Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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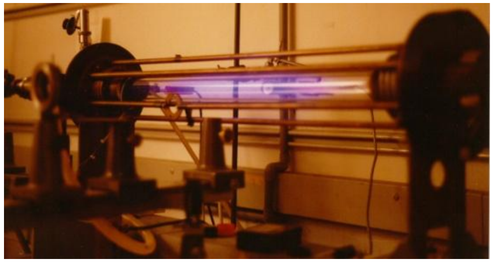
Equilibrium: 1-D Configurations

- **Introduction**

- The MHD equilibrium problem separates into two parts for most configurations of interest:
  - a. radial pressure balance: zero order in a/R
  - b. toroidal force balance: first order in a/R
- We examine radial pressure balance in a 1-D geometry: straight cylinder (no problems with toroidal force balance)
- The 1-D radial pressure balance relation is valid for tokamaks, stellarators, RFP’s, pinches, and EBT’s.
- We introduce toroidal effects as an aspect ratio expansion to see what must be done to achieve toroidal force balance. This requires 2-D calculations.

θ pinch



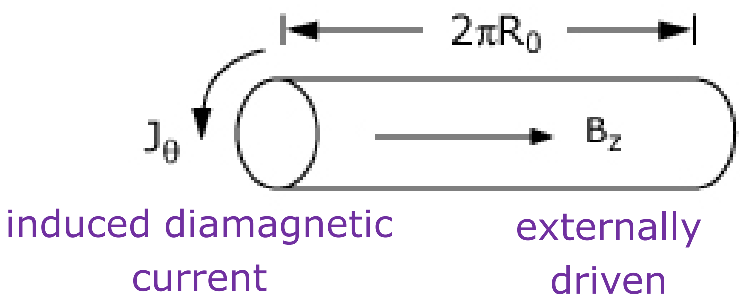
Z pinch

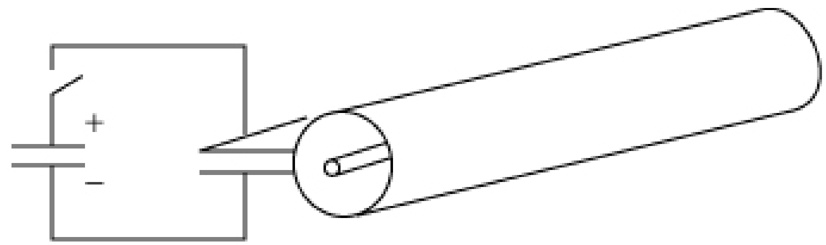
screw pinch

Equilibrium: 1-D Configurations

- **The θ Pinch**

  - 1-D analog of the toroidal configuration with purely toroidal field





- Sequence of solution of the MHD equilibrium equations

1. The $\nabla \cdot \mathbf{B} = 0$
2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$
3. The momentum equation: $\mathbf{J} \times \mathbf{B} = \nabla p$

Equilibrium: 1-D Configurations

- **The θ Pinch**
  - Sequence of solution of the MHD equilibrium equations

1. The $\nabla \cdot \mathbf{B} = 0$

$$\frac{\partial B_z}{\partial z} = 0$$

2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$

$$J_\theta = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial r}$$

3. The momentum equation: $\mathbf{J} \times \mathbf{B} = \nabla p$

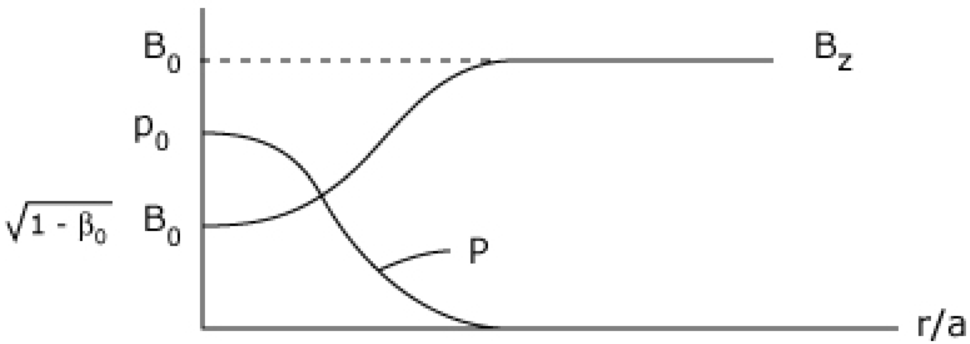
$$J_\theta B_z = \frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0} \right) = 0 \qquad p + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

- At any local value of *r*, the sum of the particle pressure and the magnetic pressure is a constant, equal to the externally applied magnetic pressure.
- The plasma is confined radially by the pressure of the externally applied magnetic field.

