Topics in Fusion and Plasma Studies 2

(459.667, 3 Credits)

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Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

    Analytic Model: large aspect ratio, circular cross section
    small average β but high β' in a small region

$$\frac{\partial}{\partial\chi}\left(f\frac{\partial X}{\partial\chi}\right)-gX=0$$

$$f=\frac{k_n^2+k_t^2}{JB^2},\quad g=-\frac{2\mu_0 JRB_p}{B^2}\frac{dp}{d\psi}(k_t^2\kappa_n-k_t k_n\kappa_t)$$

$$\hat{\chi}\equiv\int_{\chi_0}^{\chi}\frac{\partial\hat{q}}{\partial\psi}d\chi'$$

$$\hat{q}(\psi,\chi)=JB_\phi/R$$

$$\frac{\partial}{\partial\chi}\left[\left(a_0\hat{\chi}^2+a_1\right)\frac{\partial X}{\partial\chi}\right]+\left(\frac{\partial c_0}{\partial\chi}\hat{\chi}+c_1\right)X=0$$

$$q(\psi)=\frac{1}{2\pi}\int_0^{2\pi}\hat{q}d\chi$$

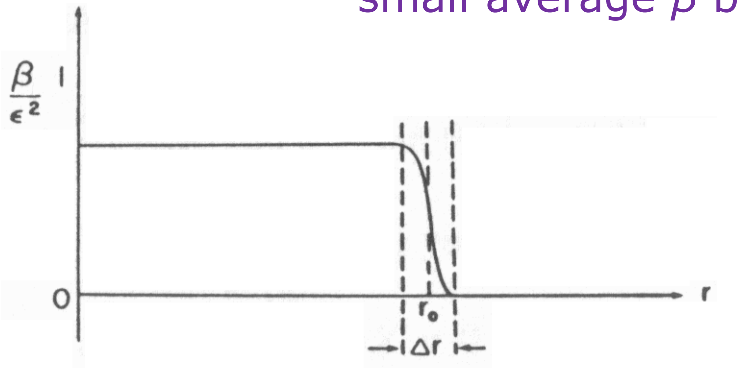
$$a_0=\frac{R^2B_p^2}{JB^2},\quad a_1=\frac{1}{JR^2B_p^2},\quad c_0=\frac{\mu_0 p'F}{B^2},\quad c_1=\frac{2\mu_0 Jp'\kappa_n}{RB_p}$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

Analytic Model: large aspect ratio, circular cross section

small average β but high β' in a small region



$$\beta \sim \varepsilon^2$$

$$\Delta r \sim \varepsilon$$

$$r_0 \beta' \sim \varepsilon$$

- Leading order in the inverse aspect ratio expansion
- Cylindrical values used

$$\chi \to \theta$$

$$R \to R_0$$

$$B_p \to B_\theta(r)$$

$$J \to r / B_\theta$$

$$B \to B_0$$

$$\frac{dp}{d\psi} \to \frac{1}{rB_\theta}\frac{dp}{dr} = \frac{1}{rB_\theta} p'$$

$$\kappa_n \to -\frac{\cos\theta}{R_0}$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

Analytic Model: large aspect ratio, circular cross section

small average β but high β′ in a small region

$$a_0 = \frac{R^2 B_p^2}{JB^2}, \quad a_1 = \frac{1}{JR^2 B_p^2}, \quad c_0 = \frac{\mu_0 p' F}{B^2}, \quad c_1 = \frac{2\mu_0 J p' \kappa_n}{RB_p}$$

$$a_0 = \frac{R_0^2 B_\theta^3}{B_0^2 r}$$

$$a_1 = \frac{1}{r B_\theta R_0^2}$$

$$c_0 \approx \left(\frac{\mu_0}{F} \frac{dp}{d\psi} \right) R^2 \approx \left(\frac{\mu_0}{F} \frac{dp}{d\psi} \right) \left(R_0^2 + 2R_0 r \cos\theta \right)$$

$$c_1 = -\left(\frac{2\mu_0 r p'}{R_0^2 B_\theta^3} \right) \cos\theta$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

    Analytic Model: large aspect ratio, circular cross section

    small average β but high β' in a small region

$$B \approx B_\phi \approx F / R$$

$$\frac{\partial \hat{q}}{\partial \psi} \propto \frac{\partial \hat{q}}{\partial r} = q' + (K_1\psi + K_2\psi_1' + K_3\psi_1'')\cos\theta$$

$$\frac{\partial c_0}{\partial \chi} \approx -\left(\frac{2\mu_0 r p'}{R_0 B_0 B_\theta}\right)\sin\theta$$

$$\hat{\chi} \equiv \int_{\chi_0}^{\chi} \frac{\partial \hat{q}}{\partial \psi} d\chi'$$

$$\hat{\chi} \approx \frac{1}{R_0 B_\theta}\int_{\theta_0}^{\theta} d\hat{\theta}\left(q' - \frac{q\psi_1''}{R_0 B_\theta}\cos\hat{\theta}\right)$$

$$\hat{q}(\psi,\chi) = JB_\phi / R$$

$$\psi_1'' \approx -\frac{2\mu_0 r p'}{B_\theta}$$

$$q(\psi) = \frac{1}{2\pi}\int_0^{2\pi} \hat{q} d\chi$$

$$\hat{\chi} = \frac{1}{R_0 B_\theta}\left[q'(\theta - \theta_0) + \frac{2\mu_0 r q p'}{R_0 B_\theta^2}(\sin\theta - \sin\theta_0)\right] = \frac{q}{r R_0 B_\theta}\Lambda$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

    Analytic Model: large aspect ratio, circular cross section

    small average β but high β' in a small region

$$\frac{\partial}{\partial\chi}\left[\left(a_0\hat{\chi}^2+a_1\right)\frac{\partial X}{\partial\chi}\right]+\left(\frac{\partial c_0}{\partial\chi}\hat{\chi}+c_1\right)X=0$$

$$\frac{\partial}{\partial\theta}\left[(1+\Lambda^2)\frac{\partial X}{\partial\theta}\right]+\alpha(\Lambda\sin\theta+\cos\theta)X=0$$

desired form of the ballooning mode equation for the model equilibrium (s, a)

$$\Lambda(\theta)=s(\theta-\theta_0)-\alpha(\sin\theta-\sin\theta_0)$$

$$s=\frac{rq'}{q}$$ average shear

$$\alpha=-\frac{2\mu_0 r^2 p'}{R_0 B_\theta^2}=-\frac{r^2B_0^2}{R_0^2B_\theta^2}\cdot R_0\cdot\frac{p'}{B_0^2/2\mu_0}=-q^2R_0\beta'$$

