Numerical Solutions of Tokamak Equilibria (2/2)

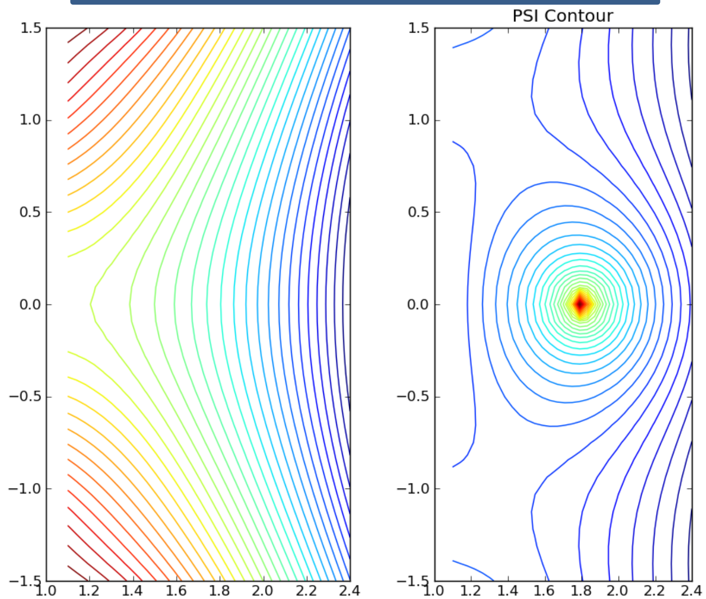
- 핵융합 플라즈마 특강 2 -

2009. 2nd Semester

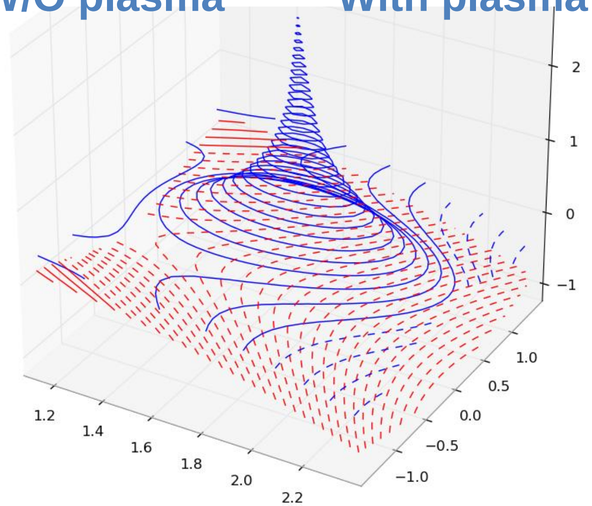
YoungMu Jeon

Homework - Discussion

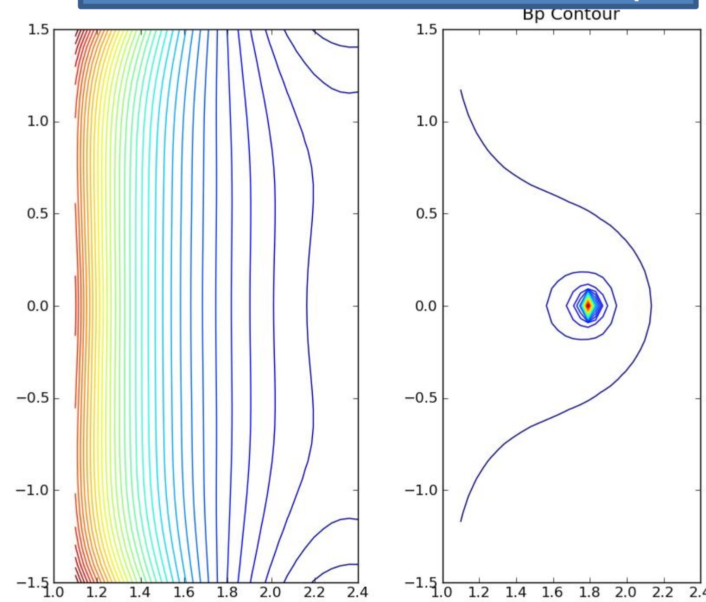
Poloidal mag. flux, ψ



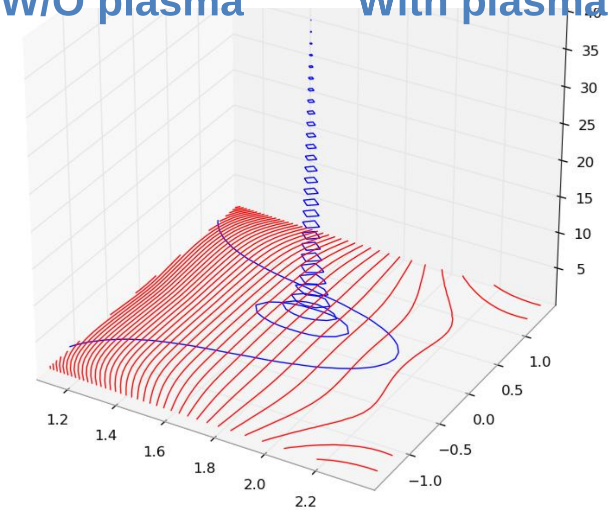
W/O plasma With plasma



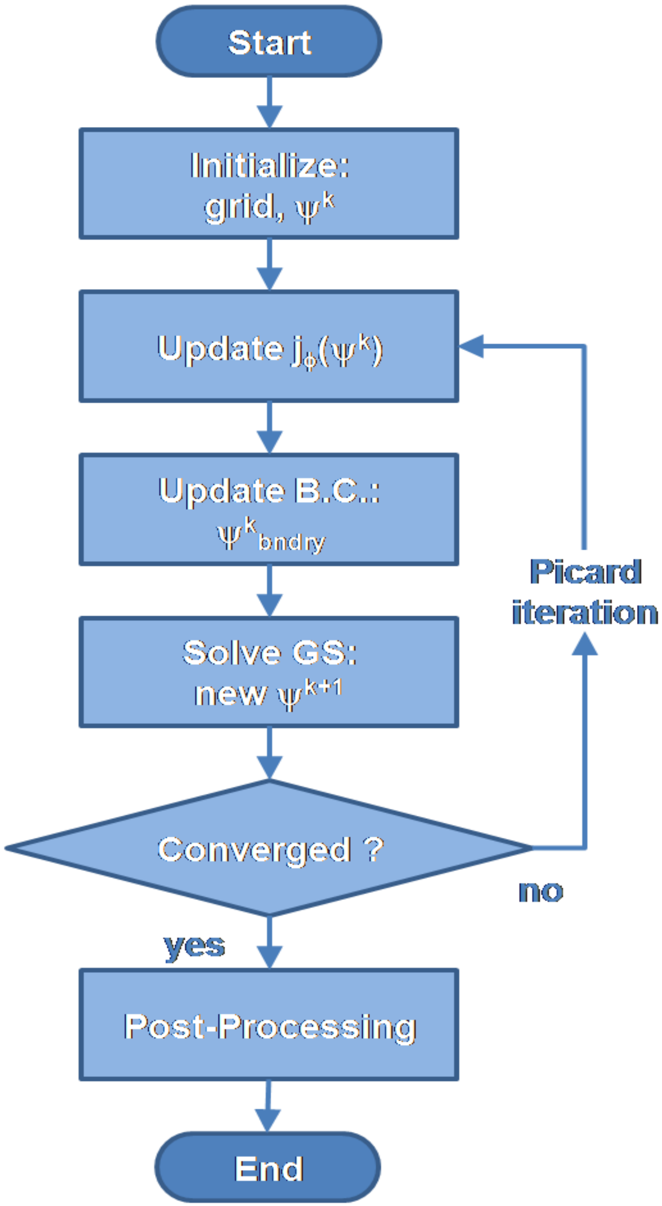
