Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

Prof. Dr. Yong-Su Na

(32-206, Tel. 880-7204)

Contents

Contents

Stability: Multidimensional Configurations

- **Introduction**

- Ballooning mode stability is one of important general aspects of multidimensionality.
- It arises because of the existence of alternating regions of favourable and unfavourable field line curvature.
- Ballooning mode perturbations are concentrated in the unfavourable curvature regions and play a crucial role as they set upper limits on the maximum achievable β.
- Toroidicity produces qualitative changes in the stability of internal pressure-driven modes: Mercier criterion
- When toroidicity is included, external kinks develop a strong ballooning mode component to the perturbation, setting limits on the maximum achievable β.

Stability: General Considerations

- **The Intuitive Form of δW_F**

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\underbrace{\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2}_{\text{stabilising}} \overbrace{- 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp}^{\text{destabilising}}\right]$$

Pressure-driven modes (+ or -)

current-driven (kinks) modes (+ or -)

Energy required to bend magnetic field lines: dominant potential energy contribution to the shear Alfvén wave

Energy necessary to compress the magnetic field: major potential energy contribution to the compressional Alfvén wave

Energy required to compress the plasma: main source of potential energy for the sound wave

Stability: Multidimensional Configurations

- **General Reduction of δW for Ballooning Modes**
  - The most dangerous modes, in terms of setting β limits, are usually characterized $k_\perp \to \infty$ perturbation.
  - In order to exploit the $k_\perp \to \infty$ limit, an eikonel representation is used.

$$\xi_\perp = \eta_\perp e^{iS} \qquad \eta_{\perp 0} = Y\vec{b} \times k_\perp$$

$$k_\perp = \nabla S \qquad X \equiv YB$$

$$B \cdot \nabla S = 0$$

$$\delta W_F = \frac{1}{2\mu_0}\int d\vec{r}[\left|\nabla \times (\eta_\perp \times \vec{B})_\perp\right|^2 + B^2\left|i\vec{k}_\perp \cdot \eta_\perp + \nabla \cdot \eta_\perp + 2\vec{\kappa} \cdot \eta_\perp\right|^2$$
$$-2\mu_0(\eta_\perp \cdot \nabla p)(\eta_\perp^* \cdot \vec{\kappa}) - \mu_0 J_{||}(\eta_\perp^* \times \vec{b}) \cdot \nabla \times (\eta_\perp \times \vec{B})_\perp]$$

$$\delta W_2 = \frac{1}{2\mu_0}\int d\vec{r}\left[k_\perp^2\left|\vec{b} \cdot \nabla X\right|^2 - \frac{2\mu_0}{B^2}(\vec{b} \times k_\perp \cdot \nabla p)(\vec{b} \times k_\perp \cdot \vec{\kappa})\left|X\right|^2\right]$$

Competition between the stabilising effects of line bending and the destabilising effects of unfavourable curvature

Stability: Multidimensional Configurations

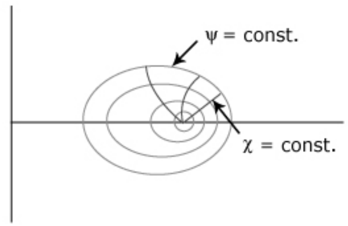
- **Tokamaks**
  - Tokamak Flux Coordinates: Hamada coordinate
  - Poloidal flux ψ as a "radial-like" coordinate
  - χ as a "angle-like" coordinate