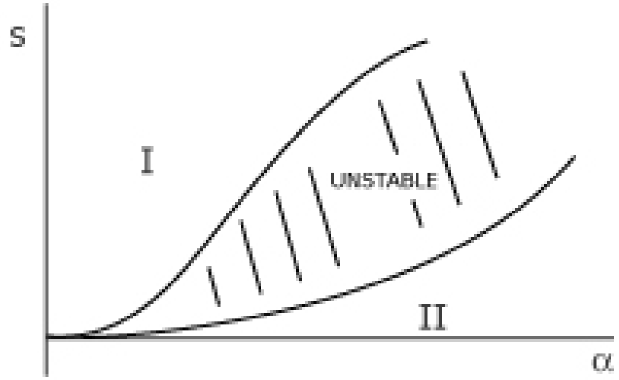
measure of the pressure gradient

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

    Numerical Solution

(s, a) diagram

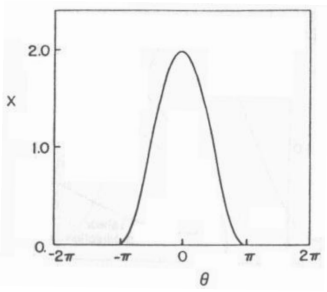


- As sufficiently high pressure gradient, the destabilising contribution from the unfavourable curvature region overcomes the shear → unstable
- When the shear increases, the maximum allowable pressure gradient increases.
- Second region of stability: sufficiently large values of the pressure gradient are stabilised even at low values of shear → possibility of high β operation

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

    Analytic Solution: Energy Principle



$$\frac{\partial}{\partial\theta}\left[(1+\Lambda^{2})\frac{\partial X}{\partial\theta}\right]+\alpha(\Lambda\sin\theta+\cos\theta)X=0$$

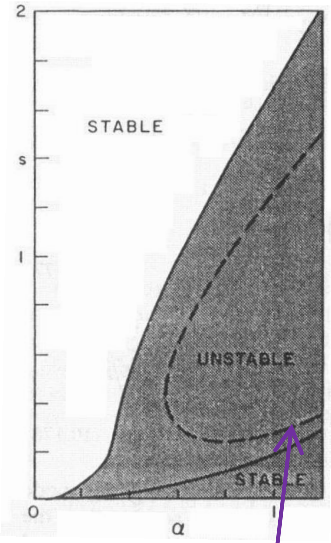
$$W=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\theta\left[(1+\Lambda^{2})\left(\frac{\partial X}{\partial\theta}\right)^{2}-\alpha(\Lambda\sin\theta+\cos\theta)X^{2}\right]$$

simple trial function

$$X=\begin{cases}1+\cos\theta, & -\pi<\cos\theta<\pi\\ 0, & |\theta|>\pi\end{cases}$$

θ = 0: largest X - unfavourable toroidal curvature

θ = π: X = 0 favourable curvature



$$W=1.39s^{2}-2.17s\alpha+\alpha^{2}-\alpha+0.5$$

θ_0 = 0

$$s=0.78\alpha\pm(0.72\alpha-0.36-0.11\alpha^{2})^{1/2}$$

W = 0 for marginal stability

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

Analytic Solution: Energy Principle

Why does the second region of stability exist?

$$\frac{r}{q}\frac{\partial \hat{q}}{\partial r} = s - \alpha\cos\theta$$

local shear

pressure-driven modulation

average shear

If the modulation neglected, the local shear = the average shear

Variational analysis repeated

$$\alpha = \frac{0.5+1.39s^2}{1+0.83s} \longleftarrow \Lambda \approx s(\theta-\theta_0),\ \ \theta_0 = 0$$

Without the pressure-driven modulation, no second region of stability

Stability: Multidimensional Configurations

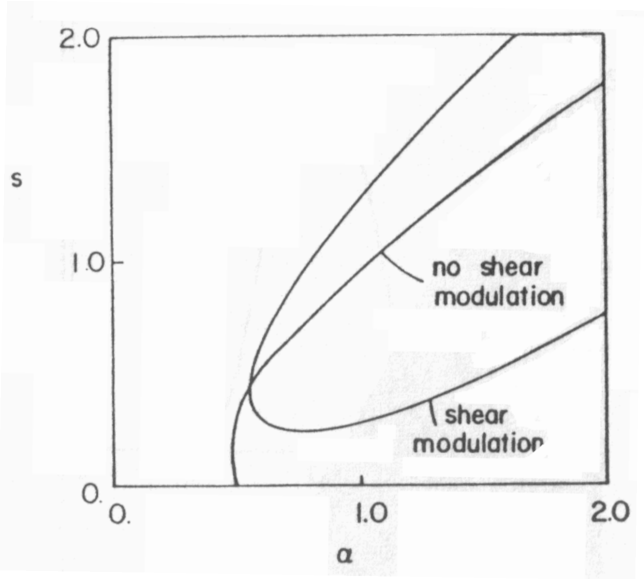
- **Tokamaks**
  - Ballooning Modes

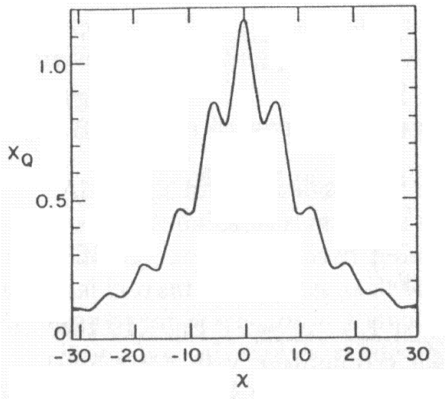
    Analytic Solution: Energy Principle

    - Full variational solution

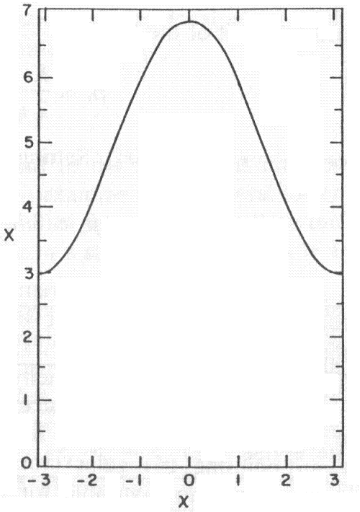
$$s = 0.78\alpha \pm (0.72\alpha - 0.36 - 0.11\alpha^2)^{1/2}$$