Poloidal mag. field, abs(B_p)



W/O plasma With plasma



Solving Tokamak Plasma Equilibrium



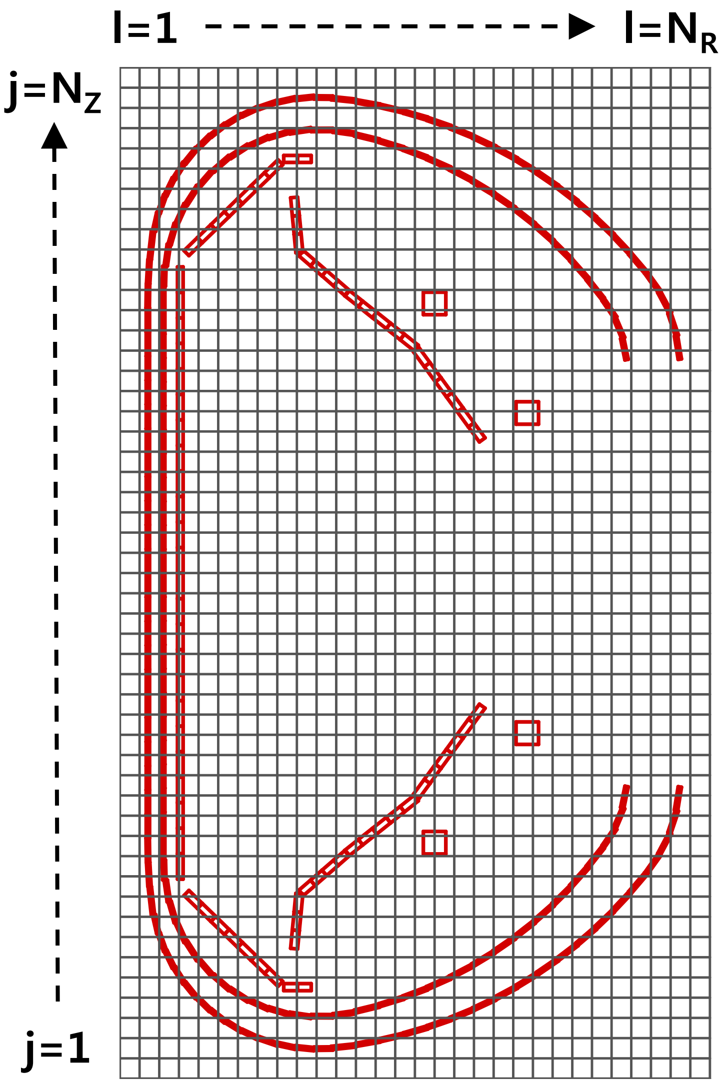
- We are solving the following G.S. equation on the **rectangular grid** of ($r_{min} \le r \le r_{max}$) and ($z_{min} \le z \le z_{max}$) with a given current density profile as a **free-boundary** problem.

