Computer Aided Ship Design Part.3 Grillage Analysis of Midship Cargo Hold

2009 Fall

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Summary

u: Axial Displacement G: Shear Modulus A: Sectional Area G: Shear Modulus E: Young's Modulus

w: Vertical Displacement θ: Angle of Twist l: Length J: Polar Moment of Inertia I: Moment of Inertia

Element	Differential Equation	Variational Method	•Discretization •Approximation	Finite Element Method	Behavior → Structure
	$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F}$, where $\ddot{\mathbf{x}} = 0$			$\mathbf{Kd} = \mathbf{F}$	•Superposition of Stiffness Matrix •Coordinates Transformation
Bar	$EA\dfrac{d^2u(x)}{dx^2} + f(x) = 0$	$\delta\int_0^l \left[\dfrac{EA}{2}\left(\dfrac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$	$u(x) = a_0 + a_1 x$	$\dfrac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$	Tension → Truss, Frame
Beam	$EI\dfrac{\partial^4 w(x)}{\partial x^4} + f(x) = 0$	$\delta\int_0^l \left[\dfrac{EA}{2}\left(\dfrac{d^2w}{dx^2}\right)^2 + (f\,w) \right] dx = 0$	$w(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$	$\dfrac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{bmatrix}$	Bending → Frame, Grillage
Shaft	$GJ\dfrac{\partial^2 \theta(x)}{\partial x^2} + f(x) = 0$	$\delta\int_0^l \left[\dfrac{GJ}{2}\left(\dfrac{d\theta}{dx}\right)^2 - (f\,\theta) \right] dx = 0$	$\theta(x) = c_0 + c_1 x$	$\dfrac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$	Torsion → Grillage

Application

Midship Cargo Hold → (Engineering Concept !) → Grillage Modeling → Solution

Grillage Modeling
- 3D→ 2D
- Equivalent Force & Moment
- Boundary condition

Solution
- programming
- visulaization

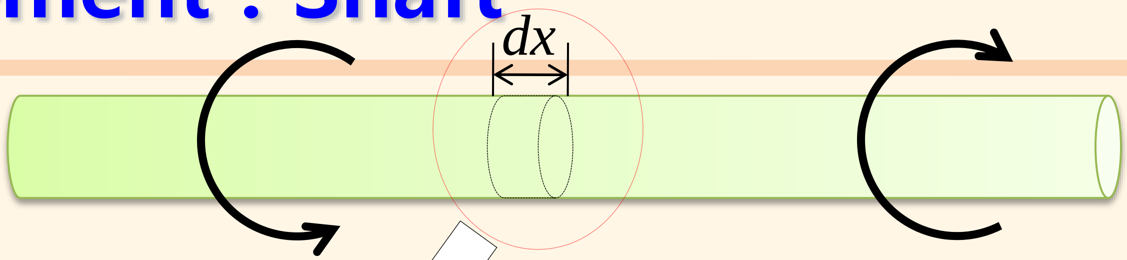
Beam Theory : Sign Convention, Deflection of Beam

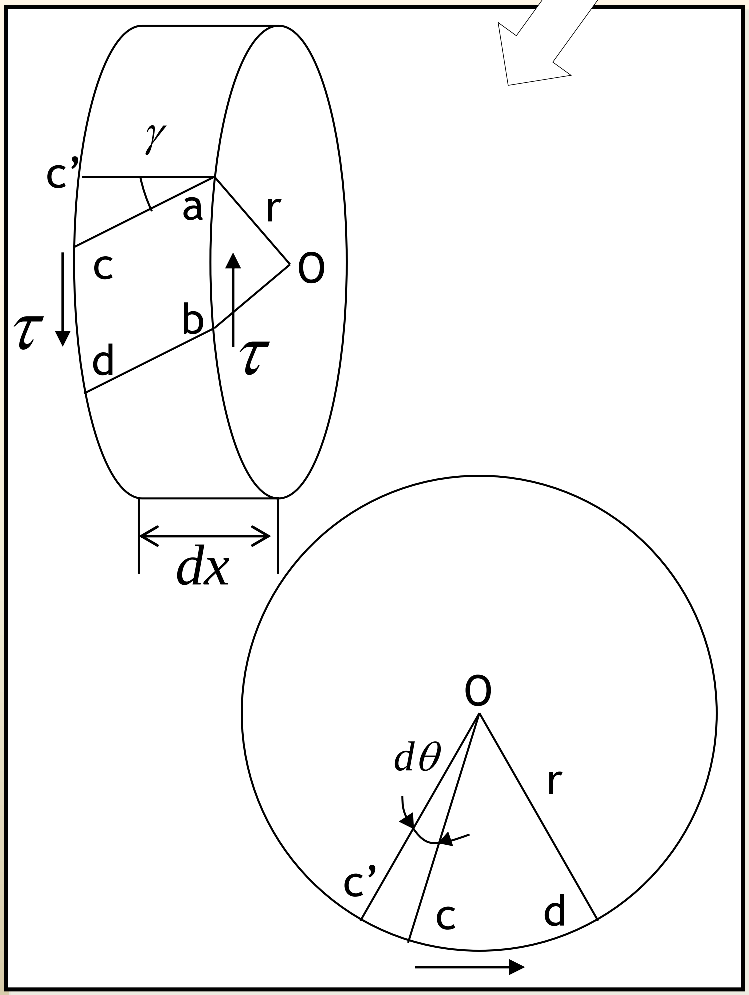
Elasticity : Displacement, Strain, Stress, Force Equilibrium, Compatibility, Constitutive Equation

