Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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Textbook

- J. Freidberg,
  "Ideal Magnetohydrodynamics
  (Modern Perspectives in Energy)", Springer,
  1st edition (1987)

- T. Takeda and S. Tokuda,
  "Computation of MHD Equilibrium of Tokamak Plasma",
  Journal of Computational Physics,
  Vol. 93, p1~107 (1991)

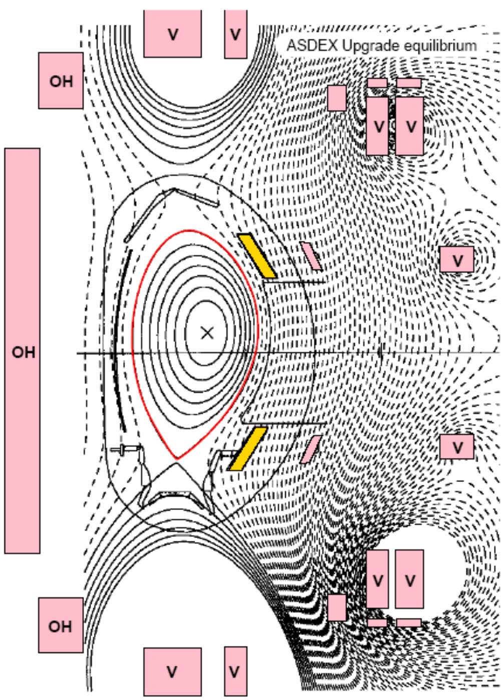
Contents

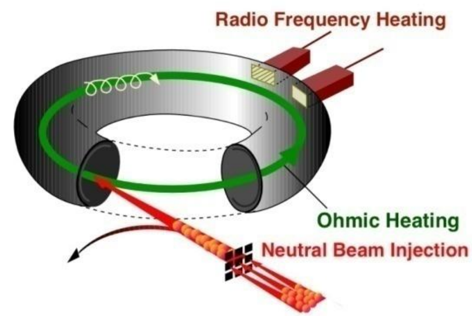
Contents

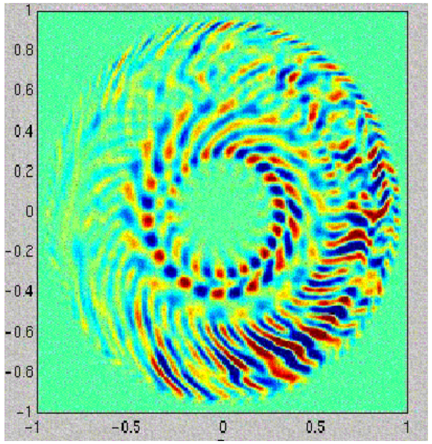
Introduction

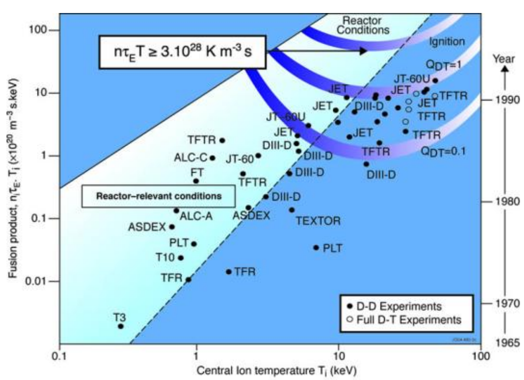
- **Basic physics areas in magnetic fusion**

- Equilibrium and stability: plasma beta
- Heating and current drive: Lawson criterion
- Transport: Lawson criterion