$$\Delta^* \psi^{k+1} = -\mu_0 r J_\phi\left(\psi^k\right)$$

- To resolve non-linearity, we use "**Picard iteration**"

- To solve linearized GS equation, we can use **iterative methods** such as SOR (Successive Over-Relaxation) or MGM (Multi-Grid Method) methods. In this case, we will have two iteration loops; One is inner loop to solve GS eq. and the other is outer loop for Picard iteration.
- Of course, we can use a matrix inverse directly.

(1) Initialize



- Construct computational grids : (R_l, Z_j), l=1,…,N_R, j=1,…,N_Z

- Initial setup of ψ matrix on grid points if necessary
  - Initial plasma region and corresponding flux distributions

- And do other initializations required

(2) Update $j_\phi(\psi)$

- A basic approach for current profile is to assume an explicit form as like

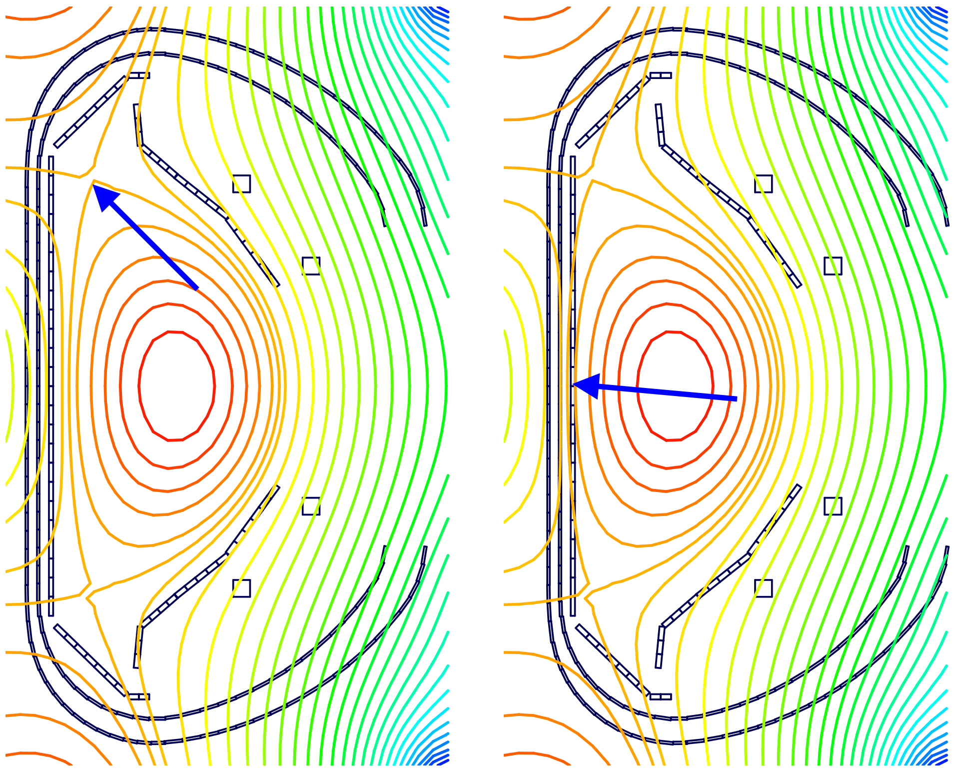
$$J_\phi(\psi) = -rp'(\psi) - \frac{F(\psi)F'(\psi)}{\mu_0 r}$$

$$\Rightarrow \begin{cases} J_\phi(\psi) \equiv \lambda_0 \left(\beta_0 \dfrac{r}{r_0} + (1-\beta_0)\dfrac{r_0}{r} \right)\left(1-\psi_s^m\right)^n \\ \psi_s \equiv \dfrac{\psi - \psi_a}{\psi_b - \psi_a} \end{cases}$$

- At first, find the plasma region (i.e. boundary) ➔ gives ψ_b
    - ψ_b could be obtained from the fluxes on limiter points or X-points
    - X-points could be defined as points on which $d\psi/dr = d\psi/dz = 0$
- And then, find ψ_a ➔ gives ψ_s
- Then, update $p(\psi_s)$, $F(\psi_s)$, and $j_\phi(\psi_s)$ on the plasma region

- Probably we need some constraints on $j_\phi(\psi_s)$ to make it converge
    - It will be discussed later in a separate section

Find Plasma Boundary



Diverted plasma

Limited plasma

- **Tokamak plasma has two types of boundaries : “limited” or “diverted”**
- **Therefore, the mag. flux at the plasma boundary should be defined from both $\psi_{x\text{-point}}$ and $\psi_{limiter}$ together**
- **After obtaining ψ_b, one can define ψ_s and the plasma region in which $p(\psi_s)$, $F(\psi_s)$, and $j_\phi(\psi_s)$ are defined**

(3) Update B.C.

