Naval Architecture & Ocean Engineering

Computer Aided Ship Design
Part.3 Grillage Analysis of Midship Cargo Hold

2009 Fall

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Summary

u : Axial Displacement　G : Shear Modulus　A : Sectional Area　G : Shear Modulus　E : Young's Modulus

w : Vertical Displacement　θ : Angle of Twist　l : Length　J : Polar Moment of Inertia　I : Moment of Inertia

Element | Differential Equation → Variational Method → (•Discretization •Approximation) → Finite Element Method | •Superposition of Stiffness Matrix •Coordinates Transformation — **Behavior ⇒ Structure**

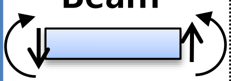
$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F}$, where $\ddot{\mathbf{x}} = 0$　　　　$\mathbf{Kd} = \mathbf{F}$

Bar

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0 \Rightarrow \delta\int_0^l\left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0 \Rightarrow u(x) = a_0 + a_1 x \Rightarrow \frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

Tension → **Truss**, **Frame**

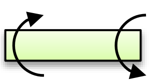
Beam



$$EI\frac{\partial^4 w(x)}{\partial x^4} + f(x) = 0 \Rightarrow \delta\int_0^l\left[\frac{EA}{2}\left(\frac{d^2w}{dx^2}\right)^2 + (f\,w)\right]dx = 0 \Rightarrow w(x) = b_0 + b_1x + b_2x^2 + b_3x^3 \Rightarrow \frac{2EI}{l^3}\begin{bmatrix}6 & 3l & -6 & 3l\\ 3l & 2l^2 & -3l & l^2\\ -6 & -3l & 6 & -3l\\ 3l & l^2 & -3l & 2l^2\end{bmatrix}\begin{bmatrix}u_1\\ \theta_1\\ u_2\\ \theta_2\end{bmatrix} = \begin{bmatrix}f_1\\ M_1\\ f_2\\ M_2\end{bmatrix}$$

Bending → **Frame**, **Grillage**

Shaft



$$GJ\frac{\partial^2\theta(x)}{\partial x^2} + f(x) = 0 \Rightarrow \delta\int_0^l\left[\frac{GJ}{2}\left(\frac{d\theta}{dx}\right)^2 - (f\,\theta)\right]dx = 0 \Rightarrow \theta(x) = c_0 + c_1x \Rightarrow \frac{GJ}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}M_1\\ M_2\end{bmatrix}$$

Torsion → **Grillage**

Application

Midship Cargo Hold → (Engineering Concept !) → **Grillage Modeling** → **Solution**

Grillage Modeling:
- 3D→ 2D
- Equivalent Force & Moment
- Boundary condition

Solution:
- programming
- visulaization

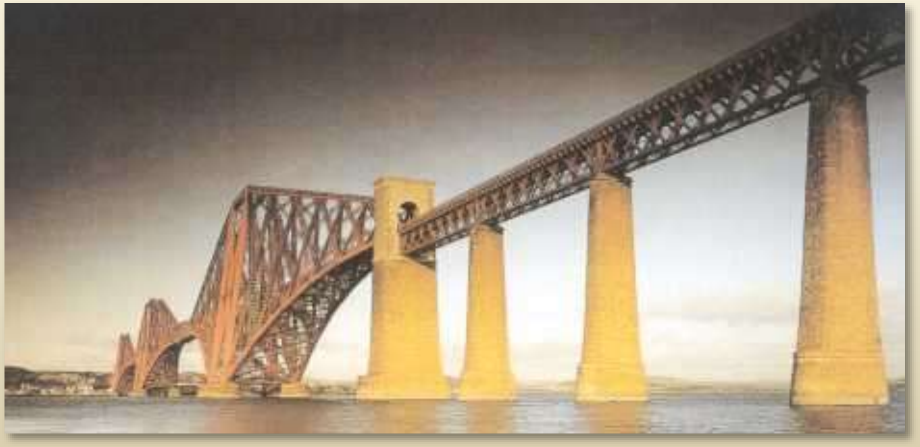
Beam Theory : Sign Convention, Deflection of Beam

Elasticity : Displacement, Strain, Stress, Force Equilibrium, Compatibility, Constitutive Equation

Chapter 5. Truss

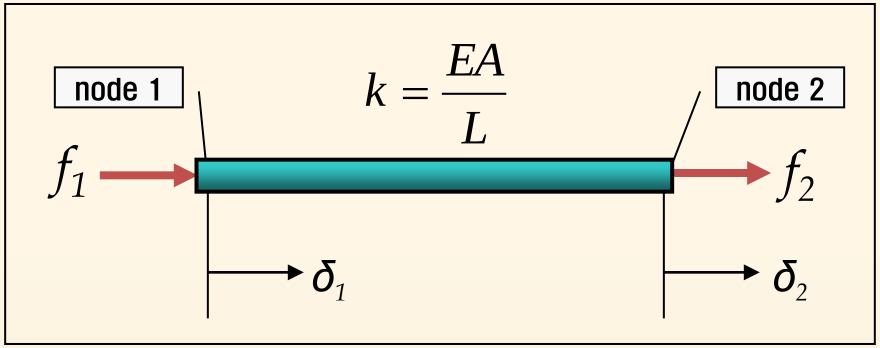
Truss

- Plane Truss (2-Dimensional Truss) : a structure composed of bar elements that all lie in a common plane and are connected by frictionless pins. The plane truss also must have loads acting only in the common plane*

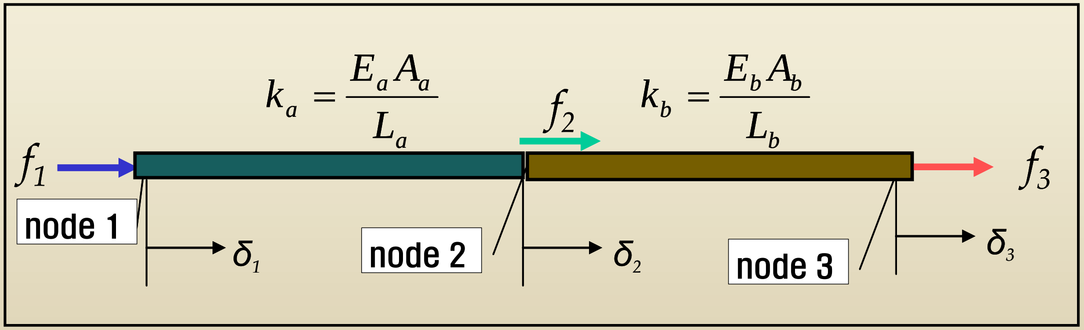


*Daryl L. Logan, A first Course in the Finite Element Method, 2nd edition, PWS Publishing,1993, p.80