Chapter 3. Element : Shaft

Element : Shaft*

τ : shear stress





① $cc' = rd\theta$

② **assume,** $\gamma \ll 1$

$$\gamma \approx \tan\gamma = \frac{cc'}{ac'} = \frac{rd\theta}{dx}$$

③ **let** $\frac{d\theta}{dx} = \phi$ **, [** ϕ **: angle of twist per unit length]**

$$\gamma = r\phi$$

④ Hooke's law in shear for the linearly elastic material

$$\tau = G\gamma = Gr\phi$$

G : shear modulus of elasticity

γ : shear strain

Cf.] $\sigma = E\varepsilon$

*Gere, J.M., Mechanics of Materials, 6th Edition, Thomson, 2006, pp 189~195

Element : Shaft*

$$\frac{d\theta}{dx} = \phi \quad , \tau = G\gamma = Gr\phi$$

G : Shear Modulus

⑤ the shear stress at an interior point (radius ρ)

$$\tau = G\rho\phi$$

⑥ we consider an element of area dA located at radial distance ρ

☞ shear force acting on the element : $\tau\, dA = G\rho\phi\, dA$

☞ moment : $\rho \times \tau\, dA = G\varphi\rho^2\, dA$

⑦ the resultant moment equal to the torque

$$T = \int_A G\phi\rho^2 dA = G\phi\int_A \rho^2 dA = G\phi J \quad \Rightarrow \quad T = GJ\frac{d\theta}{dx}$$

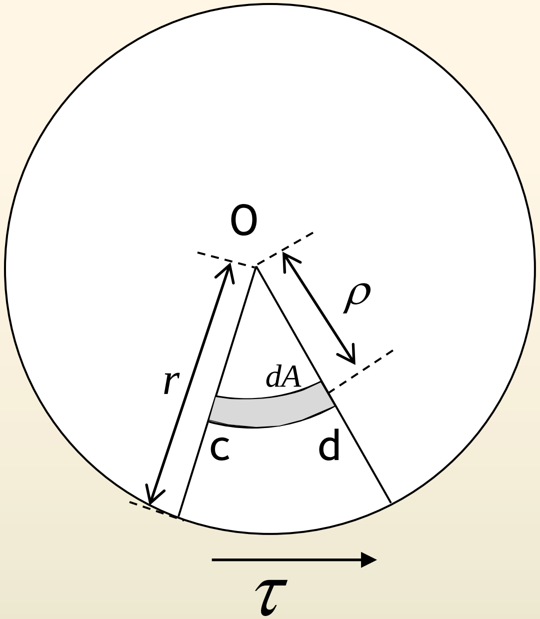
※ J : Polar Moment of Inertia $\left(J = \int_A \rho^2 dA\right)$

⑧ for a bar in pure torsion, the total angle of twist θ, equal to the rate of twist times the length of the bar