- **Use Green function formula to calculate the mag. flux at the computational boundary**

$$\psi_{\text{bndry}} = \sum_l \left((\Delta S_l) J_{\phi,\text{plasma}}(\mathbf{r}_l') \right) G(\mathbf{r}_{\text{bndry}};\mathbf{r}_l') + \sum_c I_c G(\mathbf{r}_{\text{bndry}};\mathbf{r}_c')$$

- **Only one time calculation is required for the contribution from the coils, while one need to recalculate the contribution from the plasma whenever the current density is updated**

(4) Solve G.S. Equation

- **Solve the linearized G.S. equation.**

$$\psi_{j,l} = \frac{1}{2}\left[\left(\frac{h_R^2}{h_R^2+h_Z^2}\right)\psi_{j-1,l} + \left(\frac{h_R^2}{h_R^2+h_Z^2}\right)\psi_{j+1,l} + \left(\frac{h_Z^2}{h_R^2+h_Z^2}\right)\left(1+\frac{h_R}{2R_l}\right)\psi_{j,l-1} \right.$$
$$\left. + \left(\frac{h_Z^2}{h_R^2+h_Z^2}\right)\left(1-\frac{h_R}{2R_l}\right)\psi_{j,l+1} + \left(\frac{h_R^2 h_Z^2}{h_R^2+h_Z^2}\right)\mu_0 R_l J_{\phi\, j,l} \right]$$

- **In this class, we will use SOR(Successive Over-Relaxation) method (cf: also refer to Jacobi and Gauss-Seidel methods)**

$$\left(\psi_{j,l}^{k+1}\right)^* = \frac{1}{2}\left[\left(\frac{h_R^2}{h_R^2+h_Z^2}\right)\psi_{j-1,l}^{k+1} + \left(\frac{h_R^2}{h_R^2+h_Z^2}\right)\psi_{j+1,l}^{k} + \left(\frac{h_Z^2}{h_R^2+h_Z^2}\right)\left(1+\frac{h_R}{2R_l}\right)\psi_{j,l-1}^{k+1} \right.$$
$$\left. + \left(\frac{h_Z^2}{h_R^2+h_Z^2}\right)\left(1-\frac{h_R}{2R_l}\right)\psi_{j,l+1}^{k} + \left(\frac{h_R^2 h_Z^2}{h_R^2+h_Z^2}\right)\mu_0 R_l {J_{\phi}}_{j,l}^{k} \right]$$

$$\Delta\left(\psi_{j,l}^{k+1}\right)^* \equiv \left(\psi_{j,l}^{k+1}\right)^* - \psi_{j,l}^{k}, \qquad \psi_{j,l}^{k+1} = \psi_{j,l}^{k} + \omega\Delta\left(\psi_{j,l}^{k+1}\right)^*$$

here $\omega = \text{relaxation factor}, 0 < \omega < 2$

(5) Convergence

- **In our case, the convergence criteria can be defined as follows (for outer loop: Picard iteration)**

$$\mathrm{E}_{\mathrm{total}} \equiv \frac{1}{N_R \times N_Z} \sum_{j,l}^{N_R, N_Z} \left\{ \Delta \left(\psi_{j,l}^{k+1} \right)^* \right\}^2 < \varepsilon \ \ (\mathrm{ex}: 1.\mathrm{e}-5)$$

- **One can define a similar criterion for inner loop (iterative method like SOR)**

(6) Post-Processing

- Based on calculated mag. poloidal flux, ψ_s, we can obtain many of useful and important equilibrium quantities such as κ(elongation), δ(triangularity), ι(rotational transform) or q(safety factor), β(plasma beta) so on. Please refer to any text for detail formula.

$$\beta_p = \frac{\langle p \rangle}{\langle B_p^2 \rangle_{\psi_a} / 2\mu_0} \qquad \beta_N = \frac{\langle \beta \rangle}{I_p[\mathrm{MA}] / aB_{T0}}$$

$$l_i \equiv \frac{\psi}{\mu_0 r_0 (I_p)^2} \int \frac{B_p^2}{2\mu_0} dV \qquad l_i(3) \equiv 2V \langle B_p^2 \rangle / r_0 (\mu_0 I_p)^2$$

$$q(\psi) = \frac{1}{2\pi} \oint_\psi \frac{B_\phi}{rB_\theta} dl = \frac{F}{2\pi} \oint_\psi \frac{dl}{r|\nabla \psi|}$$

(7) Constraints On Current Profile

$$J_\phi(\psi) \equiv \lambda_0 \left(\beta_0 \frac{r}{r_0} + (1-\beta_0)\frac{r_0}{r} \right)(1-\psi_s^m)^n$$

$$= -rp'(\psi) - \frac{F(\psi)F'(\psi)}{\mu_0 r}$$

1) Plasma current : λ_0 is updated to match given I_p

$$I_p = \int_{\Omega_s} J_\phi(\psi) ds \qquad \Rightarrow \qquad \text{update } \lambda_0$$

2) Plasma beta : β_0 is updated to match given β_p or $p(\psi_a)$

$$\beta_p = \frac{\langle p \rangle}{\left\langle B_p^2 \right\rangle_{\psi_a} / 2\mu_0} \qquad \Rightarrow \qquad \text{update } \beta_0$$

Variations & Applications Of Equilibrium Solution

App-1. Solve Coil Currents For Equilibrium

- If the plasma boundary is given, then we can calculate the equilibrium PF coil currents. In this case, the problem become a sort of fixed boundary problem.

