Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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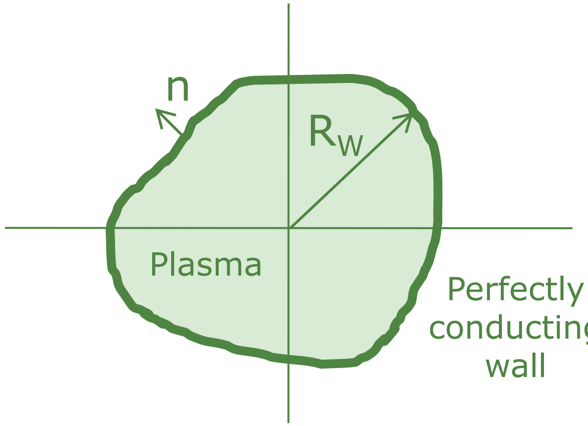
Equilibrium: 2-D Configurations

- **The Ohmically Heated Tokamak**
  - Simple configuration: low β, circular cx, simple BCs

Assumption: Perfectly conducting wall



$$\vec{n}\times\vec{E}\Big|_{R_W}=0$$

$$\vec{n}\cdot\vec{B}\Big|_{R_W}=0$$

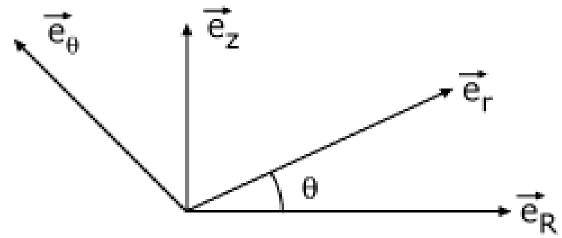
$$\vec{n}\cdot\vec{v}\Big|_{R_W}=0 \quad \longleftarrow \quad \vec{n}\times\vec{E}+(\vec{n}\cdot\vec{B})\vec{v}-(\vec{n}\cdot\vec{v})\vec{B}=0$$

ideal Ohm's law

$$\vec{n}\cdot\vec{B}\Big|_{R_W}=B_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}\bigg|_{r=b}=0 \qquad (\psi(b,\theta)=const)$$

Equilibrium: 2-D Configurations

Transform from R, ϕ', Z to r, θ, ϕ



$$R = R_0 + r\cos\theta, \quad Z = r\sin\theta, \quad \phi' = \phi$$

$$e_R = e_r\cos\theta - e_\theta\sin\theta$$

$$e_Z = e_r\sin\theta + e_\theta\cos\theta$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{R}\frac{\partial\psi}{\partial\phi}e_\phi$$

$$\nabla\cdot V = \frac{1}{r}\frac{\partial rV_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{1}{R}\frac{\partial V_\phi}{\partial\phi} + \frac{1}{R}(V_r\cos\theta - V_\theta\sin\theta)$$

$$\Delta^*\psi = \nabla\cdot\nabla\psi - \frac{2}{R}\frac{\partial\psi}{\partial R}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right)$$

$$= \nabla^2\psi - \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right)$$

Equilibrium: 2-D Configurations

The Grad-Shafranov equation

$$\nabla^2\psi = -\mu_0 (R_0 + r\cos\theta)^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} + \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial r}\sin\theta\right)$$

In the finite aspect ratio limit:

- $R \to \infty$
- $r/R \to 0$
- $\psi(r,\theta) \to \psi(r)$: cylindrically symmetric

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = -\mu_0 R_0^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} \qquad \frac{d}{dr}\left(p + \frac{B_\phi^2}{2\mu_0}\right) + \frac{B_\theta}{\mu_0 r}\frac{d}{dr} rB_\theta = 0$$

$$T_1 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = \frac{1}{r}\frac{d}{dr} rR_0 B_\theta = \frac{R_0}{r}\frac{d}{dr} rB_\theta$$

$$T_2 = -\mu_0 R_0^2 \frac{dp}{d\psi} = -\mu_0 R_0^2 \frac{dp}{dr}\left(\frac{d\psi}{dr}\right)^{-1} = -\mu_0 \frac{R_0}{B_\theta}\frac{dp}{dr}$$

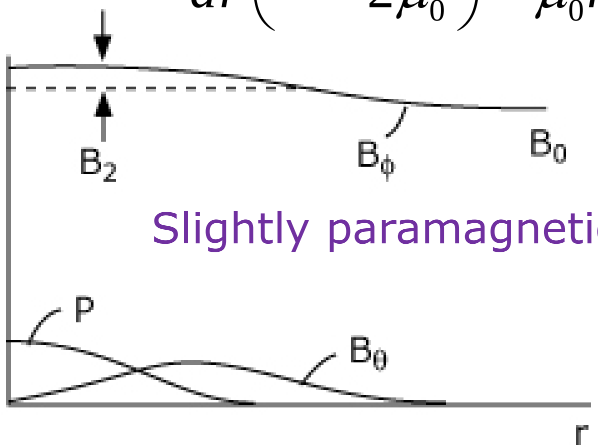
$$\longleftarrow \quad B_\theta = \frac{1}{R_0}\frac{d\psi}{dr}$$

$$T_3 = -F\frac{dF}{d\psi} = -\left(\frac{d\psi}{dr}\right)^{-1}\frac{d}{dr}\frac{F^2}{2} = -\frac{R_0^2}{R_0 B_\theta}\frac{d}{dr}\frac{B_\phi^2}{2} = -\frac{R_0}{B_\theta}\frac{d}{dr}\frac{B_\phi^2}{2}$$