1-Dimensional Truss



$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$



$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{E_a A_a}{L_a} & -\frac{E_a A_a}{L_a} & 0 \\ -\frac{E_a A_a}{L_a} & \frac{E_a A_a}{L_a} + \frac{E_b A_b}{L_b} & -\frac{E_b A_b}{L_b} \\ 0 & -\frac{E_b A_b}{L_b} & \frac{E_b A_b}{L_b} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

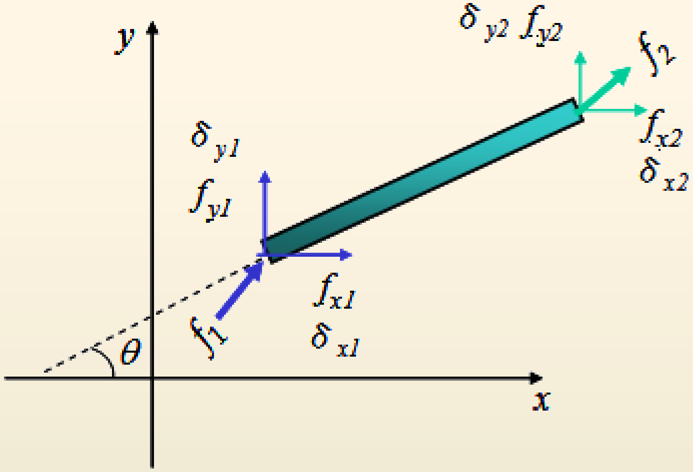
2-Dimensional Truss

$$[\mathbf{T}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$[\mathbf{F}_{xy}] = [\mathbf{T}]^T [\mathbf{K}_{pq}][\mathbf{T}][\delta_{xy}] = [\mathbf{K}_{xy}][\delta_{xy}]$$

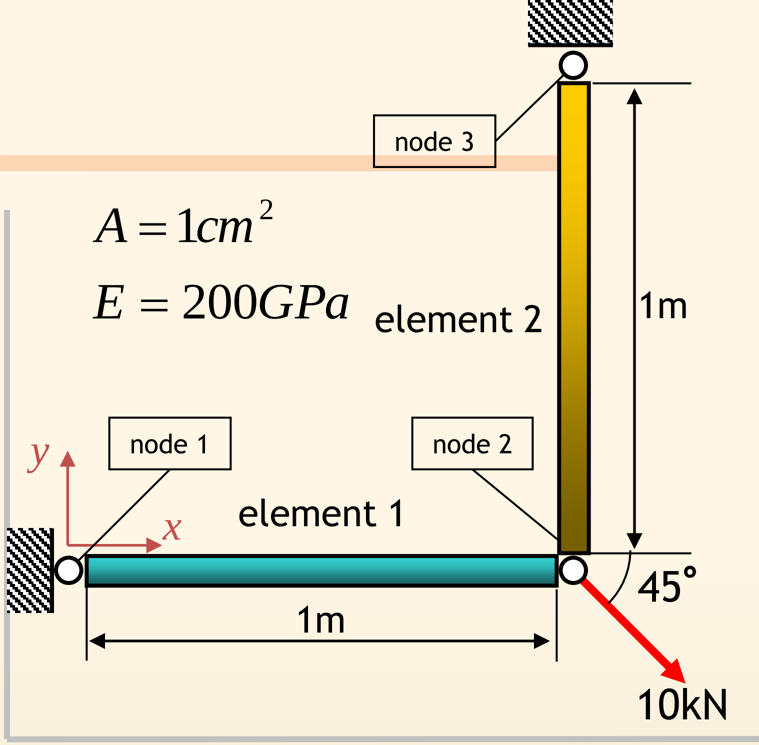
$$[\mathbf{F}_{xy}] = [\mathbf{K}_{xy}][\delta_{xy}]$$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{x2} \\ \delta_{y2} \end{bmatrix}$$



Ex.) 2-Dimenstional Truss

ex.) Find displacements and reaction force at each nodes of frame in the following figure.



Step1. Input Data

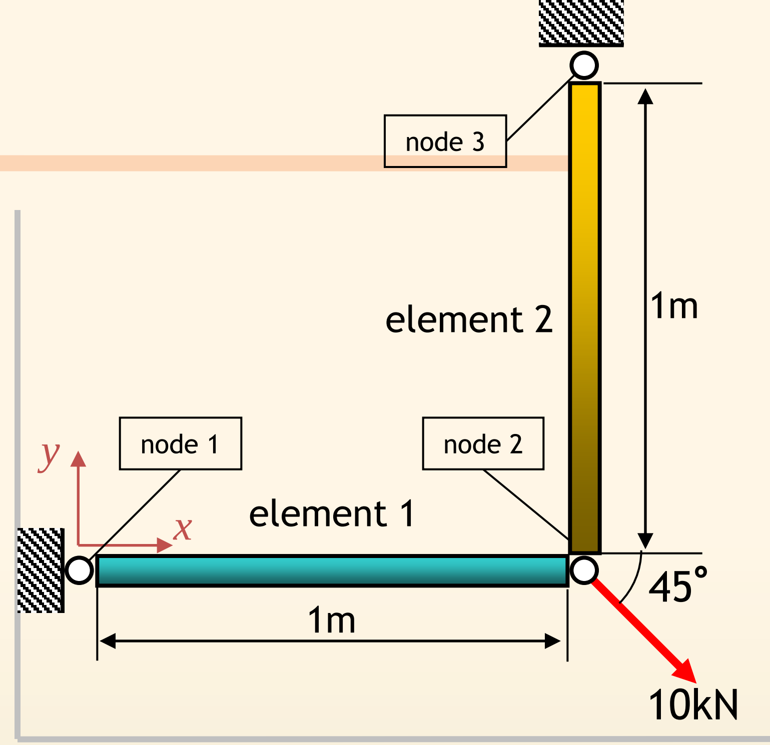
- constants($\theta_1 = 0$,$\theta_2 = 90$)

$$200GPa = 200 \times 10^9 N/m^2$$
$$= 200 \times 10^5 N/cm^2$$

$$\frac{EA}{L} = \frac{200 \times 10^5 N/cm^2 \cdot 1cm^2}{100cm} = 2 \times 10^5 N/cm$$

element	cos θ	sin θ	length (cm)	Sectional area (㎠)	Young's modulus (GPa)
1	1	0	100	1	200
2	0	1			

