Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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Week 15. Project Presentation and Summary

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Week 13-14. Stability: Multidimensional Configurations

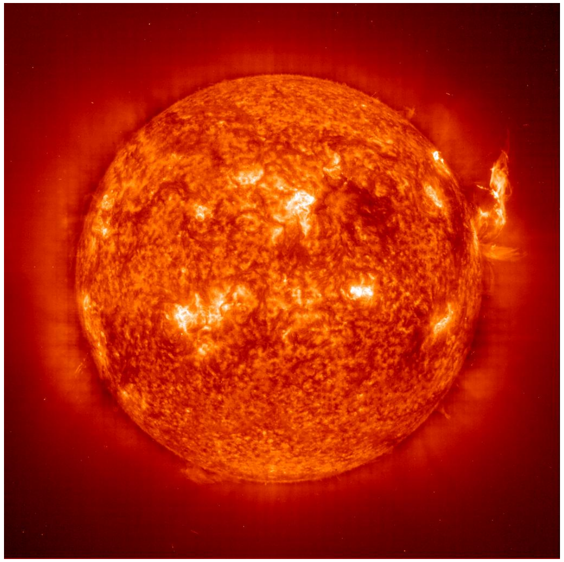
Week 15. Project Presentation and Summary

Textbook

- J. Freidberg,
  "Ideal Magnetohydrodynamics
  (Modern Perspectives in Energy)", Springer,
  1st edition (1987)

- T. Takeda and S. Tokuda,
  "Computation of MHD Equilibrium of Tokamak Plasma",
  Journal of Computational Physics,
  Vol. 93, p1~107 (1991)

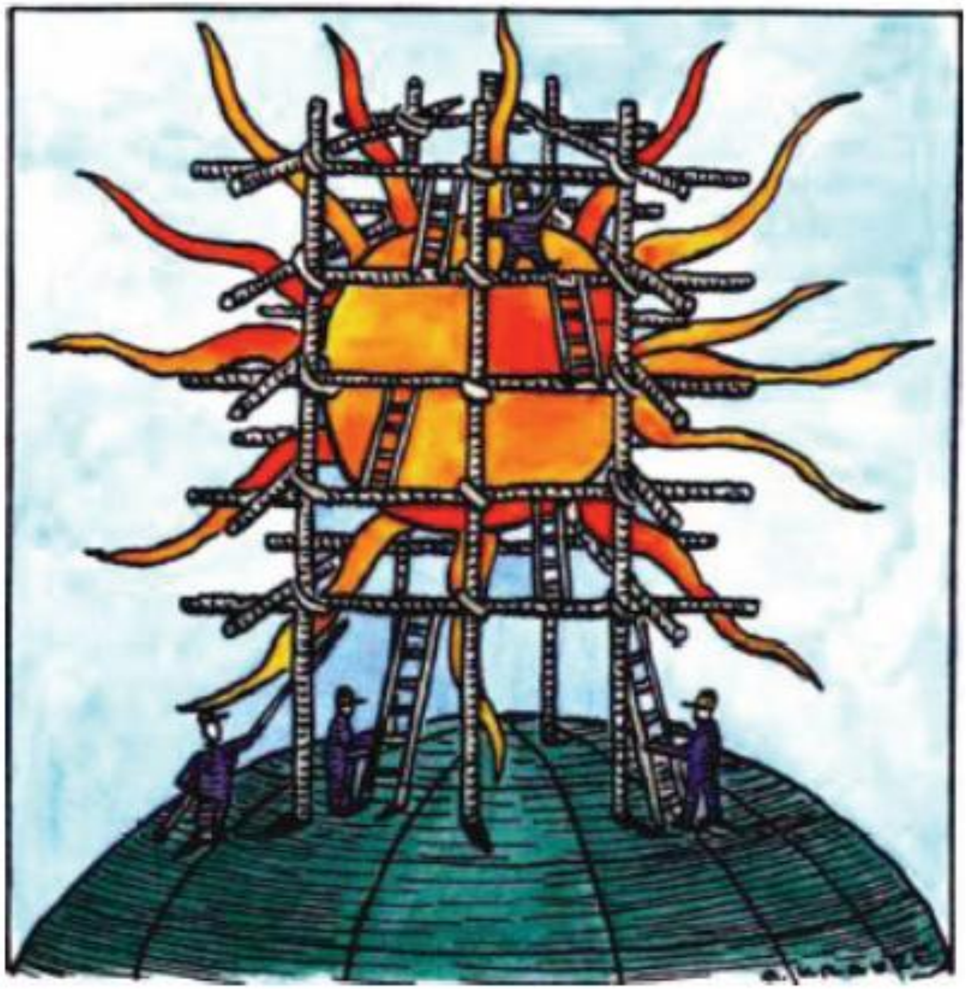
The Origin of the Star Energy



Nobel prize in physics 1967
"for his contribution to the theory of nuclear reactions, especially his discoveries concerning the energy production in stars"

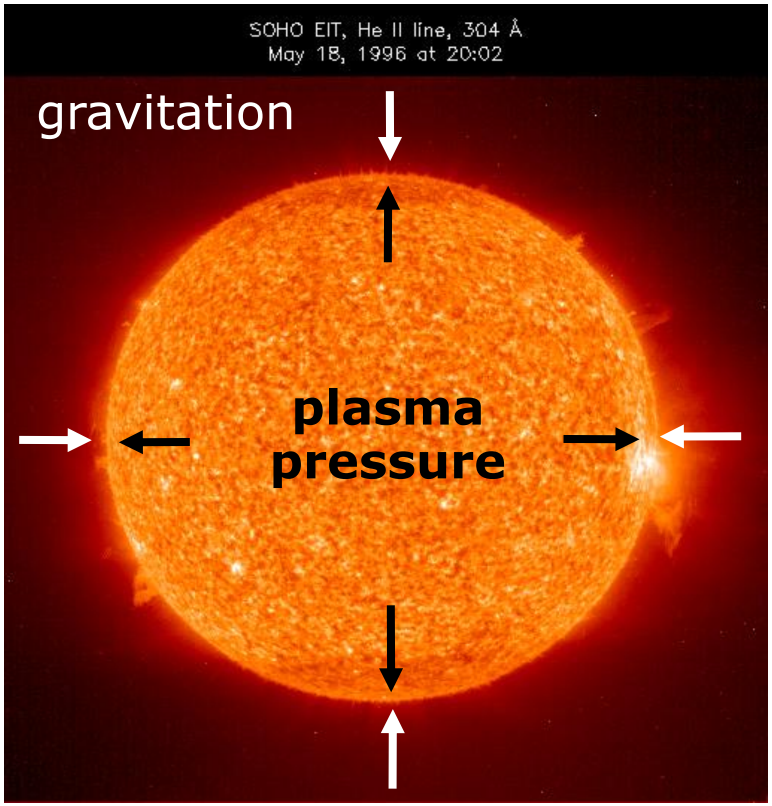
Hans Albrecht Bethe
(1906. 7. 2 – 2005. 3. 6)

To build a sun on earth

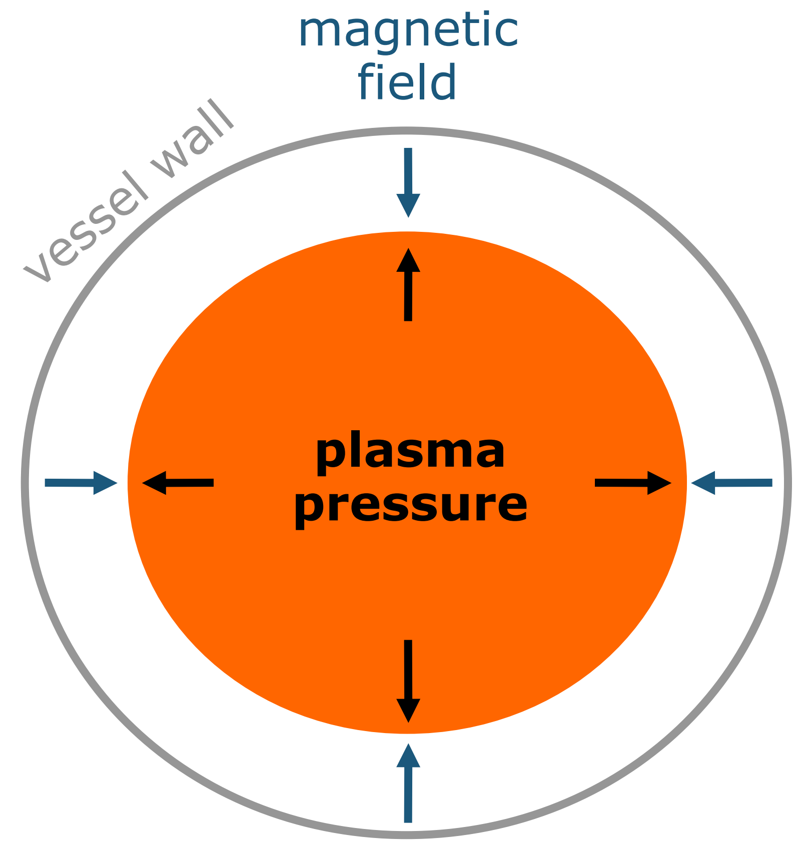


Magnetic confinement

- **Imitation of the Sun on Earth**



Equilibrium in the sun



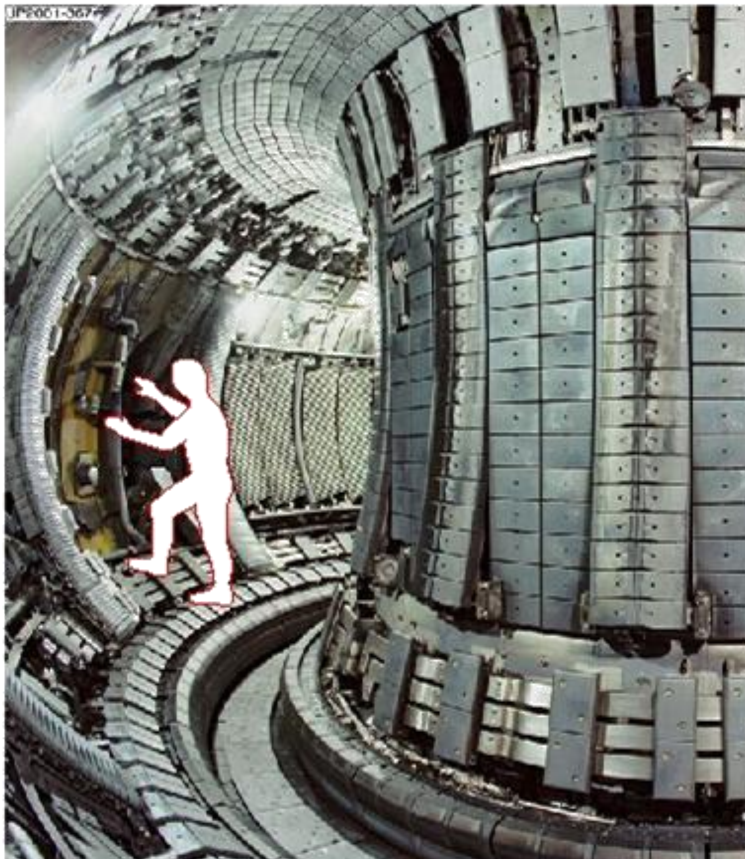
Plasma on earth
much, much smaller & tiny mass!