$$\vec{n} = \frac{\nabla\psi}{|\nabla\psi|}$$

$$\vec{t} = \frac{B_\phi}{B}\vec{b}_p - \frac{B_p}{B}\vec{e}_\phi$$

$$\vec{b} = \frac{B_p}{B}\vec{b}_p + \frac{B_\phi}{B}\vec{e}_\phi$$

$$\vec{B} = \nabla\psi \times \nabla\chi$$



General form of the ballooning mode energy integral

$$\delta W_2 = \frac{\pi}{\mu_0}\int d\psi W(\psi)$$

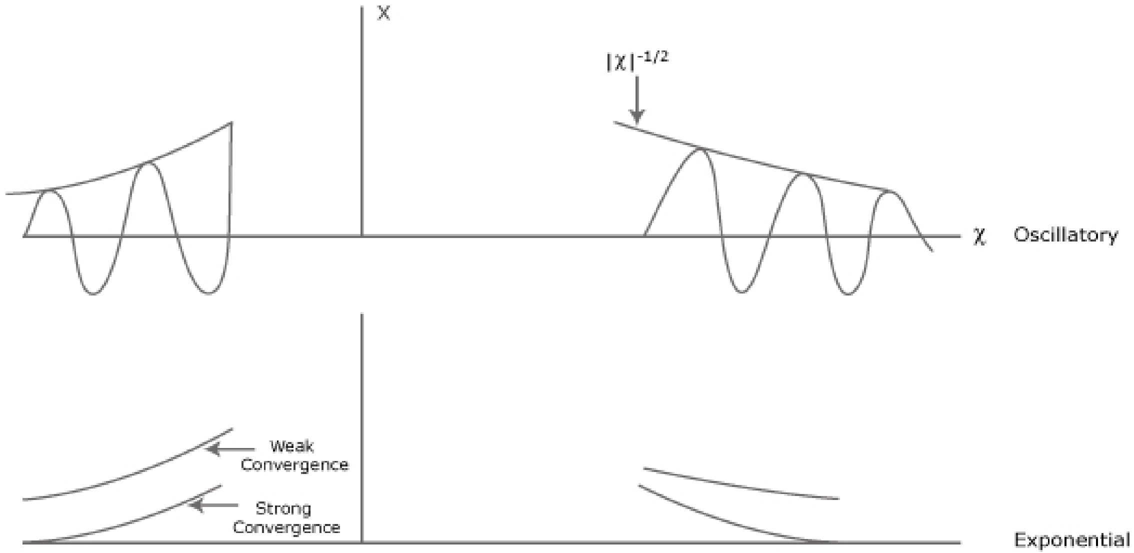
$$W(\psi) = \int_0^{2\pi} J d\chi \left[(k_n^2 + k_t^2)\left(\frac{1}{JB}\frac{\partial X}{\partial \chi}\right)^2 - \frac{2\mu_0 R B_p}{B^2}\frac{dp}{d\psi}(k_t^2\kappa_n - k_t k_n \kappa_t) X^2 \right]$$

$$k_n = n\left[R B_p \int_{\chi_0}^{\chi} \frac{\partial}{\partial\psi}\left(\frac{JB_\phi}{R}\right) d\chi' \right], \quad k_t = \frac{nB}{RB_p} \qquad X(-\infty) = X(\infty) = 0$$

One-dimensional form of the ballooning mode potential energy
- Only χ derivatives appears on X.
- Stability can be tested one surface at a time.

Stability: Multidimensional Configurations

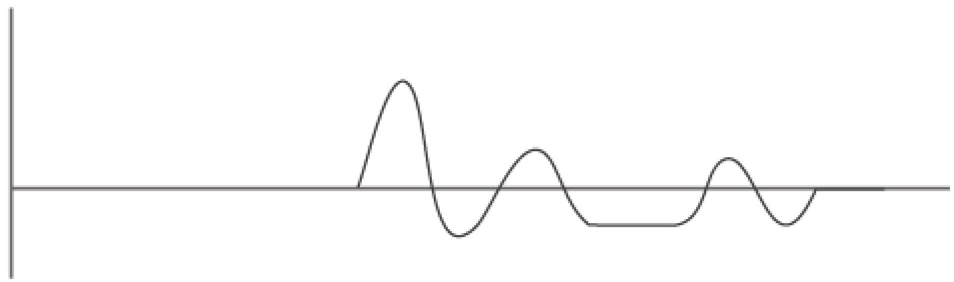
- **Tokamaks**
  - Interchange Stability: The Mercier Criterion
    - In using the quasimode representation we have had to assume the solution X converges sufficiently rapidly as $\chi \rightarrow +\infty, -\infty$.
    - Whether or not convergence is acceptable depends upon equilibrium profiles and parameters.
    - Analysis of Euler-Lagrange equation for X indicates that there are two classes of solutions for large $|\chi|$ depending on profiles.



Stability: Multidimensional Configurations

- **Tokamaks**
  - Interchange Stability: The Mercier Criterion

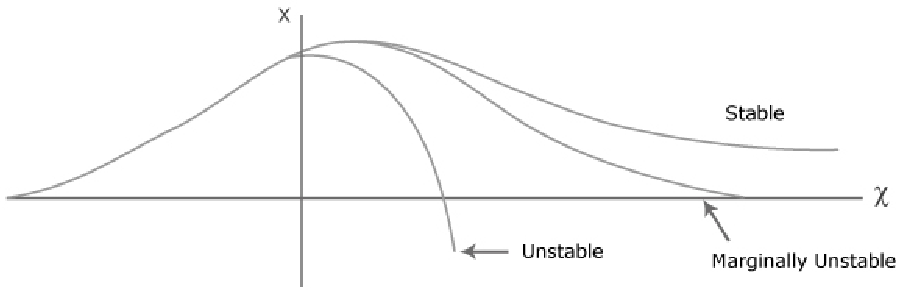
    - Oscillating solutions give rise to unbounded energy: $W \to \infty$. Strong convergence gives rise to bounded energy.
    - Oscillatory case implies that ballooning mode formation is not valid as $\chi \to |\infty|$. However, for this case a trial function of the form



leads to $\delta W < 0$ (instability).