Ex.) 2-Dimenstional Truss



Step2. Stiffness Equation

$$[\mathbf{F}_{xy}] = [\mathbf{K}_{xy}][\delta_{xy}]$$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = k \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{x2} \\ \delta_{y2} \end{bmatrix}$$

☞ **element 1**

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} 2\times10^5 & 0 & -2\times10^5 & 0 \\ 0 & 0 & 0 & 0 \\ -2\times10^5 & 0 & 2\times10^5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{x2} \\ \delta_{y2} \end{bmatrix}$$

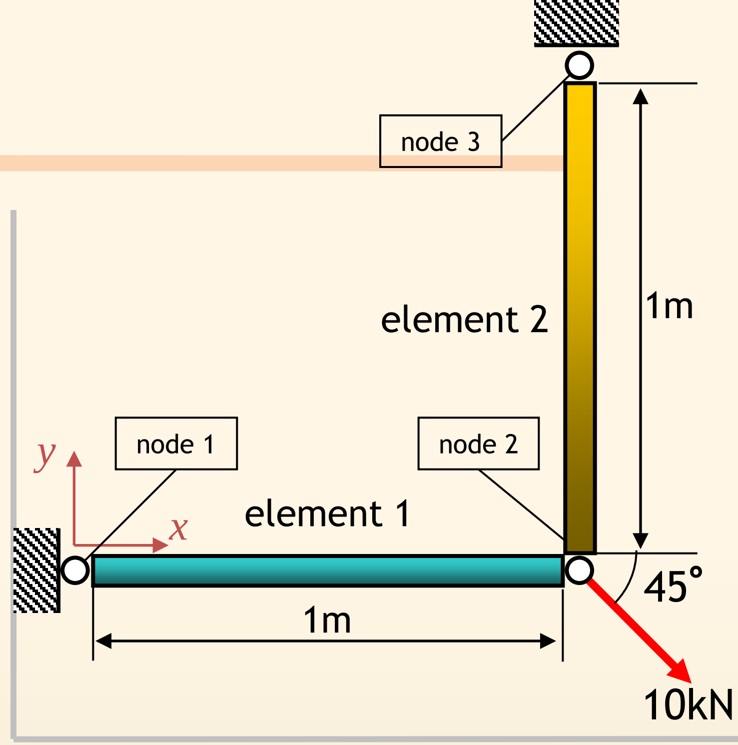
☞ **element 2**

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\times10^5 & 0 & -2\times10^5 \\ 0 & 0 & 0 & 0 \\ 0 & -2\times10^5 & 0 & 2\times10^5 \end{bmatrix} \begin{bmatrix} \delta_{x2} \\ \delta_{y2} \\ \delta_{x3} \\ \delta_{y3} \end{bmatrix}$$

☞ stiffness equation

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 2\times10^5 & 0 & -2\times10^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2\times10^5 & 0 & 2\times10^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\times10^5 & 0 & -2\times10^5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\times10^5 & 0 & 2\times10^5 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{x2} \\ \delta_{y2} \\ \delta_{x3} \\ \delta_{y3} \end{bmatrix}$$

Ex.) 2-Dimenstional Truss

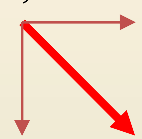


Step3. Find Displacements

- known/unknown displacements
  - ✓ known : $\delta_{x1}, \delta_{y1}, \delta_{x3}, \delta_{y3}$ (=0)
  - ✓ unknown : δ_{x2}, δ_{y2}
- known/unknown forces
  - ✓ known : $f_{x2}(=7{,}070N)$, $f_{y2}(=-7{,}070N)$
  - ✓ unknown : $f_{x1}, f_{y1}, f_{x3}, f_{y3}$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2}=7070 \\ f_{y2}=-7070 \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 2\times10^5 & 0 & -2\times10^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2\times10^5 & 0 & 2\times10^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\times10^5 & 0 & -2\times10^5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\times10^5 & 0 & 2\times10^5 \end{bmatrix} \begin{bmatrix} \delta_{x1}=0 \\ \delta_{y1}=0 \\ \delta_{x2} \\ \delta_{y2} \\ \delta_{x3}=0 \\ \delta_{y3}=0 \end{bmatrix}$$

$7{,}070N = 7.07kN$



$7{,}070N$
$= 7.07kN$

$$\begin{bmatrix} 5000\sqrt{2} \\ -5000\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2\times10^5 & 0 \\ 0 & 2\times10^5 \end{bmatrix} \begin{bmatrix} \delta_{x2} \\ \delta_{y2} \end{bmatrix}$$

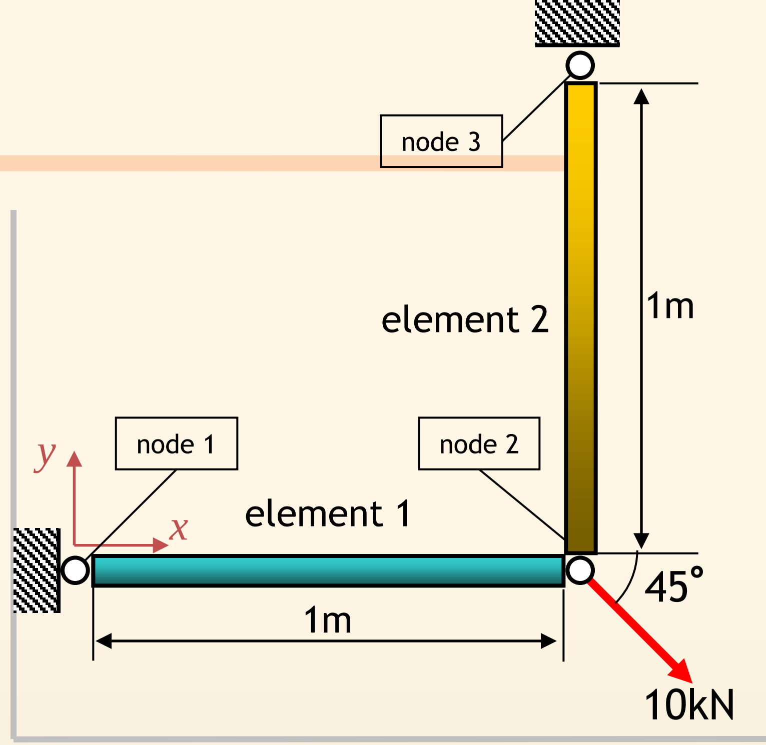
$$\begin{bmatrix} \delta_{x2} \\ \delta_{y2} \end{bmatrix} = \begin{bmatrix} 2\times10^5 & 0 \\ 0 & 2\times10^5 \end{bmatrix}^{-1} \begin{bmatrix} 5000\sqrt{2} \\ -5000\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.03535 \\ -0.03535 \end{bmatrix}$$