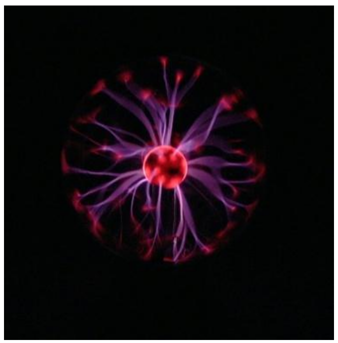


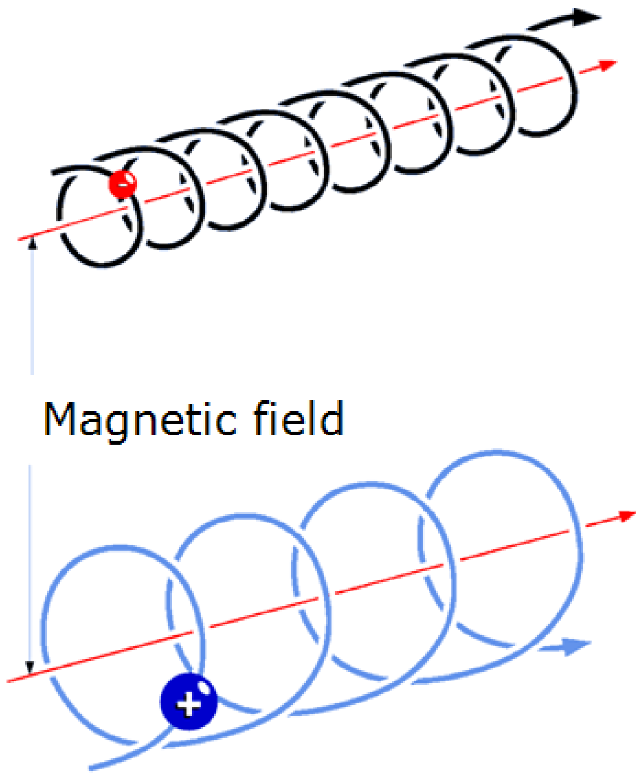


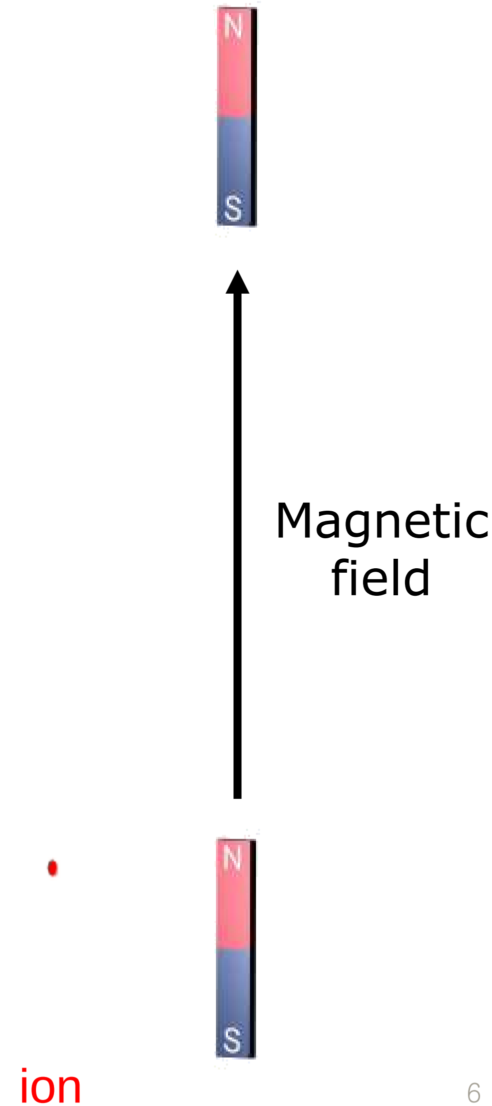


Introduction

- **Single particle motion of the plasma**

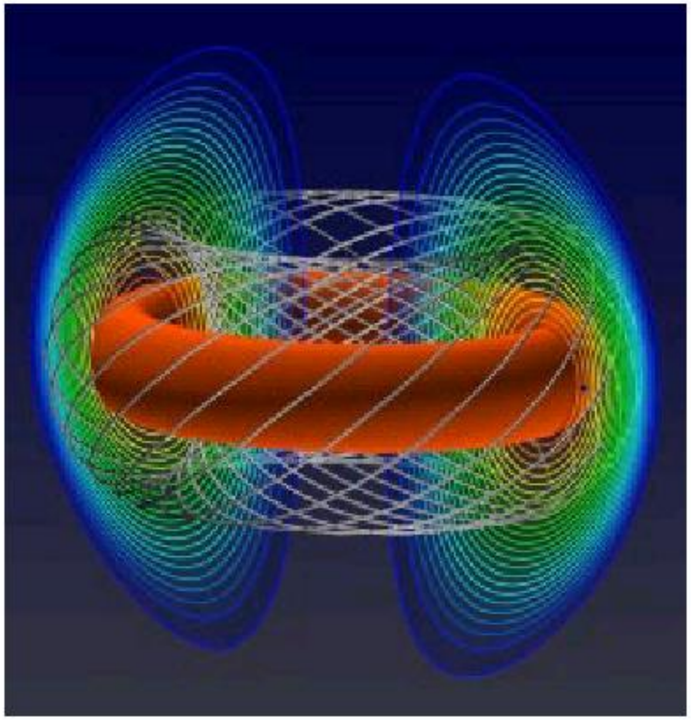


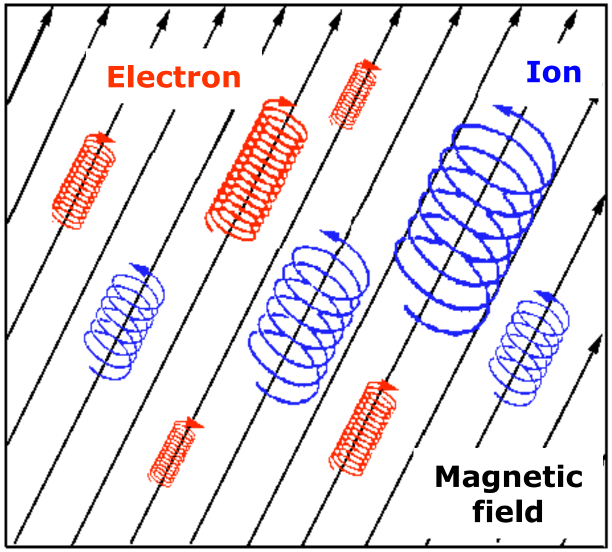




Introduction

- **Single particle motion of the plasma**





Strong magnetic field:
$r_L << a$

- Magnetic field lines trace out “magnetic surfaces”, to particles stay on these surfaces.

Introduction

- **Plasmas as fluids**

  - The single particle approach gets to be complicated.

  - A more statistical approach can be used because we cannot follow each particle separately.

