Normal Vector of a Sphere (구의 단위법선벡터)

$$g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$$

$$\mathbf{n}(x, y, z) = \left[\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right] = \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}$$

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

- **Ex. 5 Unit Normal Vector of a Cone**

$$g(x, y, z) = -z + \sqrt{x^2 + y^2} = 0$$

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

- **Ex. 5 Unit Normal Vector of a Cone**

$$g(x, y, z) = -z + \sqrt{x^2 + y^2} = 0$$

$$\mathbf{n}(x, y, z) = \left[\frac{x}{\sqrt{2(x^2 + y^2)}}, \frac{y}{\sqrt{2(x^2 + y^2)}}, \frac{-1}{\sqrt{2}} \right]$$

10.5 Surfaces for Surface Integrals (면적분에서의 곡면)

PROBLEM SET 10.5

HW: 9, 18

10.6 Surface Integrals (면적분)

- **Surface Integrals**

곡면 S : $\mathbf{r}(u,v)=[x(u,v),y(u,v),z(u,v)]=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$는 구분적으로 매끄럽다.

$*$ 법선벡터 : $\mathbf{N}=\mathbf{r}_u\times\mathbf{r}_v$

$*$ 단위법선벡터 : $\mathbf{n}=\dfrac{1}{|\mathbf{N}|}\mathbf{N}=\dfrac{1}{|\mathbf{r}_u\times\mathbf{r}_v|}\mathbf{r}_u\times\mathbf{r}_v$

벡터함수 $\mathbf{F}$에 대해 S에서 면적분 : $\iint_S \mathbf{F}\bullet\mathbf{n}\,dA=\iint_R \mathbf{F}(\mathbf{r}(u,v))\bullet\mathbf{N}(u,v)\,du\,dv$

$*$ $\mathbf{n}\,dA=\mathbf{n}|\mathbf{N}|\,du\,dv=\mathbf{N}\,du\,dv$

$*$ $\mathbf{F}\bullet\mathbf{n}$: $\mathbf{F}$의 법선성분

$$\cos\alpha=\mathrm{n}\cdot\mathrm{i}/|\mathrm{n}||\mathrm{i}|$$

$$\cos\alpha\, dA=dy\,dz$$

$\mathbf{F}=[F_1,F_2,F_3]$, $\mathbf{N}=[N_1,N_2,N_3]$, $\mathbf{n}=[\cos\alpha,\cos\beta,\cos\gamma]$ (α, β, γ는 $\mathbf{n}$과 좌표축 사이의 각도)

$$\Rightarrow \iint_S \mathbf{F}\bullet\mathbf{n}\,dA=\iint_S (F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma)\,dA$$

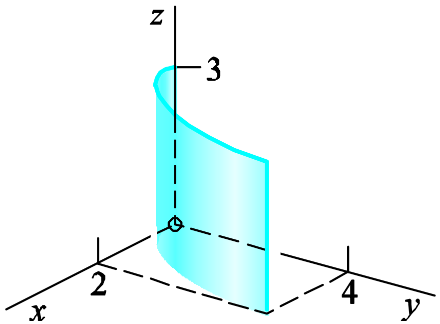
$$=\iint_R (F_1N_1+F_2N_2+F_3N_3)\,du\,dv=\iint_S (F_1\,dy\,dz+F_2\,dz\,dx+F_3\,dx\,dy)$$

Surface integral → double integral

10.6 Surface Integrals (면적분)

■ **Ex. 1 Flux Through a Surface (곡면을 통과하는 유출량)**

Compute the flux of water through the parabolic cylinder $S\!: \; y = x^2, \; 0 \le x \le 2, \; 0 \le z \le 3$ if the velocity vector is, $\mathbf{v} = \mathbf{F} = [3z^2, \; 6, \; 6xz]$, speed being measured in m/s. (Generally, $\mathbf{F}=\rho\mathbf{v}$, but water has the density ρ=1 gm/cm^3=1ton/m^3)



10.6 Surface Integrals (면적분)

■ Ex. 1 Flux Through a Surface (곡면을 통과하는 유출량)