Ex.) 2-Dimenstional Truss

Step4. Find Reaction Forces

$$\begin{bmatrix}\delta_{x2}\\ \delta_{y2}\end{bmatrix}=\begin{bmatrix}0.03535\\ -0.03535\end{bmatrix}$$

$$\begin{bmatrix}f_{x1}\\ f_{y1}\\ f_{x2}\\ f_{y2}\\ f_{x3}\\ f_{y3}\end{bmatrix}=\begin{bmatrix}2\times10^5 & 0 & -2\times10^5 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -2\times10^5 & 0 & 2\times10^5 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\times10^5 & 0 & -2\times10^5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -2\times10^5 & 0 & 2\times10^5\end{bmatrix}\begin{bmatrix}\delta_{x1}=0\\ \delta_{y1}=0\\ \delta_{x2}\\ \delta_{y2}\\ \delta_{x3}=0\\ \delta_{y3}=0\end{bmatrix}$$



$$\begin{bmatrix}f_{x1}\\ f_{y1}\\ f_{x3}\\ f_{y3}\end{bmatrix}=\begin{bmatrix}-2\times10^5 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 2\times10^5\end{bmatrix}\begin{bmatrix}0.03535\\ -0.03535\end{bmatrix}\approx\begin{bmatrix}-7070\\ 0\\ 0\\ 7070\end{bmatrix}(N)$$

- reaction forces at node 2

$$\begin{bmatrix}f_{x1}\\ f_{y1}\\ f_{x2}\\ f_{y2}\\ f_{x3}\\ f_{y3}\end{bmatrix}=\begin{bmatrix}-7.07\\ 0\\ 7.07\\ -7.07\\ 0\\ 7.07\end{bmatrix}(kN) \qquad \begin{bmatrix}\delta_{x1}\\ \delta_{y1}\\ \delta_{x2}\\ \delta_{y2}\\ \delta_{x3}\\ \delta_{y3}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0.03535\\ -0.03535\\ 0\\ 0\end{bmatrix}(cm)$$

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Summary

u: Axial Displacement G: Shear Modulus A: Sectional Area G: Shear Modulus E: Young's Modulus

w: Vertical Displacement θ: Angle of Twist l: Length J: Polar Moment of Inertia I: Moment of Inertia

Element | Differential Equation ⇨ Variational Method ⇨ •Discretization •Approximation ⇨ Finite Element Method | •Superposition of Stiffness Matrix •Coordinates Transformation **Behavior ⇨ Structure**

$$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F}\ , where\ \ddot{\mathbf{x}} = 0$$

$$\mathbf{Kd} = \mathbf{F}$$

Bar

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0 \quad ⇨ \quad \delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0 \quad ⇨$$

$$u(x) = a_0 + a_1 x$$

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Tension → **Truss**, **Frame**

Beam

$$EI\frac{\partial^4 w(x)}{\partial x^4} + f(x) = 0 \quad ⇨ \quad \delta\int_0^l \left[\frac{EA}{2}\left(\frac{d^2w}{dx^2}\right)^2 + (f\,w)\right]dx = 0 \quad ⇨$$

$$w(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$\frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{bmatrix}$$

Bending → **Frame**, **Grillage**

Shaft

$$GJ\frac{\partial^2 \theta(x)}{\partial x^2} + f(x) = 0 \quad ⇨ \quad \delta\int_0^l \left[\frac{GJ}{2}\left(\frac{d\theta}{dx}\right)^2 - (f\,\theta)\right]dx = 0 \quad ⇨$$

$$\theta(x) = c_0 + c_1 x$$

$$\frac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Torsion → **Grillage**

Application

Midship Cargo Hold ⟶ (Engineering Concept !) **Grillage Modeling** ⟶ **Solution**

Grillage Modeling:
- 3D→ 2D
- Equivalent Force & Moment
- Boundary condition

Solution:
- programming
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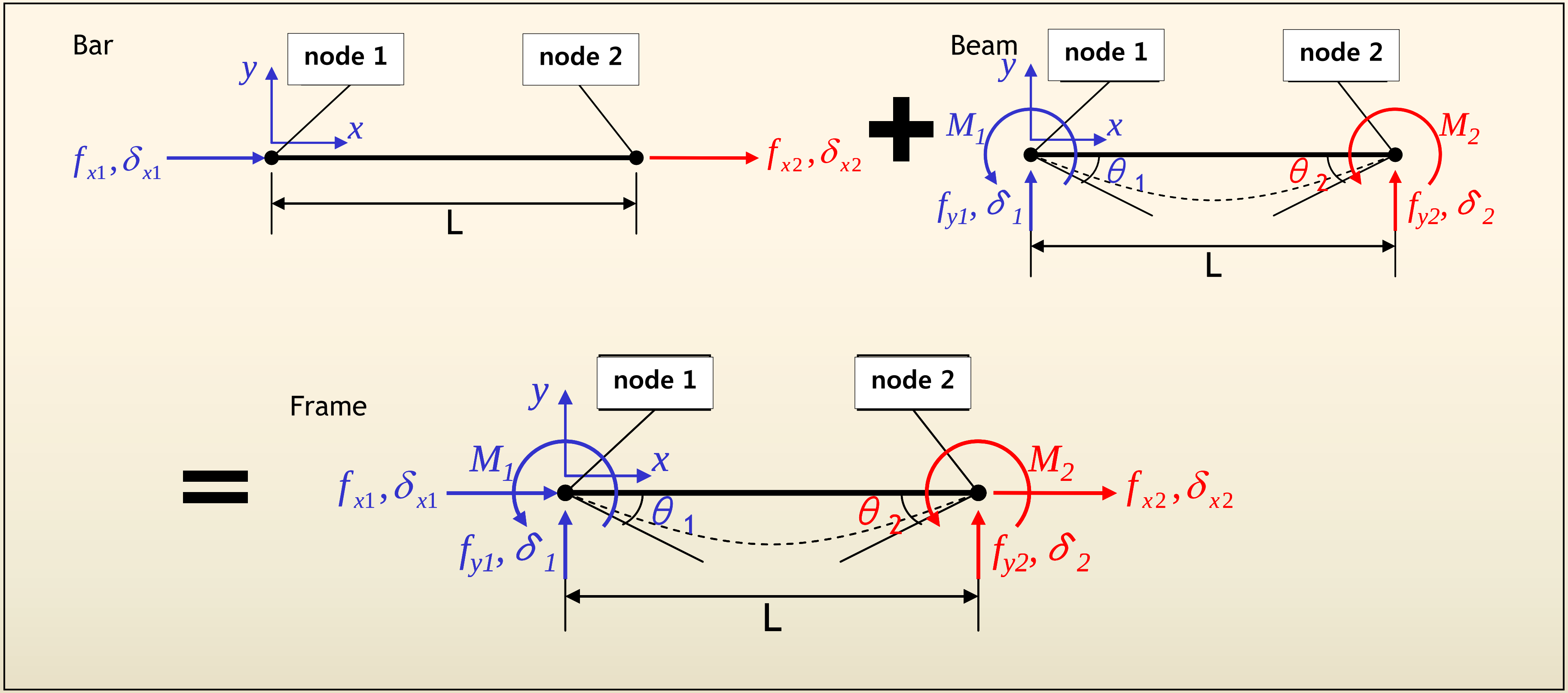
Beam Theory : Sign Convention, Deflection of Beam