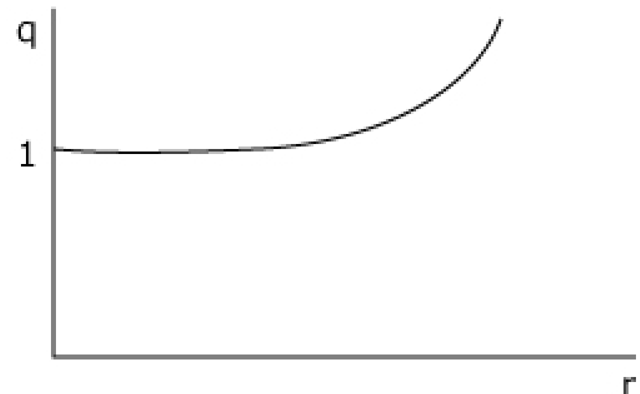
Equilibrium: 2-D Configurations

Radial pressure balance

$$\frac{d}{dr}\left(p+\frac{B_\phi^2}{2\mu_0}\right)+\frac{B_\theta}{\mu_0 r}\frac{d}{dr}rB_\theta=0$$



Slightly paramagnetic



- Since the toroidal magnetic field is slightly paramagnetic in experiments with no auxiliary heating, the entire plasma pressure is held in radial pressure balance by the poloidal magnetic field.
- The plasma pressure and current are small but the safety factor is of order of unity.
- The small toroidal current provides both radial balance and toroidal force balance.

Equilibrium: 2-D Configurations

Expand in terms of $\varepsilon = a/R_0$, assume $\varepsilon << 1$.

$$q \sim \frac{rB_\phi}{RB_\theta} \sim 1 \quad \left(\frac{B_\phi}{B_\theta} \sim \frac{R}{r} \sim \frac{1}{\varepsilon} \right)$$

$$\beta_p \sim \frac{2\mu_0 p}{B_\theta^2} \sim 1, \quad \beta_t \sim \frac{2\mu_0 p}{B_\phi^2} \sim \varepsilon^2$$

Assume the plasma is confined in radial pressure balance by the poloidal field due to OH current.

$$\beta \sim \frac{2\mu_0 p}{B_\phi^2 + B_\theta^2} \sim \frac{2\mu_0 p}{B_\phi^2} \sim \beta_t \sim \frac{2\mu_0 p}{B_\theta^2}\frac{B_\theta^2}{B_\phi^2} \sim \varepsilon^2$$

$$\frac{\delta B_\phi}{B_\phi} \sim \frac{\mu_0 p}{B_\phi^2} \sim \varepsilon^2$$

- The ohmically heated tokamak is dominated by a large toroidal field.
- The toroidal field plays little or no role in either radial pressure balance or toroidal force balance.
- Its primary function is to provide stability at high OH currents by generating safety factors on the order of unity.

Equilibrium: 2-D Configurations

Solution of Grad-Shafranov equation

Expand

$$\psi(r,\theta) = \psi_0(r) + \psi_1(r,\theta) + \cdots$$

$$\psi_1 / \psi_0 \sim \varepsilon$$

$$\psi_0 \sim r R_0 B_\theta$$

B_0: vacuum toroidal field

$$F(\psi) = R B_\phi = R_0\{B_0 + B_2(\psi)\} = R_0\left\{B_0 + B_2(\psi) + \frac{dB_2}{d\psi_0}\psi_1 + \cdots\right\}$$

$$\frac{B_2}{B_0} \sim \varepsilon^2$$ small dia/paramagnetism $\longleftarrow$ $$\frac{\delta B_\phi}{B_\phi} \sim \frac{\mu_0 p}{B_\phi^2} \sim \varepsilon^2$$

$$p(\psi) = p(\psi_0) + \frac{dp}{d\psi_0}\psi_1 + \cdots$$

$$R = R_0\left(1 + \frac{r}{R_0}\cos\theta\right)$$

($1 \rightarrow \varepsilon^0$, $\frac{r}{R_0}\cos\theta \rightarrow \varepsilon^1$)

Equilibrium: 2-D Configurations

Expansion of the Grad-Shafranov equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=-\mu_0(R_0+r\cos\theta)^2\frac{dp}{d\psi}-F\frac{d}{d\psi}\left(\frac{F^2}{2}\right)+\frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\psi}{\partial r}\sin\theta\right)$$

$$\varepsilon^0:\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_0}{\partial r}\right)=-\mu_0R_0^2\frac{dp}{d\psi_0}-\frac{d}{d\psi_0}\left(R_0^2B_0B_2\right)$$

$$\varepsilon^1:\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}=-2\mu_0R_0r\cos\theta\frac{dp}{d\psi_0}-\mu_0R_0^2\psi_1\frac{d^2p}{d\psi_0^2}-\psi_1\frac{d^2}{d\psi_0^2}\left(R_0^2B_0B_2\right)+\frac{1}{R_0}\frac{d\psi_0}{dr}\cos\theta$$

