Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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Contents

Contents

Stability: One-Dimensional Configurations

- **Introduction**

- Describing the application of the Energy Principle to 1-D cylindrical configurations: The θ pinch, the Z pinch and the general screw pinch
- Showing that the θ pinch has inherently favourable stability properties while the Z pinch is strongly unstable.
- A rather high level of complexity is exhibited in the general screw pinch: A general 2^{nd} order equation is derived for the plasma displacement with BCs corresponding to a perfectly conducting wall, an isolating vacuum region and a resistive wall.

  → Suydam's criterion for localised interchanges,
  Newcomb's general procedure for testing stability,
  The oscillation theorem describing the eigenvalue behaviour for the full linearised stability equations.

Stability: One-Dimensional Configurations

- **The θ Pinch**

- Sequence of solution of the MHD equilibrium equations

1. The $\nabla \cdot \mathbf{B} = 0$

$$\frac{\partial B_z}{\partial z} = 0$$

2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$

$$J_\theta = -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r}$$

3. The momentum equation: $\mathbf{J} \times \mathbf{B} = \nabla p$

$$J_\theta B_z = \frac{dp}{dr}$$

$$\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0}\right) = 0 \qquad p + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$\xi(\vec{r}) = \xi(r)\exp[i(m\theta + kz)]$$ Fourier analysed

m, *k*: poloidal and toroidal wave numbers, respectively

Stability: One-Dimensional Configurations

- **The θ Pinch**
  - The first step in the minimisation of δ*W* is to check the incompressibility.

$$\nabla\cdot\xi = \nabla\cdot\xi_\perp + \nabla\cdot(\xi_{||}\vec{e}_z) = \nabla\cdot\xi_\perp + ik\xi_{||} = 0 \qquad \xi(\vec{r}) = \xi(r)\exp[i(m\theta + kz)]$$

$$\xi_{||} \equiv \xi_z = \frac{i}{k}\nabla\cdot\xi_\perp = \frac{i}{kr}[(r\xi_r)' + im\xi_\theta] \qquad k \neq 0$$

  - The next step is the evaluation of δW_F.

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

$$\vec{Q}_\perp = ikB_z\xi_\perp = ikB_z(\xi_r\vec{e}_r + \xi_\theta\vec{e}_\theta)$$

$$\vec{\kappa} = \vec{b}\cdot\nabla\vec{b} = \frac{\partial}{\partial z}\vec{e}_x = 0 \qquad \text{no pressure driven terms}$$

$$\vec{J}_{||} = \vec{J}\cdot\vec{b} = J_0\vec{e}_\theta\cdot\vec{e}_x = 0 \qquad \text{no current driven terms}$$

Stability: One-Dimensional Configurations

- **The θ Pinch**

$$\frac{\delta W_F}{2\pi R_0} = \frac{\pi}{\mu_0}\int_0^a W(r)rdr$$

$$W(r) = B_z^2[k^2(|\xi_r|^2 + |\xi_\theta|^2) + \frac{1}{r^2}|(r\xi_r)'|^2 + \frac{m^2}{r^2}|\xi_\theta|^2 + \frac{im}{r^2}(r\xi_r^*)'\xi_\theta - \frac{im}{r^2}(r\xi_r)'\xi_\theta^*]$$

ξ_θ terms combined by completing the squares

$$W(r) = B_z^2\left\{\left|k_0\xi_\theta - \frac{im}{k_0 r^2}(r\xi_r)'\right|^2 + \frac{k^2}{k_0^2 r^2}[|(r\xi)'|^2 + k_0^2 r^2|\xi|^2]\right\}$$

minimised by choosing

$$\xi_\theta = \frac{im}{m^2 + k^2 r^2}(r\xi_r)' \qquad k_0^2 = k^2 + m^2/r^2$$

$$\frac{\delta W_F}{2\pi R_0} = \frac{\pi}{\mu_0}\int_0^a rdr\frac{k^2 B_z^2}{m^2 + k^2 r^2}[|(r\xi)'|^2 + (m^2 + k^2 r^2)|\xi|^2] \qquad \xi \equiv \xi_r$$

Stability: One-Dimensional Configurations

- **The θ Pinch**

$$\frac{\delta W_F}{2\pi R_0} = \frac{\pi}{\mu_0}\int_0^a r dr \frac{k^2 B_z^2}{m^2 + k^2 r^2}\left[\left|(r\xi)'\right|^2 + (m^2 + k^2 r^2)\left|\xi\right|^2\right]$$

$$\delta W_S = 0, \quad \delta W_V \geq 0$$

- $\delta W_F > 0$ for any nonzero k^2 and $\delta W_F \to 0$ as $k^2 \to 0$:
- At any value of β the θ pinch is positively stable for finite wavelengths and approaches marginal stability for very long wavelengths.
- Current-driven modes cannot be excited due to no parallel currents.
- Since the field lines are straight, their curvature is zero and pressure-driven modes cannot be excited.
- Any perturbation to the equilibrium either bends or compresses the magnetic field lines, and both are stabilising influences.

Stability: One-Dimensional Configurations

- **The *Z* Pinch**

- Sequence of solution of the MHD equilibrium equations

1. The ▽·**B** = 0

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

2. Ampere's law: μ_0**J** = ▽x**B**

$$J_z = \frac{1}{\mu_0 r}\frac{d}{dr}(rB_\theta)$$

3. The momentum equation: **J**x**B** = ▽p

$$J_z B_\theta = -\frac{dp}{dr}$$

$$\frac{dp}{dr} + \frac{B_\theta}{\mu_0 r}\frac{d}{dr}(rB_\theta) = 0 \qquad \frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

$$\xi(\vec{r}) = \xi(r)\exp[i(m\theta + kz)]$$ Fourier analysed

m, *k*: poloidal and toroidal wave numbers, respectively

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - The first step in the minimisation of δ*W* is to check the incompressibility.

$$\nabla\cdot\xi = \nabla\cdot\xi_\perp + \nabla\cdot(\xi_{||}\vec{e}_z) = \nabla\cdot\xi_\perp + ik\xi_{||} = 0 \qquad \xi(\vec{r}) = \xi(r)\exp[i(m\theta + kz)]$$

$$\xi_{||} \equiv \xi_\theta = \frac{ir}{m}\nabla\cdot\xi_\perp = \frac{i}{m}[(r\xi_r)' + ikr\xi_z] \qquad m \neq 0$$

$m = 0$

Stability: General Considerations

- **Incompressibility**

Several minimising condition

If $\nabla \cdot \xi \neq 0$

- Existing sufficient equilibrium symmetry

$$\vec{B} \cdot \nabla \frac{\xi_{||}}{B} = \frac{B_\theta}{r} \frac{\partial}{\partial \theta} \frac{\xi_{||}}{B_\theta} = \frac{im\xi_{||}}{r} \quad = 0 \text{ for } m = 0$$

$$\nabla \cdot \xi = \nabla \cdot \xi_\perp + \nabla \cdot \frac{\xi_{||}}{B} \vec{B} = \nabla \cdot \xi_\perp + \vec{B} \cdot \nabla \frac{\xi_{||}}{B}$$

$$= \nabla \cdot \xi_\perp$$

$\xi_{||}$ does not appear.
The term must be maintained for the rest of the minimisation.

$$\int_P \gamma p |\nabla \cdot \xi|^2 dr = \int_P \gamma p |\nabla \cdot \xi_\perp|^2 dr$$

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - The first step in the minimisation of δ*W* is to check the incompressibility.

$$\nabla\cdot\xi=\nabla\cdot\xi_\perp+\nabla\cdot(\xi_{||}\vec{e}_\theta)=\nabla\cdot\xi_\perp+\frac{im}{r}\xi_{||}=0 \qquad \xi(\vec{r})=\xi(r)\exp[i(m\theta+kz)]$$

$$\xi_{||}\equiv\xi_\theta=\frac{ir}{m}\nabla\cdot\xi_\perp=\frac{i}{m}[(r\xi_r)'+ikr\xi_z] \qquad m\neq 0$$

- $m \neq 0$ Modes

$$\delta W_F=\frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0}+\frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp+2\xi_\perp\cdot\vec{\kappa}\right|^2+\gamma p\left|\nabla\cdot\xi\right|^2-2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*)-J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

$$\vec{\kappa}=\vec{b}\cdot\nabla\vec{b}=\vec{e}_\theta\cdot\nabla\vec{e}_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta=-\frac{\vec{e}_r}{r}$$

$$\vec{J}_{||}=\vec{J}\cdot\vec{b}=J_z\vec{e}_z\cdot\vec{e}_\theta=0$$ no current driven terms