Elasticity : Displacement, Strain, Stress, Force Equilibrium, Compatibility, Constitutive Equation

Chapter 6. Frame

Frame

- Frame = Bar + Beam



Frame

Stiffness Equation of Frame = Stiffness Equation of Bar + Stiffness Equation of Beam

Bar

$$\begin{bmatrix} f_{x1} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

+

Beam

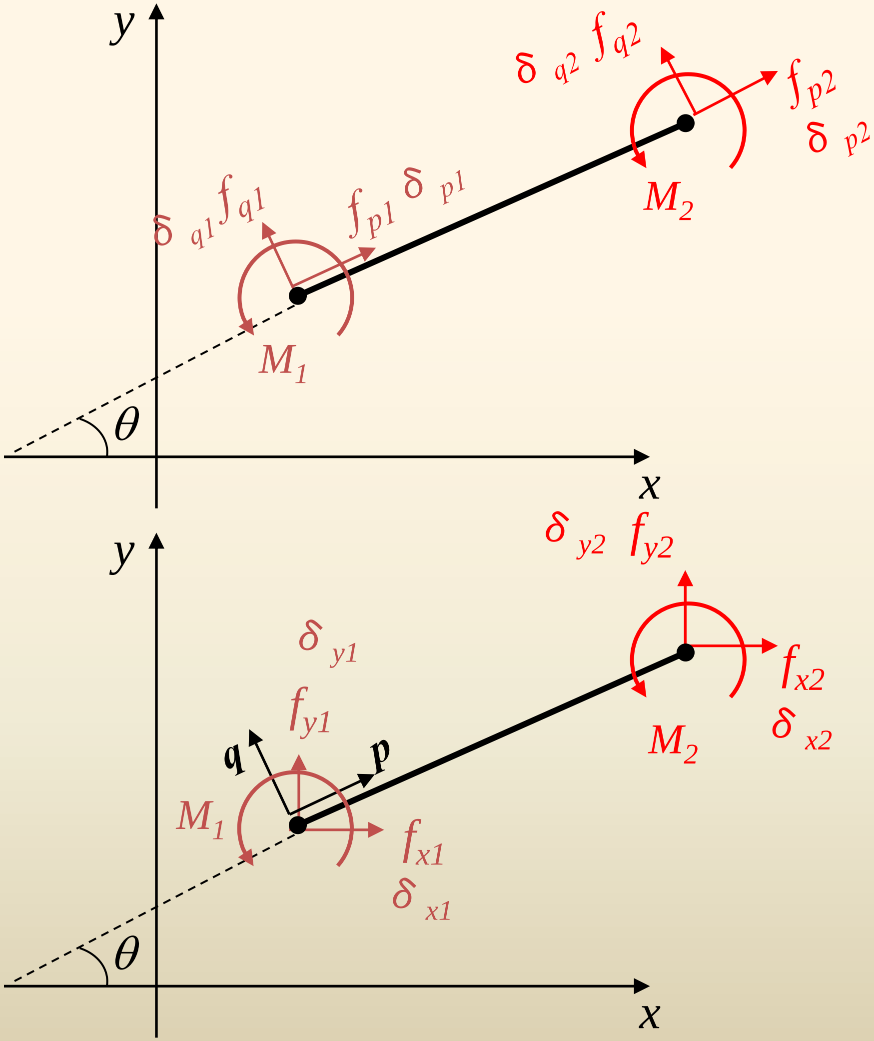
$$\begin{bmatrix} f_{y1} \\ M_1 \\ f_{y2} \\ M_2 \end{bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \delta_{y1} \\ \theta_1 \\ \delta_{y2} \\ \theta_2 \end{bmatrix}$$

Frame

=

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x2} \\ f_{y2} \\ M_2 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \theta_1 \\ \delta_{x2} \\ \delta_{y2} \\ \theta_2 \end{bmatrix}$$

2-Dimensional Frame



- transformation between the coordinate systems

(1) displacements and forces : transformation

(2) moments : no transformation

$$[\delta_{pq}] = [\mathbf{T}][\delta_{xy}]$$

$$\begin{bmatrix} \delta_{p1} \\ \delta_{q1} \\ \theta_1 \\ \delta_{p2} \\ \delta_{q2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \theta_1 \\ \delta_{x2} \\ \delta_{y2} \\ \theta_2 \end{bmatrix}$$

$$[f_{pq}] = [\mathbf{T}][f_{xy}]$$

$$\begin{bmatrix} f_{p1} \\ f_{q1} \\ M_1 \\ f_{p2} \\ f_{q2} \\ M_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x2} \\ f_{y2} \\ M_2 \end{bmatrix}$$