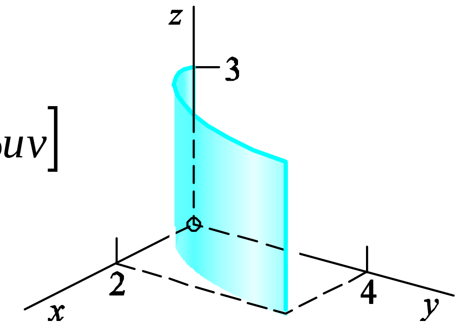
Compute the flux of water through the parabolic cylinder $S:\ y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$ if the velocity vector is, $\mathbf{v} = \mathbf{F} = [3z^2,\ 6,\ 6xz]$, speed being measured in m/s. (Generally, **F**=ρ**v**, but water has the density ρ=1 gm/cm³=1ton/m³)

$$x = u,\ z = v \quad \Rightarrow \quad y = x^2 = u^2$$

$$\Rightarrow \quad S\ :\ \mathbf{r} = [u, u^2, v] \quad (0 \le u \le 2,\ 0 \le v \le 3), \quad \mathbf{F}(S) = [3v^2,\ 6,\ 6uv]$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [1,\ 2u,\ 0] \times [0,\ 0,\ 1] = [2u,\ -1,\ 0]$$

$$\therefore\ \mathbf{F}(S) \bullet \mathbf{N} = 6uv^2 - 6$$

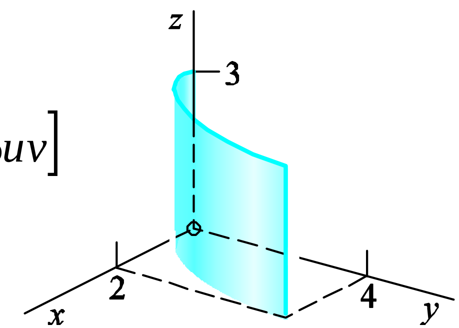


$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv = \int_0^3 (3u^2v^2 - 6u)\Big|_{u=0}^{2} dv = \int_0^3 (12v^2 - 12)\, dv = (4v^3 - 12v)\Big|_{v=0}^{3} = 72 \left[\mathrm{m}^3/\mathrm{sec}\right]$$

10.6 Surface Integrals (면적분)

■ Ex. 1 Flux Through a Surface (곡면을 통과하는 유출량)

Compute the flux of water through the parabolic cylinder $S:\ y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$ if the velocity vector is, $\mathbf{v} = \mathbf{F} = [3z^2,\ 6,\ 6xz]$, speed being measured in m/s.
(Generally, **F**=ρ**v**, but water has the density ρ=1 gm/cm³=1ton/m³)



$$x = u,\ z = v \quad \Rightarrow \quad y = x^2 = u^2$$

$$\Rightarrow \quad S\ :\ \mathbf{r} = [u, u^2, v] \quad (0 \le u \le 2,\ 0 \le v \le 3), \quad \mathbf{F}(S) = [3v^2,\ 6,\ 6uv]$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [1,\ 2u,\ 0] \times [0,\ 0,\ 1] = [2u,\ -1,\ 0]$$

$$\therefore\ \mathbf{F}(S) \bullet \mathbf{N} = 6uv^2 - 6$$

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv = \int_0^3 (3u^2v^2 - 6u)\Big|_{u=0}^{2} dv = \int_0^3 (12v^2 - 12)\, dv = (4v^3 - 12v)\Big|_{v=0}^{3} = 72 \left[\mathrm{m}^3 / \mathrm{sec} \right]$$

다른 해법

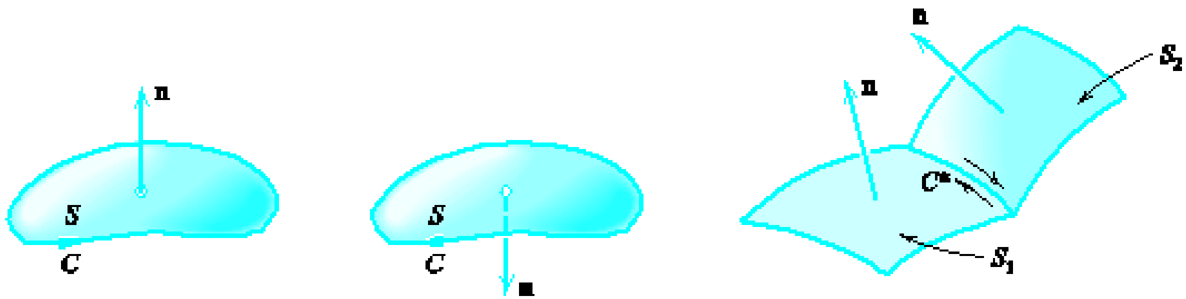
$$\mathbf{N} = |\mathbf{N}|\mathbf{n} = |\mathbf{N}|[\cos\alpha, \cos\beta, \cos\gamma] = [2u,\ -1,\ 0] = [2x,\ -1,\ 0] \quad \Rightarrow \quad \cos\alpha > 0,\ \cos\beta < 0,\ \cos\gamma = 0$$