ε^0 consequences

$$\frac{d}{dr}\left(p+\frac{B_0B_2}{\mu_0}\right)+\frac{B_\theta}{\mu_0 r}\frac{d}{dr}rB_\theta=0 \qquad \frac{d}{dr}\left(p+\frac{B_\phi^2}{2\mu_0}\right)+\frac{B_\theta}{\mu_0 r}\frac{d}{dr}rB_\theta=0$$

Equilibrium: 2-D Configurations

- ε^0 consequences

$$q(r)=\frac{rB_0}{R_0B_\theta}+O(\varepsilon)$$

$$B_\theta(a)=\frac{\mu_0 I}{2\pi a} \qquad q_a=\frac{2\pi a^2 B_0}{\mu_0 R_0 I}$$

$$\beta_t=\frac{4\mu_0}{a^2B_0^2}\int_0^a prdr=\frac{4\mu_0}{a^2B_0^2}\frac{\mu_0 I^2}{16\pi^2}\beta_p=\left(\frac{\mu_0 I}{2\pi a B_0}\right)^2\beta_p \longleftarrow \beta_p=\frac{16\pi^2}{\mu_0 I^2}\int_0^a prdr$$

$$=\frac{a^2}{R_0^2}\left(\frac{\mu_0 R_0 I}{2\pi a^2 B_0}\right)^2\beta_p=\frac{\varepsilon^2}{q_a^2}\beta_p$$

Equilibrium: 2-D Configurations

ε^0 consequences

$$q_0 = \left.\frac{rB_0}{R_0 B_\theta}\right|_{r=0} = \frac{B_0}{R_0 B'_{\theta 0}} = \frac{2B_0}{\mu_0 R_0 J_0}$$

$$B_\theta(r) \approx B_\theta(0) + B'_\theta(0)r + B''_\theta(0)r^2/2 \approx B'_\theta(0)r$$

$$\frac{1}{r}(rB_\theta)' = \mu_0 J_0 \quad \rightarrow \quad B_\theta \approx \frac{\mu_0 J_0 r}{2} \quad \rightarrow \quad B'_\theta(0) = \frac{\mu_0 J_0}{2}$$

Equilibrium: 2-D Configurations

Expansion of the Grad-Shafranov equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=-\mu_0(R_0+r\cos\theta)^2\frac{dp}{d\psi}-F\frac{d}{d\psi}\left(\frac{F^2}{2}\right)+\frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\psi}{\partial r}\sin\theta\right)$$

$$\varepsilon^0:\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_0}{\partial r}\right)=-\mu_0R_0^2\frac{dp}{d\psi_0}-\frac{d}{d\psi_0}\left(R_0^2B_0B_2\right)$$

$$\varepsilon^1:\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}=-2\mu_0R_0r\cos\theta\frac{dp}{d\psi_0}-\mu_0R_0^2\psi_1\frac{d^2p}{d\psi_0^2}-\psi_1\frac{d^2}{d\psi_0^2}\left(R_0^2B_0B_2\right)+\frac{1}{R_0}\frac{d\psi_0}{dr}\cos\theta$$

ε^1 consequences

$$B_\theta=\frac{1}{R_0}\frac{d\psi_0}{dr}=\frac{\psi_0'}{R_0},\quad \psi_0=\psi_0(r)$$

$$R_0\frac{dp}{d\psi_0}=R_0\frac{dr}{d\psi_0}\frac{dp}{dr}=\frac{R_0}{\psi_0'}\frac{dp}{dr}=\frac{1}{B_\theta}\frac{dp}{dr}$$

Equilibrium: 2-D Configurations

ε^1 consequences - simplification

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\left(\mu_0 R_0^2\frac{d^2p}{d\psi_0^2}+R_0^2\frac{d^2}{d\psi_0^2}B_0B_2\right)\psi_1=-2\mu_0R_0\frac{dp}{d\psi_0}r\cos\theta+\frac{1}{R_0}\frac{d\psi_0}{dr}\cos\theta$$

$$-2\mu_0R_0\frac{dp}{d\psi_0}r\cos\theta+\frac{1}{R_0}\frac{d\psi_0}{dr}\cos\theta=\left(B_\theta-\frac{2\mu_0}{B_\theta}r\frac{dp}{dr}\right)\cos\theta$$

$$\mu_0R_0^2\frac{d^2p}{d\psi_0^2}+R_0^2\frac{d^2}{d\psi_0^2}(B_0B_2)=R_0^2\frac{1}{R_0B_\theta}\frac{d}{dr}\left\{\frac{1}{R_0B_\theta}\frac{d}{dr}(\mu_0p+B_0B_2)\right\}$$

$$=-\frac{1}{B_\theta}\frac{d}{dr}\left(\frac{1}{B_\theta}\frac{B_\theta}{r}\frac{d}{dr}rB_\theta\right)=-\frac{1}{B_\theta}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}rB_\theta\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}-\frac{1}{B_\theta}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}rB_\theta\right)\psi_1=\left(-\frac{2\mu_0r}{B_\theta}\frac{dp}{dr}+B_\theta\right)\cos\theta$$

The inhomogeneous terms all have a θ dependence proportional to $\cos\theta$.