    - Complete ballooning mode perturbation

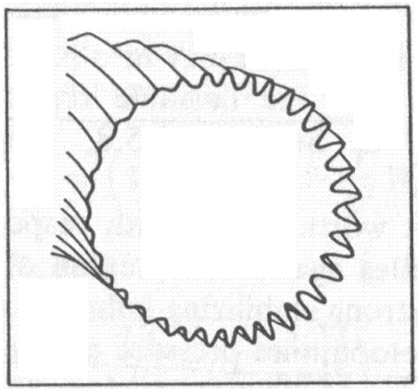




quasimode



eigenfunction

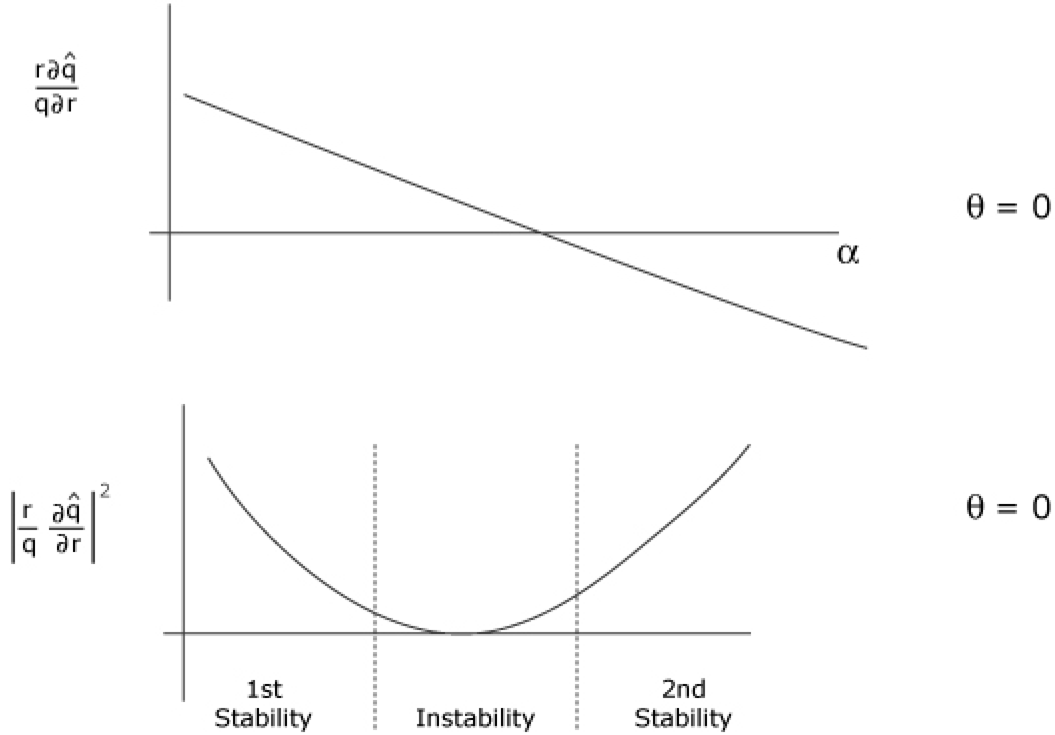


ballooning mode displacement (perturbed plasma surface)

Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

    Analytic Solution: Energy Principle



Stability: Multidimensional Configurations

- **Tokamaks**
  - Ballooning Modes

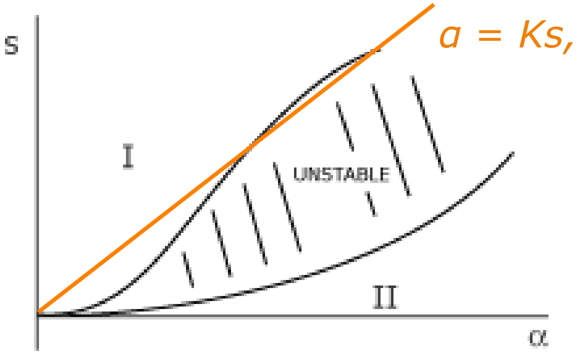
Application of the Ballooning Mode Stability Criterion: 1st stability region

If one integrates the critical pressure gradient over the entire pressure profile, there is a maximum allowable overall average β.

For stability, $\alpha \le Ks$

$$-R_0\beta' \le K(rq'/q^3)$$

Multiply by r^2 and integrates over the plasma volume



$\alpha = Ks$, $K \approx 0.6$

$q = q_0(1 + r^3/r_0^3)$ relatively flat profile

$$\beta_t \le \frac{K}{2}\frac{\varepsilon}{q_0 q_a}\left(1 - \frac{q_0}{q_a}\right) \longrightarrow \beta_t \le 0.3\frac{\varepsilon}{q_a}\left(1 - \frac{1}{q_a}\right)$$

$$q = q_0(1 + a^3/r_0^3)$$
$$K = 0.6$$
$$q_0 = 1$$

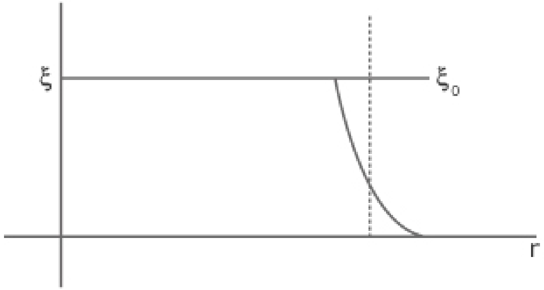
$\beta_t \le 2\%$ for $\varepsilon = 1/3$, $q_a = 3$: somewhat optimistic

Stability: One-Dimensional Configurations

- **The "Straight" Tokamak**
  - Internal Current-Driven Modes

$$\frac{\delta W_F}{2\pi R_0 / \mu_0} = \int_0^a (f\xi'^2 + g\xi^2)dr$$

$$\frac{\delta W_4}{W_0} = \frac{\xi_0^2}{a^2}\int_0^{r_s}\left[r\beta' + \frac{r^2}{R_0^2}\left(1-\frac{1}{q}\right)\left(3+\frac{1}{q}\right)\right]rdr$$



Use same trial function as before

- Contributions from both the pressure and parallel current are destabilising → unstable
- Often referred to as the *m* = 1 internal kink mode. But the mode can also be driven by the pressure gradient, particularly in the high-*β* tokamak regime.
- The nonlinear evolution of the *m* = 1 internal kink mode, including the effects of dissipation is believed to be an important component in sawtooth oscillations observed in many tokamaks.
- If the energetic particle effect is included, a modified *m* = 1 mode thought to be responsible for the so-called fishbone oscillations.