Tokamak

JET (Joint European Torus): R_0=3m, a=0.9m, 1983-today

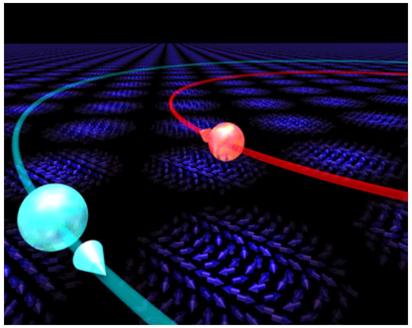
Tokamak

JET (Joint European Torus): R_0=3m, a=0.9m, 1983-today



How to describe fusion plasmas?

What is Ideal MHD?



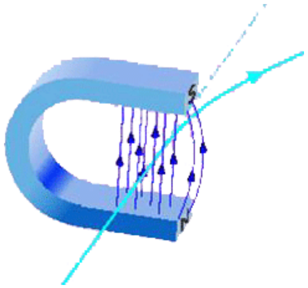
- **Ideal MHD**

- Ideal:

  Perfect conductor with zero resistivity

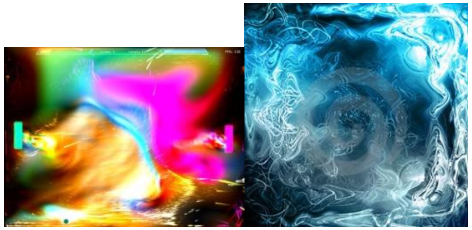
- MHD:

  Magnetohydrodynamic (magnetic fluid dynamic)



- Single-fluid model:

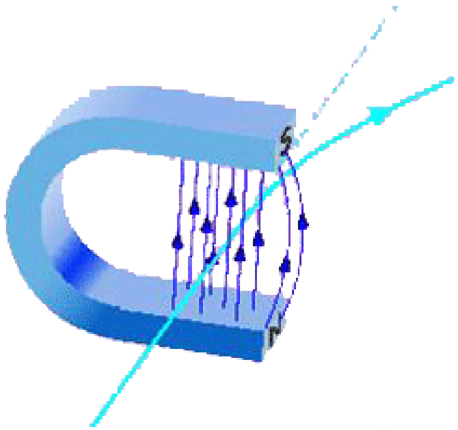
  electrically charged current-carrying fluid.



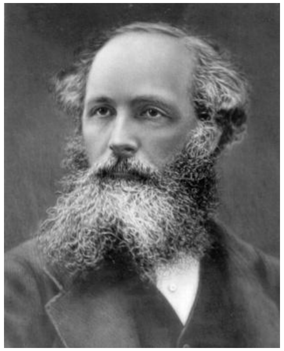
Derivation of the Ideal MHD Model

- **Starting Equations**

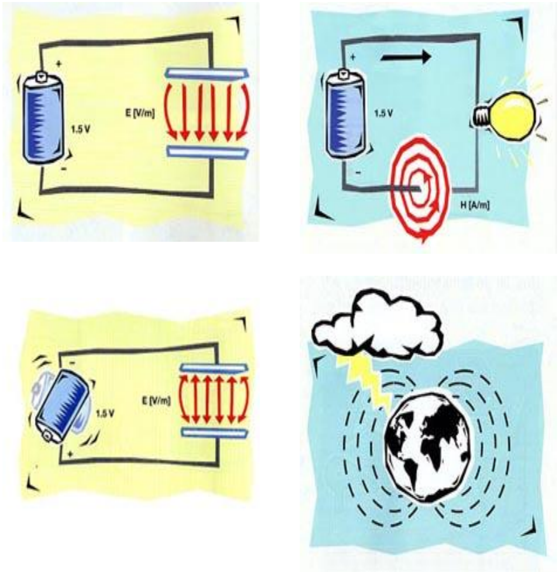
$$\frac{\partial f_\alpha}{\partial t} + \vec{u}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E}+\vec{u}\times\vec{B})\cdot\nabla_u f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$



Ludwig Boltzmann
(1844-1906)



James Clark Maxwell
(1831-1879)



Derivation of the Ideal MHD Model

- **Assumptions**
  - First asymptotic assumption: $\varepsilon_0 \to 0$
    (Full → low-frequency Maxwell's equations)

$$\nabla \times \vec{B} = \mu_0 e\left(n_i \vec{v}_i - n_e \vec{v}_e\right) + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \approx \mu_0 \vec{J}$$

$$n_i - n_e = \frac{\varepsilon_0}{e}\nabla \cdot \vec{E} \approx 0$$

quasineutrality → displacement current, net charge neglected

  - Second asymptotic assumption: $m_e \to 0$
    (electron inertia neglected: electrons have an infinitely fast response time because of their small mass)

$$n_e m_e\left(\frac{d\vec{v}_e}{dt}\right) = -e n_e(\vec{E} + \vec{v}_e \times \vec{B}) - \nabla \cdot \overleftrightarrow{P}_e + \vec{R}_e \approx 0$$

  - Third assumption: collision dominated plasma

ions $\qquad \omega \tau_{ii} \sim V_{T_i} \tau_{ii} / a << 1$

electrons $\qquad \omega \tau_{ee} \sim (m_e / m_i)^{1/2} V_{T_i} \tau_{ii} / a << 1$

Derivation of the Ideal MHD Model

- **Ideal MHD model**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Mass continuity equation

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Energy equation (equation of state): adiabatic evolution

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

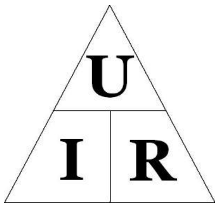
Ohm's law: perfect conductor → "ideal" MHD

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Maxwell equations

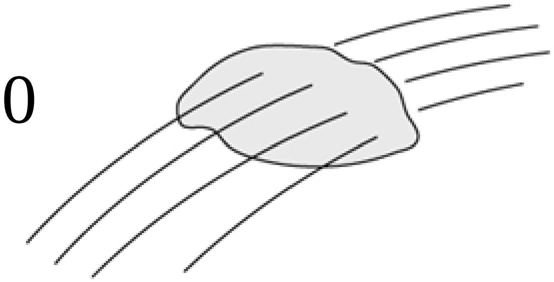


Ohm's triangle

General Properties of Ideal MHD

- **Conservation of Flux: "Frozen" Field Line Picture**
  - A consequence of the perfect conductivity Ohm's law, is that the magnetic flux passing through any arbitrary open surface area moving with the plasma is constant.

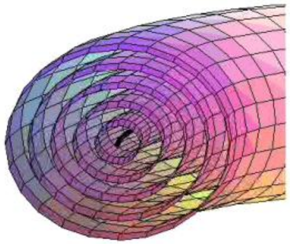
$$\frac{d\psi}{dt} = 0$$

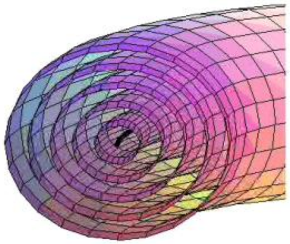


time rate of change of the flux passing through any moving surface, S

  - Magnetic lines move with the plasma; they are "frozen" into the fluid.

Ideal MHD:
$\eta = 0$





Resistive MHD:
$\eta \neq 0$

General Properties of Ideal MHD

- **Conservation of Flux: "Frozen" Field Line Picture**
  - A consequence of the perfect conductivity Ohm's law, is that the magnetic flux passing through any arbitrary open surface area moving with the plasma is constant.

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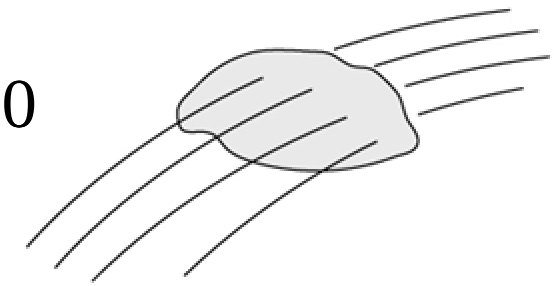

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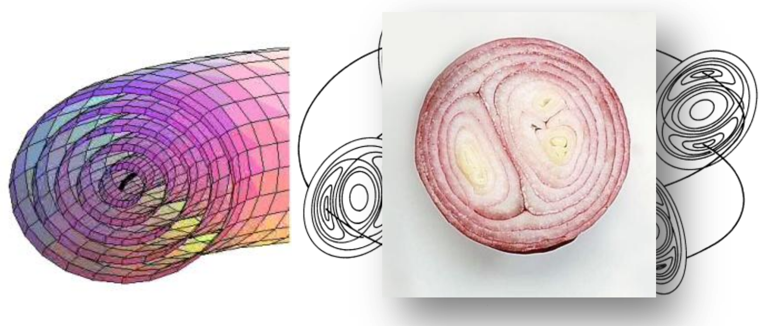


Resistive MHD:
$\eta \neq 0$

Limitations

- **Major discoveries of ... physics**
  - Maxwell's equat... wave propagation
  - Relativi...
  - ... mechanics

Eliminated in the derivation of MHD

- **Important processes in fusion ... mas**
  - Radiation
  - RF heating
  - Resonant ...
  - M...
  - ...sport
  - A...alous transport
  - Resistive instabilities
  - alpha-particle behaviour

Not adequately described by ideal MHD

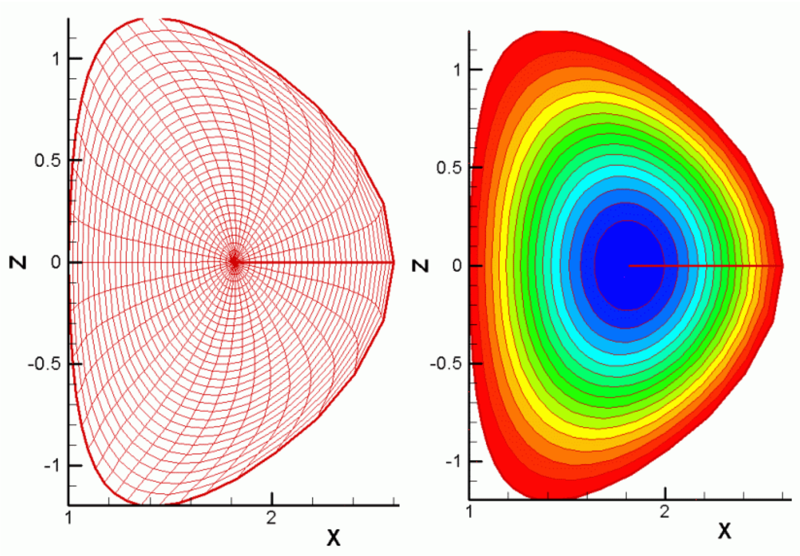
Applications

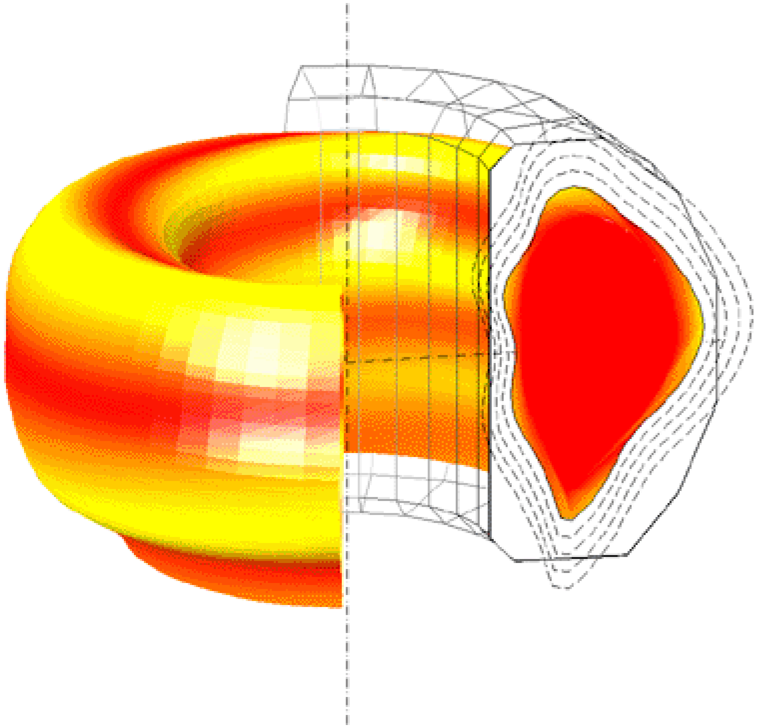
How does a given magnetic geometry provide forces to hold a plasma in equilibrium?