Equilibrium: 2-D Configurations

Solve

Boundary conditions for a circle of radius *b*

$$\psi(r,\theta)=\psi_0(r)+\psi_1(r)\cos\theta$$

$$\psi_1(b)=0$$

$$\frac{1}{r}\left(r\psi_1'\right)'-\frac{\psi_1}{r^2}-\frac{1}{B_\theta}\left(B_\theta''+\frac{B_\theta'}{r}-\frac{B_\theta}{r^2}\right)\psi_1=-\frac{2\mu_0 rp'}{B_\theta}+B_\theta$$

$$\frac{1}{r}\left(r\psi_1'\right)'-\frac{(rB_\theta')'}{rB_\theta}\psi_1=B_\theta-\frac{2\mu_0 r}{B_\theta}p'$$

$$\left\{rB_\theta^2\left(\frac{\psi_1}{B_\theta}\right)'\right\}'=\left(rB_\theta\psi_1'-r\psi_1B_\theta'\right)'=B_\theta(r\psi_1')'+rB_\theta'\psi_1'-\psi_1(rB_\theta')'-r\psi_1'B_\theta'$$

$$=rB_\theta\left\{\frac{1}{r}(r\psi_1')'-\frac{(rB_\theta')'}{rB_\theta}\psi_1\right\}$$

Equilibrium: 2-D Configurations

$$\frac{d}{dr}\left(rB_\theta^2\frac{d}{dr}\frac{\psi_1}{B_\theta}\right)=rB_\theta^2-2\mu_0 r^2\frac{dp}{dr}$$

$$\frac{d}{dr}\frac{\psi_1}{B_\theta}=\frac{1}{rB_\theta^2}\int_0^r\left(yB_\theta^2-2\mu_0 y^2\frac{dp}{dy}\right)dy$$

$$\psi_1=-B_\theta\int_r^b\frac{dx}{xB_\theta^2}\int_0^x\left(yB_\theta^2-2\mu_0 y^2\frac{dp}{dy}\right)dy$$

- The plasma were surrounded by a perfectly conducting shell located at $r = b$.
- This expression represents the toroidal correction to the equilibrium solution.

Equilibrium: 2-D Configurations

Consequences of Toroidicity

- outward shift of the flux surfaces

$$\psi(r,\theta) = \psi_0(r) + \psi_1(r)\cos\theta = const$$

$$r = r_0 + r_1(r_0,\theta)$$

$$\psi_0(r_0) + \psi_0'(r_0)r_1 + \psi_1(r_0)\cos\theta \cdots = const$$

$$r_1(r,\theta) = -\frac{\psi_1(r)\cos\theta}{\psi_0'(r)} = \Delta(r)\cos\theta$$

$$\Delta(r) \equiv -\frac{\psi_1(r)}{\psi_0'(r)}$$

$$r = r_0 + \Delta(r_0)\cos\theta \quad (\Delta << r_0)$$

The boundary is a slightly shifted circle.

Equilibrium: 2-D Configurations

Equation of a shifted circle

$$(x-\Delta)^2 + y^2 = r_0^2$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 - 2r\Delta\cos\theta \approx r_0^2$$

$$r\left(1 - \frac{\Delta}{r}\cos\theta\right) \approx r_0$$

$$r \approx r_0 + \Delta\cos\theta$$

$$\Delta(a) = \Delta_a \equiv -\frac{\psi_1(r)}{\psi_0'(r)} = ?$$

Equilibrium: 2-D Configurations

The Shafranov Shift

$$\psi_1(a) = -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x\left(yB_\theta^2 - 2\mu_0 y^2\frac{dp}{dy}\right)dy$$

$$-B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x yB_\theta^2 dy$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left(\int_0^a yB_\theta^2 dy + \int_a^x yB_\theta^2 dy\right)$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left(\int_0^a yB_\theta^2 dy + B_\theta^2(a)a^2\int_a^x \frac{dy}{y}\right)$$

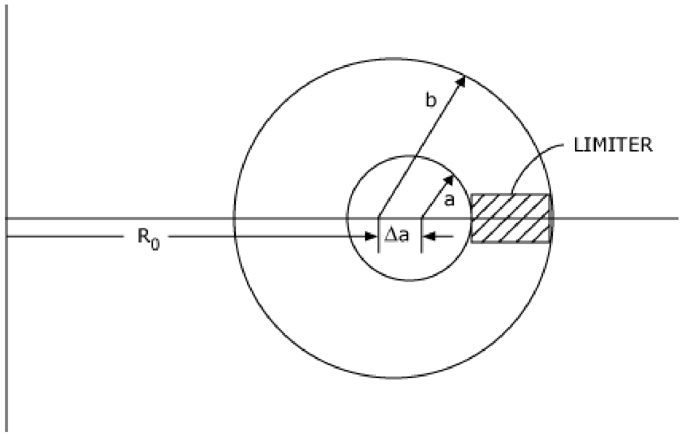
$$B_\theta = B_\theta(a)\frac{a}{x} \quad (x > a)$$