① Solve the G.S. equation as a fixed boundary problem with $\psi_b = \psi(r_{bndry}, z_{bndry}) = 0$.

② Find out the PF coil currents which satisfies following

$$\begin{cases} \psi_{\text{plasma}}(r_{\text{bndry}}, z_{\text{bndry}}) = \int_{\Omega_{\text{plasma}}} J_\phi(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\Omega \\ \psi_{\text{PF-coils}}(r_{\text{bndry}}, z_{\text{bndry}}) = \sum_{k}^{\mathrm{N_{coil}}} I_k G(\mathbf{r}_{\text{bndry}}, \mathbf{r}_k) \end{cases}$$

$$\min_{I_k} \sum_{l}^{\mathrm{N_{bndry}}} \left\{ \psi_{\text{plasma}} + \psi_{\text{PF-coils}}(I_k) \right\}^2$$

App-2. Reconstructions (EFIT)

- **Similar with our equilibrium solution method described in this class except the form of current density profile and the constraints on that.**
- **In EFIT, polynomial or spline representations for $j_\phi(\psi)$ are used according to user's choice.**

$$\begin{cases} p(\psi) = \alpha_0 + \alpha_1\psi + \cdots + \alpha_{n-1}\psi^{n-1} \\ F(\psi) = \beta_0 + \beta_1\psi + \cdots + \beta_{n-1}\psi^{n-1} \end{cases}$$

$$J_\phi(\psi) = -rp'(\psi) - \frac{F(\psi)F'(\psi)}{\mu_0 r}$$

- **Constraints using experimental measurements (MD) can specify the coefficients, α_0, α_1, …, β_0, β_1, …, so that current density can be identified from measurements.**