2-Dimensional Frame

(3) stiffness equation

$$[f_{pq}] = [\mathbf{K}_{pq}][\delta_{pq}] \quad \Rightarrow \quad [\mathbf{T}][f_{xy}] = [\mathbf{K}_{pq}][\mathbf{T}][\delta_{xy}]$$

$$[f_{pq}] = [\mathbf{T}][f_{xy}] \qquad [\delta_{pq}] = [\mathbf{T}][\delta_{xy}]$$

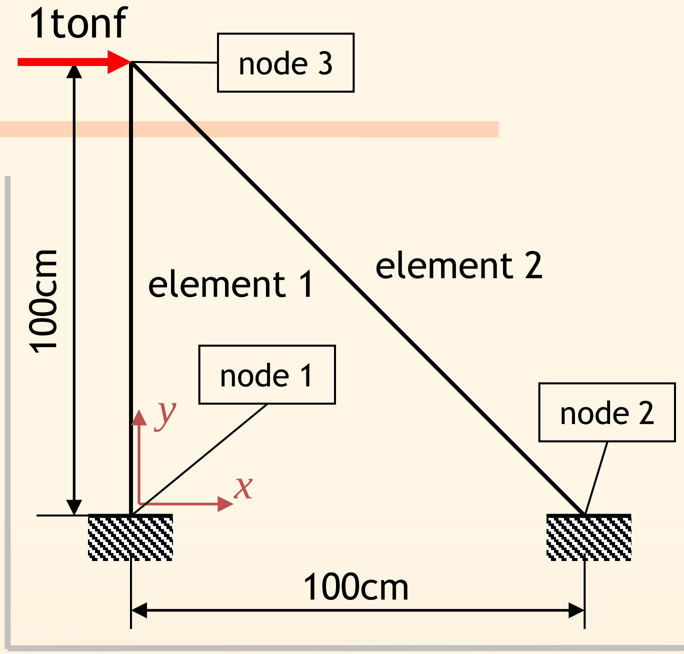
multiply $[\mathbf{T}]^{-1} = [\mathbf{T}]^T$

$$[f_{xy}] = [\mathbf{T}]^T[\mathbf{K}_{pq}][\mathbf{T}][\delta_{xy}]$$

$$[\mathbf{T}]^T[\mathbf{K}_{pq}][\mathbf{T}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{EA}{L}\cos^2\theta + \frac{12EI}{L^3}\sin^2\theta & \frac{EA}{L}\sin\theta\cos\theta - \frac{12EI}{L^3}\sin\theta\cos\theta & -\frac{6EI}{L^2}\sin\theta & -\frac{EA}{L}\cos^2\theta - \frac{12EI}{L^3}\sin^2\theta & -\frac{EA}{L}\sin\theta\cos\theta + \frac{12EI}{L^3}\sin\theta\cos\theta & -\frac{6EI}{L^2}\sin\theta \\ - & \frac{EA}{L}\sin^2\theta + \frac{12EI}{L^3}\cos^2\theta & \frac{6EI}{L^2}\cos\theta & \frac{EA}{L}\sin\theta\cos\theta - \frac{12EI}{L^3}\sin\theta\cos\theta & -\frac{EA}{L}\sin^2\theta - \frac{12EI}{L^3}\cos^2\theta & \frac{6EI}{L^2}\cos\theta \\ - & - & \frac{4EI}{L} & \frac{6EI}{L^2}\sin\theta & \frac{6EI}{L^2}\cos\theta & \frac{2EI}{L} \\ - & \textit{symmetric} & - & \frac{EA}{L}\cos^2\theta + \frac{12EI}{L^3}\sin^2\theta & \frac{EA}{L}\sin\theta\cos\theta - \frac{12EI}{L^3}\sin\theta\cos\theta & \frac{6EI}{L^2}\sin\theta \\ - & - & - & - & \frac{EA}{L}\sin^2\theta + \frac{12EI}{L^3}\cos^2\theta & -\frac{6EI}{L^2}\cos\theta \\ - & - & - & - & - & \frac{4EI}{L} \end{bmatrix}$$

Ex.) Frame



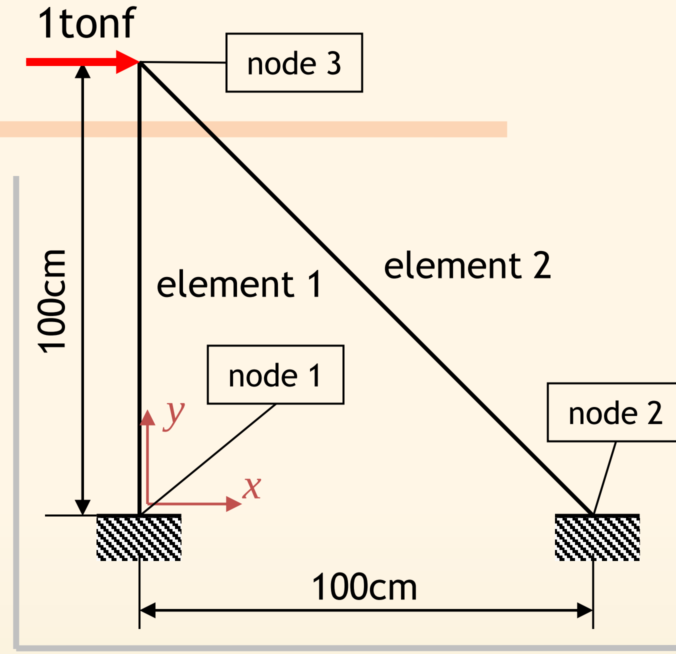
ex) Find displacements and reaction force at each nodes of frame in the following figure.