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \int_0^3 \int_0^4 3z^2\, dy\, dz - \int_0^2 \int_0^3 6\, dz\, dx = \int_0^3 4(3z^2)\, dz - \int_0^2 (6 \cdot 3)\, dx = 4 \cdot 3^3 - 6 \cdot 3 \cdot 2 = 72$$

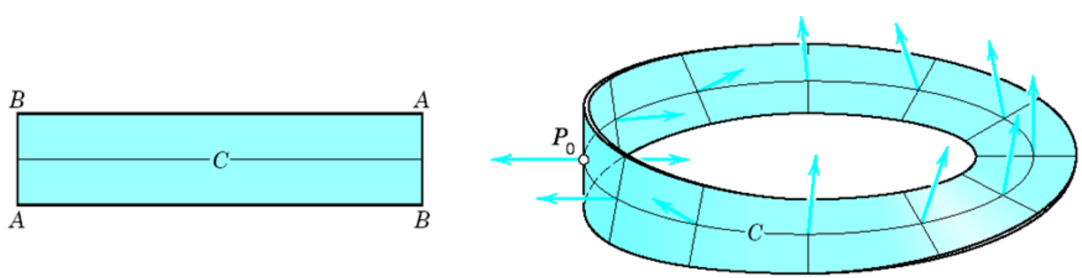
10.6 Surface Integrals (면적분)

- **Orientation of Surfaces (곡면의 방향)**

- 방향의 변경: **n**을 –**n**으로 대체하는 것은 면적분에 -1을 곱하는 것과 일치함.
- 매끄러운 곡면은 방향을 가질 수 있다 (Orientable).
- 구분적으로 매끄러운 곡면도 방향을 가질 수 있다.



- 매끄러운 곡면의 충분히 작은 조각도 항상 방향을 가진다.
  그러나 전체 곡면에 대해서는 성립하지 않을 수도 있다. Ex) Möbius strip



10.6 Surface Integrals (면적분)

- **Surface Integrals Without Regard to Orientation (방향을 고려하지 않는 면적분)**

• 면적분의 다른 형식 : $\iint_S G(\mathbf{r})dA = \iint_R G(\mathbf{r}(u,v))|\mathbf{N}(u,v)|dudv$

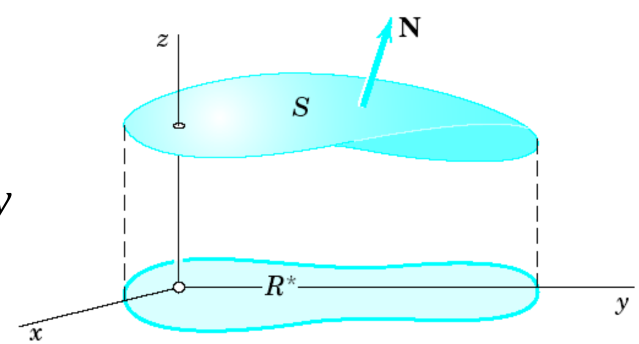
$S:\ z = f(x,y) \quad \Rightarrow \quad u = x,\ v = y,\ \mathrm{r} = [u,\ v,\ f]$

$\Rightarrow\ |\mathbf{N}| = |\mathbf{r}_u \times \mathbf{r}_v| = |[1,\ 0,\ f_u] \times [0,\ 1,\ f_v]| = |[-f_u,\ -f_v,\ 1]| = \sqrt{1 + f_u{}^2 + f_v{}^2}$

$\Rightarrow\ \iint_S G(\mathbf{r})dA = \iint_{R^*} G(x, y, f(x,y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$

• S의 면적$(G=1)$: $A(S) = \iint_S dA = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| dudv$

$S:\ z = f(x,y) \quad \Rightarrow \quad A(S) = \iint_{R^*} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$



10.6 Surface Integrals (면적분)

- **Ex. 4 Area of a Sphere**

$$r(u,v)=[a\cos v\cos u,\ a\cos v\sin u,\ a\sin v],\ 0\le u\le 2\pi,\ -\pi/2\le v\le \pi/2$$