$$\theta = \phi l$$

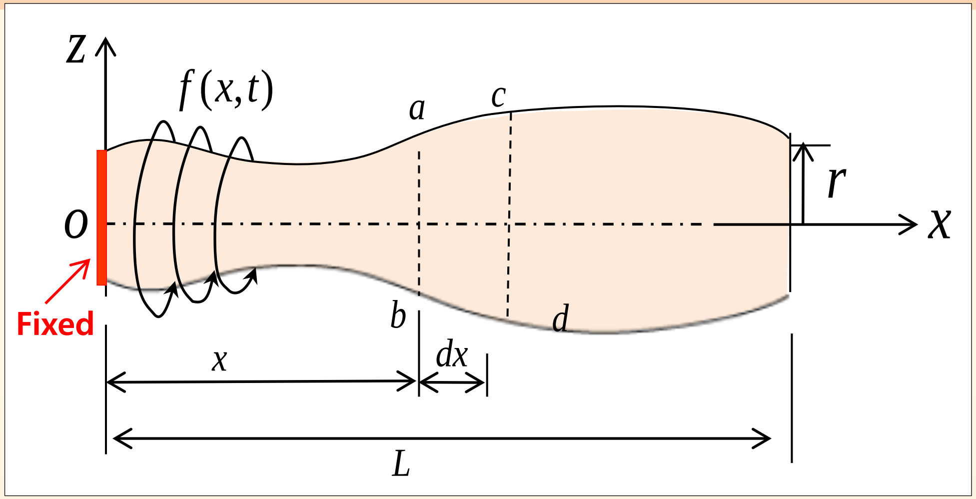
$$T = G\phi J = \frac{GJ}{l}\theta$$

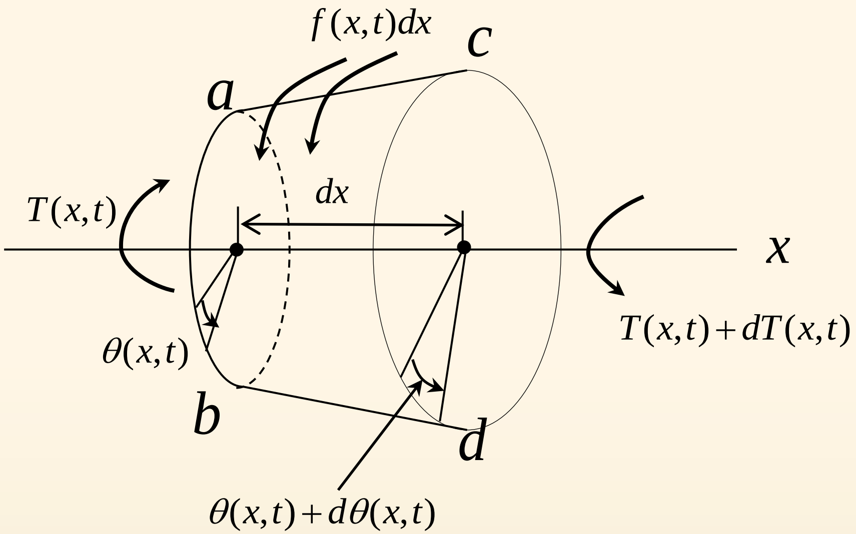
$$\theta = \frac{Tl}{GJ}$$



*Gere, J.M., Mechanics of Materials, 6th Edition, Thomson, 2006, pp 189~195

Equation of Motion for Torsional Vibration of a Shaft





Inertia torque action on an element of length dx

$$I\,dx\frac{\partial^2\theta}{\partial t^2} \qquad , I: \text{ mass polar moment of inertia of the shaft per unit length}$$

The application of Newton's second law yields the equation of motion

$$(T+dT)+fdx-T=I\,dx\frac{\partial^2\theta}{\partial t^2}$$

$$dT=\frac{\partial T}{\partial x}dx \quad \hookrightarrow \quad \frac{\partial T(x,t)}{\partial x}dx+f(x,t)dx=I\;dx\frac{\partial^2\theta(x,t)}{\partial t^2}$$

divided by dx $\quad \hookrightarrow \quad \dfrac{\partial T(x,t)}{\partial x}+f(x,t)=I\;\dfrac{\partial^2\theta(x,t)}{\partial t^2}$

Rao S.S., Mechanical Vibrations, Fourth Edition, Prentice-Hall, 2004, pp.606-607

Equation of Motion for Torsional Vibration of a Shaft

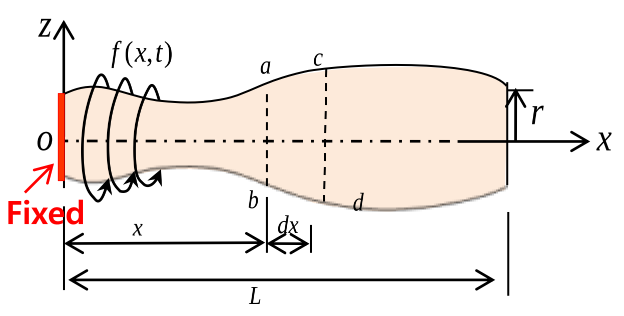
$$J = \int_A \rho^2 dA \qquad T = GJ\frac{d\theta}{dx}$$

Equation of motion

homogeneous

$$c.f. \quad U_{tt} = c^2 U_{xx}$$

$$\frac{\partial T(x,t)}{\partial x} + f(x,t) = I\frac{\partial^2\theta(x,t)}{\partial t^2}$$ ← **nonhomogeneous**