Step1. Input Data

- constants ($\theta_1 = 90$, $\theta_2 = 135$)

element	cos θ	sin θ	length (cm)	sectional area (cm²)	Young's modulus (kgf/cm²)	moment of inertia (cm⁴)
1	0	1	100	1	10^6	0.5
2	$-\sqrt{2}/2$	$\sqrt{2}/2$	$100\sqrt{2}$			

Ex.) Frame



Step2. Stiffness Equation

$$[\mathbf{F}_{xy}]=[\mathbf{K}_{xy}][\delta_{xy}]$$

☞ element 1

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -300 & -6 & 0 & -300 \\ 0 & 10000 & 0 & 0 & -10000 & 0 \\ -300 & 0 & 20000 & 300 & 0 & 10000 \\ -6 & 0 & 300 & 6 & 0 & 300 \\ 0 & -10000 & 0 & 0 & 10000 & 0 \\ -300 & 0 & 10000 & 300 & 0 & 20000 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \theta_1 \\ \delta_{x3} \\ \delta_{y3} \\ \theta_3 \end{bmatrix}$$

☞ element 2

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ M_2 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} 3536.59 & -3534.47 & -106.07 & -3536.59 & 3534.47 & -106.07 \\ -3534.47 & 3536.59 & -106.07 & 3534.47 & -3536.59 & -106.07 \\ -106.07 & -106.07 & 14142.14 & 106.07 & 106.07 & 7071.07 \\ -3536.59 & 3534.47 & 106.07 & 3536.59 & -3534.47 & 106.07 \\ 3534.47 & -3536.59 & 106.07 & -3534.47 & 3536.59 & 106.07 \\ -106.07 & -106.07 & 7071.07 & 106.07 & 106.07 & 14142.14 \end{bmatrix} \begin{bmatrix} \delta_{x2} \\ \delta_{y2} \\ \theta_2 \\ \delta_{x3} \\ \delta_{y3} \\ \theta_3 \end{bmatrix}$$

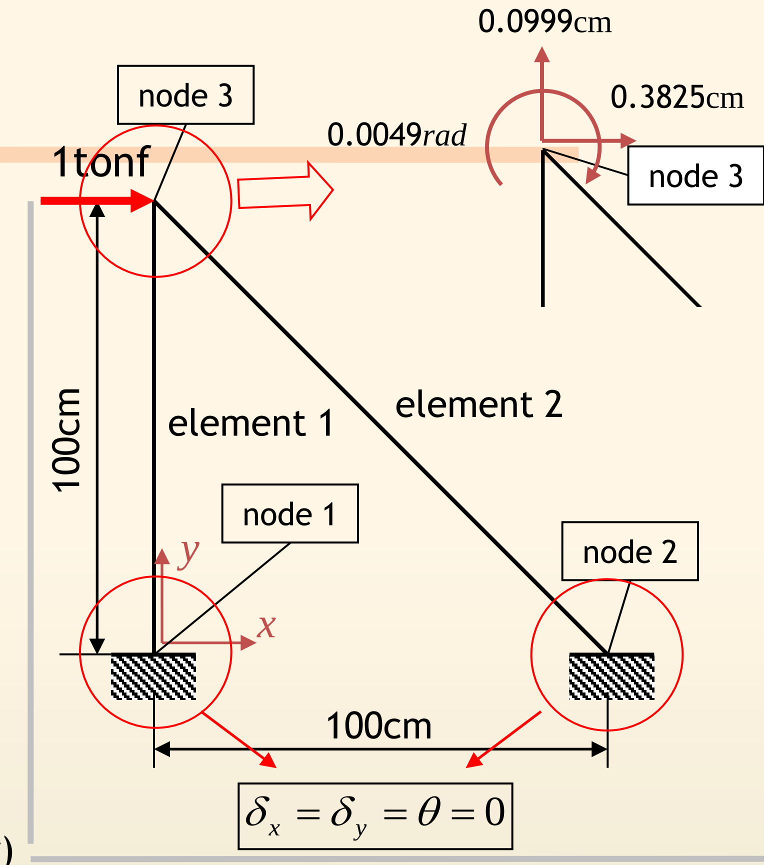
※New Recommanded Method

: 강성 방정식의 크기가 커지므로, known, unknown을 구별한 뒤 필요한 부분만을 모아서 강성 방정식을 구성

Ex.) Frame

Step3. Find Displacements

- known/unknown displacements
  - ✓ known : $\delta_{x1}, \delta_{y1}, \theta_1, \delta_{x2}, \delta_{y2}, \theta_2 (=0)$
  - ✓ unknown : $\delta_{x3}, \delta_{y3}, \theta_3$
- known/unknown forces
  - ✓ known : f_{x3}(=1tonf), f_{y3}(=0), M_3(=0)
  - ✓ unknown : f_{x1}, f_{y1}, M_1, f_{x2}, f_{y2}, M_2



$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -300 & -6 & 0 & -300 \\ 0 & 10000 & 0 & 0 & -10000 & 0 \\ 0-300 & 0 & 20000 & 300 & 0 & 10000 \\ -6 & 0 & 300 & 6 & 0 & 300 \\ 0 & -10000 & 0 & 0 & 10000 & 0 \\ -300 & 0 & 10000 & 300 & 0 & 20000 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \theta_1 \\ \delta_{x3} \\ \delta_{y3} \\ \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ M_2 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} 3536.59 & -3534.47 & -106.07 & -3536.59 & 3534.47 & -106.07 \\ -3534.47 & 3536.59 & -106.07 & 3534.47 & -3536.59 & -106.07 \\ -106.07 & -106.07 & 14142.14 & 106.07 & 106.07 & 7071.07 \\ -3536.59 & 3534.47 & 106.07 & 3536.59 & -3534.47 & 106.07 \\ 3534.47 & -3536.59 & 106.07 & -3534.47 & 3536.59 & 106.07 \\ -106.07 & -106.07 & 7071.07 & 106.07 & 106.07 & 14142.14 \end{bmatrix} \begin{bmatrix} \delta_{x2} \\ \delta_{y2} \\ \theta_2 \\ \delta_{x3} \\ \delta_{y3} \\ \theta_3 \end{bmatrix}$$

(box in blue color)

$$\begin{bmatrix} f_{x3} = 1000kgf \\ f_{y3} = 0 \\ M_3 = 0 \end{bmatrix} = \begin{bmatrix} 3542.59 & -3534.47 & 406.07 \\ -3534.47 & 13536.59 & 10607 \\ 406.07 & 106.07 & 34142.14 \end{bmatrix} \begin{bmatrix} \delta_{x3} \\ \delta_{y3} \\ \theta_3 \end{bmatrix}$$