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *m* ≠ 0 Modes

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

$$\vec{B}_1 = \nabla\times(\xi_\perp\times\vec{B}) = \vec{B}\cdot\nabla\xi_\perp - \xi_\perp\cdot\nabla\vec{B} - \vec{B}\nabla\cdot\xi_\perp$$

$$B_{1r} = \frac{imB_\theta}{r}\xi r - \frac{B_\theta\xi_\theta}{r} + \frac{B_\theta\xi_\theta}{r} = \frac{imB_\theta}{r}\xi r$$

$$B_{1z} = \frac{imB_\theta}{r}\xi_z$$

$$\left|\vec{B}_{1\perp}\right|^2 = \frac{m^2B_\theta^2}{r^2}[\left|\xi r\right|^2 - \left|\xi_z\right|^2]$$

$\nabla\cdot\xi = 0$ for *m* ≠ 0

$\nabla\cdot\xi = \frac{(r\xi_r)'}{r} + ik\xi_z$ for *m* = 0

Stability: One-Dimensional Configurations

- **The Z Pinch**
  - $m \neq 0$ Modes

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

$$\vec{Q}_\perp = \frac{imB_\theta}{r}(\xi_r\vec{e}_r + \xi_z\vec{e}_z) \qquad \vec{Q} \equiv \vec{B}_1 = \nabla\times(\xi\times\vec{B})$$

$$B^2\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 = B_\theta^2\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2\left|\xi_z\right|^2 + ikr\xi_z\left(\frac{\xi_r^*}{r}\right)' - ikr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right]$$

$$\gamma p\left|\nabla\cdot\xi\right|^2 = \gamma p\left[\left|\frac{(r\xi_r)'}{r}\right|^2 + k^2\left|\xi_z\right|^2 + \frac{ik\xi_z}{r}(r\xi_r^*)' - \frac{ik\xi_z^*}{r}(r\xi_r)'\right]$$ for $m = 0$

$$-2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) = (-2\xi_r p')\left(-\frac{\xi_r^*}{r}\right) = \frac{2p'}{r}\left|\xi_r\right|^2$$ < 0 if $p' < 0$

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *m* ≠ 0 Modes

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

for *m* ≠ 0

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left\{\frac{m^2B_\theta^2}{\mu_0 r^2}\left[\left|\xi_r\right|^2 + \left|\xi_z\right|^2\right] + \frac{B_\theta^2}{\mu_0}\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2\left|\xi_z\right|^2 + ikr\xi_z\left(\frac{\xi_r^*}{r}\right)' - ikr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right] + 0 + \frac{2p'}{r}\left|\xi_r\right|^2 + 0\right\}$$

Minimise δW_F: ξ_z terms

$$\frac{B_\theta^2}{\mu_0}\left[\left(\frac{m^2}{r^2} + k^2\right)\left|\xi_z\right|^2 + ikr\xi_z\left(\frac{\xi_r^*}{r}\right)' - ikr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right]$$

$$\xi_z = \frac{ikr}{k_0^2}\left(\frac{\xi_r}{r}\right)'$$

$$\frac{B_\theta^2 k_0^2}{\mu_0}\left[\left|\xi_z - \frac{ikr}{k_0^2}\left(\frac{\xi_r}{r}\right)'\right|^2 - \frac{k^2r^2}{k_0^4}\left|\left(\frac{\xi_r}{r}\right)'\right|^2\right] \qquad k_0^2 = k^2 + m^2/r^2$$

minimising condition

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *m* ≠ 0 Modes

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left\{\frac{m^2 B_\theta^2}{\mu_0 r^2}\left[|\xi_r|^2 + |\xi_z|^2\right] + \frac{B_\theta^2}{\mu_0}\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2|\xi_z|^2 + ikr\xi_z\left(\frac{\xi_r^*}{r}\right)' - ikr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right] + 0 + \frac{2p'}{r}|\xi_r|^2 + 0\right\}$$

$$\delta W_F = \frac{1}{2\mu_0}\int_P d\vec{r}\left[|\xi|^2\left(\frac{m^2 B_\theta^2}{r^2} + \frac{2\mu_0 p'}{r}\right) + B_\theta^2\left|r\left(\frac{\xi}{r}\right)'\right|^2\left(1 - \frac{k^2}{k_0^2}\right)\right] \qquad \xi \equiv \xi_r$$

$$= \frac{1}{2\mu_0}\int_P d\vec{r}\left[|\xi|^2\left(\frac{m^2 B_\theta^2}{r^2} + \frac{2\mu_0 p'}{r}\right) + \frac{m^2 B_\theta^2}{m^2 + k^2 r^2}\left|r\left(\frac{\xi}{r}\right)'\right|^2\right]$$

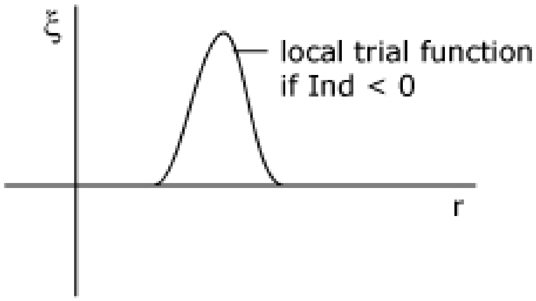
δW_F minimised by letting $k^2 \to \infty$

$$\delta W_F = \frac{1}{2\mu_0}\int_P d\vec{r}|\xi|^2\left(\frac{m^2 B_\theta^2}{r^2} + \frac{2\mu_0 p'}{r}\right) \qquad \frac{\delta W_F}{2\pi R_0} = \frac{\pi}{\mu_0}\int_0^a r\,dr(2\mu_0 r p' + m^2 B_\theta^2)\frac{|\xi|^2}{r^2}$$

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *m* ≠ 0 Modes

$$2r\frac{dp}{dr}+\frac{m^2B_\theta^2}{\mu_0}>0$$ stability condition



A trial function localised within the region with negative value could be constructed that would make $\delta W_F < 0$, implying instability.

$$B_\theta(rB_\theta)'<\frac{m^2B_\theta^2}{2} \longleftarrow \frac{dp}{dr}+\frac{B_\theta}{\mu_0 r}\frac{d}{dr}(rB_\theta)=0 \qquad p'=-\frac{B_\theta}{\mu_0 r}(rB_\theta)'$$

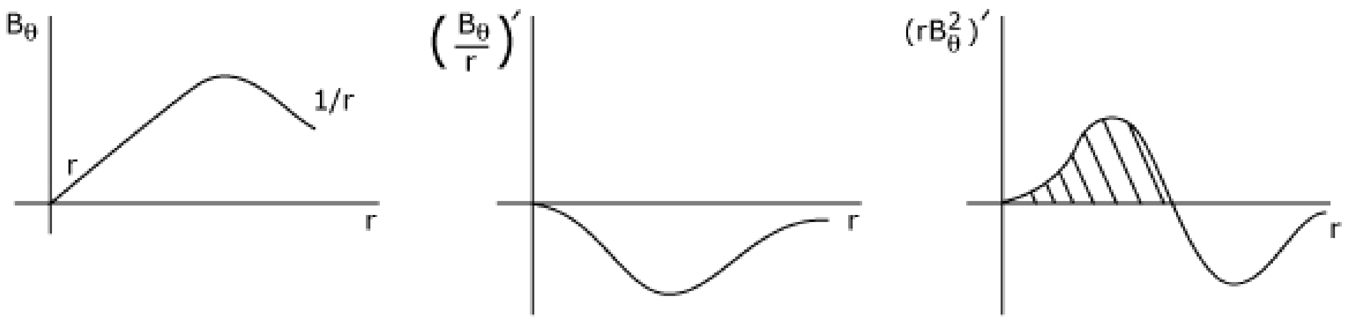
$$\frac{r^2}{B_\theta}\left(\frac{B_\theta}{r}\right)'<\frac{1}{2}(m^2-4) \longleftarrow B_\theta(rB_\theta)'=B_\theta\left(r^2\frac{B_\theta}{r}\right)'=r^2B_\theta\left(\frac{B_\theta}{r}\right)'+2B_\theta^2$$

$$\frac{\left(rB_\theta^2\right)'}{B_\theta^2}<m^2-1 \longleftarrow B_\theta(rB_\theta)'=r\left(\frac{B_\theta^2}{2}\right)'+B_\theta^2=\left(\frac{rB_\theta^2}{2}\right)'+\frac{B_\theta^2}{2}$$