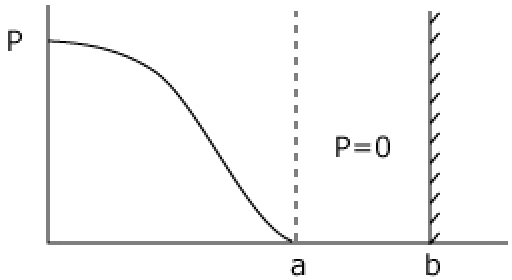
$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left\{\int_0^a yB_\theta^2 dy + a^2B_\theta^2(a)(\ln x - \ln a)\right\}$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^a yB_\theta^2 dy - B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left\{a^2B_\theta^2(a)(\ln x - \ln a)\right\}$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^a yB_\theta^2 dy - B_\theta(a)\left\{-\left(\frac{b^2-a^2}{2}\right)\ln a + \frac{b^2}{2}\ln b - \frac{b^2}{4} - \frac{a^2}{2}\ln a + \frac{a^2}{4}\right\}$$

$$= \boxed{-B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^a yB_\theta^2 dy} + B_\theta(a)\frac{b^2}{4}\left(1 - \frac{a^2}{b^2} - 2\ln\frac{b}{a}\right)$$





Equilibrium: 2-D Configurations

The Shafranov Shift

Introduce the normalised internal inductance per unit length

$$\frac{1}{2}L_i I^2 = \int_p \frac{B_\theta^2}{2\mu_0}dr = (2\pi R_0)(2\pi)\frac{1}{2\mu_0}\int B_\theta^2 r dr = \frac{2\pi^2 R_0}{\mu_0}\int_0^a B_\theta^2 r dr$$ volume integral

$$l_i \equiv \frac{L_i}{2\pi R_0} \Big/ \frac{4\pi}{\mu_0}$$

Internal inductance per unit length normalised to $\mu_0/4\pi$

$$-B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x yB_\theta^2 dy$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^a yB_\theta^2 dy + B_\theta(a)\frac{b^2}{4}\left(1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right)$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left(\frac{\mu_0}{2\pi^2 R_0}\right)\frac{1}{2}(2\pi R_0 l_i)\frac{\mu_0}{4\pi}\left(\frac{2\pi a B_\theta(a)}{\mu_0}\right)^2 + B_\theta(a)\frac{b^2}{4}\left(1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right)$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\frac{1}{2}a^2 B_\theta^2(a) l_i + B_\theta(a)\frac{b^2}{4}\left(1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right) \longleftarrow B_\theta = B_\theta(a)\frac{a}{x} \quad (x>a)$$

$$= -B_\theta(a)\frac{l_i}{4}b^2\left(1-\frac{a^2}{b^2}\right) + B_\theta(a)\frac{b^2}{4}\left(1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right)$$

Equilibrium: 2-D Configurations

The Shafranov Shift

$$\psi_1(a) = -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x \left(yB_\theta^2 - 2\mu_0 y^2 \frac{dp}{dy}\right)dy$$

$$-B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x \left(-2\mu_0 y^2 \frac{dp}{dy}\right)dy$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^a \left(-2\mu_0 y^2 \frac{dp}{dy}\right)dy$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left(-2\mu_0 y^2 p\Big|_0^a + 4\int_0^a \mu_0 p y dy\right)$$

$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left(\frac{\mu_0^2 I^2}{4\pi^2}\beta_p\right)$$

$$\beta_p = \frac{16\pi^2}{\mu_0 I^2}\int_0^a prdr$$

$$B_\theta(a) = \frac{\mu_0 I}{2\pi a}$$

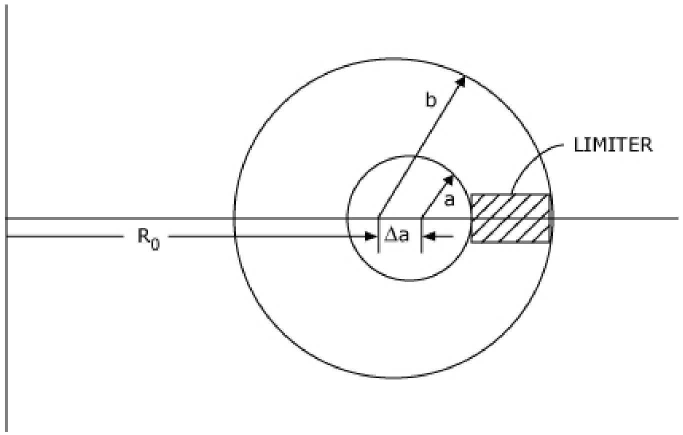
$$= -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\left\{ a^2 B_\theta^2(a)\beta_p\right\}$$