10.6 Surface Integrals (면적분)

■ **Ex. 4 Area of a Sphere**

$$r(u,v)=[a\cos v\cos u,\ a\cos v\sin u,\ a\sin v],\ 0\le u\le 2\pi,\ -\pi/2\le v\le \pi/2$$

$$\mathbf{r}_u\times\mathbf{r}_v=[a^2\cos^2 v\cos u,\ \ a^2\cos^2 v\sin u,\ \ a^2\cos v\sin v]$$

$$\cos^2 u+\sin^2 u=1,\quad \cos^2 v+\sin^2 v=1$$

$$\Rightarrow\quad |\mathbf{r}_u\times\mathbf{r}_v|=a^2\left(\cos^4 v\cos^2 u+\cos^4 v\sin^2 u+\cos^2 v\sin^2 v\right)^{1/2}=a^2|\cos v|$$

$$\Rightarrow\quad A(S)=a^2\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}|\cos v|\,du\,dv=2\pi a^2\int_{-\pi/2}^{\pi/2}\cos v\,dv=4\pi a^2$$

10.6 Surface Integrals (면적분)

- **Ex. 5 Torus Surface (Doughnut Surface): Representation and Area**

$$r(u,v) = [(a + b\cos v)\cos u,\ (a + b\cos v)\sin u,\ b\sin v],\ 0 \le u \le 2\pi,\ 0 \le v \le 2\pi$$

10.6 Surface Integrals (면적분)

- **Ex. 5 Torus Surface (Doughnut Surface): Representation and Area**

$$r(u,v) = [(a + b\cos v)\cos u,\ (a + b\cos v)\sin u,\ b\sin v],\ 0 \le u \le 2\pi,\ 0 \le v \le 2\pi$$

$$\mathbf{r}_u = [-(a+b\cos v)\sin u,\ \ (a+b\cos v)\cos u,\ \ 0]$$

$$\mathbf{r}_v = [-b\sin v\cos u,\ \ -b\sin v\sin u,\ \ b\cos v]$$

$$\mathbf{r}_u \times \mathbf{r}_v = [b(a+b\cos v)\cos u\cos v,\ \ b(a+b\cos v)\sin u\cos v,\ \ b(a+b\cos v)\sin v]$$

$$\Rightarrow \quad |\mathbf{r}_u \times \mathbf{r}_v| = b(a+b\cos v)$$

$$\Rightarrow \quad A(S) = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos v)\,du\,dv = 4\pi^2 ab$$

10.6 Surface Integrals (면적분)

PROBLEM SET 10.6

HW: 30 (a), (b), (c), (d)

10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

- **Triple Integrals (삼중적분):** 공간의 닫힌 유한한 3차원 영역에서 함수의 적분
- 좌표평면(삼차원)에 평행한 평면으로 T를 분할한다.
- 각 상자에서 한 점 (x_k, y_k, z_k)을 택하여

  $J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k$와 같은 형태의 합을 만든다.($\Delta V_k$: k번째 상자의 부피)

- $f(x,y,z)$가 T를 포함하는 영역에서 연속이고

  T는 유한개의 매끄러운 곡선에 의해 제한된다고 가정

  $\Rightarrow$ 수열 J_n이 수렴, 극한을 영역 T에서의 $f(x,y,z)$의 삼중적분이라한다.

$$\iiint_T f(x,y,z)\,dxdydz = \iiint_T f(x,y,z)\,dV$$

10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

- **Divergence Theorem of Gauss**
  (Transformation Between Triple and Surface Integrals))

T: 닫혀있고 유한한 입체

S:경계가 구분적으로 매끄러우며방향을 가지는 곡면으로 T의 표면

$\mathbf{F}$:연속이며 T 를 포함하는 영역에서 연속인 1차 편도함수를 가지는 벡터함수

$$\Rightarrow \quad \iiint_T \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \bullet \mathbf{n} dA$$

$\mathbf{F} = [F_1,\ F_2,\ F_3]$이고 S의 외향 법선벡터 $\mathbf{n} = [\cos\alpha,\ \cos\beta, \cos\gamma]$이면

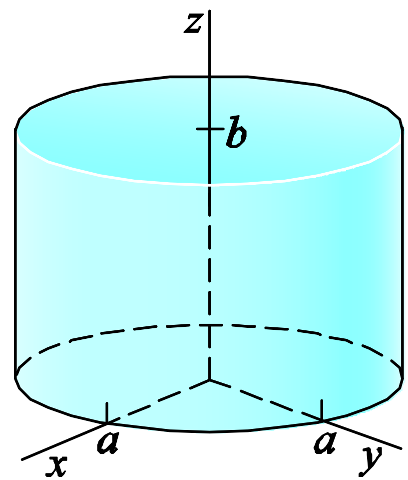
$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