Equilibrium: 1-D Configurations

- **The θ Pinch**
  - Typical example



$$p = p_0 \exp(-r^2/a^2)$$

$$B_z = B_0[1 - \beta_0 \exp(-r^2/a^2)]^{1/2}$$

$$\beta_0 \equiv 2\mu_0 p / B_0^2$$

$$\beta_t = \frac{4\mu_0}{a^2 B_0^2}\int_0^a prdr = \frac{2}{a^2}\int_0^a \left(1 - \frac{B_z^2}{B_0^2}\right) rdr \qquad 0 < \beta_t < 1$$

$$\beta_p = \infty \quad (B_\theta = 0)$$

$q_* = \infty$ favourable stability against current-driven kinks

$\iota = 0$

$s = 0$ not available to stabilise MHD instabilities

Equilibrium: 1-D Configurations

- **The θ Pinch**

$$\widehat{W} = 2\frac{V}{\left\langle B^2 \right\rangle}\frac{d}{dV}\left\langle \frac{B^2}{2} \right\rangle = \frac{r}{B_z}\frac{dB_z}{dr} = 0$$

$$W = 2\frac{V}{\left\langle B^2 \right\rangle}\frac{d}{dV}\left\langle \mu_0 p + \frac{B^2}{2} \right\rangle = \frac{\mu_0 r}{B_z^2}\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0} \right) = 0$$ at finite β

- Configuration is neutral w.r.t. magnetic well stabilisation.
- "A plasma cannot dig its own magnetic well by means of its diamagnetic confinement currents."

Equilibrium: 1-D Configurations

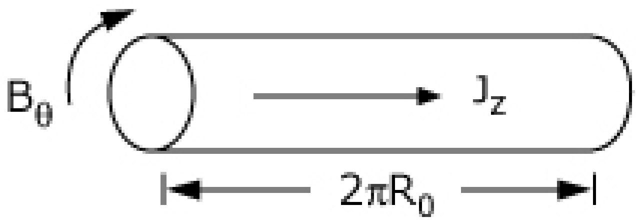
- **The θ Pinch**

  - One of the early successes of the fusion program in terms of the performance.
  - $T_i \sim$ 1-4 keV, $n \sim$ 1-2x10^{22} m^{-3}, $\beta(0) \sim$ 0.7-0.9, $\beta_t \sim$ 0.05
  - No indication of macroscopic instability (neutrally stable)
  - Severe end loss
  - Cannot be bent into a toroidal equilibrium.

Equilibrium: 1-D Configurations

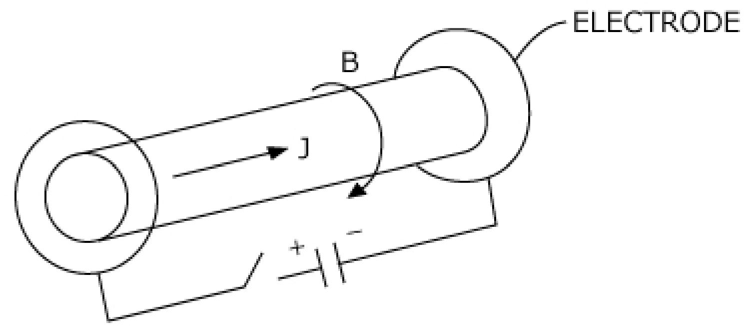
- **The *Z* Pinch**

  - 1-D analog of the toroidal configuration with purely poloidal field



self field induced by J_z

longitudinal plasma current



  - Sequence of solution of the MHD equilibrium equations

  1. The $\nabla \cdot \mathbf{B} = 0$
  2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$
  3. The momentum equation: $\mathbf{J} \times \mathbf{B} = \nabla p$

Equilibrium: 1-D Configurations

- **The *Z* Pinch**

  - Sequence of solution of the MHD equilibrium equations

  1. The $\nabla \cdot \mathbf{B} = 0$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

  2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$

$$J_z = \frac{1}{\mu_0 r}\frac{d}{dr}(rB_\theta)$$

  3. The momentum equation: $\mathbf{J} \times \mathbf{B} = \nabla p$

$$J_z B_\theta = -\frac{dp}{dr}$$

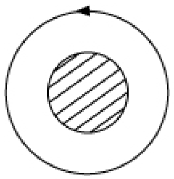
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_0 r}\frac{d}{dr}(rB_\theta) = 0 \qquad \frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