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *m* ≠ 0 Modes

Typical profile



$$\frac{r^2}{B_\theta}\left(\frac{B_\theta}{r}\right)' < \frac{1}{2}(m^2 - 4)$$

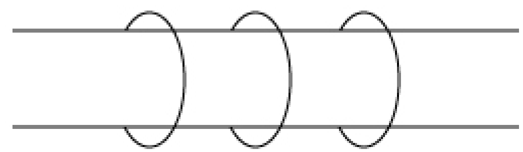
Stability for *m* ≥ 2

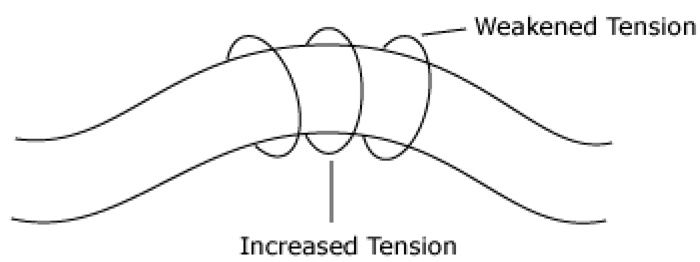
$$\frac{\left(rB_\theta^2\right)'}{B_\theta^2} < m^2 - 1$$

- At large radii where the current is low: stability for *m* = 1
- Near the origin: instability for *m* = 1

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - $m \neq 0$ Modes

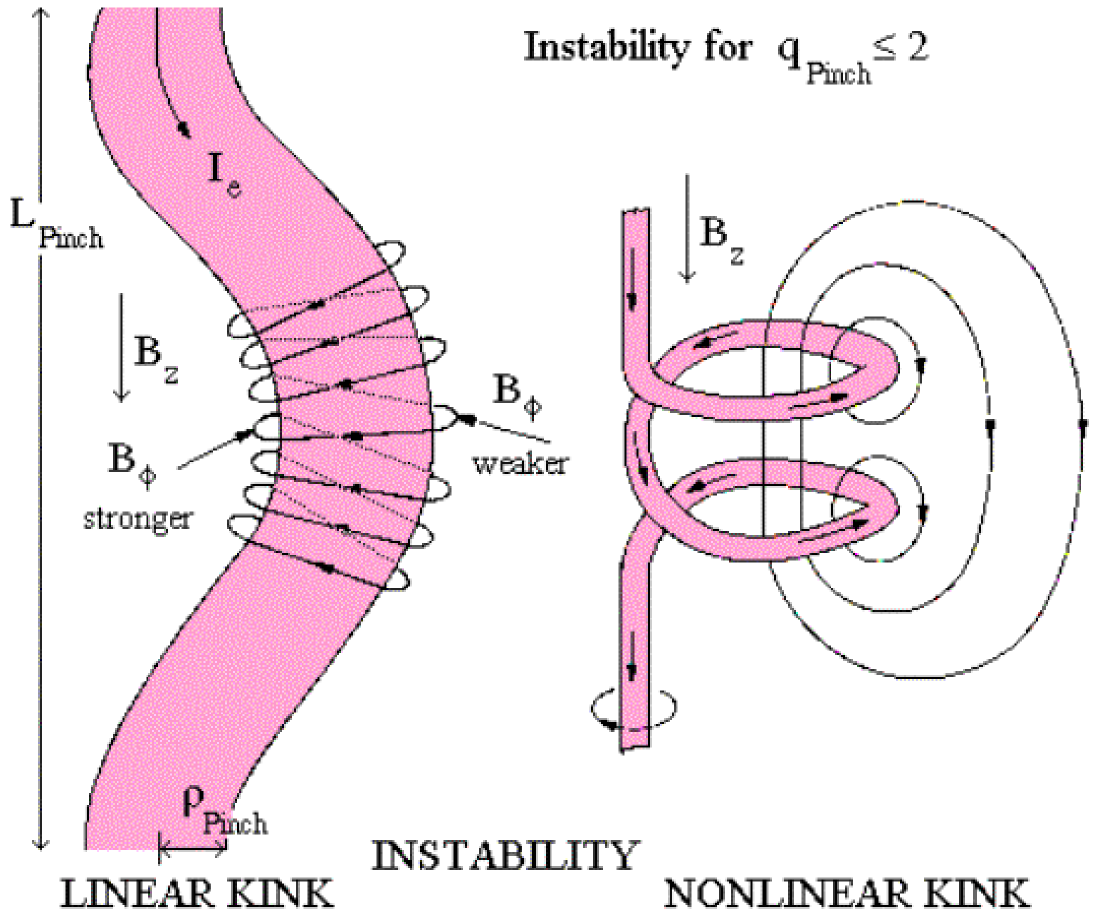




- As the plasma undergoes an $m = 1$ deformation the magnetic lines concentrate in the tighter portion of the column, raising the value of B_θ.
- The corresponding increased magnetic tension produces a force in the direction to further increase the $m = 1$ deformation; hence, instability.
- Although the plasma distortion has the appearance of a helix, it does not correspond to a kink mode since the current is zero.
- The minimising perturbation is best described as a competition between line bending and unfavourable curvature, with the magnetic compression making a negligibly small contribution.

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - $m \neq 0$ Modes



Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *m* = 0 Mode

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

for *m* ≠ 0

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left\{\frac{m^2B_\theta^2}{\mu_0 r^2}\left[\left|\xi_r\right|^2 + \left|\xi_z\right|^2\right] + \frac{B_\theta^2}{\mu_0}\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2\left|\xi_z\right|^2 + ikr\xi_z\left(\frac{\xi_r^*}{r}\right)' - ikr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right] + 0 + \frac{2p'}{r}\left|\xi_r\right|^2 + 0\right\}$$

for *m* = 0

$$\delta W_F = \frac{1}{2\mu_0}\int_P d\vec{r}\left\{0 + B_\theta^2\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2\left|\xi_z\right|^2 + ikr\xi_z\left(\frac{\xi_r^*}{r}\right)' - ikr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right] + \mu_0\gamma p\left[\left|\frac{(r\xi_r)'}{r}\right|^2 + k^2\left|\xi_z\right|^2 + \frac{ik\xi_z}{r}\left(r\xi_r^*\right)' - \frac{ik\xi_z^*}{r}\left(r\xi_r\right)'\right] + \frac{2\mu_0 p'}{r}\left|\xi_r\right|^2 + 0\right\}$$

$$\vec{Q}_\perp = \frac{imB_\theta}{r}(\xi_r\vec{e}_r + \xi_z\vec{e}_z)$$

$$\gamma p\left|\nabla\cdot\xi\right|^2 = \gamma p\left[\left|\frac{(r\xi_r)'}{r}\right|^2 + k^2\left|\xi_z\right|^2 + \frac{ik\xi_z}{r}(r\xi_r^*)' - \frac{ik\xi_z^*}{r}(r\xi_r)'\right]$$ for *m* = 0

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - m = 0 Mode

$$\delta W_F = \frac{1}{2\mu_0}\int_P d\vec{r}\left\{0 + B_\theta^2\left[\left|r\left(\frac{\xi_r}{r}\right)'\right|^2 + k^2|\xi_z|^2 + ikr\xi_z\left(\frac{\xi_r^*}{r}\right)' - ikr\xi_z^*\left(\frac{\xi_r}{r}\right)'\right] + \mu_0\gamma p\left[\left|\frac{(r\xi_r)'}{r}\right|^2 + k^2|\xi_z|^2 + \frac{ik\xi_z}{r}(r\xi_r^*)' - \frac{ik\xi_z^*}{r}(r\xi_r)'\right] + \frac{2\mu_0 p'}{r}|\xi_r|^2 + 0\right\}$$

$\xi_{||} \equiv \xi_\theta$ does not appear.
The term must be maintained for the rest of the minimisation.

Minimise δW_F: ξ_z terms

$$k^2(B_\theta^2 + \mu_0\gamma p)|\xi_z|^2 + \left[B_\theta^2 r\left(\frac{\xi_r^*}{r}\right)' + \mu_0\gamma p\frac{(r\xi_r^*)'}{r}\right](ik\xi_z) + \text{c.c.}$$

$$(B_\theta^2 + \mu_0\gamma p)\left|k\xi_z - \frac{i\left[B_\theta^2 r\left(\frac{\xi_r}{r}\right)' + \mu_0\gamma p\frac{(r\xi_r)'}{r}\right]}{B_\theta^2 + \mu_0\gamma p}\right|^2 - \frac{1}{B_\theta^2 + \mu_0\gamma p}\left|B_\theta^2 r\left(\frac{\xi_r}{r}\right)' + \mu_0\gamma p\frac{(r\xi_r)'}{r}\right|^2$$