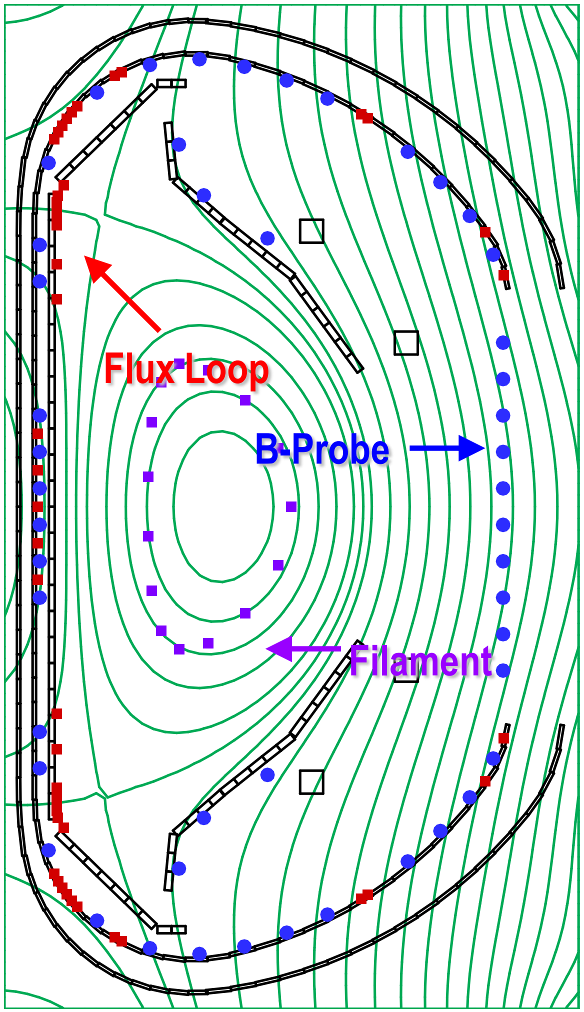
Constraints On Current Density Profile

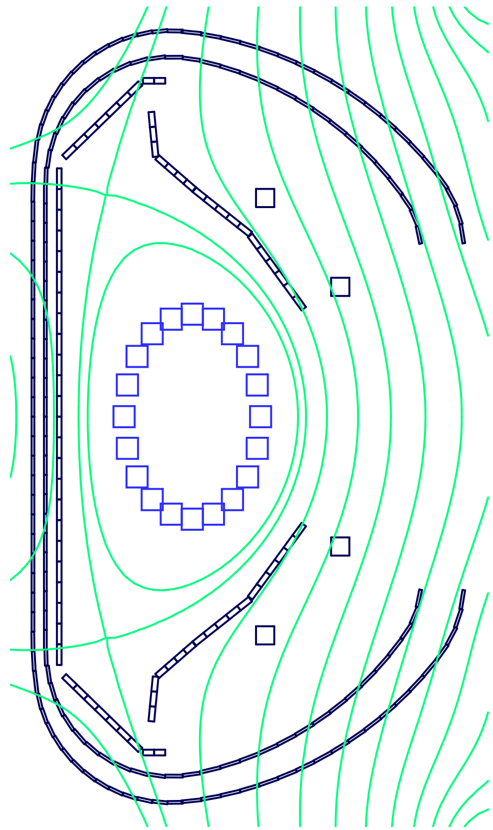
- As a constraint on current profile, the experimental measurements can be used in order to configure the coefficients of p(ψ) and F(ψ).
- Many of diagnostics can be used such as magnetic measurements and MSE (Motional Stark Effect) so on.

- Primarily, with magnetic diagnostics (MD), we can find out the appropriate coefficients by solving the following equation using “least-squre method” or “SVD(Singular Value Decomposition)”.

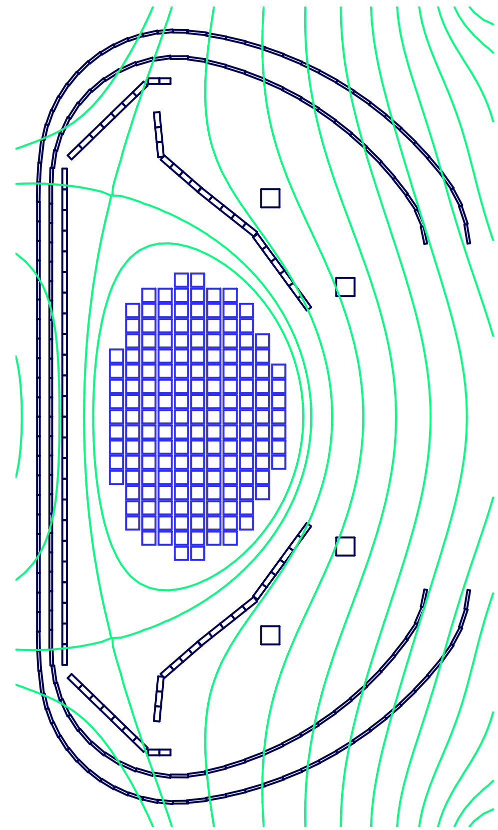
$$\min_{\alpha_l,\ \beta_l} \sum_{l}^{N_{MD}} \left\{ \tilde{\psi}_{l,measured} - \tilde{\psi}_{l,calculated}(\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \cdots) \right\}^2$$