Stability: Multidimensional Configurations

- **Tokamaks**
  - Interchange Stability: The Mercier Criterion
    - For exponential solutions, one starts with the strongly converging solution as $\chi = -\infty$ and integrates to the right.



    - The condition of oscillatory solutions is known as the Mercier criterion. When the Mercier criterion is violated, the solutions oscillate for large χ. The ballooning mode equation is not valid but this does not matter as the system is already unstable to interchanges.
    - When the solution's do not oscillate the Mercier criterion is satisfied and the system is stable to interchanges, and the ballooning mode formalism is solid. In this case one integrates the equation and looks to see if there is a zero-crossing (Newcomb's analysis). If there is one, the system is unstable to ballooning modes.

Stability: Multidimensional Configurations

- **Tokamaks**
  - Interchange Stability: The Mercier Criterion

The minimising Euler-Lagrange equation for the general ballooning mode Energy Principle

$$\frac{\partial}{\partial \chi}\left(f \frac{\partial X}{\partial \chi} \right) - gX = 0$$

$$f = \frac{k_n^2 + k_t^2}{JB^2}, \quad g = -\frac{2\mu_0 JRB_p}{B^2} \frac{dp}{d\psi} (k_t^2 \kappa_n - k_t k_n \kappa_t)$$

$$X = \hat{\chi}^{\alpha} \left[X_0 + \frac{X_1}{\hat{\chi}} + \frac{X_2}{\hat{\chi}^2} + \cdots \right]$$

$$\alpha^2 + \alpha + D_M = 0$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - Interchange Stability: The Mercier Criterion

$$D_s = -\left[\frac{2\mu_0 p' q^2}{r B_z^2 q'^2}\right]_{r_s} < \frac{1}{4}$$

Suydam's criterion

$$D_M < \frac{1}{4}$$

$$D_M = \frac{\mu_0 p'}{q'^2}\left[2\left\langle\frac{RB_p\kappa_n}{B^2}\right\rangle + \left\langle\frac{\Lambda}{B^4}\right\rangle - \left\langle\frac{1}{B^2}\right\rangle\left\langle\frac{\Lambda}{B^2}\right\rangle\right] / \left\langle\frac{R^2 B_p^2}{JB^2}\right\rangle$$

$$\Lambda = F\left(\mu_0 p' F - \frac{R^2 B_p^2}{J}\frac{\partial \hat{q}}{\partial \psi}\right)$$

$$\langle Q\rangle = \int_0^{2\pi}\left(\frac{QB^2}{R^2 B_p^2}\right) J d\chi \Big/ \int_0^{2\pi}\left(\frac{B^2}{R^2 B_p^2}\right) J d\chi$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - Interchange Stability: The Mercier Criterion

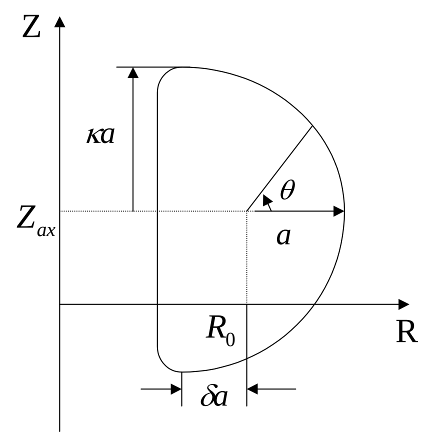
$$\left(\frac{rq'}{q}\right)^2+\frac{8\mu_0 rp'}{B_\phi^2}>0$$

Suydam's criterion

For a circular cross section, large aspect ratio with $\beta_p \sim 1$

$$\left(\frac{rq'}{q}\right)^2+4r\beta'(1-q^2)>0$$

Mercier criterion



$$1<q_0^2\left\{1-\frac{4}{1+3\kappa^2}\left[\frac{3}{4}\frac{\kappa^2-1}{\kappa^2+1}\left(\kappa^2-\frac{2\delta}{\varepsilon}\right)+\frac{(\kappa-1)^2\beta_{p0}}{\kappa(\kappa+1)}\right]\right\}$$