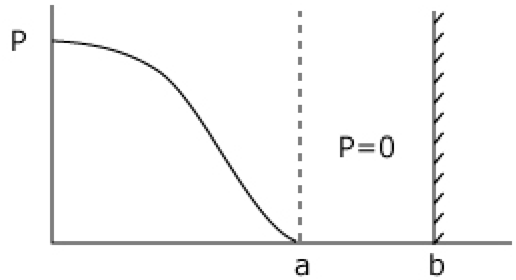
$$= -a^2 B_\theta^3(a)\beta_p \int_a^b \frac{dx}{xB_\theta^2}$$

$$B_\theta = B_\theta(a)\frac{a}{x} \quad (x > a)$$

$$= -B_\theta(a)\beta_p \int_a^b x dx$$

$$= -\frac{B_\theta(a)\beta_p b^2}{2}\left(1 - \frac{a^2}{b^2}\right)$$





Equilibrium: 2-D Configurations

The Shafranov Shift

$$\psi_1(a) = -B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x \left(yB_\theta^2 - 2\mu_0 y^2 \frac{dp}{dy}\right) dy$$

$$-B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x yB_\theta^2 dy = -B_\theta(a)\frac{l_i}{4}b^2\left(1-\frac{a^2}{b^2}\right) + B_\theta(a)\frac{b^2}{4}\left(1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right)$$

$$-B_\theta(a)\int_a^b \frac{dx}{xB_\theta^2}\int_0^x \left(-2\mu_0 y^2 \frac{dp}{dy}\right) dy = -\frac{B_\theta(a)\beta_p b^2}{2}\left(1-\frac{a^2}{b^2}\right)$$

$$\psi_1(a) = -B_\theta(a)\frac{l_i}{4}b^2\left(1-\frac{a^2}{b^2}\right) + B_\theta(a)\frac{b^2}{4}\left(1-\frac{a^2}{b^2}-2\ln\frac{b}{a}\right) - \frac{B_\theta(a)\beta_p b^2}{2}\left(1-\frac{a^2}{b^2}\right)$$

$$\Delta_a = -\frac{\psi_1(a)}{\psi_0'(a)} = -\frac{\psi_1(a)}{R_0 B_\theta(a)} \qquad \frac{\Delta_a}{b} = \frac{b}{2R_0}\left\{\left(\beta_p + \frac{l_i}{2} - \frac{1}{2}\right)\left(1-\frac{a^2}{b^2}\right) + \ln\frac{b}{a}\right\}$$

Equilibrium: 2-D Configurations

Properties of the Shafranov Shift

$$\frac{\Delta_a}{b} \sim \frac{b}{R_0} \sim \varepsilon << 1$$

The shift is small, implying that our approximations are consistent.

$$\frac{\Delta_a^{(1)}}{b} \propto \beta_p$$

outward shift due to the tire tube force and the $1/R$

$$\frac{\Delta_a^{(2)}}{b} \propto \frac{l_i}{2}\left(1-\frac{a^2}{b^2}\right)+\ln\frac{b}{a}-\frac{1}{2}\left(1-\frac{a^2}{b^2}\right)$$

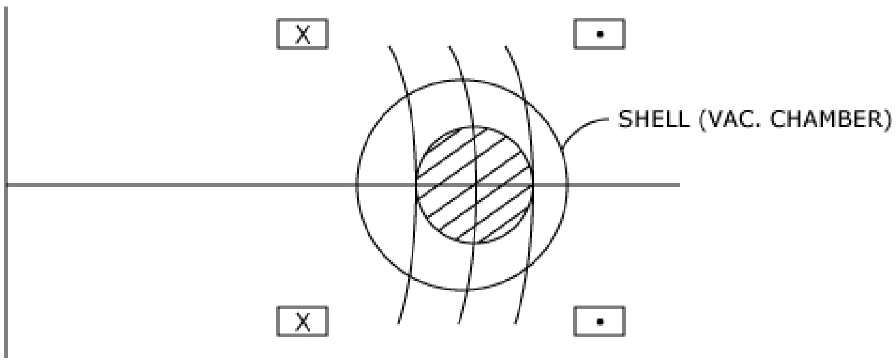
outward shift due to the hoop force

internal field external field

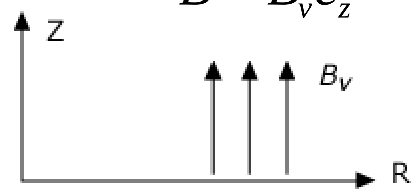
hoop force

Equilibrium: 2-D Configurations

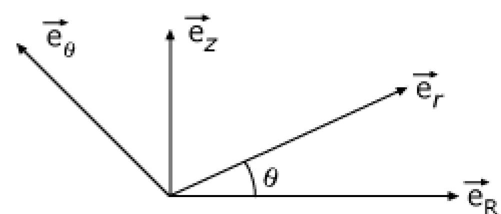
Toroidal Force Balance by Means of a Vertical Field



$$B = B_v e_z$$



The vertical field causes a shift of the plasma surface with respect to R_0.



$$\frac{B_v}{B_\theta} \sim \varepsilon, \quad \frac{B_v}{B_0} \sim \varepsilon^2$$

$$B_v = B_v e_z = B_v\left(e_r \sin\theta + e_\theta \cos\theta\right) \equiv \frac{1}{R_0}\left(\frac{\partial \psi_v}{\partial r} e_\theta - \frac{1}{r}\frac{\partial \psi_v}{\partial \theta} e_r\right)$$