■ **Ex. 1 Evaluation of a Surface Integral by the Divergence Theorem**

$$I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy\right)$$

$$S: \quad x^2 + y^2 = a^2 (0 \le z \le b), \; z = 0, \; z = b\left(x^2 + y^2 \le a^2\right)$$



10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

- **Ex. 1 Evaluation of a Surface Integral by the Divergence Theorem**

$$I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy\right)$$

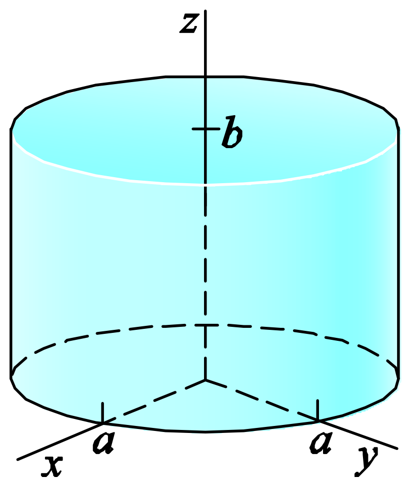
$$S: \ x^2 + y^2 = a^2 (0 \le z \le b),\ z = 0,\ z = b\left(x^2 + y^2 \le a^2\right)$$

$$F_1 = x^3,\ F_2 = x^2 y,\ F_3 = x^2 z \quad \Rightarrow \quad \operatorname{div}\mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$$

극좌표$(x = r\cos\theta,\ y = r\sin\theta)$ 적용 $\quad \Rightarrow \quad dxdydz = rdrd\theta dz$

$$I = \iiint_T 5x^2 dxdydz = \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \left(5r^2 \cos^2\theta\right) rdrd\theta dz$$

$$= 5\int_{z=0}^{b} \int_{\theta=0}^{2\pi} \frac{a^4}{4} \cos^2\theta d\theta dz = 5\int_{z=0}^{b} \frac{a^4 \pi}{4} dz = \frac{5\pi}{4} a^4 b$$



10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

- **Ex. 2 Verification of the Divergence Theorem**

$$\iint_S (7x\mathrm{i} - z\mathrm{k}) \cdot \mathrm{n}\, dA$$

$$S: \quad x^2 + y^2 + z^2 = 4$$

10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

- **Ex. 2 Verification of the Divergence Theorem**

$$\iint_S (7x\mathrm{i} - z\mathrm{k}) \cdot \mathrm{n}\, dA$$

$$S: \quad x^2 + y^2 + z^2 = 4$$

(a) $\operatorname{div} \mathbf{F} = 6 \quad \Rightarrow \quad 6 \cdot (4/3)\pi \cdot 2^3 = 64\pi$

(b) $S: \ \mathrm{r} = [2\cos v \cos u, \ \ 2\cos v \sin u, \ \ 2\sin v]$

$\mathrm{N} = \mathrm{r}_u \times \mathrm{r}_v = [4\cos^2 v \cos u, \ \ 4\cos^2 v \sin u, \ \ 4\cos v \sin v]$

$$\iint_S \mathbf{F}(S) \cdot \mathrm{N}\, dA = 64\pi$$

10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

- **Coordinate Invariance of the Divergence (발산의 좌표계 불변)**

- **Mean Value Theorem for Triple Integrals (삼중적분의 평균값 정리)**

  유한하고 단순연결된 영역 T의 연속함수 $f(x, y, z)$에 대해, T에서

  $$\iiint_T f(x, y, z)\,dV = f(x_0, y_0, z_0)\,V(T) \quad (V(T) : T\text{의 체적})$$

  만족하는 점 (x_0, y_0, z_0)가 있다.

- **Invariance of the Divergence**

  영역에서 1차 편도함수가 연속인 벡터함수의 발산은 직각 좌표계의 선택에 독립이다.

10.7 Triple Integrals. Divergence Theorem of Gauss (삼중적분. Gauss의 발산정리)

PROBLEM SET 10.7

HW: 8, 24

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

■ Ex. 1 **Fluid Flow. Physical Interpretation of the Divergence**

We consider the flow of an incompressible fluid of constant density ρ=1 which is steady, that is, does not vary with time.