Stability: Multidimensional Configurations

- **Tokamaks**
  - Low-*n* Internal Modes
  - setting an upper limit on the toroidal current density on axis
  - large aspect ratio, circular cross section, $\beta_t \sim \varepsilon^2$, $\beta_p \sim 1$

$$W_c = \int_0^{r_s} \left[r\beta' + \frac{n^2 r^2}{R_0^2}\left(1 - \frac{1}{nq}\right)\left(3 + \frac{1}{nq}\right)\right] r\,dr$$

$$\frac{\delta W_F}{W_0} = \left(1 - \frac{1}{n^2}\right) W_c + \frac{1}{n^2} W_t$$

$3n^2r^4$ (13)

W

, $q_0 < 1$

$nq(r_s) - 1$

- Numerical results: sensitive to *q* profile near the axis and $q_0 > 1$ required for stability

$$\beta_p = -\frac{R_0^2}{n^2 r_s^2}\int_0^{r_s} r^2 \beta' dr$$

- $n >> 1$: $\delta W_F / W_0 \approx W_c$, identical to that in the straight tokamak, requiring $nq_0 > 1$
- $n = 1$: $\delta W_F / W_0 \approx W_t$, stable in the limit if $\beta_p \to 0$ if $q_0 < 1$
- In both cases, increasing β_p is destabilising and instability for $\beta_p > (13/144)^{1/2} \approx 0.3$

Stability: Multidimensional Configurations

- **Tokamaks**
  - External Modes

  - More severe β_t limit than $n \to \infty$ internal ballooning modes
  - In the regime of the ohmically heated tokamak, $\beta \sim \varepsilon^2$, ballooning effects are unimportant on external modes: identical to the straight tokamak
  - The new stability limit appears in the high regime $\beta \sim \varepsilon$ and is associated with toroidal ballooning effects.
  - Combination of ballooning and kinking - the most unstable modes: driven by a combination of both the pressure gradient and the parallel current, in contrast to $n \to \infty$ internal ballooning modes driven solely by the pressure gradient.
  - $q_a \neq q_*$ in high β tokamaks in contrast to low β circular system

$$q_* \equiv \frac{aB_0}{R_0\overline{B}_p} = \frac{2\pi a^2 \kappa B_0}{\mu_0 R_0 I_0} = \frac{2AB_0}{\mu_0 R_0 I_0}, \quad q_a = \frac{q_*}{(1-\nu^2)^{1/2}}\left(\frac{1+\kappa^2}{2\kappa}\right)$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - External Modes

Sharp Boundary Model: surface current model

- Within the plasma **J** = 0, *p* = const.

- Circular cross section

$$b_\theta(\theta) \equiv \frac{B_\theta(a,\theta)}{\varepsilon B_0} = \frac{1}{q_*}\left(\frac{\pi}{2E}\right)[1-k^2\sin^2(\theta/2)]^{1/2}$$

$$\frac{\beta_t q_*^2}{\varepsilon} = \left(\frac{\pi k}{4E}\right)^2$$

E: complete elliptic integral of the second kind

If *k* → 1, equilibrium limit

$$\frac{\beta_t}{\varepsilon} \le \frac{\pi^2}{16 q_*^2}$$

$$\frac{1}{q_*} \to \frac{4}{\pi}\left(\frac{\beta_t}{\varepsilon}\right)^{1/2}, \quad \varepsilon\beta_p \to \frac{\pi^2}{16}$$

$$\varepsilon\beta_p \le \frac{\pi^2}{16}$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - External Modes

    Sharp Boundary Model: surface current model

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

$$\frac{\delta W_S}{W_0} = -\frac{1}{2\pi}\int_0^{2\pi} d\theta \left|\xi\right|^2 [b_\theta^2 + (\beta_t / \varepsilon)\cos\theta]$$ for high β

$$W_0 = 2\pi^2 \varepsilon^2 B_0^2 R_0 / \mu_0, \quad b_\theta = B_\theta / \varepsilon B_0$$

toroidal field curvature

high β ballooning effect (pressure-driven term)

destabilising effect due to the parallel current (kink term)

- For low β, the ballooning contribution negligible

Stability: Multidimensional Configurations

- **Tokamaks**
  - External Modes

    Sharp Boundary Model: surface current model

    Stability analysis $\xi(\theta) = \exp(-in\phi)\sum_m \xi_m \exp(im\theta)$

$$\frac{\delta W}{n^2 W_0} = \xi \cdot W \cdot \xi$$

$$W = W^{(1)} + \frac{W^{(2)}}{nq_*} + \frac{W^{(3)}}{n^2 q_*^2}$$

$$W_{mp}^{(1)} = \frac{2\delta_{m-p}}{|m|}, \quad W_{mp}^{(2)} = -\frac{\pi}{2E}\left(\frac{m}{|m|} + \frac{p}{|p|}\right) G_{mp}$$

$$W_{mp}^{(3)} = \frac{\pi^2}{4E^2}\left[-\left(1 - \frac{k^2}{2}\right)\delta_{m-p} - \frac{3k^2}{8}(\delta_{m-p-1} + \delta_{p-m-1}) + \sum_l |l| G_{lm} G_{lp}\right]$$

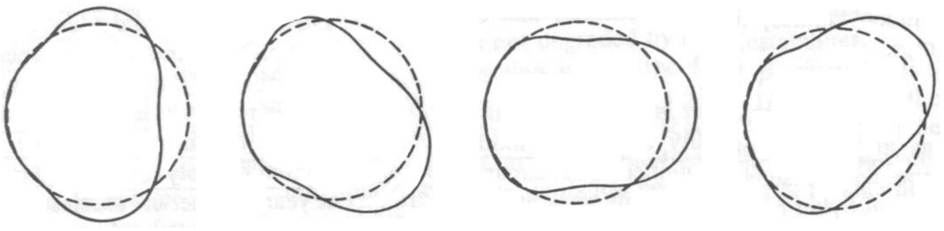
$$G_{lm} = \frac{1}{\pi}\int_0^\pi d\theta [1 - k^2 \sin^2(\theta/2)]^{1/2} \cos(l-m)\theta$$

Stability: Multidimensional Configurations

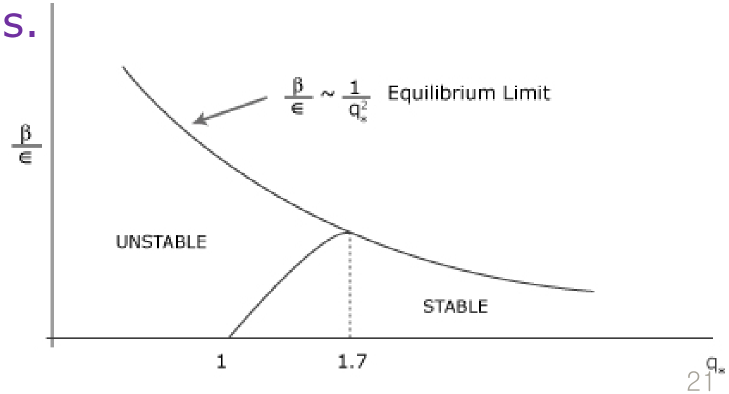
- **Tokamaks**
  - External Modes

    Sharp Boundary Model: surface current model

    - Ballooning instabilities do not in general set limits on β_t or I_0, but only on the ratio $\beta_t q_*^2/\varepsilon$ (plus shape factors). In contrast, external kinks set individual limits on both β_t/ε and q_*.
    - The system is unstable along the equilibrium boundary for $q_* < 1.7$: the kink mode is unstable even though $q_a = \infty$ → q_* rather than q_a is the critical parameter.
    - External ballooning-kink modes require both a current limit $q_* > 1$ and a pressure limit $\beta_t/\varepsilon < 0.21$ for stability: most dangerous ideal MHD instabilities.



structure of pressure-driven kinks



Stability: Multidimensional Configurations

- **Tokamaks**
  - Numerical Results: The Sykes Limit, the Troyon Limit

Once an equilibrium is established, the following stability tests are made.