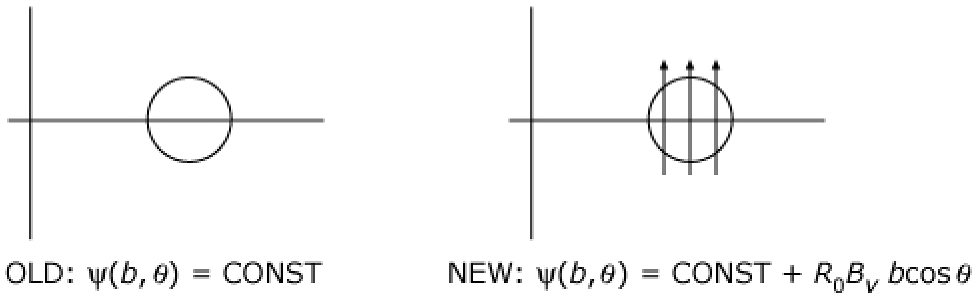
$$\vec{B}_p = \frac{1}{R} \nabla \psi \times \vec{e}_\phi$$

$\psi_v = R_0 B_v r \cos\theta$ vacuum vertical field flux function

Equilibrium: 2-D Configurations

Toroidal Force Balance by Means of a Vertical Field

The new boundary condition including the vertical field



Solve the Grad-Shafranov equation

- Zeroth order: same as before (radial pressure balance)
- First order: same equation as before but with a new BC.

$$\psi_1(b,\theta) = R_0 B_v b\cos\theta$$

- Let $\psi(r,\theta) = \psi_0(r) + \psi_1(r)\cos\theta, \quad \psi_1(b) = R_0 B_v b$
- The solution is found as follows:

$$\frac{d}{dr}\left(rB_\theta^2 \frac{d}{dr}\frac{\psi_1}{B_\theta} \right) = rB_\theta^2 - 2\mu_0 r^2 \frac{dp}{dr}$$

Equilibrium: 2-D Configurations

Toroidal Force Balance by Means of a Vertical Field

$$\frac{d}{dr}\left(rB_\theta^2\frac{d}{dr}\frac{\psi_1}{B_\theta}\right)=rB_\theta^2-2\mu_0 r^2\frac{dp}{dr}$$

$$\psi_1=\psi_1^{old}+\psi_1^{\mathrm{hom}}$$

$$\psi_1^{old}(b)=0,\quad \psi_1^{\mathrm{hom}}(b)=R_0B_vb$$

Homogeneous solution

$$\frac{d}{dr}\left(rB_\theta^2\frac{d}{dr}\frac{\psi_1}{B_\theta}\right)=0$$

$$\frac{d}{dr}\left(\frac{\psi_1}{B_\theta}\right)=\frac{c_1}{rB_\theta^2}$$

$$\psi_1=c_2B_\theta+c_1B_\theta\int_0^r\frac{dr}{rB_\theta}$$

c_1=0 for regularity

$$\psi_1^{\mathrm{hom}}=c_2B_\theta(r)=\left(\frac{R_0bB_v}{B_\theta(b)}\right)B_\theta(r)\longleftarrow\psi_1^{\mathrm{hom}}(b)=R_0B_vb$$

Equilibrium: 2-D Configurations

Toroidal Force Balance by Means of a Vertical Field

$$\psi_1(r) = -B_\theta \int_r^b \frac{dx}{xB_\theta^2} \int_0^x \left(yB_\theta^2 - 2\mu_0 y^2 \frac{dp}{dy} \right) dy + \frac{R_0 B_v b}{B_\theta(b)} B_\theta(r)$$

new shafranov shift

$$\Delta_a = -\frac{\psi_1(a)}{\psi_0'(a)} = -\frac{\psi_1^{old}(a)}{\psi_0'(a)} - \frac{\psi_1^{\hom}(a)}{\psi_0'(a)}$$

$$= \Delta_{old} - \left(\frac{R_0 B_v b}{B_\theta(b)} B_\theta(a) \right) \frac{1}{R_0 B_\theta(a)}$$

$$= \Delta_{old} - \frac{bB_v}{B_\theta(b)}$$

$$\frac{\Delta_a}{b} = \frac{b}{2R_0} \left\{ \left(\beta_p + \frac{l_i}{2} - \frac{1}{2} \right) \left(1 - \frac{a^2}{b^2} \right) + \ln \frac{b}{a} \right\} - \frac{B_v}{B_\theta(b)}$$

Equilibrium: 2-D Configurations

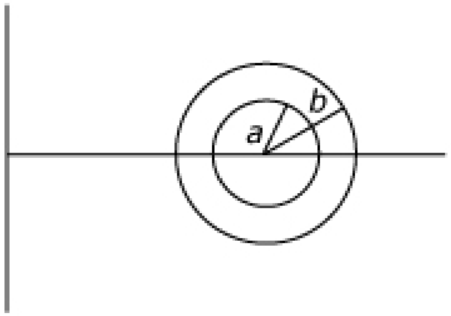
Toroidal Force Balance by Means of a Vertical Field

$$\frac{\Delta_a}{b} = \frac{b}{2R_0}\left\{\left(\beta_p + \frac{l_i}{2} - \frac{1}{2}\right)\left(1 - \frac{a^2}{b^2}\right) + \ln\frac{b}{a}\right\} - \frac{B_v}{B_\theta(b)}$$