Why are certain magnetic geometries much more stable against macroscopic disturbances than others?

Why do fusion configurations have such technologically undesirable shapes as a torus, a helix, or a baseball seam?

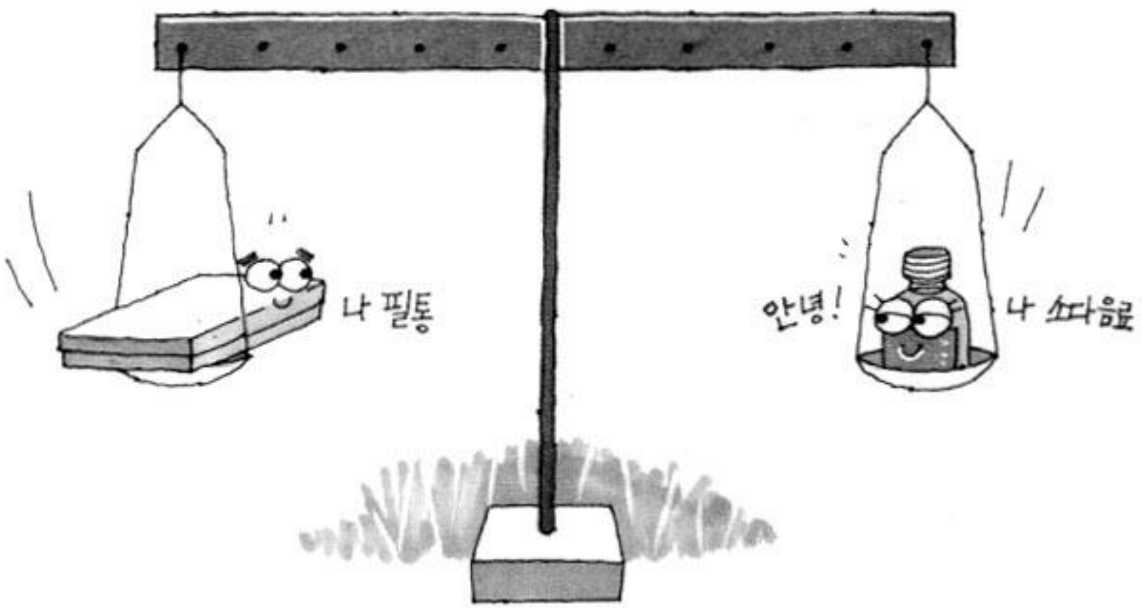
Applications





Plasma Eqauilibrium and Stability

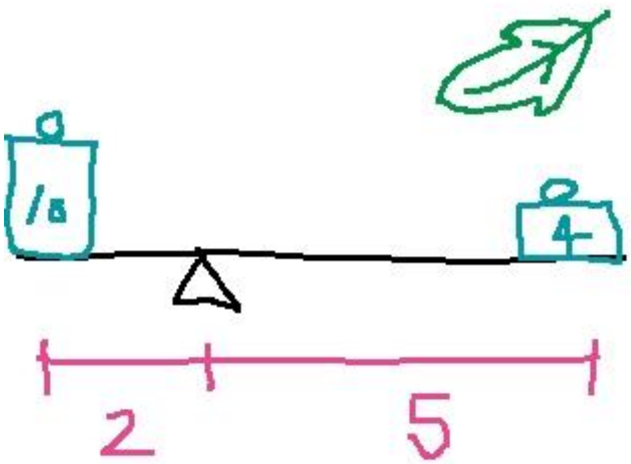
Equilibrium and Stability

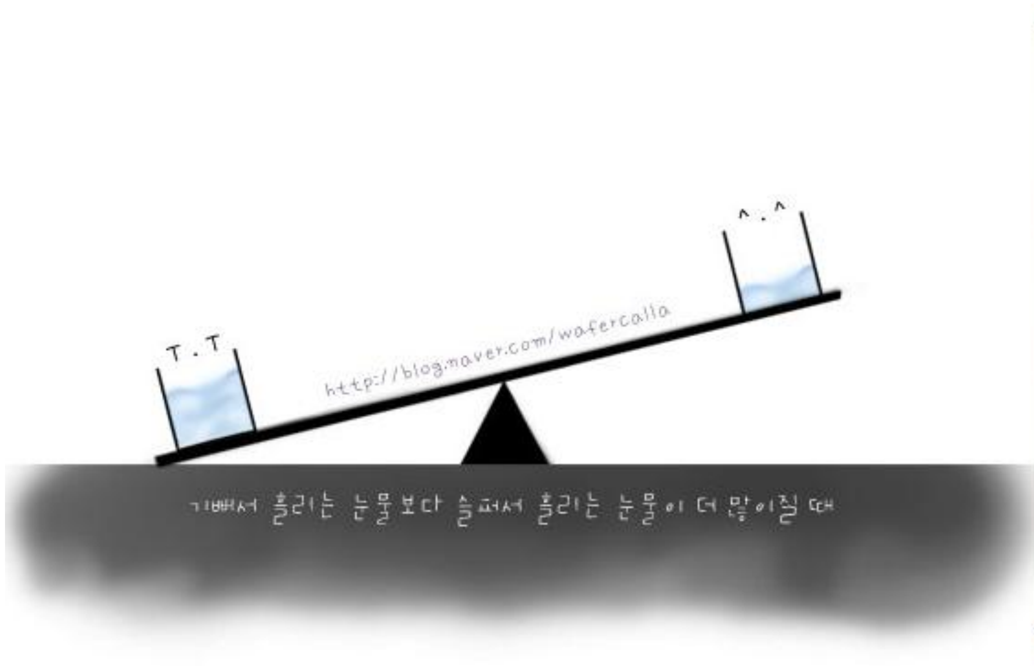


Equilibrium? Yes! Forces are balanced

Stable? No!

Equilibrium and Stability





Equilibrium? Yes! Forces are balanced

Stable? No! The system cannot recover.

Equilibrium

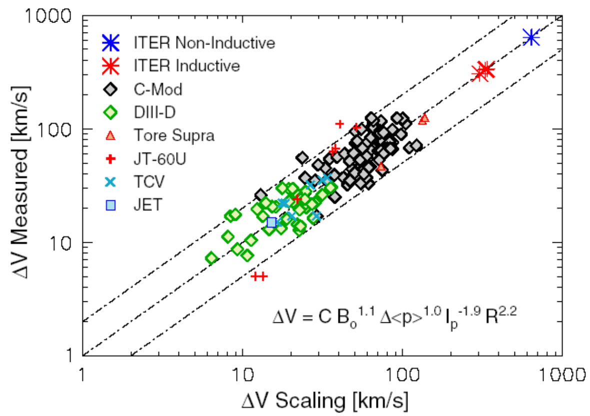
- **Basic Equations**
  - MHD equilibrium equations:
    time-independent with **v** = 0 (static)

$$\vec{J}\times\vec{B}=\nabla p$$

$$\nabla\times\vec{B}=\mu_0\vec{J}$$

$$\nabla\cdot\vec{B}=0$$

cf) stationary equilibrium with nonzero flows:
v << V_{Ti} (ideal MHD)



J. E. Rice *et al*, *Nucl. Fusion* **47** 1618 (2007)

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\vec{v}=0$$

$$\rho\frac{d\vec{v}}{dt}=\vec{J}\times\vec{B}-\nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right)=0$$

$$\vec{E}+\vec{v}\times\vec{B}=0$$

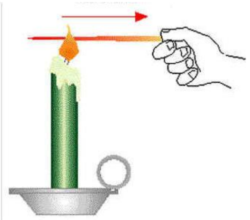
$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B}=\mu_0\vec{J}$$

$$\nabla\cdot\vec{B}=0$$

Equilibrium

- **Toroidicity**



- Why are most fusion configurations toroidal?

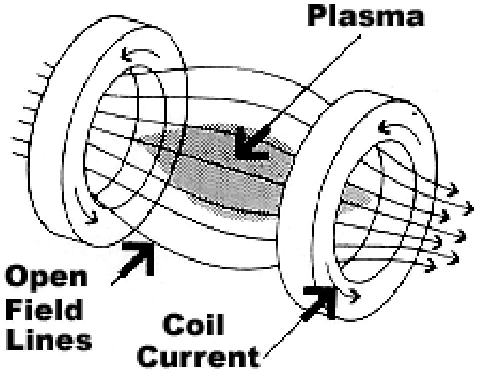
- Answer: Avoid parallel end losses
- Dominant loss mechanism is heat loss via thermal conduction.
- Heat loss is more severe along **B** than ⊥ to **B** because charged particles move freely along magnetic field lines. The magnetic field confines particles in the ⊥ direction.