(1) Mercier stability
(2) High-*n* ballooning modes
(3) Low-*n* internal modes
(4) External ballooning-kink modes

- Helpful in the design of new experiments and in the interpretation and analysis of existing experimental data
- Playing a role in the determination of optimised configurations
- Quantitative predictions for the maximum β_t or I_0 and that can be stably maintained in MHD equilibrium

Stability: Multidimensional Configurations

- **Tokamaks**
  - Numerical Results: The Sykes Limit, the Troyon Limit

    Ballooning Mode Studies: first region of stability

Sykes limit

$$\beta_t = 0.044\left(\frac{I_0}{aB_0}\right)$$

$$\beta_t = 0.22\left(\frac{\varepsilon\kappa}{q_*}\right)$$

$$\varepsilon = a / R_0, \quad q_* = 2B_0 A / \mu_0 R_0 I_0$$

- The absolute maximum value of β_t depends upon how high κ and how low q_* can be made.
- One limit is due to Mercier stability, and strong triangularity is required to delay the onset of these modes.
- The other limit is due to external kinks, although not included in the ballooning mode studies.

Stability: Multidimensional Configurations

- **Tokamaks**
  - Numerical Results: The Sykes Limit, the Troyon Limit

Full Stability Studies: against the Mercier criterion, ballooning modes,
the n = 1 internal kink, and low-n external
ballooning-kink modes

Troyon limit

$$\beta_t = 0.028\left(\frac{I_0}{aB_0}\right)$$

$$\beta_t = 0.14\left(\frac{\varepsilon\kappa}{q_*}\right)$$

- Optimised profile: q_0 slightly above 1, flat q profile with rapid rise near the plasma surface, broad pressure profile

- n = 1 external ballooning mode sets the most severe β_t limit.

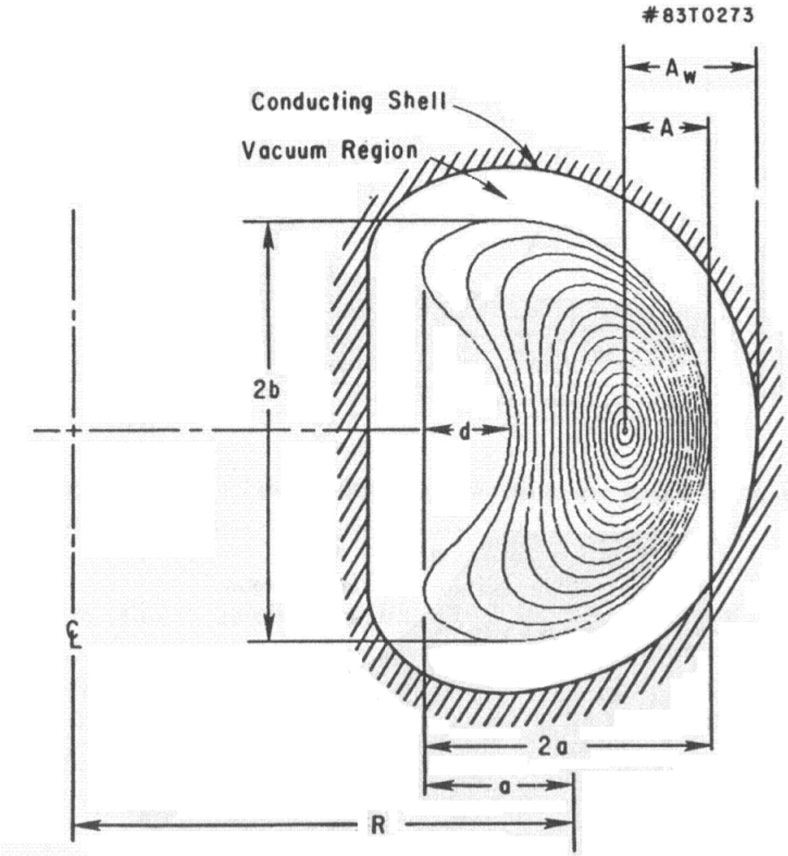
- The value of q_0 must be slightly greater than 1 to satisfy the Mercier criterion and the n = 1 internal kink condition.

- The maximum value of β_t occurs for the lowest allowable value of q_* which is, in general, a function of κ as set by external kinks.

Stability: Multidimensional Configurations

- **Tokamaks**
  - Numerical Results: The Sykes Limit, the Troyon Limit

    Second Region of Stability Studies: bean-shaped cross section

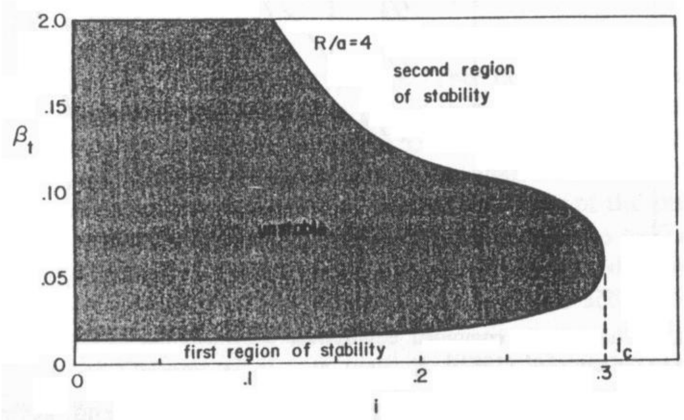


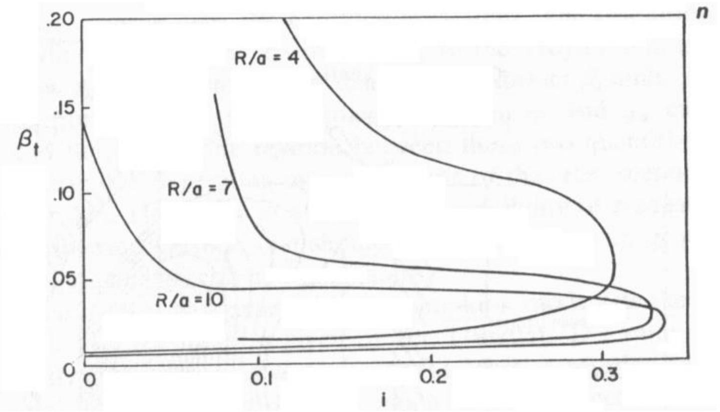
- Indentation $i \equiv d/2a$ created by adding a pusher coil in the region of indentation
- Difficult to achieve and maintain $i \sim 0.3$ technologically