  - Now introduce the concept of an **electrically charged current-carrying fluid.**

    → **Magnetohydrodynamic (magnetic fluid dynamic: MHD) equations**

Introduction

- **Ideal MHD**

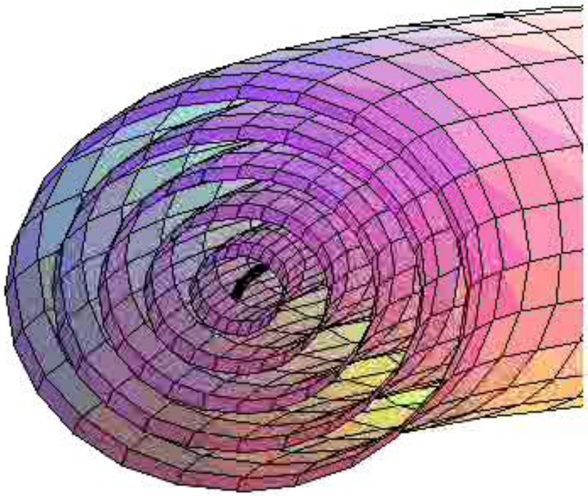
- Single-fluid model
- how magnetic, inertial, and pressure forces interact within an ideal perfectly conducting plasma in an arbitrary magnetic geometry
- Any fusion reactor must satisfy the equilibrium and stability set by ideal MHD.

- **Nonideal effects (e.g. electrical resistivity)**

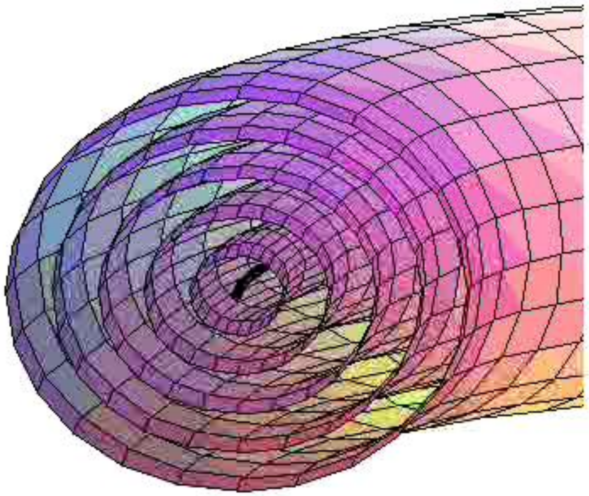
- allow the development of slower, weaker instabilities

Introduction

- Ideal MHD: $\eta = 0$

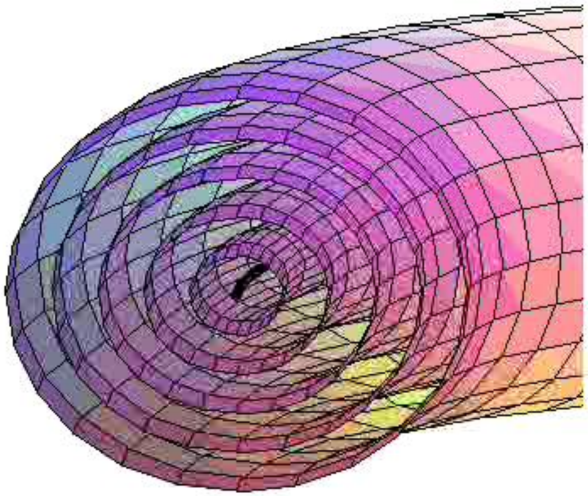


- Resistive MHD: $\eta \neq 0$



Introduction

- Ideal MHD: $\eta = 0$



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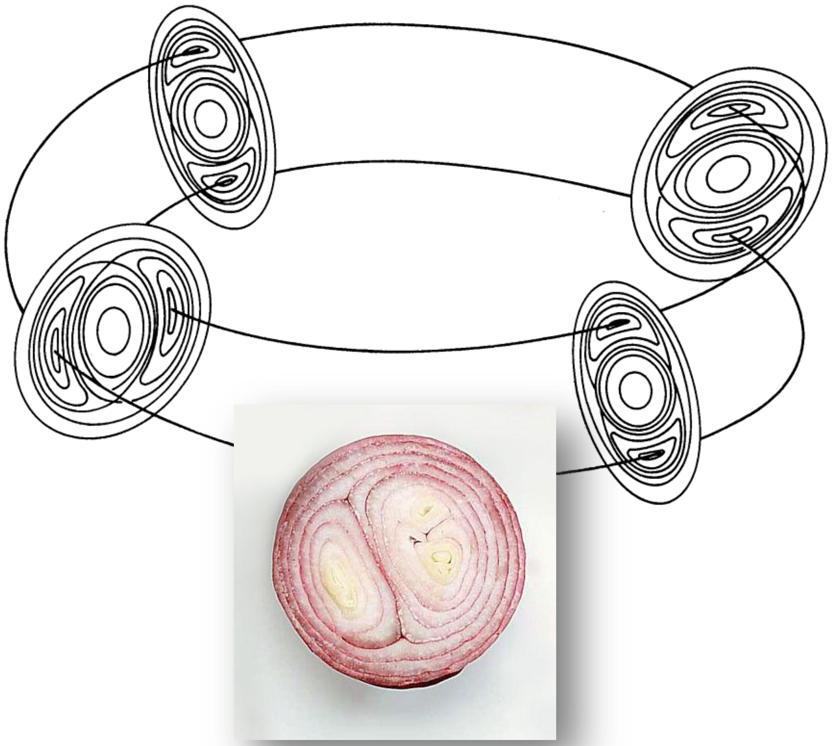


The MHD Model

- **Questions**

1. The plasma is assumed to be collision dominated.
   - Is it true in fusion plasmas?
   - The ideal MHD provides a very accurate description of most macroscopic plasma behaviour.

   cf) collisionless MHD

2. Ideal MHD must be viewed as an asymptotic model in that specific length and time scales must be assumed for the derivation to be valid.
   - Even when the collision-dominated assumption is not involved, there are many situations where MHD is used to described phenomena on time scales far beyond its strict range of applicability.