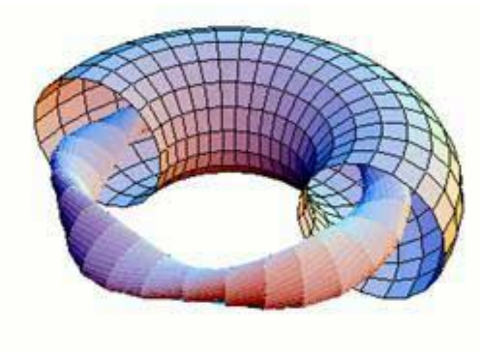
$$\frac{\kappa_{\|e}}{\kappa_{\perp i}} = 1.12\left(\frac{m_i}{m_e}\right)^{1/2}(\omega_{ci}\tau_{ii})^2 \approx 6.2\times10^{10}\left(\frac{B^2T_i^3}{n^2}\right) \longleftarrow$$

$Z = 1, \quad m_i = 2m_{proton}$

$T_e = T_i$

$\ln\Lambda = 15$



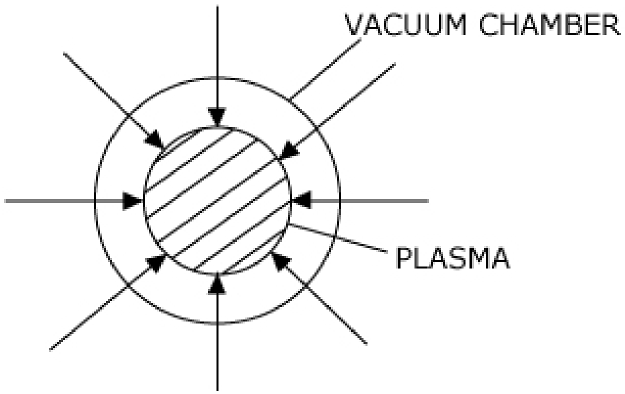


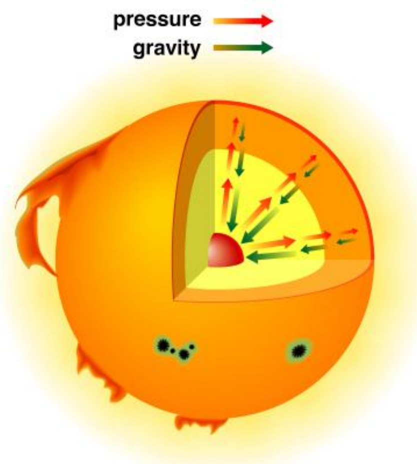
$\kappa_{\|e} / \kappa_{\perp i} = 3.1\times10^{12}$

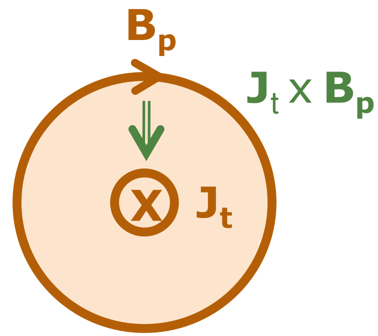
$T_i = 2\text{keV}, \quad B = 5\text{T}, \quad n = 2\times10^{20}\text{m}^{-3}$

Equilibrium - Tokamak

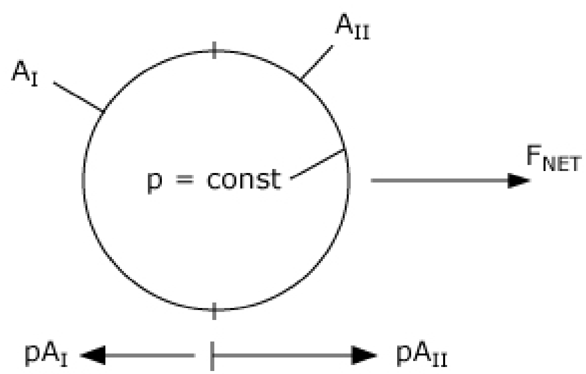
- **Basic Forces Acting on Tokamak Plasmas**
  - Radial pressure force







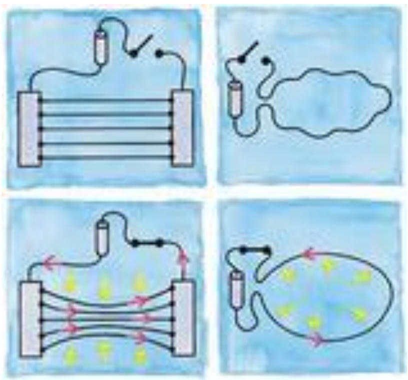
  - Tire tube force

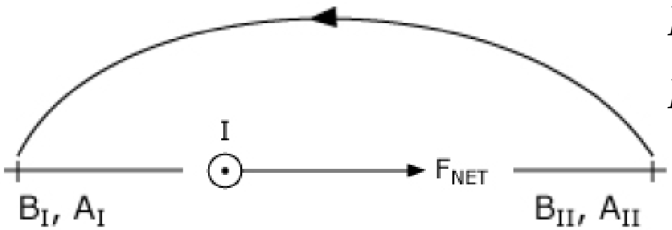


$$F_{NET} \sim -e_R(pS_1 - pS_2)$$

Equilibrium - Tokamak

- **Basic Forces Acting on Tokamak Plasmas**
  - Hoop force

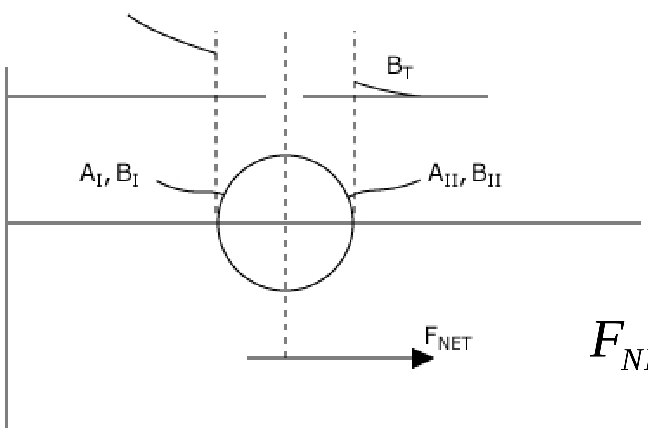




$$\phi_I = \phi_{II}$$
$$B_I > B_{II}, \quad A_I < A_{II}$$
$$B_I^2 A_I > B_{II}^2 A_{II}$$

$$F_{NET} \sim e_R (B_I^2 A_I - B_{II}^2 A_{II}) / 2\mu_0$$

  - 1/*R* force

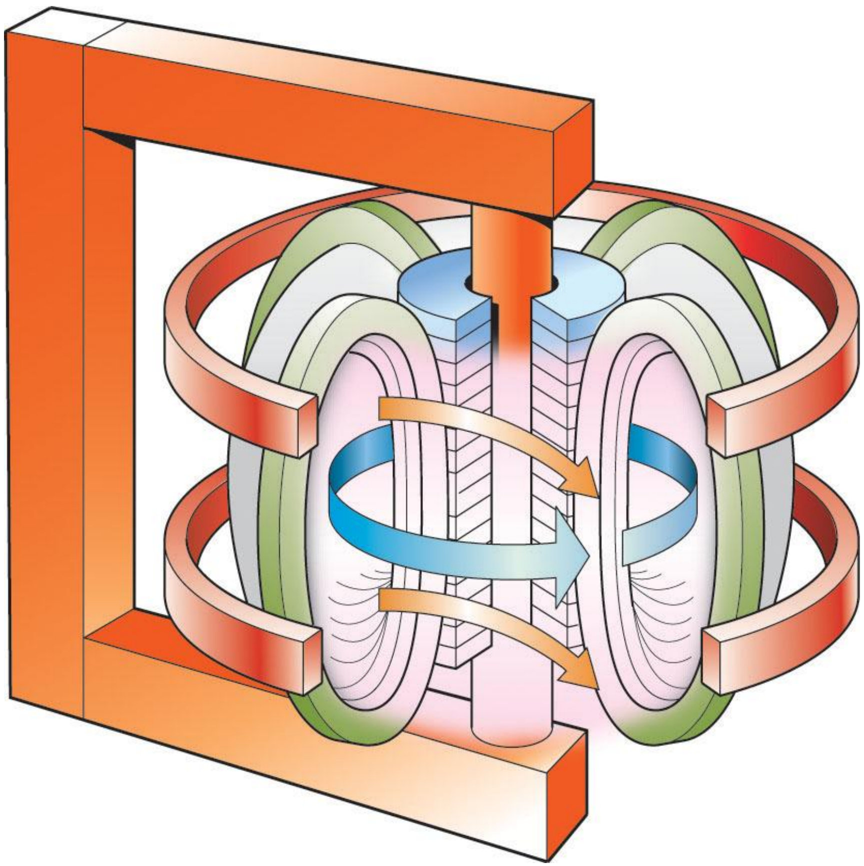


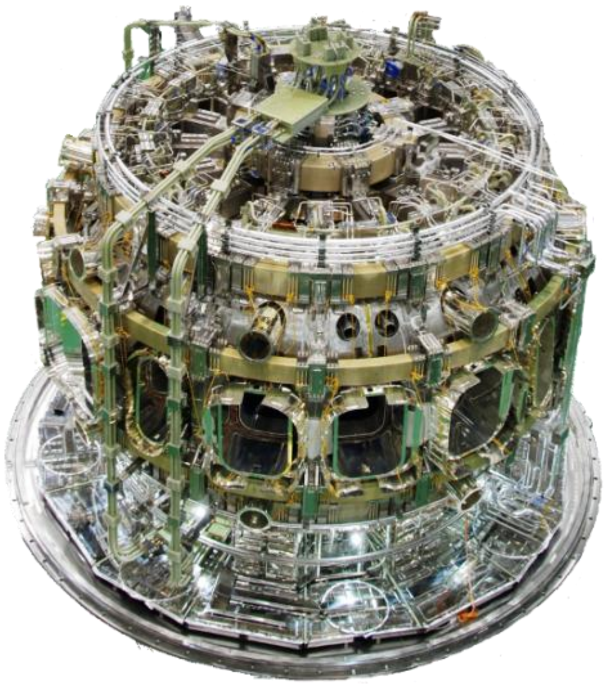
$$\phi_I = \phi_{II}$$
$$B_I > B_{II}, \quad A_I < A_{II}$$
$$B_I^2 A_I > B_{II}^2 A_{II}$$

$$F_{NET} = 2\pi^2 a^2 \frac{B^2}{2\mu_0}$$

Equilibrium - Tokamak

- **Basic Forces Acting on Tokamak Plasmas**
  - External coils required to provide the force balance

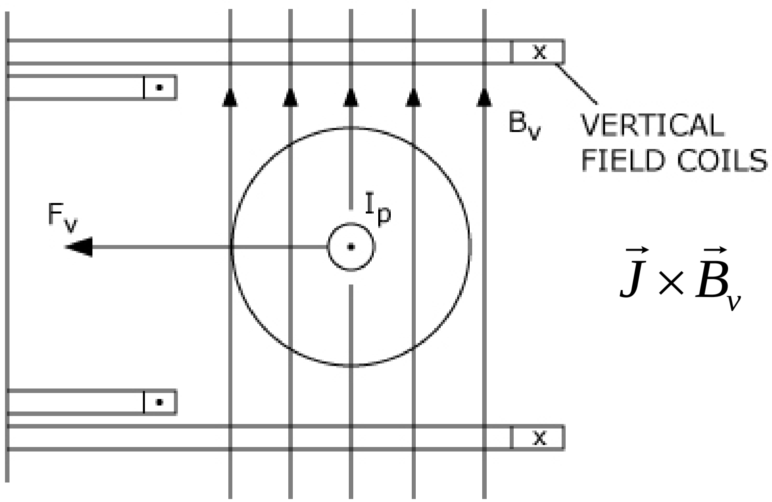




KSTAR

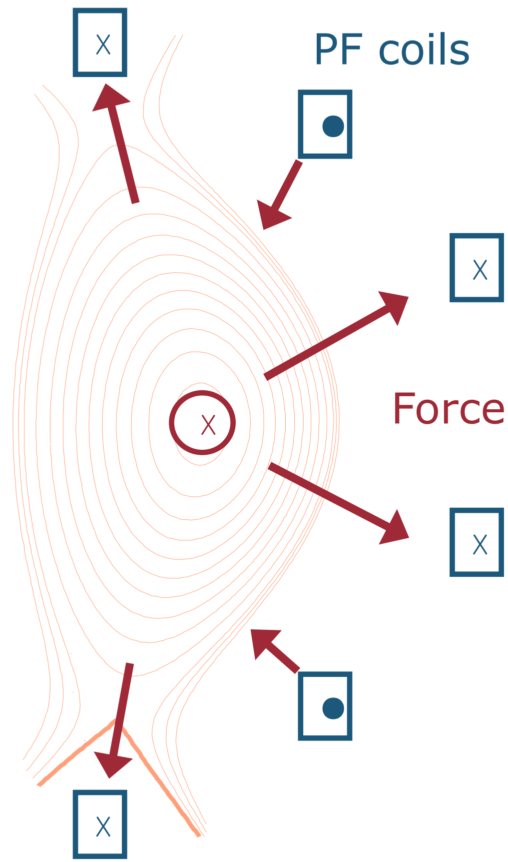
Equilibrium - Tokamak

- **Basic Forces Acting on Tokamak Plasmas**
  - External coils required to provide the force balance



$$\vec{J} \times \vec{B}_v$$

$$F_v = BIL = 2\pi R_0 I_p B_v$$



How to describe the equilibrium of plasmas?