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - $m = 0$ Mode

$$\xi_z = \frac{i}{k(B_\theta^2 + \mu_0 \gamma p)} \left[B_\theta^2 r \left(\frac{\xi_r}{r} \right)' + \mu_0 \gamma p \frac{(r\xi_r)'}{r} \right]$$

minimising condition

$$\xi \equiv \xi_r$$

$$\delta W_F = \frac{1}{2\mu_0} \int_P d\vec{r} \left\{ 0 + B_\theta^2 \left[\left| r \left(\frac{\xi_r}{r} \right)' \right|^2 + k^2 |\xi_z|^2 + ikr\xi_z \left(\frac{\xi_r^*}{r} \right)' - ikr\xi_z^* \left(\frac{\xi_r}{r} \right)' \right] + \mu_0 \gamma p \left[\left| \frac{(r\xi_r)'}{r} \right|^2 + k^2 |\xi_z|^2 + \frac{ik\xi_z}{r} (r\xi_r^*)' - \frac{ik\xi_z^*}{r} (r\xi_r)' \right] + \frac{2\mu_0 p'}{r} |\xi_r|^2 + 0 \right\}$$

$$\delta W_F = \frac{1}{2\mu_0} \int_P d\vec{r} \left\{ B_\theta^2 \left[|\xi'|^2 + \frac{|\xi|^2}{r^2} - \frac{(\xi'\xi^* + \xi^{*\prime}\xi)}{r} \right] + \mu_0 \gamma p \left| |\xi'|^2 + \frac{|\xi|^2}{r^2} + \frac{(\xi'\xi^* + \xi^{*\prime}\xi)}{r} \right|^2 + \frac{2\mu_0 p'}{r} |\xi|^2 - \frac{1}{B_\theta^2 + \mu_0 \gamma p} \left| (B_\theta^2 + \mu_0 \gamma p)\xi' - (B_\theta^2 - \mu_0 \gamma p) \frac{\xi}{r} \right|^2 \right\}$$

$$= \frac{1}{2\mu_0} \int_P d\vec{r} \left\{ \frac{|\xi|^2}{r^2} \left[B_\theta^2 + \mu_0 \gamma p - \frac{(B_\theta^2 - \mu_0 \gamma p)^2}{B_\theta^2 + \mu_0 \gamma p} \right] + \frac{(\xi'\xi^* + \xi^{*\prime}\xi)}{r} \left[\mu_0 \gamma p - B_\theta^2 + (B_\theta^2 - \mu_0 \gamma p) \right] + |\xi'|^2 \left[B_\theta^2 + \mu_0 \gamma p - (B_\theta^2 + \mu_0 \gamma p) \right] + \frac{2\mu_0 p'}{r} |\xi|^2 \right\}$$

$$\delta W_F = \frac{1}{2\mu_0} \int_P d\vec{r} \frac{|\xi|^2}{r^2} \left[2\mu_0 r p' + \frac{4\mu_0 \gamma p B_\theta^2}{B_\theta^2 + \mu_0 \gamma p} \right]$$

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *m* = 0 Mode

$$\xi_z = \frac{i}{k(B_\theta^2 + \mu_0 \gamma p)}\left[B_\theta^2 r\left(\frac{\xi_r}{r}\right)' + \mu_0 \gamma p \frac{(r\xi_r)'}{r}\right]$$

$$\longrightarrow \quad -\frac{rp'}{p} < \frac{2\gamma B_\theta^2/\mu_0}{\gamma p + B_\theta^2/\mu_0}$$ stability condition

Benett profile

$$B_\theta = \frac{\mu_0 I_0}{2\pi}\frac{r}{r^2 + r_0^2}$$

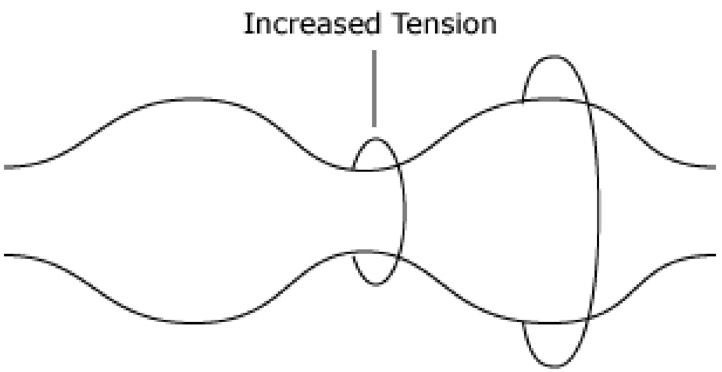
$$p = \frac{\mu_0 I_0^2}{8\pi^2}\frac{r_0^2}{(r^2 + r_0^2)^2}$$

$$\longrightarrow \quad \gamma > 2$$

- Since $\gamma = 5/3$ for ideal MHD the condition is violated.
- Instability criterion usually violated in experiments.

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
    - $m = 0$ Mode: sausage instability

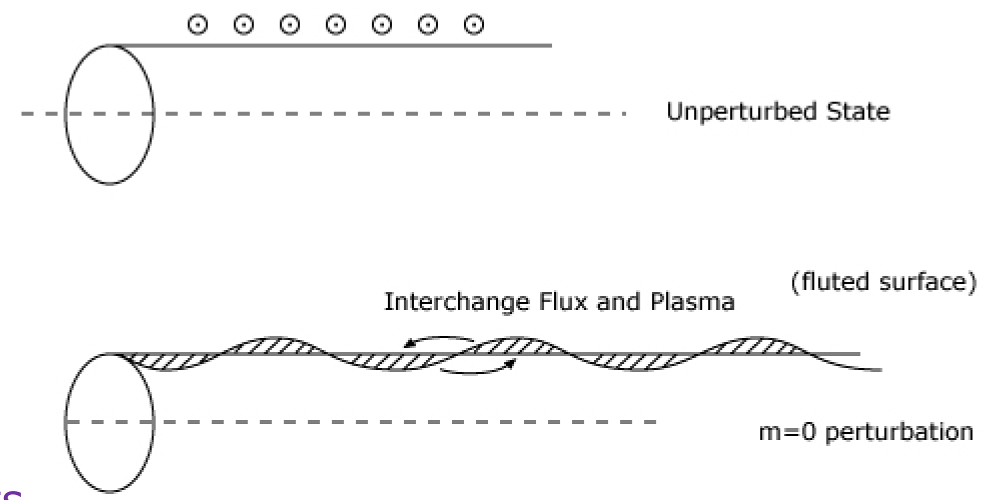


- The magnetic field in the throat region increases since the plasma carries the same current in a smaller cross section.
- The increased magnetic tension produces a force which tends to further constrict the column.
- The minimising perturbation produces a competition between unfavourable curvature and compression of the plasma (magnetic pressure and particle pressure).
- The line bending is zero.

Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - $m = 0$ Mode: sausage instability

Single particle picture



particle drifts

$$\vec{V}_{\nabla B} = \frac{v_\perp^2}{2\omega_c}\frac{\vec{B}\times\nabla B}{B^2} = -\frac{mv_\perp^2}{2e}\frac{1}{B_\theta^3}\cdot B_\theta \frac{2B_\theta}{2r}\vec{e}_z = -\frac{mv_\perp^2}{2e}\frac{B_\theta'}{B_\theta^2}\vec{e}_z$$

$$\vec{V}_\kappa = -\frac{v_\perp^2}{\omega_c}\frac{\vec{\kappa}\times\vec{B}}{B^2} = \frac{mv_{||}^2}{er}\frac{\vec{e}_r\times\vec{e}_z}{B_\theta} = \frac{mv_{||}^2}{erB_\theta}\vec{e}_z$$

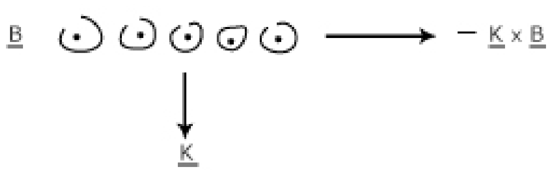
Stability: One-Dimensional Configurations

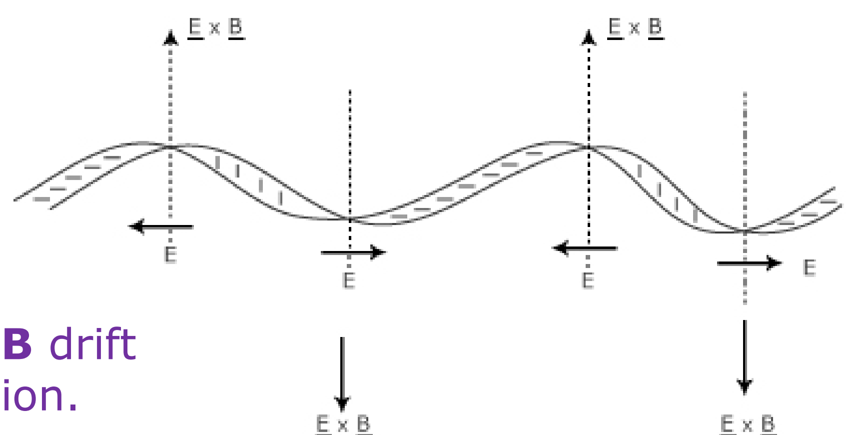
- **The *Z* Pinch**
  - *m* = 0 Mode: sausage instability

Single particle picture

$$v_{||}^2 = \frac{v_\perp^2}{2} = v^2$$ isotropic plasma

$$v_D = \frac{mv^2}{eB_\theta^2}\left(\frac{B_\theta^2}{r} - B_\theta{}'\right) = -\frac{mv^2}{eB_\theta^2} r \left(\frac{B_\theta}{r}\right)' \qquad \left(\frac{B_\theta}{r}\right)' < 0$$





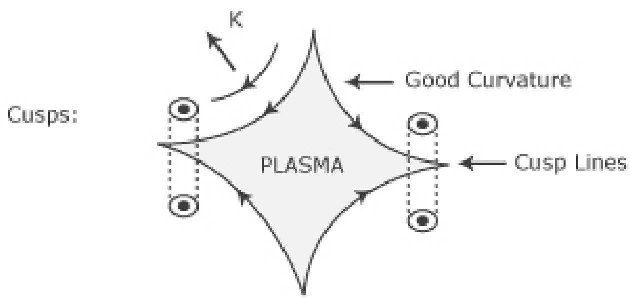
Curvature drift creates **E**x**B** drift which enhances perturbation.