Reconstruction Example Using Filament Approach



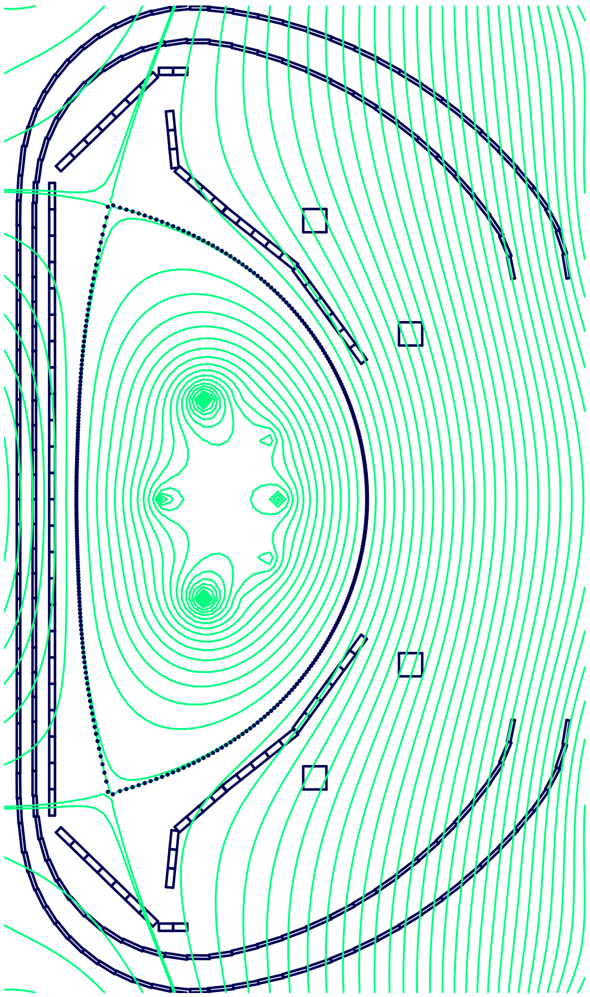


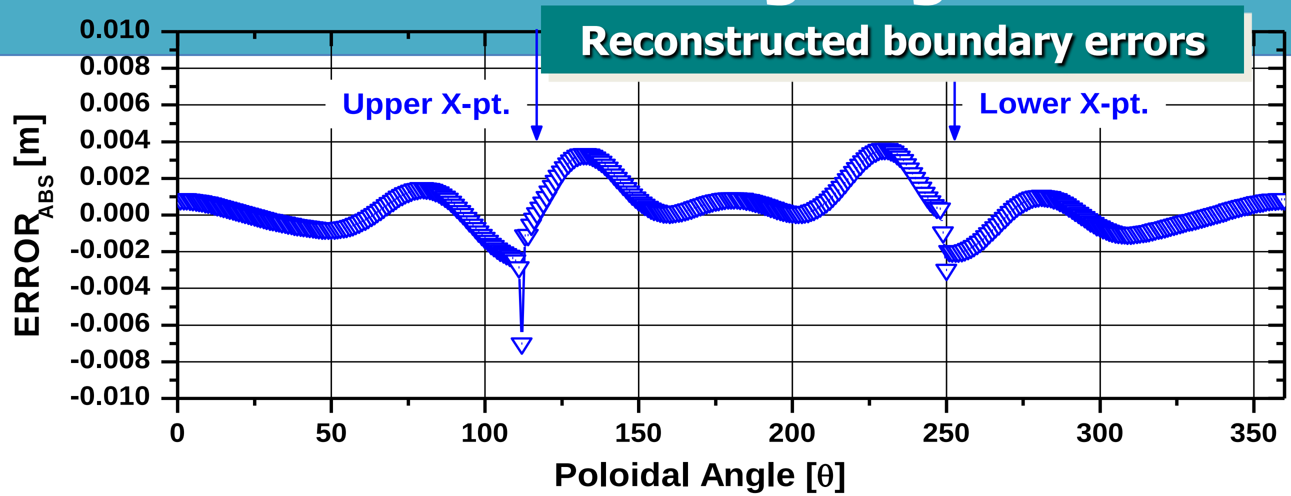
Circular Equivalent Currents (CEC)

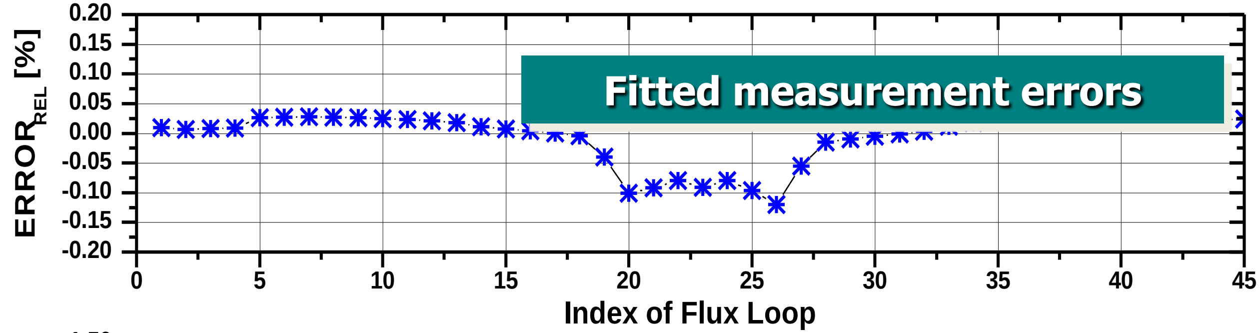


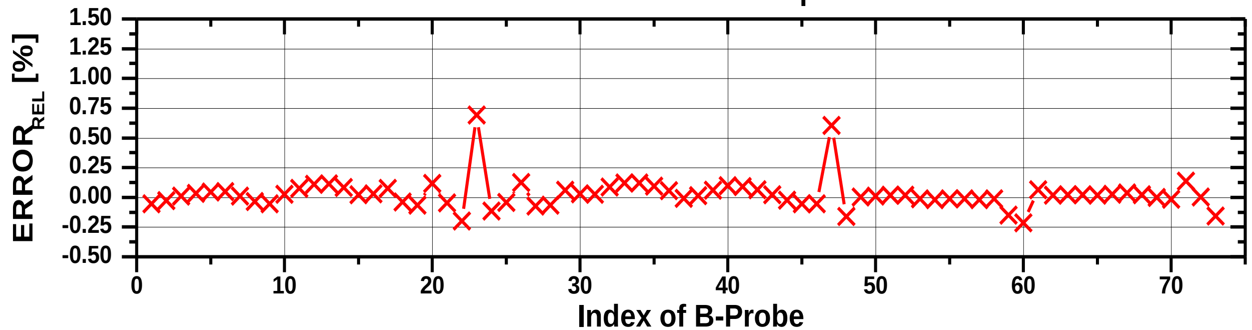
Uniform Equivalent Currents (UEC)