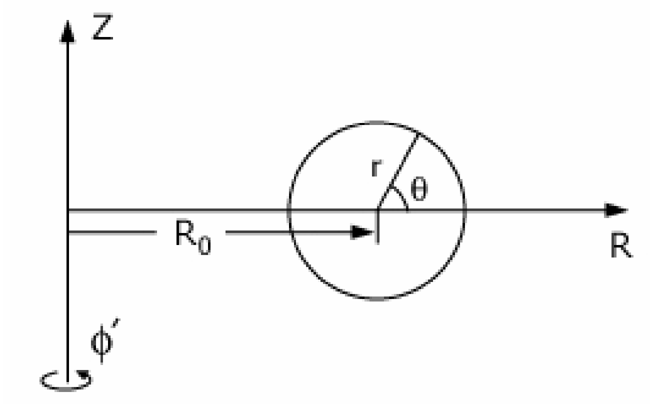
Equilibrium - Tokamak

- **The Grad-Shafranov Equation**
  - obtained from the reduction of the ideal MHD equations
  - exact (no expansion)
  - Toroidal axisymmetric $\partial/\partial\phi=0$
  - 2 dimensional
  - nonlinear
  - partial differential equation
  - elliptic characteristics
  - Grad and Rubin (1958), Shafranov (1960)

$$\vec{J}\times\vec{B}=\nabla p$$

$$\nabla\times\vec{B}=\mu_0\vec{J}$$

$$\nabla\cdot\vec{B}=0$$



$$R=R_0+r\cos\theta,\quad Z=r\sin\theta$$

$$\psi=RA_\phi$$

Equilibrium - Tokamak

- **The Grad-Shafranov Equation**

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}$$

$$\Delta^* \psi \equiv R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2} \right) = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2}$$ elliptic operator

$$p = p(\psi), \quad F = F(\psi)$$

$$\vec{B} = \frac{1}{R} \nabla \psi \times \vec{e}_\phi + \frac{F}{R} \vec{e}_\phi$$

$$\mu_0 \vec{J} = \frac{1}{R} \frac{dF}{d\psi} \nabla \psi \times \vec{e}_\phi - \frac{1}{R} \Delta^* \psi \vec{e}_\phi$$

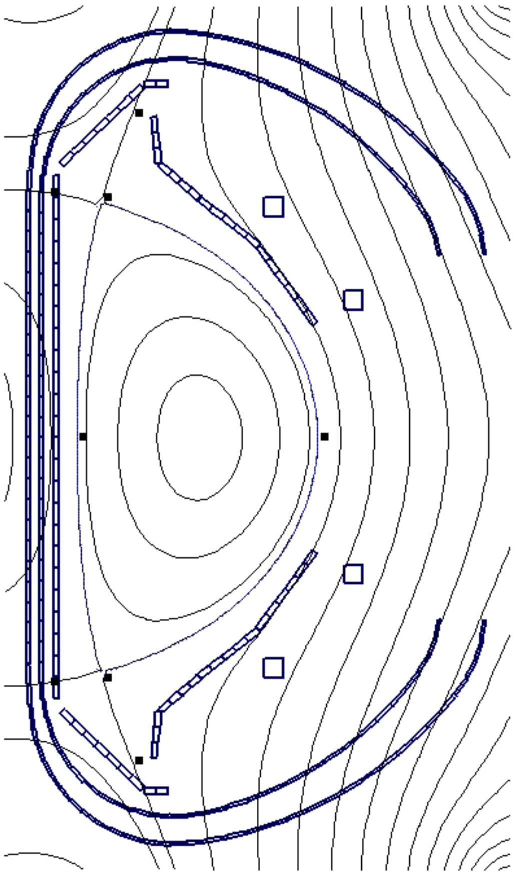
$$\psi_p = 2\pi \psi, \quad I_p = 2\pi F$$

Equilibrium - Tokamak

- **Plasma Parameters and Figures of Merit**
  - Safety factor $$q(\psi)=\frac{2\pi}{\iota}=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{rB_\phi}{RB_\theta}\right)_S d\theta=\frac{F(\psi)}{2\pi}\oint\frac{dl_p}{R^2B_p}$$
  - Kink safety factor $$q_*=\frac{aB_0}{R_0\overline{B}_p}$$
  - Plasma beta $$\beta_t=\frac{2\mu_0\langle p\rangle}{B_0^2},\quad \beta_p=\frac{2\mu_0\langle p\rangle}{\overline{B}_p^2}$$
  - Magnetic shear $$s(\psi)=2\left(\frac{V}{V'}\right)\left(\frac{q'}{q}\right)$$
  - Magnetic well $$W(\psi)=2\left(\frac{V}{V'}\right)\left(\frac{\langle B^2/2+\mu_0 p\rangle'}{\langle B^2\rangle}\right)$$

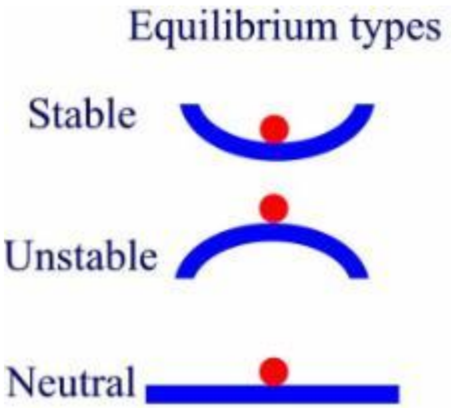
Equilibrium - Tokamak

- **Numerical Calculation of Grad-Shafranov Equation**

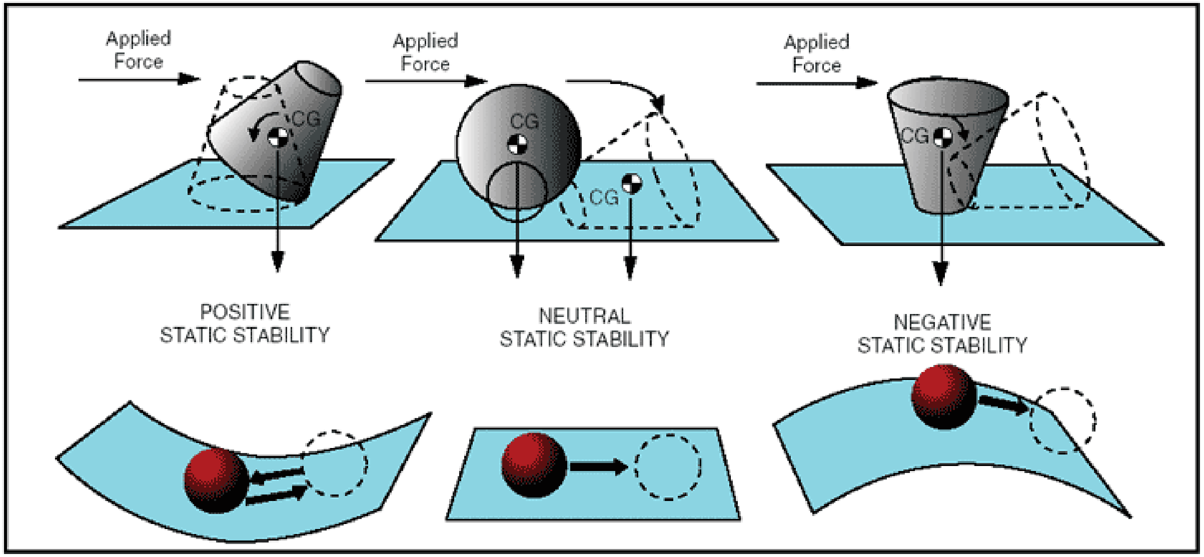


Stability

- **Definition of Stability**



Marginally stable



- assuming all quantities of interest linearised about their equilibrium values.

$$Q(\vec{r},t) = Q_0(\vec{r}) + \tilde{Q}_1(\vec{r},t)$$

small 1st order perturbation

$$\tilde{Q}_1 / |Q_0| << 1 \qquad \tilde{Q}_1(\vec{r},t) = Q_1(\vec{r})\varepsilon^{-i\omega t}$$

Im $\omega > 0$: exponential instability

Im $\omega \leq 0$: exponential stability

Stability

- **Various Approaches for Stability Analyses**
  1. Initial value problem using the general linearised equations of motion
  2. Normal-mode eigenvalue problem
  3. Variational principle
  4. Energy Principle

Stability

- Initial Value Formulation

$$\vec{J}_0 \times \vec{B}_0 = \nabla p_0$$

$$\mu_0 \vec{J}_0 = \nabla \times \vec{B}_0$$

$$\nabla \cdot \vec{B}_0 = 0$$

$$\vec{v}_0 = 0$$

$$Q(\vec{r},t) = Q_0(\vec{r}) + \tilde{Q}_1(\vec{r},t) \quad \tilde{Q}_1 / |Q_0| << 1$$

linearized

$$\tilde{v}_1 = \frac{\partial \xi}{\partial t}$$

ξ: displacement of the plasma away from its equilibrium position

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \vec{F}(\xi)$$

momentum equation

$$\vec{F}(\xi) = \vec{J} \times \tilde{B}_1 + \tilde{J}_1 \times \vec{B} - \nabla \tilde{p}_1$$

force operator

$$= \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \tilde{Q} + \frac{1}{\mu_0}(\nabla \times \tilde{Q}) \times \vec{B} + \nabla(\xi \cdot \nabla p + \gamma p \nabla \cdot \xi)$$

$$\xi(\vec{r},0) = 0, \quad \frac{\partial \xi(\vec{r},0)}{\partial t} = \tilde{v}_1(\vec{r},0)$$