The MHD Model

- **Major discoveries of ... physics**
  - Maxwell's equati... wave propagation
  - Relativi...
  - ...m mechanics

Eliminated in the derivation of MHD

- **Important processes in fusio... mas**
  - Radiation
  - RF heating
  - Resonant ...
  - Mi...
  - ...sport
  - A...alous transport
  - Resistive instabilities
  - alpha-particle behaviour

Not adequately described by ideal MHD

The MHD Model

How does a given magnetic geometry provide forces to hold a plasma in equilibrium?

Why are certain magnetic geometries much more stable against macroscopic disturbances than others?

Why do fusion configurations have such technologically undesirable shapes as a torus, a helix, or a baseball seam?

Description of the Ideal MHD Model

- **Ideal MHD model**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Mass continuity equation

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Energy equation (equation of state): adiabatic evolution

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

Ohm's law: perfect conductor → "ideal" MHD

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Maxwell equations

Description of the Ideal MHD Model

- **Validity of the ideal MHD model**

- Characteristic length: a

- Characteristic speed: $V_{Ti} = (2T_i/m_i)^{1/2}$

- Characteristic time: $\tau_M = a/V_{Ti} \sim \mu s$

Derivation of the Ideal MHD Model

- **Starting Equations**

Homework

$$\frac{\partial f_\alpha}{\partial t} + \vec{u}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E} + \vec{u}\times\vec{B})\cdot\nabla_u f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\cdot\vec{E} = \frac{\sigma}{\varepsilon_0}$$

$$\vec{J} \equiv \sum_\alpha q_\alpha \int \vec{u} f_\alpha d\vec{u}$$ current density

$$\sigma \equiv \sum_\alpha q_\alpha \int f_\alpha d\vec{u}$$ charge density

Derivation of the Ideal MHD Model

- Collision operator for elastic collisions

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_\beta C_{\alpha\beta}$$

1. Conservation of particles between like and unlike particle collisions

$$\int C_{ee} d\vec{u} = \int C_{ii} d\vec{u} = \int C_{ei} d\vec{u} = \int C_{ie} d\vec{u} = 0$$

2. Conservation of momentum and energy btw. like particle collisions

$$\int m_e \vec{u} C_{ee} d\vec{u} = \int m_i \vec{u} C_{ii} d\vec{u} = 0 \qquad \int \frac{1}{2} m_e u^2 C_{ee} d\vec{u} = \int \frac{1}{2} m_i u^2 C_{ii} d\vec{u} = 0$$

3. Conservation of total momentum and energy btw. unlike particle collisions

$$\int (m_e \vec{u} C_{ei} + m_i \vec{u} C_{ie}) d\vec{u} = 0 \qquad \int \frac{1}{2} (m_e u^2 C_{ei} + m_i u^2 C_{ie}) d\vec{u} = 0$$

Derivation of the Ideal MHD Model

- **Two-Fluid Equations**

$$\int Q_i \left[\frac{df_\alpha}{dt} - \left(\frac{\partial f_\alpha}{\partial t} \right)_c \right] d\vec{u} = 0$$

$Q_1 = 1$ mass

$Q_2 = m_\alpha \vec{u}$ momentum

$Q_3 = m_\alpha u^2 / 2$ energy

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot n_\alpha \vec{v}_\alpha = 0$$

$$\frac{\partial}{\partial t}(m_\alpha n_\alpha \vec{v}_\alpha) + \nabla \cdot \left(m_\alpha n_\alpha \langle \vec{u}\vec{u} \rangle \right) - q_\alpha n_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}) = \int m_\alpha \vec{u} C_{\alpha\beta} d\vec{u}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2} m_\alpha n_\alpha \langle u^2 \rangle \right) + \nabla \cdot \left(\frac{1}{2} m_\alpha n_\alpha \langle u^2 \vec{u} \rangle \right) - q_\alpha n_\alpha \vec{v}_\alpha \cdot \vec{E} = \int \frac{1}{2} m_\alpha u^2 C_{\alpha\beta} d\vec{u}$$

$$n_\alpha(\vec{r}, t) \equiv \int f_\alpha d\vec{u} \qquad \vec{v}_\alpha(\vec{r}, t) \equiv \frac{1}{n_\alpha} \int \vec{u} f_\alpha d\vec{u} \qquad \langle Q \rangle \equiv \frac{1}{n_\alpha} \int Q f_\alpha d\vec{u}$$

Derivation of the Ideal MHD Model

- Considering the random thermal motion of particles

$$\vec{w} = \vec{u} - \vec{v}_\alpha(\vec{r},t), \ \langle \vec{w} \rangle = 0$$

$$p_\alpha = \frac{1}{3} n_\alpha m_\alpha \langle w^2 \rangle$$ Scalar pressure

$$\overleftrightarrow{P}_\alpha \equiv n_\alpha m_\alpha \langle w^2 \rangle$$ Total pressure tensor

$$\overleftrightarrow{\Pi}_\alpha \equiv \overleftrightarrow{P}_\alpha - p_\alpha \overleftrightarrow{I}$$ Anisotropic part of the pressure tensor

$$T_\alpha \equiv p_\alpha / n_\alpha$$ Temperature

$$h_\alpha \equiv \frac{1}{2} n_\alpha m_\alpha \langle w^2 \vec{w} \rangle$$ Heat flux due to random motion

$$\vec{R}_\alpha \equiv \int m_\alpha \vec{w} C_{\alpha\beta} d\vec{w}$$ Mean momentum transfer btw. unlike particles due to the friction of collisions

$$Q_\alpha \equiv \int \frac{1}{2} m_\alpha w_\alpha^2 C_{\alpha\beta} d\vec{w}$$ Heat generated due to collisions btw. unlike particles.