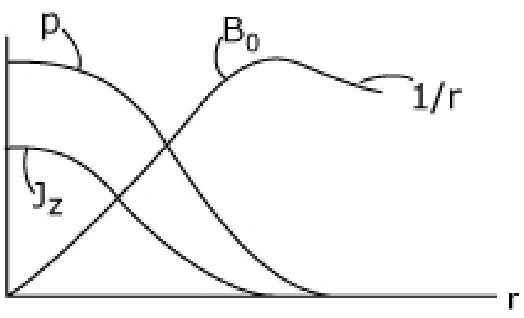
particle pressure + magnetic pressure force

tension force by the curvature of the magnetic field lines

Equilibrium: 1-D Configurations

- **The *Z* Pinch**
  - It is the tension force and not the magnetic pressure gradient that provides radial confinement of the plasma.





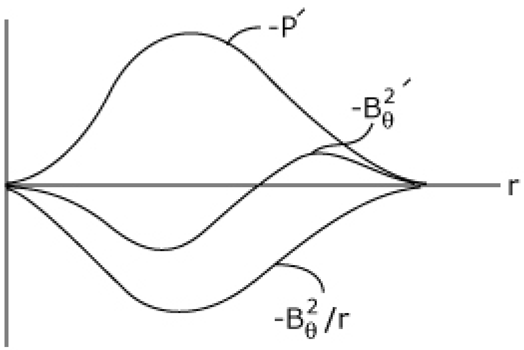
FIELDS

$$B_\theta = \frac{\mu_0 I_0}{2\pi}\frac{r}{r^2 + r_0^2}$$

$$J_z = \frac{I_0}{\pi}\frac{r_0^2}{(r^2 + r_0^2)^2}$$

$$p = \frac{\mu_0 I_0^2}{8\pi^2}\frac{r_0^2}{(r^2 + r_0^2)^2}$$

Bennett profiles
(Bennett, 1934)



FORCES

Equilibrium: 1-D Configurations

- **The *Z* Pinch**

$$\beta_p = \frac{16\pi^2}{\mu_0 I_0^2}\int_0^a prdr = 1$$: Bennett Pinch Relation

- While its high value is desirable for confinement efficiency, the lack of flexibility in achieving small to moderate β_p is a disadvantage.
- Some classes of potentially dangerous MHD modes can be stabilized if β_p is sufficiently low (poor stability).

$\beta_t = \infty \quad (B_z = 0)$

$q_* = 0$ poor stability against current-driven kinks

$\iota = \infty$

$s = 0$ Not available to stabilise MHD instabilities

$$\hat{W}(r) = \frac{r}{B_\theta}\frac{dB_\theta}{dr}, \quad W(r) = -1$$ at finite β

unfavourable stability properties w.r.t. pressure-driven MHD instabilities

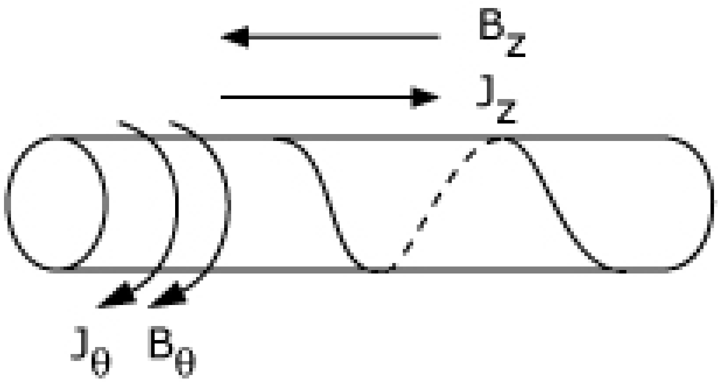
Equilibrium: 1-D Configurations

- **The *Z* Pinch**

  - A number of linear *Z*-pinch experiments were constructed during the early years of the fusion program.
  - Exhibiting disastrous instabilities, often leading to a complete quenching of the plasma after ~1-2 μs
  - Can easily be bent into a toroidal equilibrium.
  - Ohmically heated tokamak

Equilibrium: 1-D Configurations

- **The General Screw Pinch**
  - A hybrid combination of Z pinch and θ pinch
  - This combination of fields allows the flexibility to optimise configurations w.r.t. toroidal force balance and stability.