+ Boundary conditions

Formulation of the generalized stability equations as an initial value problem

Stability

- Normal-Mode Formulation

$$\tilde{Q}_1(\vec{r},t) = Q_1(\vec{r})\exp(-i\omega t)$$

$\rho_1 = -\nabla\cdot(\rho\xi)$ conservation of mass

$p_1 = -\xi\cdot\nabla p - \gamma p\nabla\cdot\xi$ conservation of energy

$\vec{Q} \equiv \vec{B}_1 = \nabla\times(\xi\times\vec{B})$ Faraday's law

$-\omega^2\rho\xi = \vec{F}(\xi)$ normal-mode formulation

$$\vec{F}(\xi) = \frac{1}{\mu_0}(\nabla\times\vec{B})\times\tilde{Q} + \frac{1}{\mu_0}(\nabla\times\tilde{Q})\times\vec{B} + \nabla(\xi\cdot\nabla p + \gamma p\nabla\cdot\xi)$$

- An eigenvalue problem for the eigenvalue ω^2

Stability

- Variational Principle

  Classic eigenvalue problem

$$\frac{d}{dx}\left(f \frac{\partial y}{\partial x} \right) + (\lambda - g)y = 0$$

λ: eigenvalue

$$y(0) = y(1) = 0$$

$$\lambda = \frac{\int (fy'^2 + gy^2)dx}{\int y^2 dx}$$

Multiplied by y
and integrated over the region $0 \le x \le 1$

Why is this variational?
- Substitute all allowable trial function $y(x)$ into the equation above.
- When resulting λ exhibits an extremum (maximum, minimum, saddle point) then λ and y are actual eigenvalue and eigenfunction.

$$\delta\lambda \approx -\frac{2\int \delta y[(fy_0')' + (\lambda_0 - g)y_0]dx}{\int y_0^2 dx}$$

Stability

- Variational Principle

$$-\omega^2 \rho \xi = \vec{F}(\xi)$$ normal-mode formulation

$$\omega^2 = \frac{\delta W(\xi^*, \xi)}{K(\xi^*, \xi)}$$ dot product with ξ^* then integrated over the plasma volume

$$\lambda = \frac{\int (fy'^2 + gy^2)dx}{\int y^2 dx}$$

$$\delta W(\xi^*, \xi) = -\frac{1}{2}\int \xi^* \cdot \vec{F}(\xi) d\vec{r}$$

$$= -\frac{1}{2}\int \xi^* \cdot [\frac{1}{\mu_0}(\nabla \times \vec{Q}) \times \vec{B} + \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{Q} + \nabla(\gamma p \nabla \cdot \xi + \xi \cdot \nabla p)] d\vec{r}$$

$$K(\xi^*, \xi) = \frac{1}{2}\int \rho |\xi|^2 d\vec{r}$$

Any allowable function ξ for which ω^2 becomes an extremum is an eigenfunction of the ideal MHD normal mode equations with eigenvalue ω^2.

Stability

- Variational Principle

$$-\omega^2 \rho \xi = \vec{F}(\xi)$$ normal-mode formulation

$$\omega^2 = \frac{\delta W(\xi^*, \xi)}{K(\xi^*, \xi)}$$ dot product with ξ^* then integrated over the plasma volume

$$\lambda = \frac{\int (fy'^2 + gy^2)dx}{\int y^2 dx}$$

$$-\omega^2 K + \delta W = 0$$ Conservation of energy

$\omega^2 K$ → Kinetic energy

δW →

- Change in potential energy associated with the perturbation
  - Equal to the work done against the force **F**(ξ) in displacing the plasma by an amount ξ.

Stability

- Energy Principle

$$\omega^2 = \frac{\delta W}{K} \geq 0 \qquad \text{stable}$$

$$\delta W \geq 0 \qquad \text{stable}$$

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}\right|^2}{\mu_0} - \xi_\perp^* \cdot (\vec{J}\times\vec{Q}) + \gamma p\left|\nabla\cdot\xi\right|^2 + (\xi_\perp\cdot\nabla p)\nabla\cdot\xi_\perp^*\right]$$

$$\delta W_S = \frac{1}{2}\int_S d\vec{S}\left|\vec{n}\cdot\xi_\perp\right|^2 \vec{n}\cdot[[\nabla\left(p + B^2/2\mu_0\right)]]$$

$$\delta W_V = \frac{1}{2}\int_V d\vec{r}\,\frac{\left|\hat{B}_1\right|^2}{\mu_0}$$

Boundary conditions on trial functions

$$\vec{n}\cdot\hat{B}_1\Big|_{r_w} = 0 \qquad \vec{n}\cdot\hat{B}_1\Big|_{r_p} = \hat{B}_1\cdot\nabla(\vec{n}\cdot\xi_\perp) - (\vec{n}\cdot\xi_\perp)[\vec{n}\cdot(\vec{n}\cdot\nabla)\hat{B}_1]\Big|_{r_p}$$

Stability

destabilising

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

stabilising

Pressure-driven modes (+ or -)

current-driven (kinks) modes (+ or -)

Energy required to bend magnetic field lines: dominant potential energy contribution to the shear Alfvén wave

Energy necessary to compress the magnetic field: major potential energy contribution to the compressional Alfvén wave

Energy required to compress the plasma: main source of potential energy for the sound wave

Stability - Tokamak

- **Ideal MHD Instabilities in a Tokamak**

1. Internal localised interchange instabilities: Mercier criterion
2. Low-n internal modes: Sawtooth
3. $m = 1$ external kink modes: Kruskal-Shafranov limit
4. Ballooning modes
5. External ballooning-kink modes
6. Resistive wall modes
7. $n = 0$ axisymmetric modes

Stability - Tokamak

- Internal localised interchange instabilities: Mercier criterion

$$\frac{d}{dx}\left(x^2\frac{d\xi}{dx}\right)+D_s\xi=0$$

Straight tokamak:
Euler-Lagrange equation

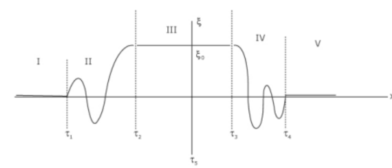
$$\xi = x^p$$

$$p(p+1)+D_s=0$$

$$p_{1,2}=-\frac{1}{2}\pm\frac{1}{2}(1-4D_s)^{1/2}$$

$$\left(\frac{rq'}{q}\right)^2+\frac{8\mu_0 rp'}{B_\phi^2}>0$$

Suydam's criterion



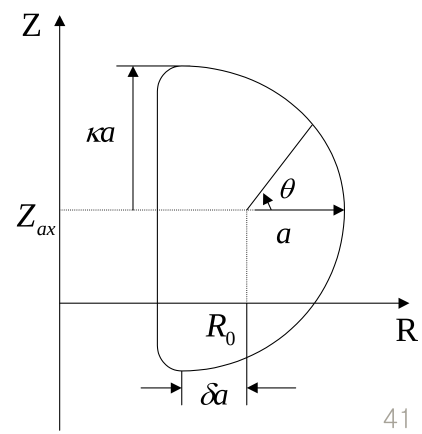
For a circular cross section, large aspect ratio with $\beta_p \sim 1$

$$\left(\frac{rq'}{q}\right)^2+4r\beta'(1-q^2)>0$$

Mercier criterion

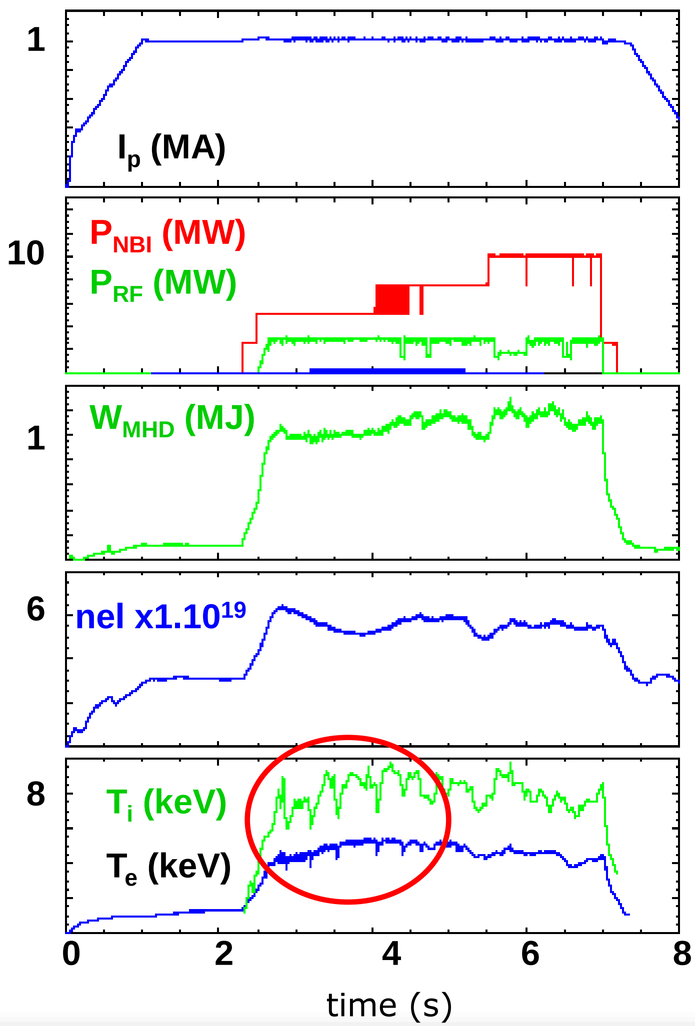
For a non-circular cross section

$$1<q_0^2\left\{1-\frac{4}{1+3\kappa^2}\left[\frac{3}{4}\frac{\kappa^2-1}{\kappa^2+1}\left(\kappa^2-\frac{2\delta}{\varepsilon}\right)+\frac{(\kappa-1)^2\beta_{p0}}{\kappa(\kappa+1)}\right]\right\}$$