Derivation of the Ideal MHD Model

$$\left(\frac{dn_\alpha}{dt}\right)_\alpha + n_\alpha \nabla \cdot \vec{v}_\alpha = 0$$

$$m_\alpha n_\alpha \left(\frac{d\vec{v}_\alpha}{dt}\right)_\alpha - q_\alpha n_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}) + \nabla \cdot \overleftrightarrow{P}_\alpha = \vec{R}_\alpha$$

$$n_\alpha \left[\frac{d}{dt}\left(\frac{m_\alpha v_\alpha^2}{2} + \frac{3}{2}T_\alpha\right)\right]_\alpha + \nabla \cdot \left(\vec{v}_\alpha \cdot \overleftrightarrow{P} + \vec{h}_\alpha\right) - q_\alpha n_\alpha \vec{v}_\alpha \cdot \vec{E} = Q_\alpha + \vec{v}_\alpha \cdot \vec{R}_\alpha$$

$$\longleftarrow \quad \left(\frac{d}{dt}\right)_\alpha \equiv \frac{\partial}{\partial t} + \vec{v}_\alpha \cdot \nabla$$

convective derivative

Derivation of the Ideal MHD Model

- Final set of two-fluid equations

$$\left(\frac{dn_\alpha}{dt}\right)_\alpha + n_\alpha \nabla \cdot \vec{v}_\alpha = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

1. Certain asymptotic orderings are introduced which eliminate the very high-frequency, short-wavelength information in the model.

2. The equations are rewritten as a set of single-fluid equations by the introduction of appropriate single-fluid variables.

3. The plasma is assumed to be collision dominated.

$$\nabla \cdot E = \frac{e}{\varepsilon_0}(n_i - n_e)$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Derivation of the Ideal MHD Model

- **Low-Frequency, Long-Wavelength, Asymptotic Expansions**
  - First asymptotic assumption: $c \to \infty$
  (Full → low-frequency Maxwell's equations)

$$\varepsilon_0 \to 0: \quad \varepsilon_0 \partial \vec{E} / \partial t = 0$$

Displacement current, net charge neglected

$$\varepsilon_0 \nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{B} = \mu_0 e\left(n_i \vec{v}_i - n_e \vec{v}_e\right) + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \approx \mu_0 \vec{J}$$

$$n_i - n_e = \frac{\varepsilon_0}{e}\nabla \cdot \vec{E} \approx 0 \qquad n_i = n_e \equiv n$$

quasineutrality

Conditions for validity:

$$\omega / k << c$$

$$V_{T_e},\ V_{T_i} << c,\ V_{T_e} \equiv (2T_\alpha / m_\alpha)^{1/2}$$

displacement current neglected

$$\omega << \omega_{pe},\ \ \omega_{pe} = (n_0 e^2 / m_e \varepsilon_0)^{1/2}$$

$$a >> \lambda_d,\ \ \lambda_d = V_{T_e} / \omega_{pe}$$

net charge neglected

Derivation of the Ideal MHD Model

- Second asymptotic assumption: $m_e \rightarrow 0$
(electron inertia neglected: electrons have an infinitely fast response time because of their small mass)

$$0 \approx -en_e(\vec{E} + \vec{v}_e \times \vec{B}) - \nabla \cdot \overleftrightarrow{P}_e + \vec{R}_e$$

Conditions for validity:

$$\omega << \omega_{pe}, \ \lambda_d << a$$

$$\omega << \omega_{ce}, \ r_{Le} << a, \ r_{Le} = V_{T_e} / \omega_{ce}$$

- Subtle effect:
- Neglect of electron inertia along **B** can be tricky.
- For long wavelengths, electrons can still require a finite response time even though m_e is small. This is region of the drift wave.
- We shall see that MHD consistently treats || motion poorly, but for MHD behavior, remarkably this does not matter!!
- To treat such behavior more sophisticated models are required. The resulting instabilities are much weaker, (and still important) than for MHD.

Derivation of the Ideal MHD Model

- **The Single-Fluid Equations**

Introduce single fluid variables

$\rho = m_i n$ mass density $m_e \to 0,\ n_i = n_e \equiv n$

$\vec{v} = \vec{v}_i$ fluid velocity

$\vec{J} = en(\vec{v}_i - \vec{v}_e)$ current density

$\vec{v}_e = \vec{v} - \vec{J} / en$

$p = nT = p_e + p_i$ total pressure

$T = T_e + T_i$ total temperature

Derivation of the Ideal MHD Model

- **The Single-Fluid Equations**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \qquad \nabla \cdot \vec{J} = 0$$ charge conservation

$$\rho \frac{d\vec{v}}{dt} - \vec{J} \times \vec{B} + \nabla p = -\nabla \cdot (\overleftrightarrow{\Pi}_i + \overleftrightarrow{\Pi}_e)$$

$$\rho[\frac{\partial u}{\partial t} + (u \cdot \nabla) u] = nq(E + u \times B) - \nabla \cdot P$$

$$\rho[\frac{\partial u}{\partial t} + (u \cdot \nabla) u] = -\nabla p + \rho \nu \nabla^2 u$$ Navier-Stokes equation

This momentum density conservation equation for species resembles in parts the one of conventional hydrodynamics, the Navier-Stokes equation. Yet, in a plasma for each species the Lorentz force appears in addition, coupling the plasma motion (via current and charge densities) to Maxwell's equation and also the various components (electrons and ions) among themselves.