* 흐름은 임의의 점 P에서의 속도벡터장 $\mathbf{v}(P)$에 의해서 결정

* S : 공간상의 영역 T의 경계면

* $\mathbf{n}$: S의 외향 단위 법선벡터

1. 단위 시간당 S를 통하여 T로부터 외부로 흐르는 유체의 전체질량 : $\iint_S \mathbf{v} \bullet \mathbf{n}\, dA$

2. T의 외부로 흐르는 평균유출량 : $\dfrac{1}{V}\iint_S \mathbf{v} \bullet \mathbf{n}\, dA$

* 비압축성 장상류의 속도벡터 $\mathbf{v}$의 발산은 대응점에서의 그 흐름의 발생강도

* T내의 발생점이 없을 필요충분조건 : $\operatorname{div} \mathbf{v} = \mathbf{0}$

$$\operatorname{div} \mathbf{v}(P) = \lim_{d(T)\to 0} \frac{1}{V(T)} \iint_{S(T)} \mathbf{v} \bullet \mathbf{n}\, dA \quad \Rightarrow \quad \iint_S \mathbf{v} \bullet \mathbf{n}\, dA = 0$$

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

■ Ex. 2 **Modeling of Heat Flow. Heat or Diffusion Equation**

Physical experiments show that in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient of the temperature. This means that the velocity v of the heat flow in a body is of the form

$$\mathbf{v} = -K \operatorname{grad} U$$

Where $U(x, u, z, t)$ is temperature, t is time, and K is called the thermal conductivity of the body; in ordinary physical circumstances K is a constant. Using this information, set up the mathematical model of heat flow, the so-called heat equation or diffusion equation.

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

■ Ex. 2 **Modeling of Heat Flow. Heat or Diffusion Equation**

Physical experiments show that in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient of the temperature. This means that the velocity v of the heat flow in a body is of the form

$$\mathbf{v} = -K \operatorname{grad} U$$

Where $U(x, u, z, t)$ is temperature, t is time, and K is called the thermal conductivity of the body; in ordinary physical circumstances K is a constant. Using this information, set up the mathematical model of heat flow, the so-called heat equation or diffusion equation.

1. 단위 시간당 T로부터 나가는 열량 : $\iint_S \mathbf{v} \bullet \mathbf{n}\, dA$

$$\operatorname{div}(\operatorname{grad} U) = \nabla^2 U = U_{xx} + U_{yy} + U_{zz} \quad \Rightarrow \quad \iint_S \mathbf{v} \bullet \mathbf{n}\, dA = -K \iiint_T \operatorname{div}(\operatorname{grad} U)\, dx\,dy\,dz = -K \iiint_T \nabla^2 U\, dx\,dy\,dz$$

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

2. T내의 열의 전체량 : $H = \iiint_T \sigma\rho U\,dxdydz$ (σ: 물체 재료의 비열, ρ: 재료의 밀도)

H가 감소하는 시간 비율 : $-\dfrac{\partial H}{\partial t} = -\iiint_T \sigma\rho \dfrac{\partial U}{\partial t} dxdydz$

H가 감소하는 시간 비율은 T로부터 나가는 열의 양과 같아야 한다.

$$\Rightarrow \quad -\iiint_T \sigma\rho \frac{\partial U}{\partial t} dxdydz = -K\iiint_T \nabla^2 U\,dxdydz$$

$$\Rightarrow \quad \iiint_T \left(\sigma\rho \frac{\partial U}{\partial t} - K\nabla^2 U\right) dxdydz = 0$$

$$\therefore \frac{\partial U}{\partial t} = \frac{K}{\sigma\rho}\nabla^2 U \qquad \frac{K}{\sigma\rho} = c^2 : \text{ thermal diffusivity}$$

Heat equation or diffusion equation

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

- **Potential Theory. Harmonic Functions (퍼텐셜 이론. 조화함수)**

- **Laplace's equation:** $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$
- **Potential theory:** 라플라스 방정식 해에 대한 이론
- **Harmonic functions:** 연속적인 2차 편도함수를 갖는 라플라스 방정식의 해

$$\iiint_T \nabla \cdot \mathrm{F}\, dV = \iiint_T \nabla^2 f\, dV$$

$$\iiint_T \nabla \cdot \mathrm{F}\, dV = \iint_S \mathrm{F} \cdot n\, dA = \iint_S \nabla f \cdot n\, dA = \iint_S \frac{\partial f}{\partial n} dA$$

- **A Basic Property of Harmonic Functions**

구분적으로 매끄럽게 닫히고 방향을 줄 수 있는 곡면에서 조화함수의 법선도함수 적분값은 0이 된다.