- Sequence of solution of the MHD equilibrium equations

1. The $\nabla \cdot \mathbf{B} = 0$
2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$
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Equilibrium: 1-D Configurations

• The General Screw Pinch

- Sequence of solution of the MHD equilibrium equations

1. The $\nabla \cdot \mathbf{B} = 0$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$

$$J_\theta = -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r} \qquad J_z = \frac{1}{\mu_0 r}\frac{d}{dr}(rB_\theta)$$

3. The momentum equation: $\mathbf{J} \times \mathbf{B} = \nabla p$

$$J_\theta B_z - J_z B_\theta = \frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

- Even though the equations are nonlinear, the forces superpose because of symmetry.
- The screw pinch has many properties of more realistic, multidimensional toroidal configurations.
- The constant pressure contours $p(r)$ = constant are circles r = constant. The flux surfaces are closed, nested, concentric circles.

Equilibrium: 1-D Configurations

- **The General Screw Pinch**

$$2\pi\int_0^a p r dr = \frac{\mu_0 I_0^2}{8\pi} + 2\pi\int_0^a \frac{B_0^2 - B_z^2}{2\mu_0} r dr$$

plasma energy line density

a measure of the energy in the poloidal magnetic field (poloidal tension)

the magnetic energy associated with the diamagnetic part of the toroidal field (toroidal diamagnetism)

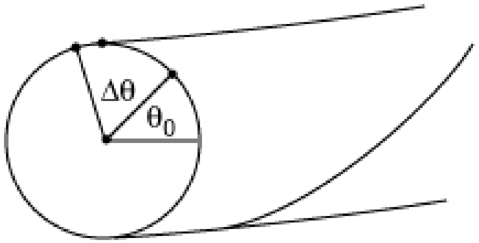
$$\beta_t = \frac{4\mu_0}{a^2 B_0^2}\int_0^a p r dr \qquad \beta_p = \frac{16\pi^2}{\mu_0 I_0^2}\int_0^a p r dr = 1$$

$$\frac{1}{\beta_p} + \frac{\alpha_t}{\beta_t} = 1 \qquad \alpha_t = \frac{2}{a^2}\int_0^a \left(1 - \frac{B_z^2}{B_0^2}\right) r dr \qquad -\infty < \alpha_t < 1$$

representing the plasma diamagetism

Equilibrium: 1-D Configurations

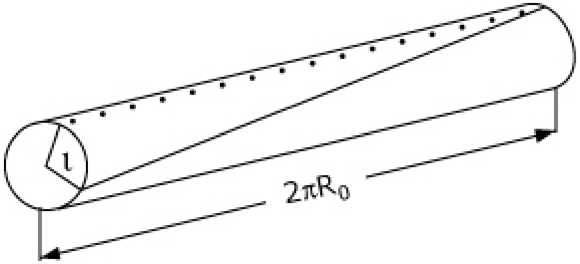
- **The General Screw Pinch**



$$q_* = \frac{2\pi a^2 B_0}{\mu_0 R_0 I_0}$$

$$\iota \equiv \Delta\theta = \int_0^{\Delta\theta} d\theta = \int_0^{2\pi R_0} \frac{d\theta}{dz} dz \qquad \frac{dr}{dz} = \frac{B_r(r)}{B_z(r)} = 0, \quad \frac{d\theta}{dz} = \frac{B_\theta(r)}{rB_z(r)}$$

$$\iota(r) = \frac{2\pi R_0 B_\theta(r)}{rB_z(r)}$$ for circular flux surface



$$q(r) \equiv \frac{2\pi}{\iota(r)} = \frac{rB_z}{R_0 B_\theta} \qquad q(a) = q_*$$

$$s(r) = \frac{r}{q}\frac{dq}{dr} \qquad V = 2\pi^2 R_0 r^2$$

measure of safety against
a variety of MHD instabilities

Equilibrium: 1-D Configurations

- **The General Screw Pinch**

$$W(r) = \frac{\mu_0 r}{B^2}\frac{d}{dr}\left(p + \frac{B^2}{2\mu_0}\right) = -\frac{B_\theta^2}{B^2} < 0$$

- Flexibility of making the magnitude of $W(r)$ small by requiring $B_\theta^2 << B_z^2$.
- The destabilising influence of $W(r)<0$ can be compensated for by shear.

Equilibrium: 1-D Configurations

- **The General Screw Pinch**

- Because of its flexibility the general screw-pinch relation describes a wide variety of configurations.
- Capable of a wide range of variation in β_t and β_p
- In general, the kink safety factor, the MHD safety factor, the shear, and the magnetic well are nonzero.
- Toroidal equilibrium is in general not difficult to achieve since B_θ is nonzero.

References

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