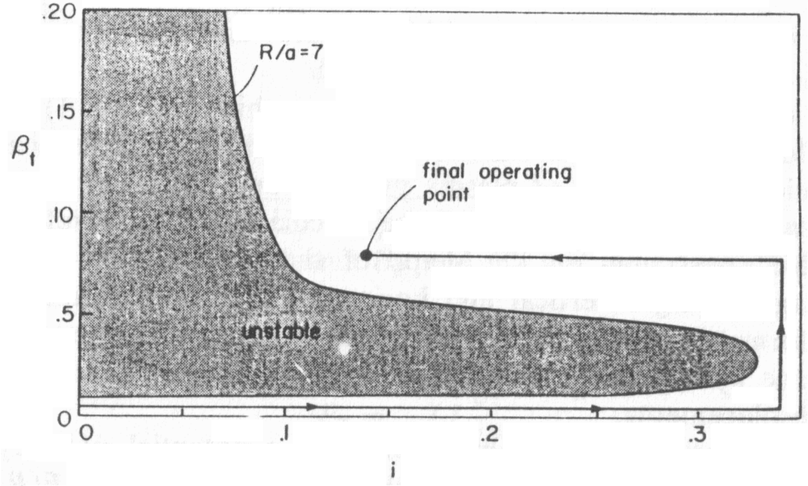
Stability: Multidimensional Configurations

- **Tokamaks**
  - Numerical Results: The Sykes Limit, the Troyon Limit

    Second Region of Stability Studies: bean-shaped cross section







Stability: Multidimensional Configurations

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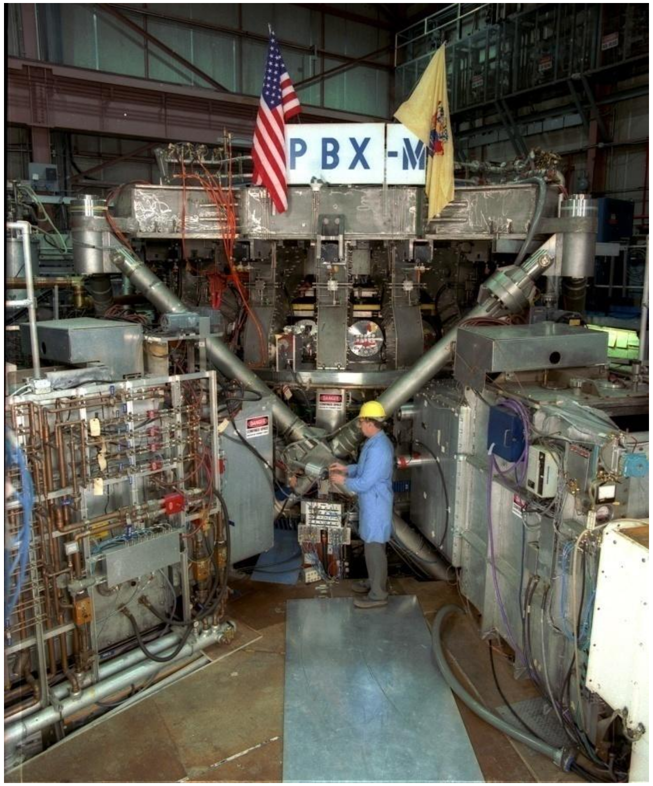
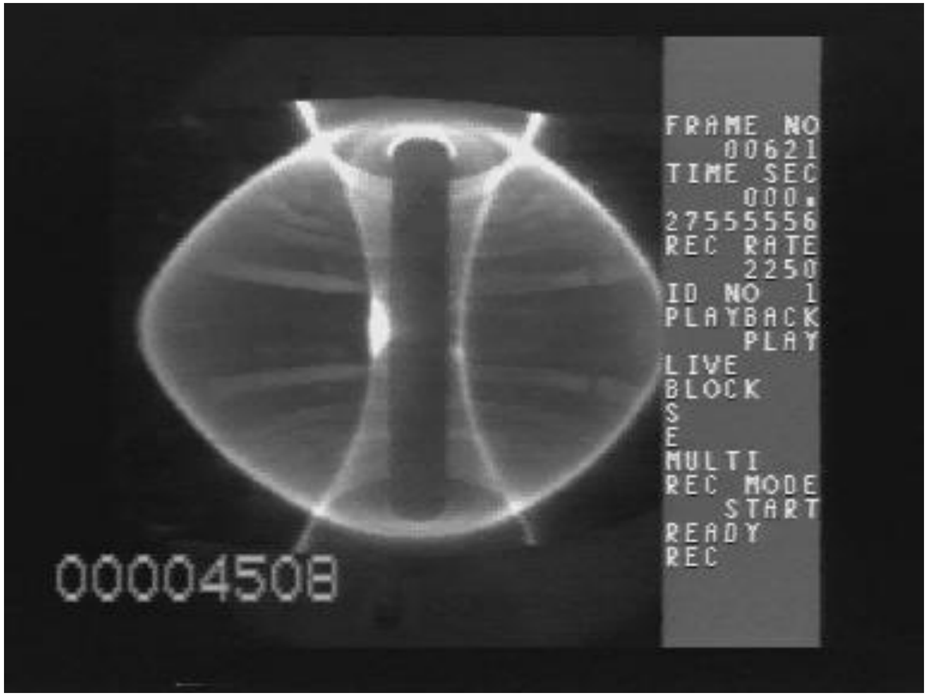
    Second Region of Stability Studies: bean-shaped cross section



Princeton Bean Experiment (PBX)

Stability: Multidimensional Configurations

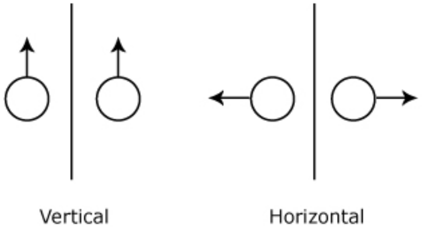
- **Tokamaks**
  - n = 0 Axisymmetric Modes