$$\iiint_T \nabla^2 f\, dV = 0 = \iint_S \frac{\partial f}{\partial n} dA$$

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

- Ex. 4 **Green's Theorem**

Let f and g be scalar functions such that F=f grad$g$ satisfies the assumption of the divergence theorem in some region T.

$$\operatorname{div}\mathbf{F} = \operatorname{div}(f \operatorname{grad} g) = \operatorname{div}\left(\left[f\frac{\partial g}{\partial x},\ f\frac{\partial g}{\partial y},\ f\frac{\partial g}{\partial z}\right]\right)$$

$$= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2}\right) = f\,\nabla^2 g + \operatorname{grad} f \bullet \operatorname{grad} g$$

f 가 스칼라 함수 $\Rightarrow \quad \mathbf{F}\bullet\mathbf{n} = \mathbf{n}\bullet\mathbf{F} = \mathbf{n}\bullet(f \operatorname{grad} g) = (\mathbf{n}\bullet\operatorname{grad} g)f$

$$\mathbf{n}\bullet\operatorname{grad} g = \frac{\partial g}{\partial n} \quad\Rightarrow\quad \iiint_T \left(f\,\nabla^2 g + \operatorname{grad} f \bullet \operatorname{grad} g\right) dV = \iint_S f\frac{\partial g}{\partial n}\,dA \quad \text{(Green의 첫 번째 공식)}$$

$$\iiint_T \left(f\,\nabla^2 g - g\nabla^2 f\right) dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dA \quad \text{(Green의 두 번째 공식)}$$

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

- **Harmonic Functions**

  f : 영역 D에서 조화함수

  S : D내의 구분적으로 매끄럽고 닫힌 방향을 줄 수 있는 곡면

  T : D에 속하는 S를 감싸는 전체영역

  f가 S의 모든점에서 값이 0이다 $\Rightarrow$ f는 T에서 동일하게 0이다.

- **Uniqueness Theorem for Laplace's Equation**

  T : 발산정리 가정을 만족하는 영역

  f : T와 T의 경계면 S를 포함하는 영역 D에서 조화함수

  $\Rightarrow$ f는 S상에서 값으로 T내에서 유일하게 결정된다.

- **Uniqueness Theorem for the Dirichlet Problem**

  위의 가정이 만족되고 라플라스 방정식에 대한 Dirichlet 문제가 T에서 해를 가진다면,

  이 해는 유일하다

10.8 Further Applications of the Divergence Theorem (발산정리의 응용)

PROBLEM SET 10.8

HW: 7, 8

10.9 Stokes's Theorem

- **Stokes' Theorem (Transformation Between Surface and Line Integrals)**

S : 공간에서 구분적으로 매끄럽고 방향을 갖는 곡면

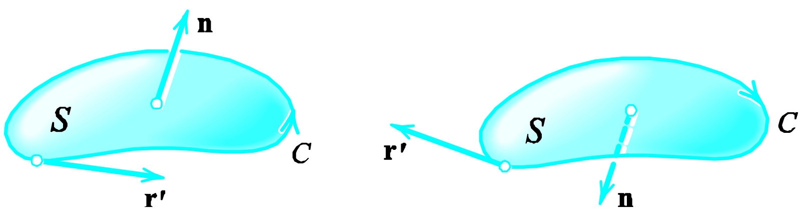
C : S의 경계로 구분적으로 매끄럽고 단순히 닫힌 곡선

$\mathbf{F}$: S를 포함하는 영역에서 연속인 편도함수를 가지는 연속인 벡터함수

$$\Rightarrow \quad \iint_S (\operatorname{curl}\mathbf{F}) \bullet \mathbf{n}\, dA = \oint_C \mathbf{F} \bullet \mathbf{r}'(s)\, ds$$

성분으로 표시하면

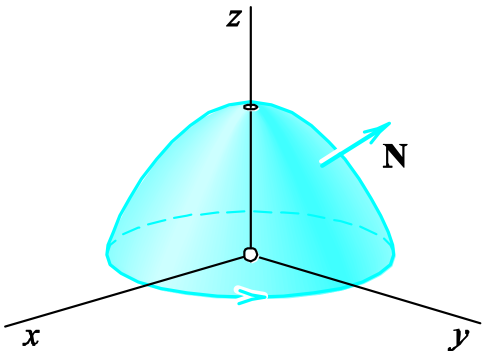
$$\Rightarrow \quad \iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \right] du\,dv = \oint_{\bar{C}} \left(F_1 dx + F_2 dy + F_3 dz \right)$$