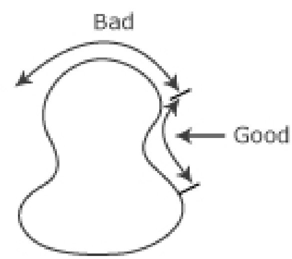
Stability: One-Dimensional Configurations

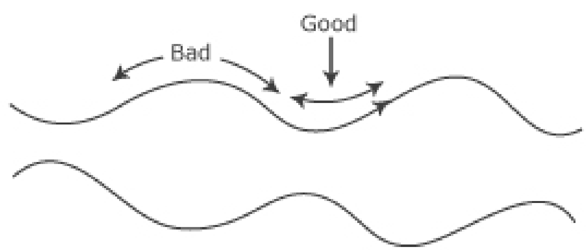
- **The *Z* Pinch**
  - *m* = 0 Mode: sausage instability

Single particle picture

If the curvature drift is in the opposite direction, **E**x**B** drift would oppose the perturbation → stability







Stability: One-Dimensional Configurations

- **The *Z* Pinch**
  - *Z* pinch is always unstable to $m = 1$ perturbations and is likely to be unstable to $m = 0$ as well.
  - The unstable modes are quite virulent and have the form of pressure-driven interchanges.

Stability: One-Dimensional Configurations

- **The General Screw Pinch**

- Sequence of solution of the MHD equilibrium equations

1. The ▽·**B** = 0

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta}+\frac{\partial B_z}{\partial z}=0$$

2. Ampere's law: μ_0**J** = ▽x**B**

$$J_\theta=-\frac{1}{\mu_0}\frac{\partial B_z}{\partial r} \qquad J_z=\frac{1}{\mu_0 r}\frac{d}{dr}(rB_\theta)$$

3. The momentum equation: **J**x**B** = ▽p

$$J_\theta B_z - J_z B_\theta = \frac{dp}{dr}$$

$$\frac{d}{dr}\left(p+\frac{B_\theta^2+B_z^2}{2\mu_0}\right)+\frac{B_\theta^2}{\mu_0 r}=0$$

$$\xi(\vec{r})=\xi(r)\exp[i(m\theta+kz)]$$ Fourier analysed

m, *k*: poloidal and toroidal wave numbers, respectively

Dual symmetry: Responsible for the algebraic elimination of two components of ξ in the minimisation procedure

Stability: One-Dimensional Configurations

- **The General Screw Pinch**

$$\xi = \xi_r \vec{e}_r + \xi_\theta \vec{e}_\theta + \xi_z \vec{e}_z = \xi_\perp + \xi_{||} \vec{b}$$

$$\vec{b} = \frac{B_\theta}{B} \vec{e}_\theta + \frac{B_z}{B} \vec{e}_z$$

$$\vec{e}_r \perp \vec{e}_\eta \perp \vec{b}$$

$$\vec{e}_\eta = \vec{b} \times \vec{e}_r = \frac{B_z}{B} \vec{e}_\theta - \frac{B_\theta}{B} \vec{e}_z$$

$$\xi_r, \quad \xi_\theta, \quad \xi_z \rightarrow \xi, \quad \eta, \quad \xi_{||}$$

$$\xi_{||} = \xi_\theta \frac{B_\theta}{B} + \xi_z \frac{B_z}{B}$$

$$\xi = \xi_\perp + \xi_{||} \vec{b}$$

$$\eta = \xi_\theta \frac{B_z}{B} - \xi_z \frac{B_\theta}{B}$$

$$\xi = \xi_r$$

$$\xi_\perp = \xi \vec{e}_r + \eta \vec{e}_\eta$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δ*W*

Incompressibility

$$\nabla\cdot\xi = \nabla\cdot\xi_\perp + \nabla\cdot\left(\frac{\xi_{||}}{B}\vec{B}\right) = \nabla\cdot\xi_\perp + \vec{B}\cdot\nabla\frac{\xi_{||}}{B} = 0$$

$$\vec{B}\cdot\nabla = \left(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} + B_z\frac{\partial}{\partial z}\right) = \left(\frac{imB_\theta}{r} + ikB_z\right) = iF$$

$$F = \frac{mB_\theta}{r} + kB_z = \vec{k}\cdot\vec{B}, \quad \vec{k} = \frac{m}{r}\vec{e}_\theta + k\vec{e}_z$$

$$\nabla\cdot\xi_\perp + iF\frac{\xi_{||}}{B} = 0$$

$$\xi_{||} = \frac{iB}{F}\nabla\cdot\xi_\perp$$

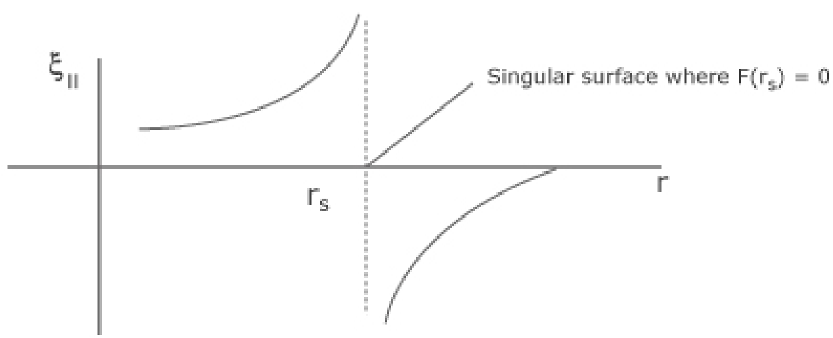
Stability: One-Dimensional Configurations

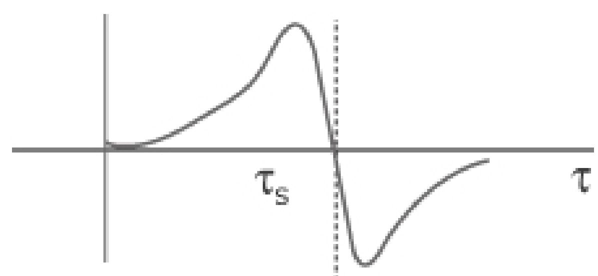
- **The General Screw Pinch**
  - Evaluation of δW

Incompressibility

$$\nabla \cdot \xi_\perp + iF\frac{\xi_{||}}{B} = 0 \qquad \xi_{||} = \frac{iB}{F}\nabla \cdot \xi_\perp$$

- Excluding the very special case of zero shear [i.e., $(B_\theta/rB_z)' = 0$], F will in general be nonzero except perhaps at a finite number of discrete radii.
- If F is nonzero everywhere a well-behaved $\xi_{||}$ can be chosen, making the plasma compressibility term vanish.
- Even when isolated $F = 0$ singular surfaces exist the compressibility term can be made negligibly small with a well-behaved $\xi_{||}$.





Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δ*W*

Incompressibility

Resolution: choose $\xi_{||} = \dfrac{iB}{F^2+\sigma^2}\nabla\cdot\xi_\perp$ now bounded, but compressibility not satisfied.

$$\nabla\cdot\xi = \nabla\cdot\xi_\perp + \frac{iF\xi_{||}}{B} = \nabla\cdot\xi_\perp + \frac{iF}{B}\left(\frac{iBF}{F^2+\sigma^2}\right)\nabla\cdot\xi_\perp = \frac{\sigma^2}{F^2+\sigma^2}\nabla\cdot\xi_\perp$$

$$F = F(r_s) + F'(r_s)(r-r_s) \approx F'(r_s)x, \quad x = r - r_s$$

$$\delta W_{||} = \frac{1}{2}\int_P \gamma p\left|\nabla\cdot\xi\right|^2 d\vec{r} = \frac{1}{2}\int_P \gamma p\left|\nabla\cdot\xi_\perp\right|^2 \frac{\sigma^4}{(F^2+\sigma^2)^2}\, r\,dr\,d\theta\,dz$$

$$= \pi L\left[\gamma pr\left|\nabla\right.\right.$$

Even for isolated singular surfaces, the plasma compressibility term makes no contribution to δ*W*

$$= \pi^2 L\left[\frac{\gamma pr\left|\nabla\cdot\xi_\perp\right|^2}{|F'|}\right]_{r_s}|\sigma|$$

→ 0 for arbitrarily small but nonzero *σ*

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δ*W*

Evaluation of δW_F

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

$$\vec{Q}_\perp = \nabla\times(\xi_\perp\times\vec{B})_\perp = Q_r\vec{e}_r + Q_\eta\vec{e}_\eta$$

$$\vec{\kappa} = \vec{b}\cdot\nabla\vec{b} = -\frac{B_\theta^2}{rB^2}\vec{e}_r$$

$$\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa} = \frac{(r\xi)'}{r} - 2\frac{B_\theta^2}{rB^2}\xi + i\frac{G}{B}\eta$$

$$(\xi_\perp\cdot\nabla p)(\xi_\perp^*\cdot\vec{\kappa}) = -\frac{B_\theta^2}{rB^2}p'\left|\xi\right|^2$$

$$\mu_0\vec{J}_{||} = \mu_0\vec{J}\cdot\vec{b} = \frac{B_z}{rB}(rB_\theta)' - \frac{B_\theta B_z'}{B}$$

$$Q_r = iF\xi$$

$$Q_\eta = iF\eta + \xi\left[\frac{B_z' B_\theta}{B} - \frac{rB_z}{B}\left(\frac{B_\theta}{r}\right)'\right]$$

$$G = \frac{mB_z}{r} - kB_\theta = \vec{e}_r\cdot(\vec{k}\times\vec{B})$$

$$F = \frac{mB_\theta}{r} - kB_z$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δW

Evaluation of δW_F

$$\frac{\delta W_F}{2\pi R_0} = \frac{\pi}{\mu_0}\int_0^a W(r)rdr$$

$$W(r) = F^2|\xi|^2 + \left|iF\eta + \xi\left[\frac{B_z' B_\theta}{B} - \frac{rB_z}{B}\left(\frac{B_\theta}{r}\right)'\right]\right|^2$$ line bending

$$+B^2\left|\frac{(r\xi)'}{r} - \frac{2B_\theta^2}{rB^2}\xi + i\frac{G}{B}\eta\right|^2$$ magnetic compression

$$+\frac{2\mu_0 p' B_\theta^2}{rB^2}|\xi|^2$$ pressure-driven

$$-\mu_0 J_{||}\left\{iF(\xi\eta^* - \xi^*\eta) - |\xi|^2\left[\frac{B_z' B_\theta}{B} - \frac{rB_z}{B}\left(\frac{B_\theta}{r}\right)'\right]\right\}$$ current-driven

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δ*W*

Evaluation of δW_F

Minimise δW_F: η terms

$$W_\eta = k_0^2 B^2 |\eta|^2 + 2\frac{ikBB_\theta}{r}(\eta\xi^* - \eta^*\xi) + \frac{iGB}{r}[\eta(r\xi^*)' - \eta^*(r\xi)']$$

$$= \left| ik_0 B\eta + \frac{2kB_\theta}{rk_0}\xi + \frac{G}{rk_0}(r\xi)' \right|^2 - \left| \frac{2kB_\theta}{rk_0}\xi + \frac{G}{rk_0}(r\xi)' \right|^2 \qquad k_0^2 = k^2 + m^2/r^2$$

$$\eta = \frac{i}{rk_0^2 B}[G(r\xi)' + 2kB_\theta\xi]$$

minimising condition

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δ*W*

Evaluation of δW_F

$$W(r) = A_1\xi'^2 + 2A_2\xi\xi' + A_3\xi^2$$

$$A_1 = \frac{F^2}{k_0^2}$$

$$A_2 = \frac{1}{rk_0^2}\left(k^2B_z^2 - \frac{m^2B_\theta^2}{r^2}\right)$$

$$A_3 = F^2 + \frac{2\mu_0 p' B_\theta^2}{rB^2} + \frac{B^2}{r^2}\left(1 - 2\frac{B_\theta^2}{B^2}\right)^2 - \frac{1}{r^2k_0^2}(G + 2kB_\theta)^2 + \frac{2B_\theta B_z}{rB}\left[\frac{B_\theta B_z'}{B} - \frac{rB_z}{B}\left(\frac{B_\theta}{r}\right)'\right]$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δW

Evaluation of δW_F

$$\frac{\delta W_F}{2\pi R_0/\mu_0} = \int_0^a (f\xi'^2 + g\xi^2)dr + \left[\frac{k^2r^2B_z^2 - m^2B_\theta^2}{k_0^2r^2}\right]_a \xi^2(a)$$

$$f = rA_1 = \frac{rF^2}{k_0^2}$$

$$g = rA_3 - (rA_2)' = \frac{2k^2}{k_0^2}(\mu_0 p)' + \left(\frac{k_0^2r^2 - 1}{k_0^2r^2}\right)rF^2 + \frac{2k^2}{rk_0^4}\left(kB_z - \frac{mB_\theta}{r}\right)F$$

- The boundary term is a consequence of an integration by parts in δW_F.
- This term vanishes for internal modes but plays an important role in external modes.
- Standard form of δW_F for the general screw pinch

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δW

Evaluation of δW_V and δW_S

Assuming no surface current: $\delta W_S = 0$

$$\delta W_V = \frac{1}{2}\int_V d\vec{r}\,\frac{\left|\hat{B}_1\right|^2}{\mu_0} \qquad \nabla\times\hat{B}_1 = \nabla\cdot\hat{B}_1 = 0$$

$$\hat{B}_1 = \nabla\hat{V}_1, \quad \nabla^2\hat{V}_1 = 0$$

Boundary condition

$$\vec{n}\cdot\hat{B}_1\Big|_{r_w} = 0$$

$$\vec{n}\cdot\hat{B}_1\Big|_{r_p} = \vec{n}\cdot\nabla\times(\xi_\perp\times\hat{B})\Big|_{r_p} = \hat{B}_1\cdot\nabla(\vec{n}\cdot\xi_\perp) - (\vec{n}\cdot\xi_\perp)[\vec{n}\cdot(\vec{n}\cdot\nabla)\hat{B}_1]\Big|_{r_p}$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δW

Evaluation of δW_V and δW_S

$$\vec{n}\cdot\hat{B}_1\Big|_b = 0 \;\rightarrow\; \frac{\partial \hat{V}_1}{\partial r}\bigg|_b = 0$$

$$\vec{n}\cdot\hat{B}_1\Big|_{a+\xi} = 0 \;\rightarrow\; \vec{n}\cdot\hat{B}_1\Big|_a = \vec{n}\cdot\nabla\times(\xi_\perp\times\vec{B})\Big|_a \qquad \frac{\partial \hat{V}_1}{\partial r}\bigg|_a = iF\xi(a)$$

$$\hat{B}_{1r}\Big|_a = [\hat{B}\cdot\nabla\xi - \xi\vec{n}\cdot(\vec{n}\cdot\nabla)\hat{B}]_a$$

Solution: $$\hat{V}_1 = A\left[K_r - \left(\frac{K_b'}{I_b'}\right)I_r\right]\exp[i(m\theta + kz)]$$

$K_z = K_m(kz),\;\; I_z = I_m(kz)$ modified Bessel functions

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δ*W*

Evaluation of δW_V and δW_S

$$\left.\frac{\partial \hat{V}_1}{\partial r}\right|_a = iF\xi(a) \qquad A = \frac{iF(a)\xi_a}{K_a}\left[1 - \left(\frac{K_b'}{I_b'}\right)\left(\frac{I_a}{K_a}\right)\right]^{-1}$$

$$\delta W_V = \frac{1}{2}\int_V d\vec{r}\,\frac{\left|\hat{B}_1\right|^2}{\mu_0} = \frac{1}{2\mu_0}\int_V \nabla\hat{V}_1^* \cdot \nabla\hat{V}_1 d\vec{r} = \frac{1}{2\mu_0}\int_V [\nabla\cdot(\hat{V}_1^*\nabla\hat{V}_1) - \hat{V}_1^*\nabla^2\hat{V}]d\vec{r}$$

$$= -\frac{1}{2\mu_0}\int_S dS\hat{V}_1^*\vec{n}\cdot\nabla\hat{V}_1 = -\frac{2\pi^2 R_0 a}{\mu_0}\left[\hat{V}_1^*\frac{\partial \hat{V}_1}{\partial r}\right]_a$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δ*W*

Evaluation of δW_V and δW_S

$$\frac{\delta W_V}{2\pi R_0} = \frac{\pi}{\mu_0}\left[\frac{r^2 F^2 \Lambda}{|m|}\right]_a \xi^2(a)$$

$$\Lambda = -\frac{|m| K_a}{ka K_a'}\left[\frac{1-(K_b' I_a)/(I_b' K_a)}{1-(K_b' I_a')/(I_b' K_a')}\right] \approx \frac{1+(a/b)^{2|m|}}{1-(a/b)^{2|m|}} \quad kb << 1$$

$$\approx \frac{|m|}{ka} \quad (ka,\ \ kb \to \infty)$$

$$\approx 1 \quad (ka \sim 1,\ \ kb \to \infty)$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Evaluation of δW

For internal modes

$$\frac{\delta W}{2\pi^2 R_0/\mu_0} = \int_0^a (f\xi'^2 + g\xi^2)dr$$

For external modes

$$\frac{\delta W}{2\pi^2 R_0/\mu_0} = \int_0^a (f\xi'^2 + g\xi^2)dr + \left[\left(\frac{krB_z - mB_\theta}{k_0^2 r^2}\right) rF + \frac{r^2 \Lambda F^2}{|m|}\right]_a \xi_a^2$$

- $f(r)$ is positive while $g(r)$ can have either sign.
- Both terms are competitive and further simplifications, as occurred for the θ pinch and Z pinch, are not possible for the general screw pinch.