known　　　　unknown

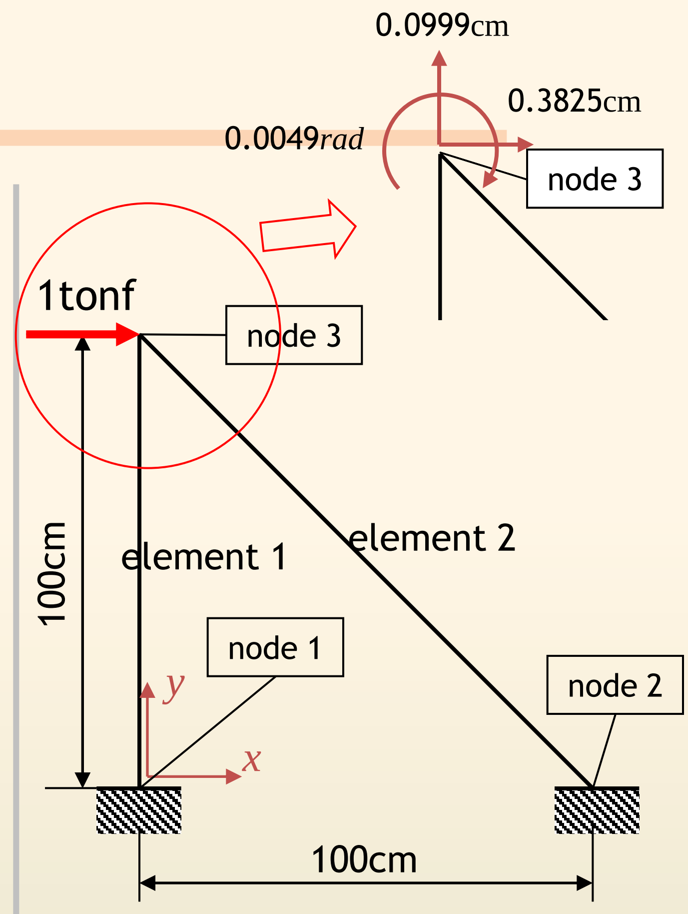
$$\begin{bmatrix} \delta_{x3} \\ \delta_{y3} \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 3542.59 & -3534.47 & 406.07 \\ -3534.47 & 13536.59 & 10607 \\ 406.07 & 106.07 & 34142.14 \end{bmatrix}^{-1} \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3825cm \\ 0.0999cm \\ -0.0049rad \end{bmatrix}$$

Ex.) Frame

Step4. Find Reaction Forces

- reaction forces for element 1

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -300 & -6 & 0 & -300 \\ 0 & 10000 & 0 & 0 & -10000 & 0 \\ 0-300 & 0 & 20000 & 300 & 0 & 10000 \\ -6 & 0 & 300 & 6 & 0 & 300 \\ 0 & -10000 & 0 & 0 & 10000 & 0 \\ -300 & 0 & 10000 & 300 & 0 & 20000 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.3825 \\ 0.0999 \\ -0.0049 \end{bmatrix} = \begin{bmatrix} -0.84 \\ -999.2 \\ 66.16 \\ 0.84 \\ 999.2 \\ 17.56 \end{bmatrix}$$



- reaction forces for element 2

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ M_2 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} 3536.59 & -3534.47 & -106.07 & -3536.59 & 3534.47 & -106.07 \\ -3534.47 & 3536.59 & -106.07 & 3534.47 & -3536.59 & -106.07 \\ -106.07 & -106.07 & 14142.14 & 106.07 & 106.07 & 7071.07 \\ -3536.59 & 3534.47 & 106.07 & 3536.59 & -3534.47 & 106.07 \\ 3534.47 & -3536.59 & 106.07 & -3534.47 & 3536.59 & 106.07 \\ -106.07 & -106.07 & 7071.07 & 106.07 & 106.07 & 14142.14 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.3825 \\ 0.0999 \\ -0.0049 \end{bmatrix} = \begin{bmatrix} -999.14 \\ 999.15 \\ 16.81 \\ 999.14 \\ -999.15 \\ -17.56 \end{bmatrix}$$

Ex.) Frame

- reaction forces for element 1

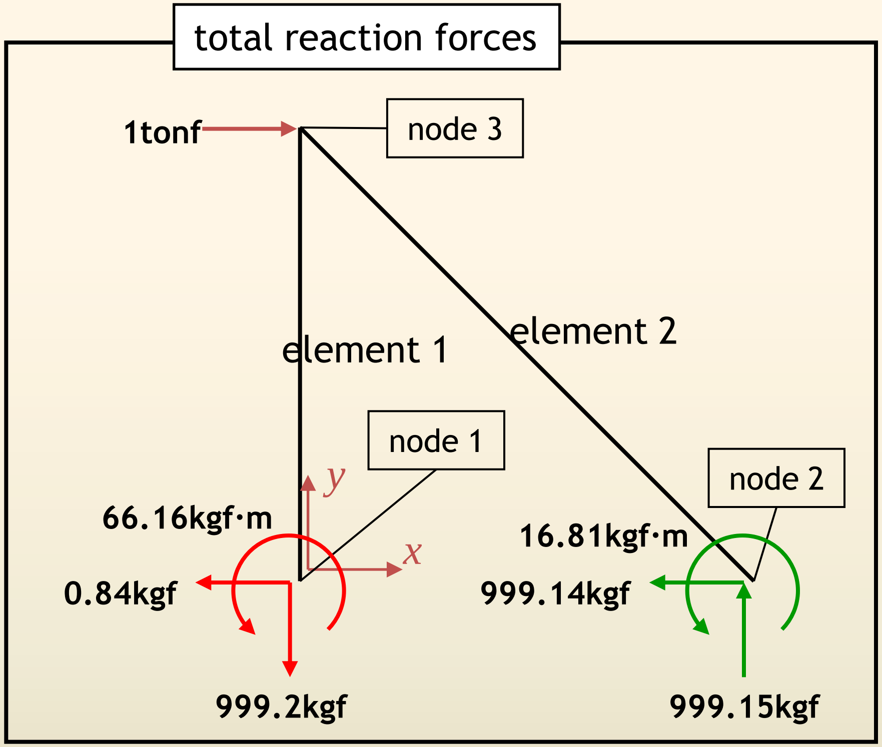
$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} -0.84 \\ -999.2 \\ 66.16 \\ 0.84 \\ 999.2 \\ 17.56 \end{bmatrix}$$

- reaction forces for element 2

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ M_2 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} -999.14 \\ 999.15 \\ 16.81 \\ 999.14 \\ -999.15 \\ -17.56 \end{bmatrix}$$

- total reaction forces

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x2} \\ f_{y2} \\ M_2 \\ f_{x3} \\ f_{y3} \\ M_3 \end{bmatrix} = \begin{bmatrix} -0.84kgf \\ -999.2kgf \\ 66.16kgf \cdot m \\ -999.14kgf \\ 999.15kgf \\ 16.81kgf \cdot m \\ 1000kgf \\ 0 \\ 0 \end{bmatrix}$$



(* the differences are caused by round off error)