How much vertical field do we need to keep the plasma centered?

$$\Delta_a = 0, \qquad B_\theta(b) = \mu_0 I_p / 2\pi b$$

$$B_v = \frac{\mu_0 I}{4\pi R_0}\left\{\left(\beta_p + \frac{l_i}{2} - \frac{1}{2}\right)\left(1 - \frac{a^2}{b^2}\right) + \ln\frac{b}{a}\right\}$$



The limit $b \to \infty$

- How much field is required for no shift if the shell is not present?
- Imagine the shell receding further and further away so that $b/a \to \infty$
- Take this limit in the expression of the vertical field.

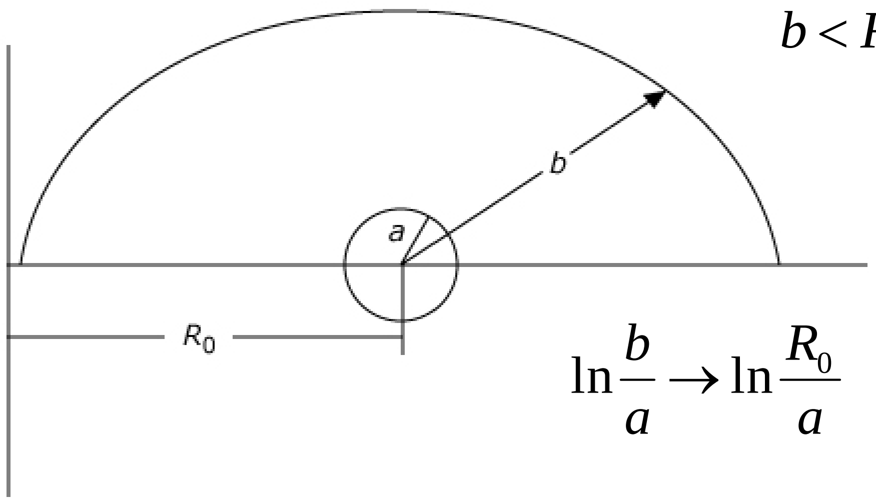
$$1 - \frac{a^2}{b^2} \to 1, \quad \ln\frac{b}{a} \to ?$$

Equilibrium: 2-D Configurations

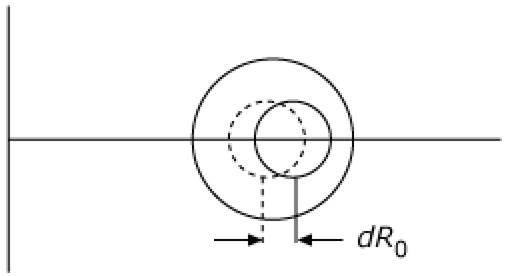
Toroidal Force Balance by Means of a Vertical Field

$$1-\frac{a^2}{b^2}\to 1,\quad \ln\frac{b}{a}\to ?$$

$$b<R_0,\quad b/R_0<<1$$



$$\ln\frac{b}{a}\to \ln\frac{R_0}{a}$$



- ln *b/a* represents the force due to the change in magnetic energy between the plasma and the wall as the plasma shifts outward by an amount $dR_0=\Delta$.
- It is the analog of the l_i term except applied to the external field.

Equilibrium: 2-D Configurations

Toroidal Force Balance by Means of a Vertical Field

The force=-∇(potential energy)=-∇(magnetic energy) as the plasma is displaced by an amount dR_0.

Note that since the plasma is a perfect conductor, the flux linking the plasma remains fixed during the displacement.

$$F = -\frac{d}{dR_0}\left(\frac{1}{2\mu_0}\int B_\theta^2 dr\right) = -\frac{d}{dR_0}\left(\frac{1}{2}L_e I^2\right) \longleftarrow \quad L_e I = \psi_e = const$$

$$= -\frac{L_e^2 I^2}{2}\frac{d}{dR}\frac{1}{L_e} = \frac{I^2}{2}\frac{dL_e}{dR_0}$$

Plasma surrounded by a shell

$$L_e = \mu_0 R_0 \ln b/a \qquad F = \frac{\mu_0 I^2}{2}\ln\frac{b}{a}$$

Plasma without a shell

$$L_e = \mu_0 R_0\left(\ln\frac{8R_0}{a} - 2\right) \qquad F = \frac{\mu_0 I^2}{a}\left(\ln\frac{8R_0}{a} - 1\right)$$

Equilibrium: 2-D Configurations

Toroidal Force Balance by Means of a Vertical Field

Therefore, the proper limit is

$$\ln\frac{b}{a} \to \ln\frac{8R_0}{a} - 1 = \ln\frac{R_0}{a} + 1.08$$

$$B_v = \frac{\mu_0 I}{4\pi R_0}\left\{\beta_p + \frac{l_i}{2} - \frac{3}{2} + \ln\frac{8R_0}{a}\right\}$$

This is a widely used formula in the design of circular tokamaks.