Derivation of the Ideal MHD Model

- **The Single-Fluid Equations**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 \qquad \nabla \cdot \vec{J} = 0$$

charge conservation

$$\rho \frac{d\vec{v}}{dt} - \vec{J} \times \vec{B} + \nabla p = -\nabla \cdot (\overleftrightarrow{\Pi}_i + \overleftrightarrow{\Pi}_e)$$

$$\vec{E} + \vec{v} \times \vec{B} = \frac{1}{en}(\vec{J} \times \vec{B} - \nabla p_e - \nabla \cdot \overleftrightarrow{\Pi}_e + \vec{R}_e)$$

Ohm's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{d}{dt}\left(\frac{p_i}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left(Q_i - \nabla \cdot \vec{h}_i - \overleftrightarrow{\Pi}_i : \nabla \vec{v}\right)$$

$$\frac{d}{dt}\left(\frac{p_e}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left[Q_e - \nabla \cdot \vec{h}_e - \overleftrightarrow{\Pi}_e : \nabla\left(\vec{v} - \frac{\vec{J}}{en}\right)\right] + \frac{1}{en}\vec{J} \cdot \nabla\left(\frac{p_e}{\rho^\gamma}\right)$$

Derivation of the Ideal MHD Model

• The Ideal MHD Limit

- Assumptions leading to ideal MHD

1. Philosophy: Ideal MHD is concerned with phenomena occurring on certain length and time scales.
2. Ordering: Using this, we can order all the terms in the one fluid equations. After ignoring small terms, we obtain ideal MHD.
3. Status: At this point only the assumptions $c \to \infty$, $m_e \to 0$ have been used in the equation.

- Characteristic length and time scales for ideal MHD

$$\frac{\partial}{\partial t} \sim \omega \sim \frac{V_{T_i}}{a}$$

$$\frac{\partial}{\partial x} \sim k \sim \nabla \sim \frac{1}{a}$$

macroscopic MHD phenomena

$$v \sim \frac{\omega}{k} \sim V_{T_i}$$

Derivation of the Ideal MHD Model

- Collision dominated limit

- Isotropic pressure is expected to arise in systems where many collisions take place on a time scale short compared to those of interest. → Maxwellian

- Evolution of full pressure tensor plays only a minor role in equilibrium and stability problems.

ions $\omega\tau_{ii} \sim V_{T_i}\tau_{ii}/a << 1$

electrons $\omega\tau_{ee} \sim (m_e/m_i)^{1/2} V_{T_i}\tau_{ii}/a << 1$ $\quad \tau_{ee} \sim (m_e/m_i)^{1/2}\tau_{ii} \quad (T_e \sim T_i)$

$\lambda_\alpha \sim V_{T_\alpha}\tau_{\alpha\alpha} << a$ mean free path

$$V_{T_i}\tau_{ii}/a \sim V_{T_e}\tau_{ee}/a << 1$$

Derivation of the Ideal MHD Model

- MHD Limit

1. Use the collision dominated assumption to obtain ideal MHD.
2. Several additional assumptions will also be required.
3. Various moments in the equations are approximated by classical transport theory of Braginskii.
4. Transport coefficients can also be derived in the homework problems.

Problems 2.2

- Reduction of single fluid equation

1. Maxwell equations – OK
2. Mass conservation – OK
3. Momentum equation

$$\rho\frac{d\vec{v}}{dt}-\vec{J}\times\vec{B}+\nabla p=-\nabla\cdot(\overleftrightarrow{\Pi}_i+\overleftrightarrow{\Pi}_e)$$

$$\overleftrightarrow{\Pi}_{jj}\sim\mu\left(2\nabla_{||}\cdot\vec{v}_{||}-\frac{2}{3}\nabla\cdot\vec{v}\right)\sim\mu V_{T_i}/a \qquad \mu\sim nT_i\tau_{ii}$$

viscosity

$$\left|\nabla\cdot\overleftrightarrow{\Pi}_i/\nabla p\right|\sim V_{T_i}\tau_{ii}/a<<1$$

Derivation of the Ideal MHD Model

- Reduction of single fluid equation

1. Maxwell equations – OK
2. Mass conservation – OK
3. Momentum equation – OK
4. Ohm's law

$$\vec{E}+\vec{v}\times\vec{B}=\frac{1}{en}(\cancel{\vec{J}\times\vec{B}}-\cancel{\nabla p_e}-\cancel{\nabla\cdot\overleftrightarrow{\Pi}_e}+\cancel{\vec{R}_e})$$

$$\frac{|\nabla p_e/en|}{|\vec{v}\times\vec{B}|}\sim\frac{r_{Li}}{a}<<1,\quad r_{Li}=V_{T_i}/\omega_{ci}\qquad \omega/\omega_{ci}\sim r_{Li}/a<<1$$

$$\frac{|\vec{R}_e/en|}{|\vec{v}\times\vec{B}|}\sim\frac{|\eta\vec{J}|}{|\vec{v}\times\vec{B}|}\sim\frac{(m_e/m_i)^{1/2}}{\omega\tau_{ii}}\left(\frac{r_{Li}}{a}\right)^2<<1$$

resistivity momentum transfer due to collisions

$$\longleftarrow\quad \frac{1}{en}\vec{R}_e\sim\eta\vec{J},\quad \eta\sim\frac{m_e}{ne^2\tau_{ei}}\qquad |\vec{J}|\sim|\nabla p|/|\vec{B}|$$

The plasma must be larger enough so that resistive diffusion does not play an important role.