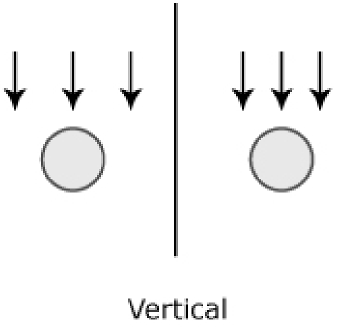


Stability: Multidimensional Configurations

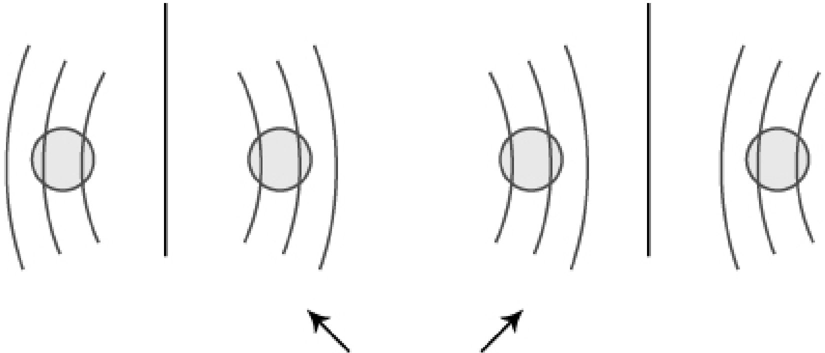
- **Tokamaks**
  - n = 0 Axisymmetric Modes

  - Macroscopic motion of the plasma towards the wall
  - Directly coupled to toroidicity and noncircularity
  - $m = 1$, $n = 0 \rightarrow \delta W = 0$ indicating neutral stability
  - Plasma treated as a thin current-carrying loop of wire with perfect conductivity embedded in an externally applied vertical field.
  - The effects of plasma pressure and the internal magnetic flux neglected
  - Objective: to determine the appropriate constraints on the shape of the vertical field to provide stability against rigid vertical and horizontal displacements





Pure vertical field is neutral by symmetry.



Which is good for stability?

Stability: One-Dimensional Configurations

- **The "Straight" Tokamak**
  - External Modes (The *m* = 1 Kruskal-Shafranov Limit)

Considering the *m* = 1 mode

- Minimising eigenfunction by $\xi(r) = \xi_a$ = const (independent of *q* profile)
  → integral contribution vanished

$$\frac{\delta W_2}{W_0} = \xi_0^2\left(n - \frac{1}{q_a}\right)\left[\left(n - \frac{1}{q_a}\right) + \left(n + \frac{1}{q_a}\right)\right] = 2\xi_0^2\left[n\left(n - \frac{1}{q_a}\right)\right]$$

$$q_a > 1$$

Kruskal-Shafranov criterion:
stability condition for the *m* = 1 external kink mode
for the worst case, *n* = 1

Imposing an important constraint on tokamak operation:
toroidal current upper limit ($I < I_{KS}$)

$$I_{KS} \equiv 2\pi a^2 B_0 / \mu_0 R_0 = 5a^2 B_0 / R_0 \text{ [MA]} \qquad q_a = \frac{aB_0}{\mu_0 R_0 I_{KS} / 2\pi a} = 1$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - *n* = 0 Axisymmetric Modes

Classical mechanics formulation $\vec{B}_v = B_v \vec{e}_z$

Force acting on plasma: $\vec{F}(R,Z) = -\nabla \phi$

Equilibrium: $F_R(R_0, Z_0) = F_Z(R_0, Z_0) = 0$

Vertical and horizontal stability: Restoring force is opposite to displacement.

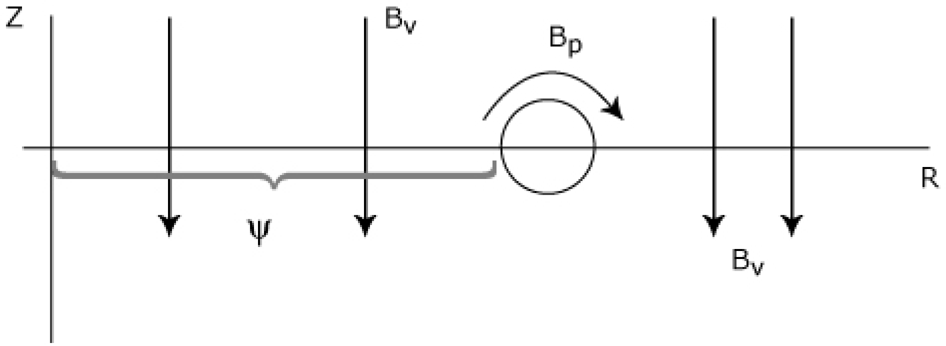
$$\partial F_Z(R_0, Z_0) / \partial Z < 0$$

$$\partial F_R(R_0, Z_0) / \partial R < 0$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - *n* = 0 Axisymmetric Modes

    Classical mechanics formulation



Potential energy: $\phi(R,Z)=\frac{1}{2}LI^2$ $\qquad L(R)=\mu_0 R[\ln(8R/a)-2]$

Flux linked by plasma: $\psi(R,Z)=LI-2\pi\int_0^R B_Z(R',Z)R'dR'=const$

Equilibrium: $F_Z=-\frac{\partial\phi}{\partial Z}=-LI\frac{\partial I}{\partial Z}=0$

$$F_R=-\frac{\partial\phi}{\partial R}=-LI\frac{\partial I}{\partial R}-\frac{I^2}{2}\frac{dL}{dR}=0$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - n = 0 Axisymmetric Modes

Constraint: $\psi = const \rightarrow \dfrac{\partial \psi}{\partial R} = \dfrac{\partial \psi}{\partial Z} = 0$

$$\begin{aligned}\frac{\partial \psi}{\partial Z} &= L\frac{\partial I}{\partial Z} - 2\pi \int_0^R \frac{\partial B_Z(R',Z)}{\partial Z} R' dR' \\ &= L\frac{\partial I}{\partial Z} + 2\pi R B_R = 0 \quad \longleftarrow \quad \nabla \cdot \vec{B} = 0: \ \frac{\partial B_Z}{\partial Z} = -\frac{1}{R'}\frac{\partial R' B_R}{\partial R'}\end{aligned}$$

$$\begin{aligned}\frac{\partial \psi}{\partial R} &= L\frac{\partial I}{\partial R} + I\frac{\partial I}{\partial R} - 2\pi \int_0^R \frac{\partial B_Z(R',Z)}{\partial R} R' dR' \\ &= L\frac{\partial I}{\partial R} + I\frac{dL}{dR} - 2\pi R B_Z = 0\end{aligned}$$