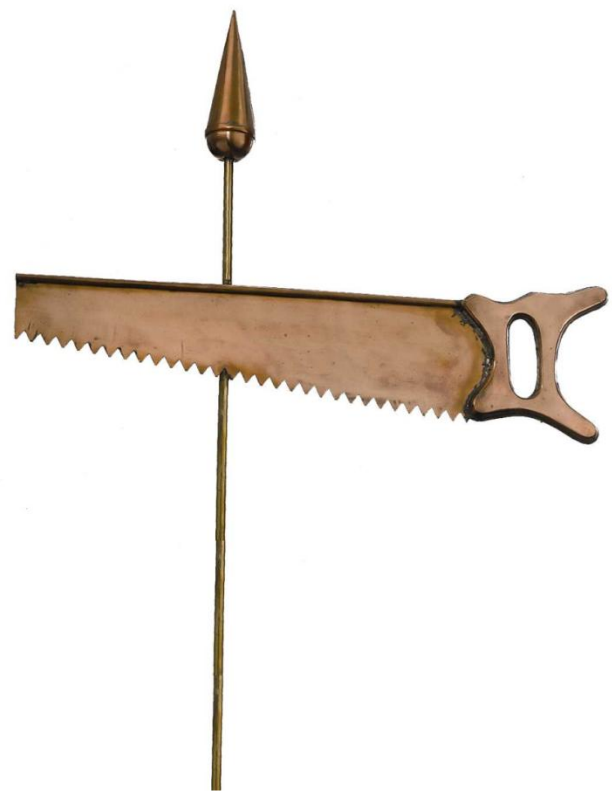


Stability - Tokamak

- Low-*n* internal modes: Sawtooth

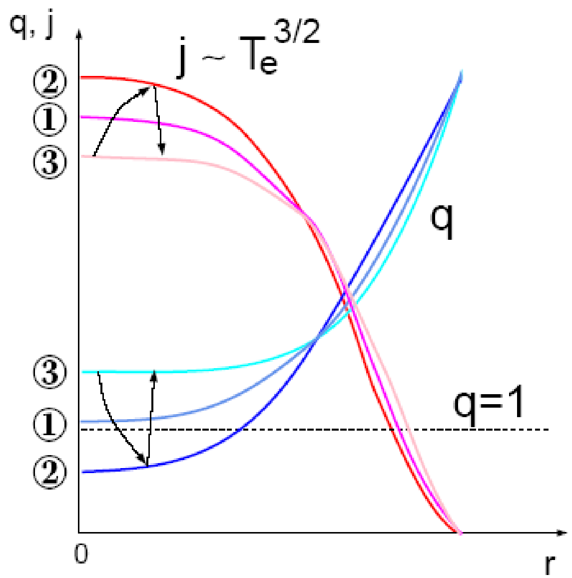


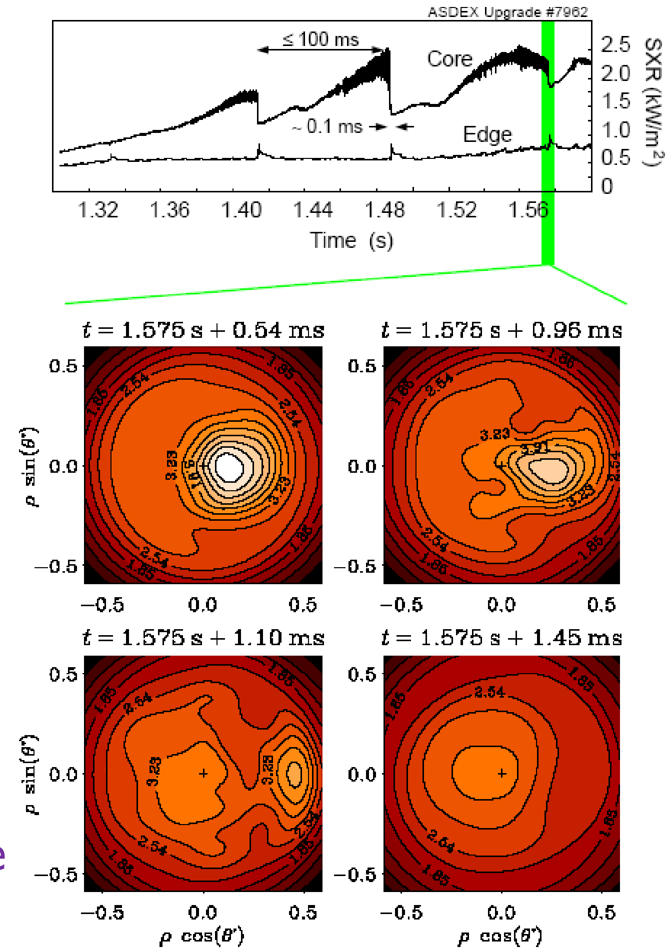
ASDEX Upgrade
pulse 20438



Stability - Tokamak

- Low-*n* internal modes: Sawtooth





ASDEX Upgrade

1. *T*(0) and *j*(0) rise
2. *q*(0) falls below 1
   → kink instability grows
3. Fast reconnection event:
   T, *n* flattened inside *q* = 1 surface
   q(0) rises slightly above 1
   kink stable

Stability - Tokamak

- m = 1 external kink mode: Kruskal-Shafranov limit

  - In the limit where the conducting wall moves to infinity

$$\frac{\delta W_2}{W_0} = \xi_0^2\left(n-\frac{1}{q_a}\right)\left[\left(n-\frac{1}{q_a}\right)+\left(n+\frac{1}{q_a}\right)\right] = 2\xi_0^2\left[n\left(n-\frac{1}{q_a}\right)\right]$$

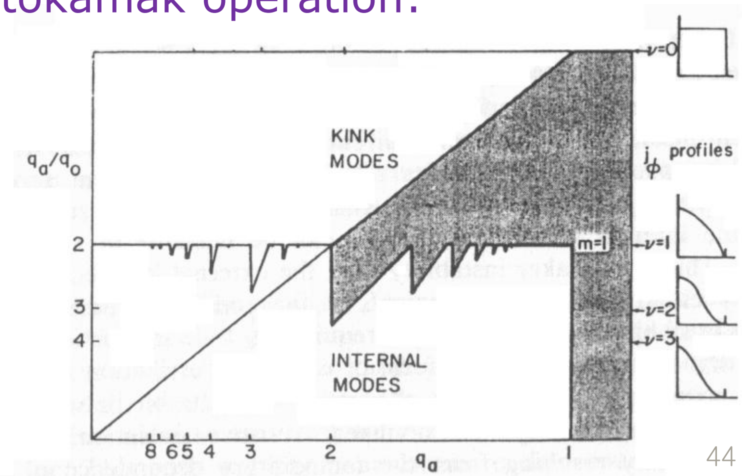
$$q_a > 1$$

Kruskal-Shafranov criterion:
stability condition for the m = 1 external kink mode for the worst case, n = 1

Imposing an important constraint on tokamak operation: toroidal current upper limit ($I < I_{KS}$)

$$I_{KS} \equiv 2\pi a^2 B_0 / \mu_0 R_0 = 5a^2 B_0 / R_0 \,[\mathrm{MA}]$$

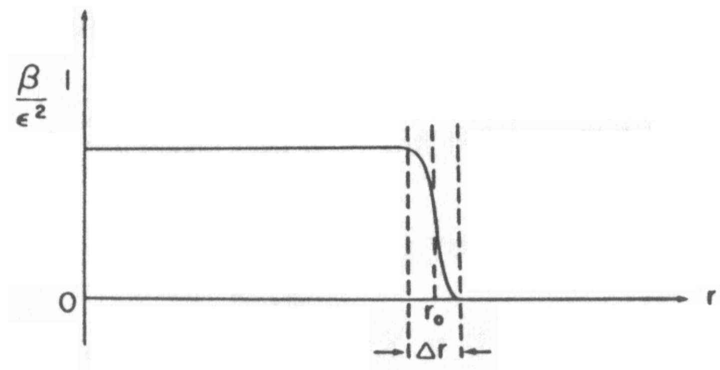
$$q_a = \frac{aB_0}{\mu_0 R_0 I_{KS} / 2\pi a} = 1$$



Stability - Tokamak

- Ballooning modes

Analytic Model



$$\frac{\partial}{\partial\theta}\left[(1+\Lambda^2)\frac{\partial X}{\partial\theta}\right]+\alpha(\Lambda\sin\theta+\cos\theta)X=0$$

desired form of the ballooning mode equation for the model equilibrium (*s*, *a*)

$$\Lambda(\theta)=s(\theta-\theta_0)-\alpha(\sin\theta-\sin\theta_0)$$

$$s=\frac{rq'}{q}$$ average shear

$$\alpha=-\frac{2\mu_0 r^2 p'}{R_0 B_\theta^2}=-\frac{r^2B_0^2}{R_0^2B_\theta^2}\cdot R_0\cdot\frac{p'}{B_0^2/2\mu_0}=-q^2R_0\beta'$$

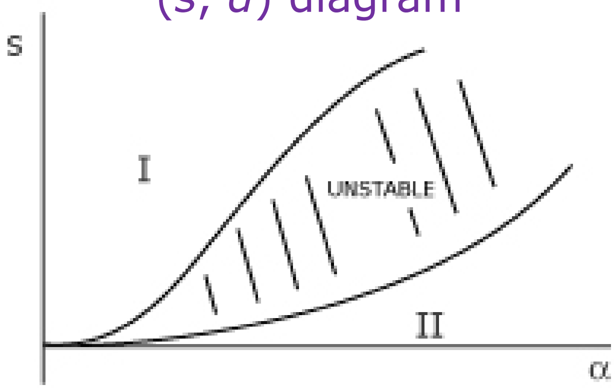
measure of the pressure gradient

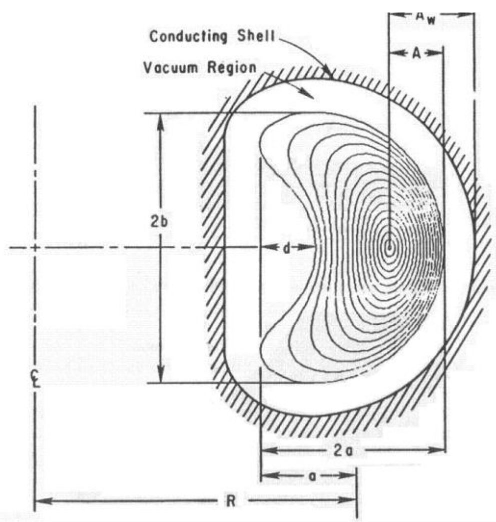
Stability - Tokamak

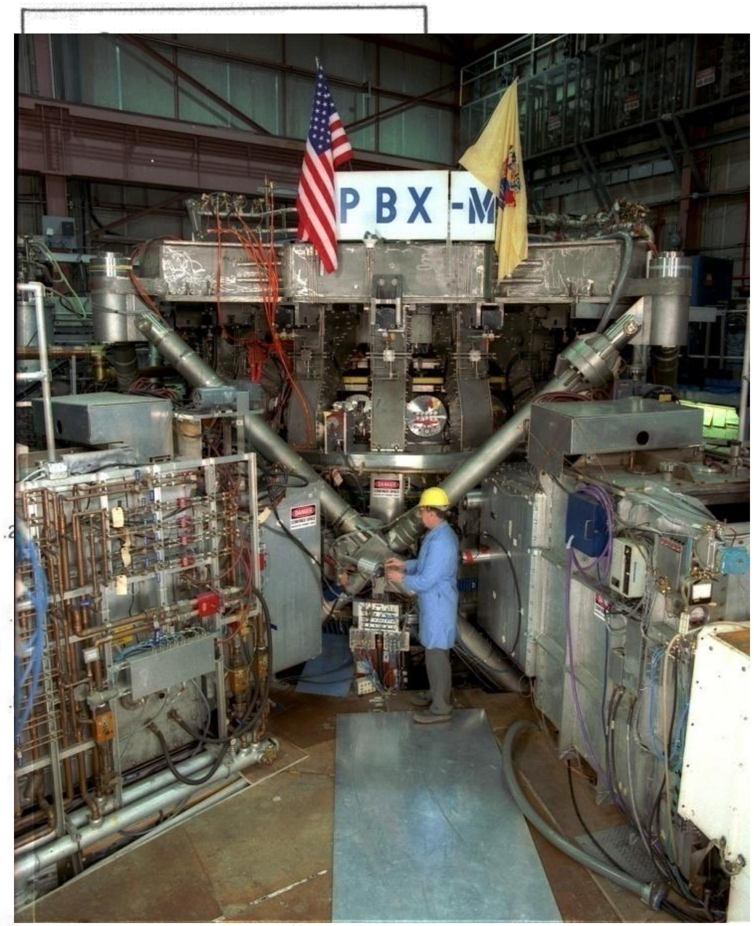
- Ballooning modes

  Numerical Solution

  (s, a) diagram







Stability - Tokamak

- External ballooning-kink modes

  Sharp Boundary Model: surface current model

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

$$\frac{\delta W_S}{W_0} = -\frac{1}{2\pi}\int_0^{2\pi} d\theta |\xi|^2 [b_\theta^2 + (\beta_t / \varepsilon)\cos\theta]$$ for high β

$$W_0 = 2\pi^2\varepsilon^2 B_0^2 R_0 / \mu_0, \quad b_\theta = B_\theta / \varepsilon B_0$$

$(\beta_t/\varepsilon)\cos\theta$: toroidal field curvature

β_t/ε: high β ballooning effect (pressure-driven term)

b_θ^2: destabilising effect due to the parallel current (kink term)

- For low β, the ballooning contribution negligible

Stability - Tokamak

- Numerical Results: The Sykes Limit, the Troyon Limit

Once an equilibrium is established, the following stability tests are made.