Derivation of the Ideal MHD Model

- Reduction of single fluid equation

1. Maxwell equations – OK
2. Mass conservation – OK
3. Momentum equation – OK
4. Ohm's law – OK
5. Energy equation

$$\frac{d}{dt}\left(\frac{p_i}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left(Q_i - \nabla\cdot\vec{h}_i - \overleftrightarrow{\Pi}_i : \nabla\vec{v}\right)$$

$$\frac{d}{dt}\left(\frac{p_e}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left[Q_e - \nabla\cdot\vec{h}_e - \overleftrightarrow{\Pi}_e : \nabla\left(\vec{v} - \frac{\vec{J}}{en}\right)\right] + \frac{1}{en}\vec{J}\cdot\nabla\left(\frac{p_e}{\rho^\gamma}\right)$$

$$\frac{(\vec{J}\cdot\nabla p_e)/en}{\partial p_e/\partial t} \sim \frac{r_{Li}}{a} << 1 \qquad \frac{\overleftrightarrow{\Pi}_e : \nabla(\vec{J}/en)}{\partial p_e/\partial t} \sim \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{r_{Li}}{a}\right)^2\left(\frac{V_{T_i}\tau_{ii}}{a}\right) << 1$$

$$\frac{\overleftrightarrow{\Pi}_e : \nabla\vec{v}}{\partial p_e/\partial t} \sim \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{V_{T_i}\tau_{ii}}{a}\right) << 1 \qquad \frac{\overleftrightarrow{\Pi}_i : \nabla\vec{v}}{\partial p_i/\partial t} \sim \left(\frac{V_{T_i}\tau_{ii}}{a}\right) << 1$$

Derivation of the Ideal MHD Model

$$\frac{d}{dt}\left(\frac{p_i}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left[\nabla_{||}\cdot(\kappa_{||i}\nabla T_i) + \frac{n(T_e - T_i)}{\tau_{eq}}\right]$$

$$\frac{d}{dt}\left(\frac{p_e}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left[\nabla_{||}\cdot(\kappa_{||e}\nabla T_e) - \frac{n(T_e - T_i)}{\tau_{eq}}\right]$$

$$\vec{h}_\alpha \approx -\kappa_{||\alpha}\nabla_{||}T_\alpha \qquad \kappa_{||} >> \kappa_\perp$$

Dominated by parallel thermal conduction

$$Q_i = -\frac{n(T_i - T_e)}{\tau_{eq}}$$

$$Q_e = -\frac{n(T_e - T_i)}{\tau_{eq}} + \frac{\vec{J}\cdot\vec{R}_e}{en}$$

Equilibration and Ohmic heating

To derive a single fluid model assuming small equilibration time

$$\omega\tau_{eq} \sim \omega\left(\frac{m_i}{m_e}\right)^{1/2}\tau_{ii} = \left(\frac{m_i}{m_e}\right)^{1/2}\frac{V_{T_t}\tau_{ii}}{a} << 1$$

If this is true, then

$$T_i \approx T_e = T/2, \quad p_i \approx p_e = p/2$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{1}{3\rho^\gamma}\nabla_{||}\cdot\left[(\kappa_{||e} + \kappa_{||i})\nabla_{||}T\right]$$

$$\kappa_{||e} \sim nT_e\tau_{ee}/m_e \approx (m_i/m_e)^{1/2}\kappa_{||i}$$

$$\frac{\nabla_{||}\cdot(\kappa_{||e}\nabla_{||}T)}{\partial p/\partial t} \sim \left(\frac{m_i}{m_e}\right)^{1/2}\frac{V_{T_i}\tau_{ii}}{a} << 1$$

Derivation of the Ideal MHD Model

- **Ideal MHD model**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Mass continuity equation

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Energy equation (equation of state):
adiabatic evolution

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

Ohm's law: perfect conductor → "ideal" MHD

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Maxwell equations

Derivation of the Ideal MHD Model

- Summary of assumptions

1. Asymptotic: $n_e \to 0,\ c \to \infty$

2. Collision dominates: $\left(\dfrac{m_i}{m_e}\right)^{1/2} \dfrac{v_{T_i}\tau_{ii}}{a} \ll 1$
   - $\pi \to p$ isotropic
   - equilibration $T_e = T_i$
   - $\kappa \to$ thermal conduction small

3. Small gyro radius: $r_{ii}/a \ll 1$
   - if all term small
   - electron diamagnetism small
   - small terms in energy equation

4. Small resistivity: $\left(\dfrac{m_e}{m_i}\right)^{1/2} \dfrac{a}{v_{T_i}\tau_{ii}} \left(\dfrac{r_{ii}}{a}\right)^2 \ll 1$
   - ηJ in ohms law small
   - Ohmic heating small