Reconstruction & Its Accuracy By CEC

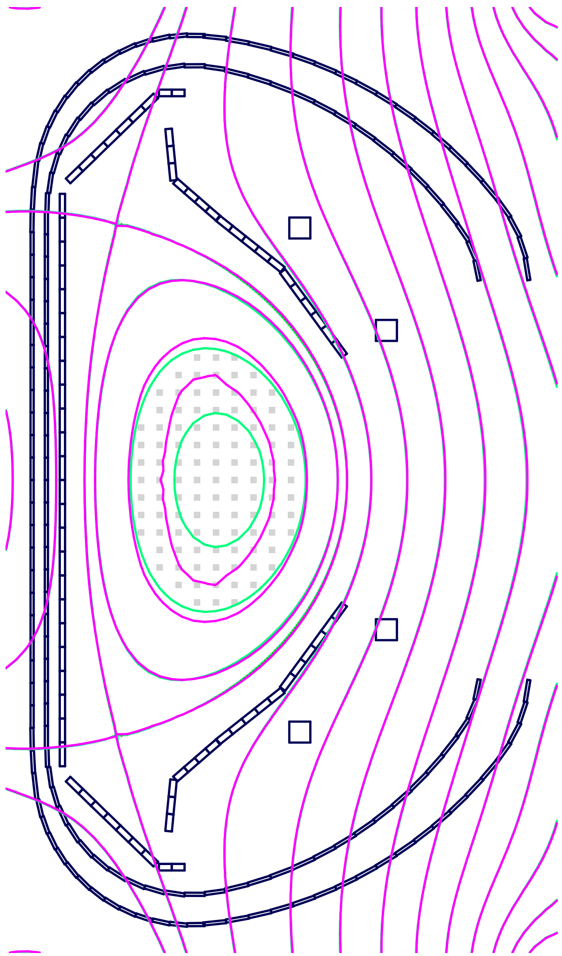




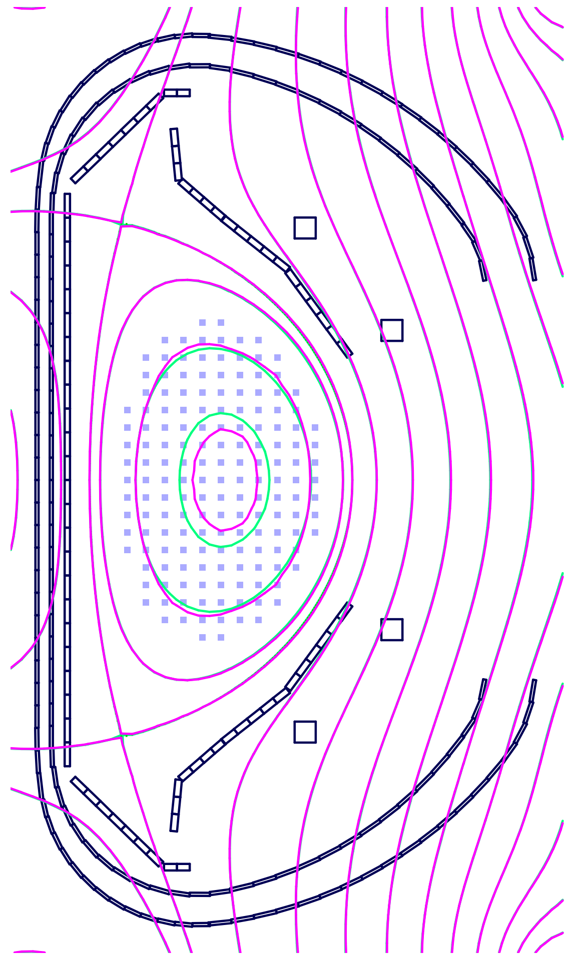




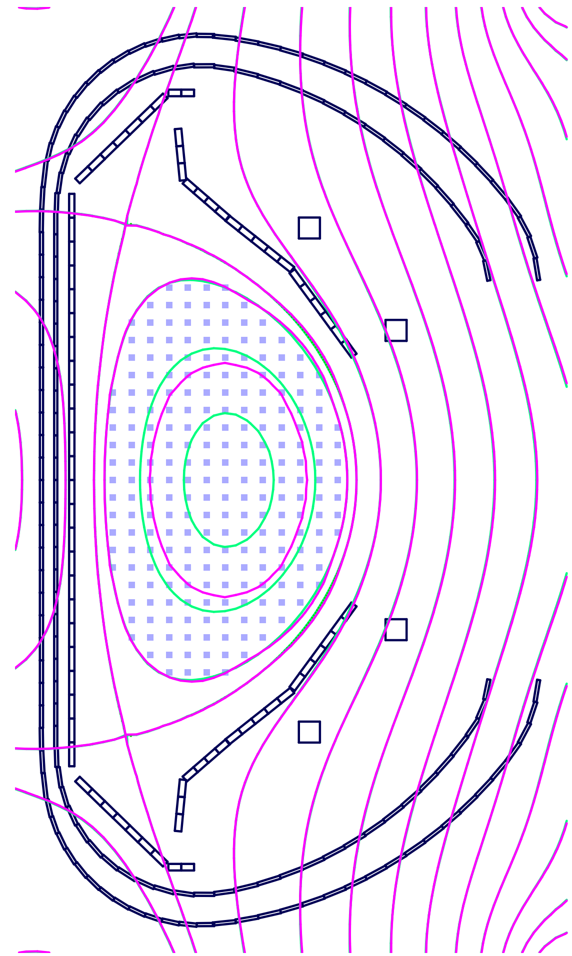
Reconstruction By UEC
With Variations Of EC Number



In 50% from SOL



In 70% from SOL



In 90% from SOL

< Term Project >

Development Of
A Free Boundary Tokamak Equilibrium Solver