(1) Mercier stability

(2) High-n ballooning modes

(3) Low-n internal modes

(4) External ballooning-kink modes

- Helpful in the design of new experiments and in the interpretation and analysis of existing experimental data
- Playing a role in the determination of optimised configurations
- Quantitative predictions for the maximum β_t or I_0 and that can be stably maintained in MHD equilibrium

Troyon limit

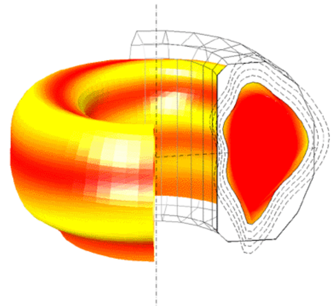
$$\beta_t = 0.028\left(\frac{I_0}{aB_0}\right)$$

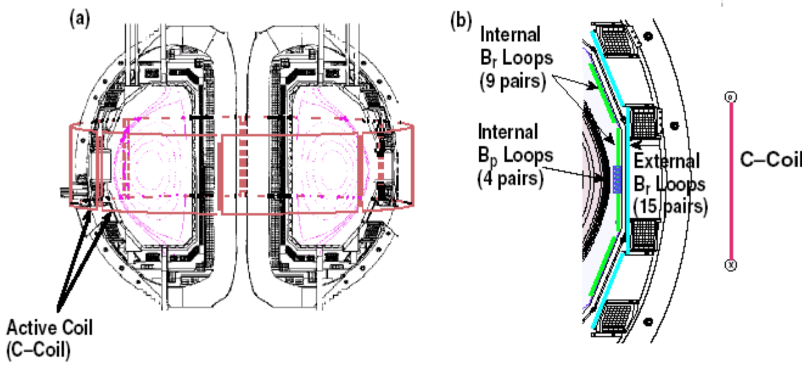
$$\beta_t = 0.14\left(\frac{\varepsilon\kappa}{q_*}\right)$$

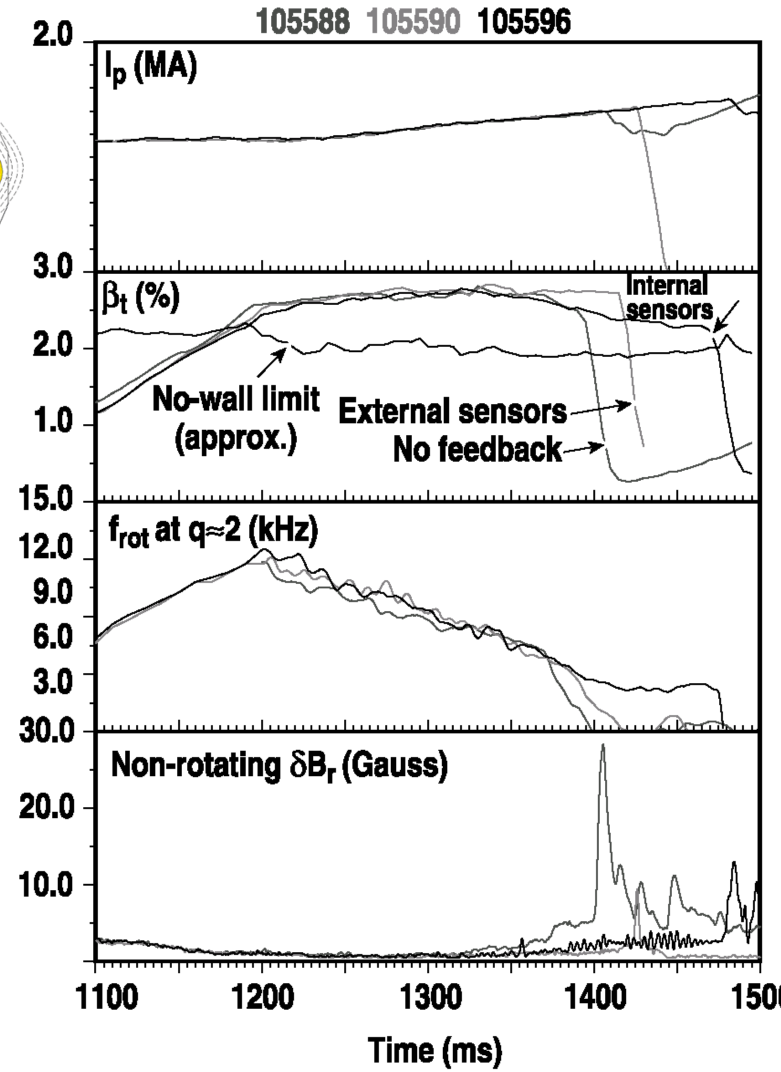
Stability - Tokamak

- Resistive wall modes

$$\omega_i \tau_D = -\frac{2|m|W^{2|m|}}{W^{2|m|}-1}\frac{\delta W_\infty}{\delta W_b}$$



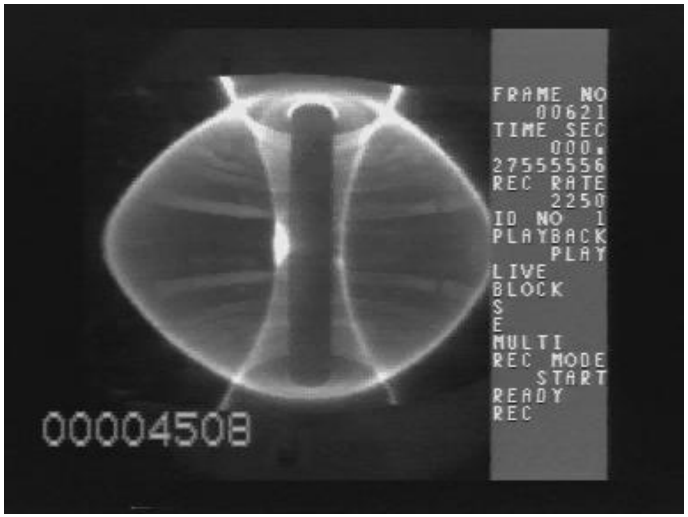


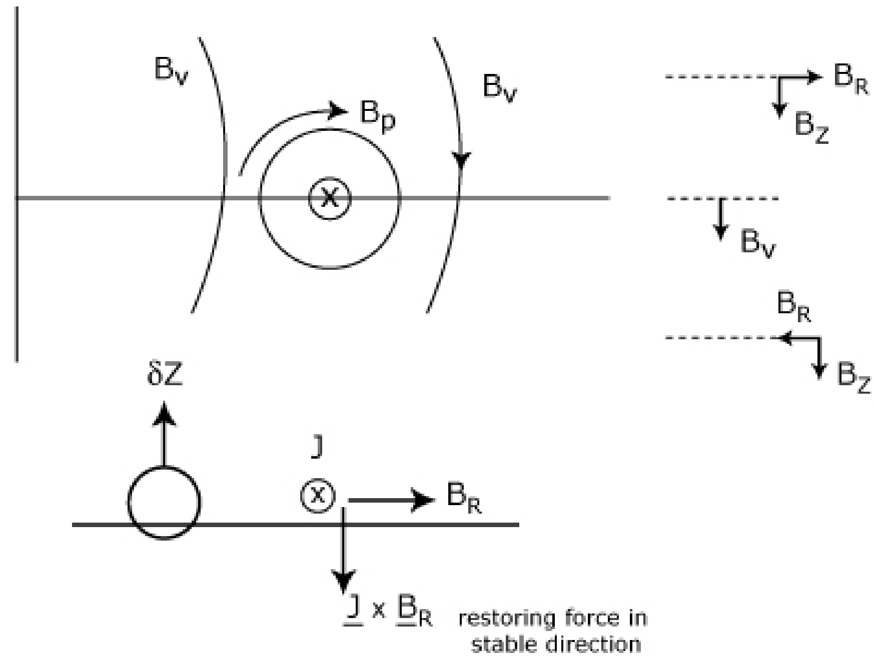


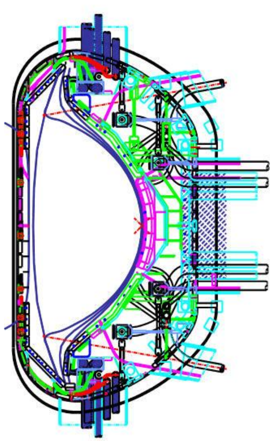
- Saddle coils for direct stabilisation
- Different feedback schemes exist
- First results look promising
- New experiments with in-vessel coils under way on DIII-D

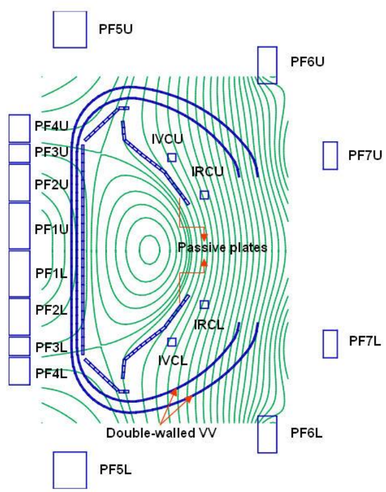
Stability - Tokamak

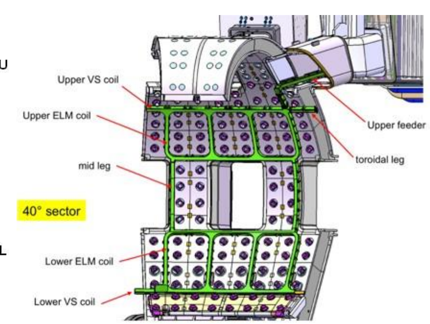
- $n = 0$ axisymmetric modes











field index

$$n(R_0, Z_0) = -\left(\frac{R}{B_Z}\frac{\partial B_Z}{\partial R}\right)_{R_0, Z_0}$$

$0 < n < 3/2$

Summary

- **Definition of Ideal MHD**
- **Equilibrium**
- **Stability**

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