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Suydam's Criterion

- Assuming that $\xi(r)$ is a highly localised function
- necessary but not sufficient condition for stability because a special localised trial function is used.
- Tests against localised interchanges:
  $F = 0$ at some radius $r = r_s \rightarrow f = g = 0$ but p' term < 0 in g (destabilising)
  $\rightarrow$ pressure-driven instability (internal localised interchange mode)
- $F = \mathbf{k} \cdot \mathbf{B} = 0 \rightarrow k_{||} \sim 0$:
  perturbations minimising the bending of the mag. lines.

$$f = \frac{rF^2}{k_0^2} \qquad g = \frac{2k^2}{k_0^2}(\mu_0 p)' + \left(\frac{k_0^2 r^2 - 1}{k_0^2 r^2}\right) rF^2 + \frac{2k^2}{rk_0^4}\left(kB_z - \frac{mB_\theta}{r}\right)F$$

$$F = \frac{mB_\theta}{r} + kB_z = \vec{k} \cdot \vec{B}$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Suydam's Criterion
    - Assuming that $\xi(r)$ is a highly localised function
    - necessary but not sufficient condition for stability because a special localised trial function is used
    - Tests against localised interchanges:
      $F = 0$ at some radius $r = r_s \rightarrow f = g = 0$ but p' term in $g < 0$ (destabilising)
      $\rightarrow$ pressure-driven instability (internal localised interchange mode)
    - $F = \mathbf{k} \cdot \mathbf{B} = 0 \rightarrow k_{\|} \sim 0$: perturbations minimising the bending of the mag. lines.
    - A localised perturbation does not automatically imply instability when $p' < 0$: if the equilibrium magnetic field has shear, then away from the resonant surface, F is no longer zero. Even though this term is small,
      ξ' is large because of localisation.
      $\rightarrow f\xi'^2$ term in δW produces a stabilising contribution.

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - Suydam's Criterion

- Interchange plasma and field: Plasma wants to expand, field lines want to contract.
- Interchange is more difficult with shear. As interchange takes place, field lines are bent from one surface to another.

Derivation

$$F \approx F(r_s) + F'(r_s)x = F'(r_s)x \qquad F(r_s) = 0 \qquad x = r - r_s$$

leading order contributions

$$f \approx \left[\frac{r^2 F'^2}{k_0^2}\right]_{r_s} x^2 = \left[\frac{r^3 F'^2}{k^2 r^2 + m^2}\right]_{r_s} x^2, \quad g \approx \left[\frac{2\mu_0 k^2 p'}{k_0^2}\right]_{r_s} = \left[\frac{2k^2 r^2 p' \mu_0}{k^2 r^2 + m^2}\right]_{r_s}$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**

$$\frac{\delta W_F}{2\pi R_0/\mu_0} = \left[\frac{r^3 F'^2}{k^2 r^2 + m^2}\right]_{r_s} \int_{-\Delta}^{\Delta} dx \left[x^2 \left(\frac{d\xi}{dx}\right)^2 - D_s \xi^2 \right] \leftarrow \frac{\delta W_F}{2\pi R_0/\mu_0} = \int_0^a (f\xi'^2 + g\xi^2) dr$$

Δ ($<<a$): measure of the localisation

$$D_s = -\left[\frac{2\mu_0 k^2 p'}{rF'^2}\right]_{r_s} = -\left[\frac{2\mu_0 p' q^2}{rB_z^2 q'^2}\right]_{r_s}$$

simplified

$$F = kB_z\left(1 + \frac{mB_\theta}{krB_z}\right) = kB_z\left(1 + \frac{m}{kR_0}\frac{1}{q}\right) \leftarrow \begin{array}{l} r = r_s, \quad F(r_s) = 0 \\ q = rB_z / R_0 B_\theta \end{array}$$

$$\frac{kR_0}{m} = \left(\frac{R_0 B_\theta}{rB_z}\right)_{r_s} = \frac{1}{q(r_s)} \qquad k = -\left[\frac{mB_\theta}{rB_z}\right]_{r_s}$$

Stability: One-Dimensional Configurations

- **The General Screw Pinch**

$$F(r) = kB_z(r)\left[1 - \frac{q(r_s)}{q(r)}\right]$$

$$F'(r)\Big|_{r_s} = kB_z'\left[1 - \frac{q(r_s)}{q(r)}\right]_{r_s} + kB_z(r_s)q(r_s)\left[\frac{q'}{q^2}\right]_{r_s} = \left(kB_z\frac{q'}{q}\right)_{r_s}$$

$$D_s = -\left[\frac{2\mu_0 k^2 p'}{rF'^2}\right]_{r_s} = -\left[\frac{2\mu_0 p' q^2}{rB_z^2 q'^2}\right]_{r_s}$$

only a function of equilibrium quantities (no *m*'s and *k*'s)

Stability: One-Dimensional Configurations

- **The General Screw Pinch**

$$\frac{\delta W_F}{2\pi R_0/\mu_0} = \left[\frac{r^3 F'^2}{k^2 r^2 + m^2}\right]_{r_s} \int_{-\Delta}^{\Delta} dx \left[x^2 \left(\frac{d\xi}{dx}\right)^2 - D_s \xi^2\right]$$

$$\delta W \propto \int_{-\Delta}^{\Delta} dx \left[x^2 \xi'^2 - D_s \xi^2\right] = \delta W_n$$ normalised form of δW_F

- $p' > 0 \rightarrow D_s < 0$: stability
- $p' < 0 \rightarrow D_s > 0$: stability?

$$D_s = -\left[\frac{2\mu_0 p' q^2}{r B_z^2 q'^2}\right]_{r_s}$$

Vary $\xi \rightarrow \xi + \delta\xi$ to determine minimising $\xi(r)$

$$\int dr (f\xi'^2 + g\xi^2) \rightarrow (f\xi')' - g\xi = 0$$

$$\int dr (x^2 \xi'^2 - D_s \xi^2) \rightarrow (x^2 \xi')' + D_s \xi = 0$$

$$\frac{d}{dx}\left(x^2 \frac{d\xi}{dx}\right) + D_s \xi = 0$$ Euler-Lagrange equation

Stability: One-Dimensional Configurations

- **The General Screw Pinch**

$$\frac{d}{dx}\left(x^2\frac{d\xi}{dx}\right)+D_s\xi=0$$

$$\xi = x^p$$

$$p(p+1)+D_s=0$$

$$p_{1,2}=-\frac{1}{2}\pm\frac{1}{2}(1-4D_s)^{1/2}$$

$$\xi = c_1x^{p_1}+c_2x^{p_2}$$

$$\int\left(x^2\xi'^2-D_s\xi^2\right)dx = x^2\xi\xi' = px^{2p+1} \longleftarrow D_s=-\frac{(x^2\xi')'}{\xi}$$

- $1-4D_s < 0$

$$\xi = x^{-\frac{1}{2}\pm\frac{1}{2}(1-4D_s)^{1/2}} = x^{-\frac{1}{2}}x^{\pm k_r(-1)^{1/2}} = x^{-\frac{1}{2}}e^{\pm k_r i\ln|x|}$$