10.9 Stokes's Theorem

- **Ex. 1 Verification of Stokes's Theorem**

$$F = [y, z, x],\ S:\ z = f(x, y) = 1 - (x^2 + y^2),\ z \geq 0$$



10.9 Stokes's Theorem

■ **Ex. 1 Verification of Stokes's Theorem**

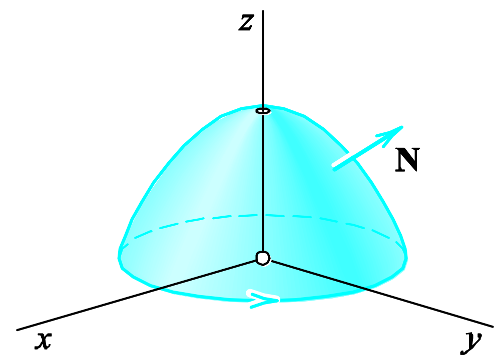
$$F = [y, z, x],\ S:\ z = f(x, y) = 1 - (x^2 + y^2),\ z \geq 0$$

Case 1 Line integral

$$C\ :\ \mathbf{r}(s) = [\cos s,\ \sin s,\ 0] \Rightarrow \text{단위 접선벡터}\ :\ \mathbf{r}'(s) = [-\sin s,\ \cos s,\ 0]$$

$$\mathbf{F}(\mathbf{r}(s)) = [\sin s,\ 0,\ \cos s]$$

$$\therefore \oint_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(s)) \bullet \mathbf{r}'(s) ds = \int_0^{2\pi} [(\sin s)(-\sin s + 0 + 0)] ds = -\pi$$



10.9 Stokes's Theorem

Ex. 1 Verification of Stokes's Theorem

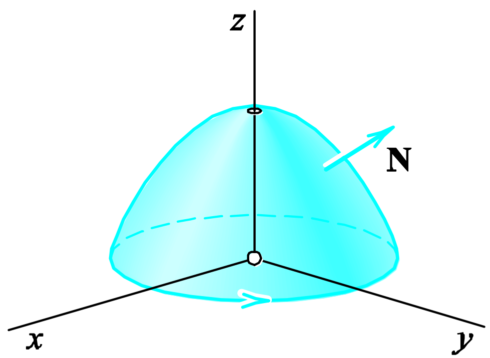
$$F=[y,z,x],\ S:\ z=f(x,y)=1-(x^2+y^2),\ z\geq 0$$

Case 1 Line integral

$C\ :\ \mathbf{r}(s)=[\cos s,\ \sin s,\ 0] \Rightarrow$ 단위 접선벡터 $:\ \mathbf{r}'(s)=[-\sin s,\ \cos s,\ 0]$

$$\mathbf{F}(\mathbf{r}(s))=[\sin s,\ 0,\ \cos s]$$

$$\therefore \oint_C \mathbf{F}\bullet d\mathbf{r}=\int_0^{2\pi}\mathbf{F}(\mathbf{r}(s))\bullet\mathbf{r}'(s)ds=\int_0^{2\pi}[(\sin s)(-\sin s+0+0)]ds=-\pi$$



Case 2 Surface integral

$F_1=y,\ F_2=z,\ F_3=x \quad \Rightarrow \quad \mathrm{curl}\mathbf{F}=\mathrm{curl}[F_1,\ F_2,\ F_3]=\mathrm{curl}[y,\ z,\ x]=[-1,\ -1,\ -1]$

S의 법선벡터 $:\ \mathbf{N}=\mathrm{grad}(z-f(x,y))=[2x,\ 2y,\ 1]$

$$\Rightarrow \quad (\mathrm{curl}\mathbf{F})\bullet\mathbf{N}=-2x-2y-1$$

$$\therefore \iint_S(\mathrm{curl}\mathbf{F})\bullet\mathbf{n}dA=\iint_R(\mathrm{curl}\mathbf{F})\bullet\mathbf{N}dxdy=\iint_R(-2x-2y-1)dxdy$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(-2r(\cos\theta+\sin\theta)-1)rdrd\theta=\int_{\theta=0}^{2\pi}\left(-\frac{2}{3}(\cos\theta+\sin\theta)-\frac{1}{2}\right)d\theta=0+0-\frac{1}{2}(2\pi)=-\pi$$

10.9 Stokes's Theorem

PROBLEM SET 10.9

HW: 19