Stability: Multidimensional Configurations

- **Tokamaks**
  - n = 0 Axisymmetric Modes

Eliminate $\partial I/\partial Z$, $\partial I/\partial R$ from force relation:

$$F_Z = -\frac{\partial \phi}{\partial Z} = -LI\,\frac{\partial I}{\partial Z} = 2\pi RIB_R = 0$$

$$F_R = -\frac{\partial \phi}{\partial R} = -LI\,\frac{\partial I}{\partial R} - \frac{I^2}{2}\frac{dL}{dR} = I\left(I\,\frac{dL}{dR} - 2\pi RB_Z\right) - \frac{I^2}{2}\frac{dL}{dR} = 0$$

$$B_Z(R_0, Z_0) = \frac{I}{4\pi R_0}\frac{dL}{dR_0} = \frac{\mu_0 I}{4\pi R_0}\left[\ln\frac{8R_0}{a} - 1\right]$$

$$\longleftarrow \quad B_v = \frac{\mu_0 I}{4\pi R_0}\left[\beta_p + \frac{l_i}{2} - \frac{3}{2} + \ln\frac{8R_0}{a}\right]$$

Shafranov result

Stability: Multidimensional Configurations

- **Tokamaks**
  - n = 0 Axisymmetric Modes

Vertical stability

$$F_Z = -\frac{\partial \phi}{\partial Z} = -LI\frac{\partial I}{\partial Z} = 2\pi RIB_R = 0$$

$$\frac{\partial F_Z}{\partial Z} = 2\pi R\left(B_R\frac{\partial I}{\partial Z} + I\frac{\partial B_R}{\partial Z}\right) = 2\pi RI\frac{\partial B_R}{\partial Z}$$ < 0 for stability

$$\longleftarrow \quad \frac{\partial \psi}{\partial Z} = L\frac{\partial I}{\partial Z} + 2\pi RB_R = 0$$

$$\nabla \times \vec{B} = 0 : \quad \frac{\partial B_R}{\partial Z} = \frac{\partial B_Z}{\partial R}$$

Define field index $n(R_0, Z_0) = -\left(\frac{R}{B_Z}\frac{\partial B_Z}{\partial R}\right)_{R_0, Z_0}$

$$\frac{\partial F_Z}{\partial Z} = 2\pi RI\frac{\partial B_R}{\partial Z} = 2\pi RI\frac{\partial B_Z}{\partial R} = -2\pi RI \cdot n\frac{B_Z}{R} = -2\pi I\frac{I}{4\pi R_0}\frac{dL}{dR}n < 0$$: $n > 0$

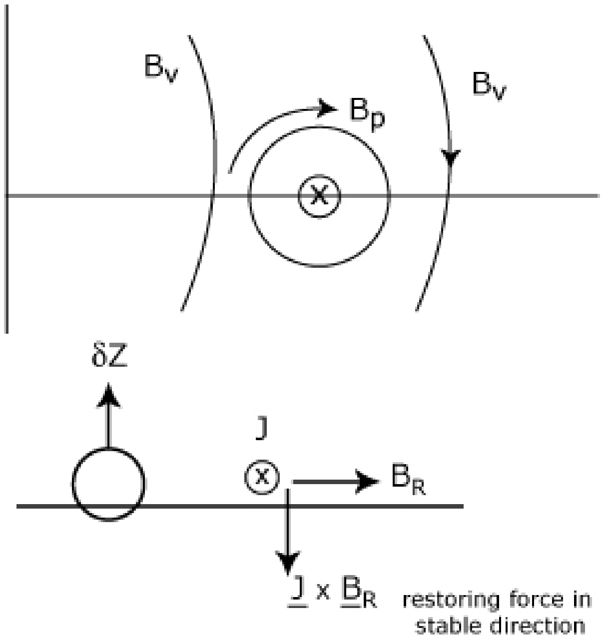
Stability: Multidimensional Configurations

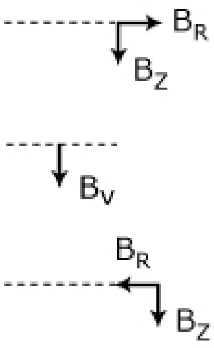
- **Tokamaks**
  - n = 0 Axisymmetric Modes

    Horizontal stability

$$\frac{\partial F_R}{\partial R} = \frac{I^2}{2R_0}\frac{dL}{dR_0}\left[n - 1 - \frac{1}{2}\frac{d\ln L}{d\ln R_0} + \frac{1}{2}\frac{d\ln(dL/dR_0)}{d\ln R_0}\right] < 0$$: $n < 3/2$

$$\ln(8R_0/a) >> 1$$





$$B_Z < 0$$

$$\frac{\partial B_R}{\partial Z} = \frac{\partial B_Z}{\partial R} > 0$$

$$n = -\frac{R}{B_Z}\frac{\partial B_Z}{\partial R} > 0$$

Stability: Multidimensional Configurations

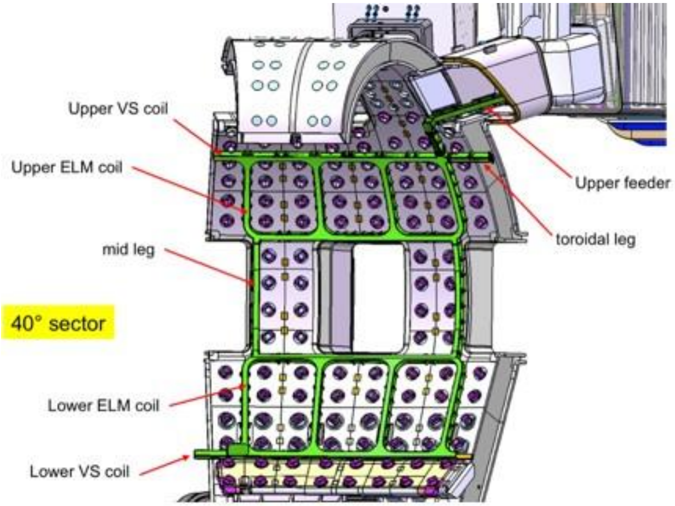
- **Tokamaks**
  - $n = 0$ Axisymmetric Modes

  - $n = 0$ axisymmetric modes can lead to potentially serious instabilities in a tokamak.

  - For a circular cross sections a moderate shaping of the vertical field should provide stability.

  - For noncircular tokamaks, vertical instabilities produce important limitations on the maximum achievable elongations.

  - Even moderate elongations require a conducting wall or a feedback system for vertical stability.



ITER: current design of the in-vessel coils to stabilize ELMs and the vertical displacement events, shown for a 40° vacuum vessel sector

References

- *http://www.truestarhealth.com/Notes/1800000.html*
- *http://burningplasma.org/enews071508.html*