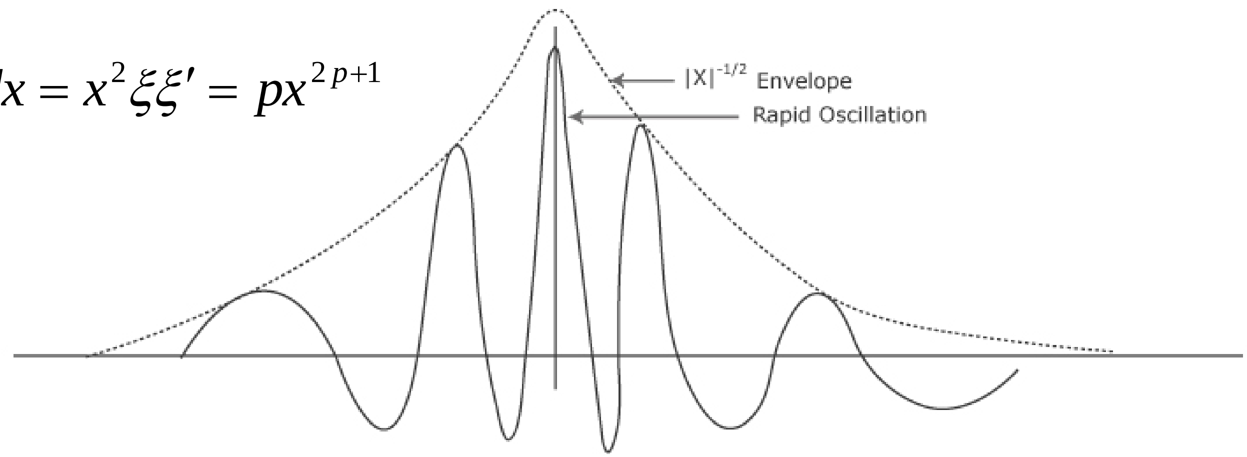
$$=\frac{1}{|x|^{1/2}}[c_1\sin(k_r\ln|x|)+c_2\cos(k_r\ln|x|)]$$

$$k_r=\frac{1}{2}(4D_s-1)^{1/2}$$

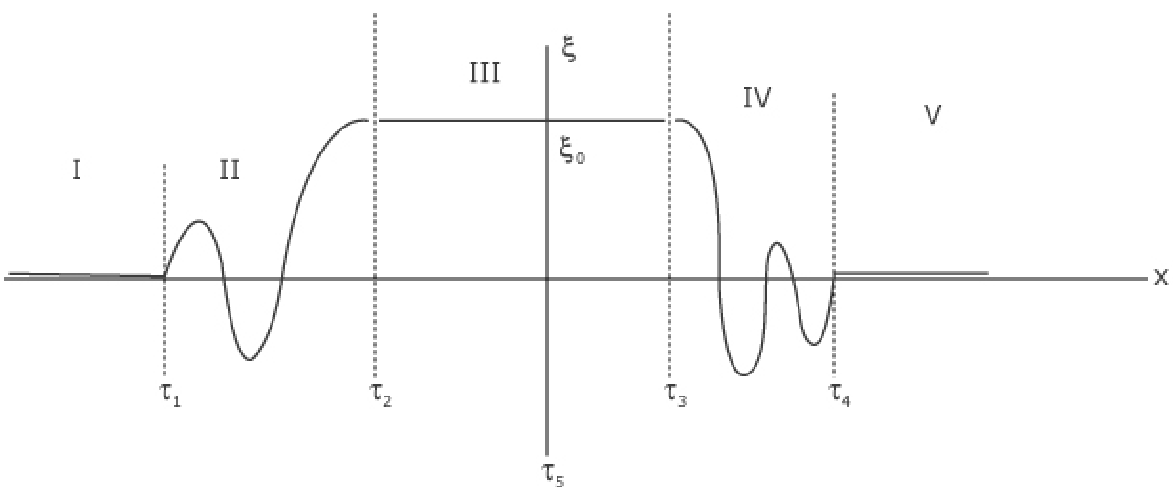
Stability: One-Dimensional Configurations

- **The General Screw Pinch**

$$\int \left(x^2 \xi'^2 - D_s \xi^2 \right) dx = x^2 \xi \xi' = p x^{2p+1}$$

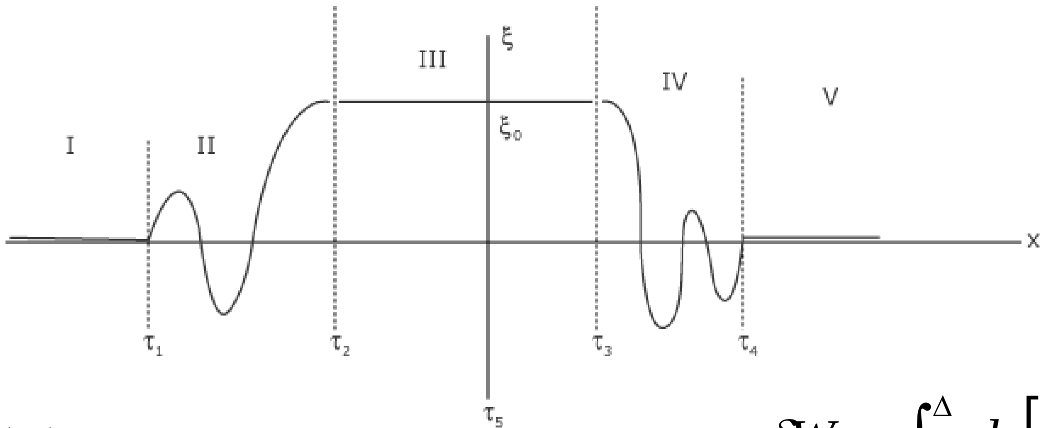


modified trial function



Stability: One-Dimensional Configurations

- **The General Screw Pinch**



$$\delta W_n(I) = \delta W_n(V) = 0$$

$$\delta W_n = \int_{-\Delta}^{\Delta} dx\left[x^2 \xi'^2 - D_s \xi^2\right]$$

$$\delta W_n(II) = \int_{x_1}^{x_2} (x^2 \xi'^2 - D_s \xi^2) dx = x^2 \xi \xi' \Big|_{x_1}^{x_2} = 0$$

$$\delta W_n(IV) = \int_{x_3}^{x_4} (x^2 \xi'^2 - D_s \xi^2) dx = x^2 \xi \xi' \Big|_{x_3}^{x_4} = 0$$

satisfying the Euler-Lagrange equation

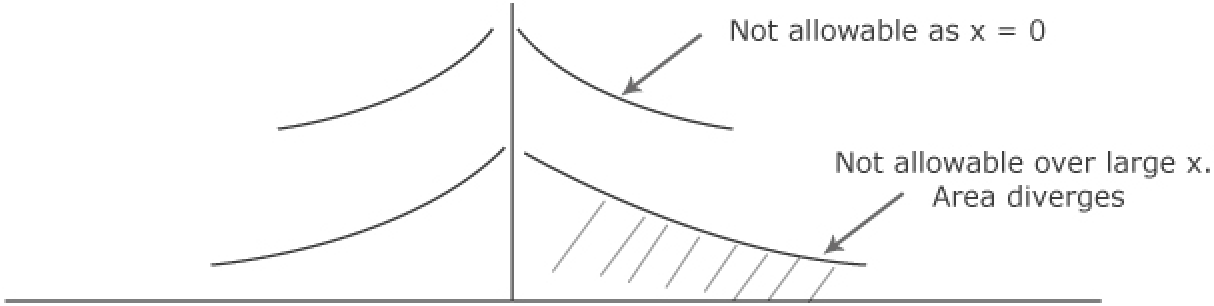
$$\delta W_n(III) = \int_{x_2}^{x_3} (x^2 \xi'^2 - D_s \xi^2) dx = -D_s \xi_0^2 (x_3 - x_2)$$

$1-4D_s < 0$ $\quad \delta W = -D_s \xi_0^2 (x_3 - x_2) < 0$: instability

Stability: One-Dimensional Configurations

- **The General Screw Pinch**
  - 1-4D_s > 0: no oscillatory solutions exist and a localised, well-behaved trial function cannot be constructed.
    → stable to localised interchange perturbations

$$\xi = c_1 x^{p_1} + c_2 x^{p_2} \qquad p_{1,2} = -\frac{1}{2} \pm \frac{1}{2}(1-4D_s)^{1/2}$$



$$D_s = -\left[\frac{2\mu_0 p' q^2}{r B_z^2 q'^2}\right]_{r_s} < \frac{1}{4} \longrightarrow \frac{r B_z^2}{\mu_0}\left(\frac{q'}{q}\right)^2 + 8p' > 0$$

Suydam's criterion

Stability: One-Dimensional Configurations

- **The General Screw Pinch**

$$\frac{rB_z^2}{\mu_0}\left(\frac{q'}{q}\right)^2 + 8p' > 0$$

Suydam's criterion:
necessary condition for stability

destabilising term: interchange drive, resulting from the combination of a negative pressure gradient and the unfavourable curvature of the B_θ field

stabilising term: work don in bending the field lines when interchanging two flux tubes in a system with shear
(shear, line bending magnetic energy)

Stability: One-Dimensional Configurations

- **The General Screw Pinch**