Region of Validity

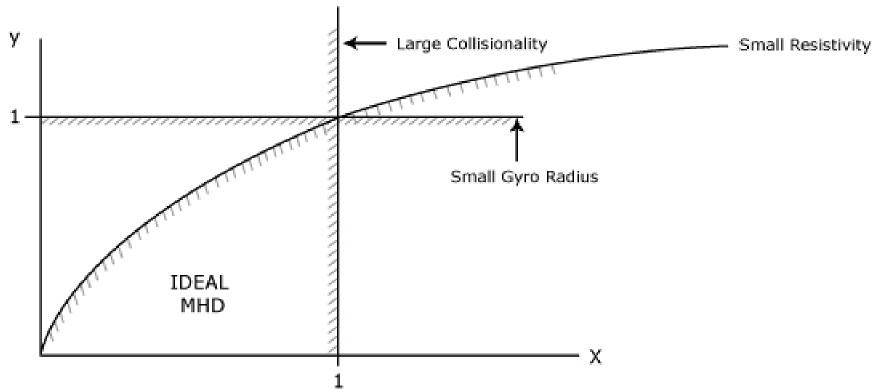
- **Overall Criteria**

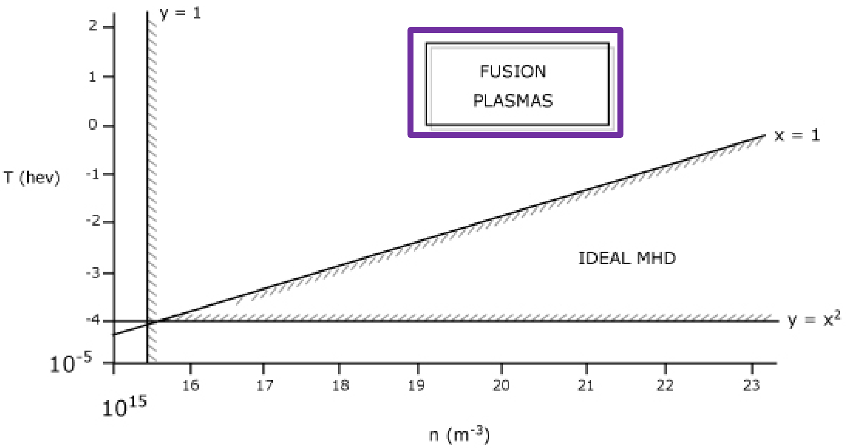
1. High collisionality: $x << 1$ $\longrightarrow$ $x = 3.0\times10^{3}(T^{2}/an) << 1$
2. Small gyro radius: $y << 1$ $\longrightarrow$ $y = 2.3\times10^{-2}(\beta/na^{2})^{1/2} << 1$
3. Small resistivity: $y^2/x << 1$ $\longrightarrow$ $y^{2}/x = 1.8\times10^{-7}\beta/aT^{2} << 1$

$$y = \frac{r_{Li}}{a}, \quad x = \left(\frac{m_i}{m_e}\right)^{1/2}\frac{v_{T_i}\tau_{ii}}{a}$$

$$10^{18}\,\mathrm{m}^{-3} < n < 10^{22}\,\mathrm{m}^{-3}$$
$$0.5\,\mathrm{keV} < T < 50\,\mathrm{keV}$$

β fixed(=0.05), a=1m, D, Λ=15





Region of Validity

- Ideal MHD model is not valid for plasmas of fusion interest.

- Reason: collision dominated assumption breaks down

- But, large empirical evidence that MHD works very well

  in describing macroscopic plasma behavior.

- Question: is this the result of some subtle and perhaps

  unexpected physics?

Region of Validity

- Where specifically does ideal MHD breakdown?

1. Momentum equation

a. $\Pi << p$ because of collision dominated assumption

b. $\Pi_{\perp} << p$ from collisionless theory

c. $\Pi_{||} \sim p$ parallel to the field, the motion of ions is kinetic.

d. $\therefore \perp$ momentum equation OK

   || momentum equation not accurate

2. Energy Equation

$$\frac{\nabla_{||}\cdot(\kappa_{||e}\nabla_{||}T)}{\partial p/\partial t} \sim \left(\frac{m_i}{m_e}\right)^{1/2}\frac{V_{T_i}\tau_{ii}}{a} << 1$$

a. collision dominated assumption

b. $\kappa_{||} \to \infty$ rather than zero in collisionless plasma

c. More accurate equation of state $\to$ $B\cdot\nabla T=0$

d. $\therefore$ energy equation not accurate

Region of Validity

- MHD errors in the momentum and energy equation do not matter, why?

1. Momentum equation $$\rho \frac{d\vec{v}_\perp}{dt} = \vec{J} \times \vec{B} - \nabla_\perp p$$

   Ohm's law and Faraday's law $$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{v}_\perp \times \vec{B}$$

   Valid for collisionless and collisional theory

   Note that $v_{||}$ does not appear.
2. Errors appear in || momentum equation and energy equation.
3. However, it turns out that for MHD equilibrium and most MHD instabilities, the parallel motion plays a small or negligible role. This is not obvious *a priori*.
4. Assuming this to be true, an incorrect treatment of parallel motion is unimportant, since no parallel motions are exerted: the motions are incompressible.
   a. $B\cdot\nabla\rho=0$ no density compression along B
   b. $B\cdot\nabla T=0$. $\kappa_{||} \rightarrow \infty$
5. The condition $B\cdot\nabla\rho=0$, Faraday's law and Ohm's law can be shown to imply $d\rho/dt=0$. Conservation of mass then implies $\nabla\cdot v=0$

What is imcompressibilty? $d\rho/dt=0$

Region of Validity

- Conclusion

- Once incompressibility is accepted as the dominant motion of unstable MHD modes, then errors in ideal MHD do not enter the calculation.
- Ideal MHD gives the “same” answer as “collisionless MHD”.