Relation between torsional deflection and the twisting moment

$$T(x,t) = GJ(x)\frac{\partial\theta(x,t)}{\partial x}$$

G : Shear modulus

GJ : Torsional stiffness

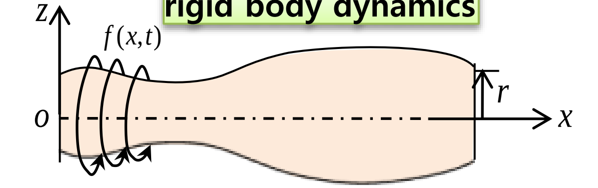
$$\frac{\partial}{dx}\left[GJ(x)\frac{\partial\theta(x,t)}{\partial x}\right] + f(x,t) = I\frac{\partial^2\theta(x,t)}{\partial t^2}$$

if J is constant

$$GJ\frac{\partial^2\theta(x,t)}{\partial x^2} + f(x,t) = I\frac{\partial^2\theta(x,t)}{\partial t^2}$$

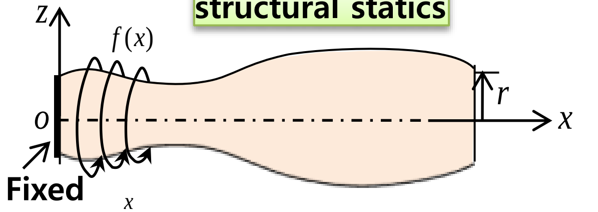
structural vibration

rigid body dynamics



$$f(x,t) = I\frac{\partial^2\theta(x,t)}{\partial t^2}$$

structural statics



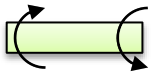
$$GJ\frac{\partial^2\theta(x)}{\partial x^2} + f(x) = 0$$

Rao S.S., Mechanical Vibrations, Fourth Edition, Prentice-Hall, 2004, pp.606-607

Element : Shaft

T : Torque
l : length
G : Shear Modulus
J : Polar Moment of Inertia

Shaft



$$GJ\frac{\partial^2\theta(x)}{\partial x^2}+f(x)=0 \Rightarrow \delta\int_0^l\left[\frac{GJ}{2}\left(\frac{d\theta}{dx}\right)^2-(f\,\theta)\right]dx=0 \Rightarrow \frac{GJ}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=\begin{bmatrix}M_1\\ M_2\end{bmatrix}$$

$$\theta(x)=c_0+c_1x$$

- Reference : Chapter.1 Bar

Bar

$$EA\frac{d^2u(x)}{dx^2}+f(x)=0 \Rightarrow \delta\int_0^l\left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2-(f\,u)\right]dx=0 \Rightarrow \frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

$$u(x)=a_0+a_1x$$

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Summary

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Element | Differential Equation → Variational Method → •Discretization •Approximation → Finite Element Method | •Superposition of Stiffness Matrix •Coordinates Transformation | **Behavior → Structure**

$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F}$, where $\ddot{\mathbf{x}} = 0$

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Bar

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0 \Rightarrow \delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0$$

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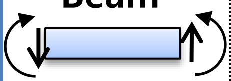
Tension → **Truss**, **Frame**

Beam



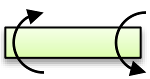
$$EI\frac{\partial^4 w(x)}{\partial x^4} + f(x) = 0 \Rightarrow \delta\int_0^l \left[\frac{EA}{2}\left(\frac{d^2w}{dx^2}\right)^2 + (f\,w)\right]dx = 0$$

$w(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$$\frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{bmatrix}$$

Bending → **Frame**, **Grillage**

Shaft



$$GJ\frac{\partial^2 \theta(x)}{\partial x^2} + f(x) = 0 \Rightarrow \delta\int_0^l \left[\frac{GJ}{2}\left(\frac{d\theta}{dx}\right)^2 - (f\,\theta)\right]dx = 0$$

$\theta(x) = c_0 + c_1 x$

$$\frac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Torsion → **Grillage**

Application

Midship Cargo Hold → **Grillage Modeling** → **Solution**

Engineering Concept !

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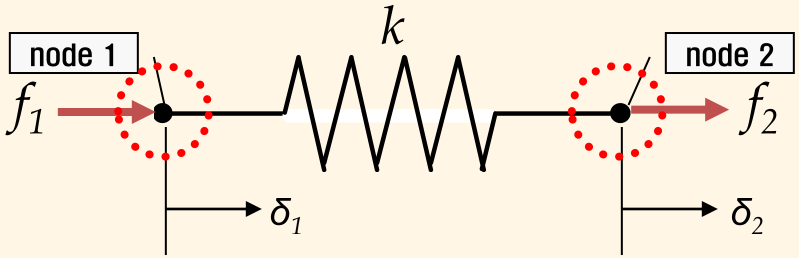
Chapter 4. Superposition of Stiffness Matrix

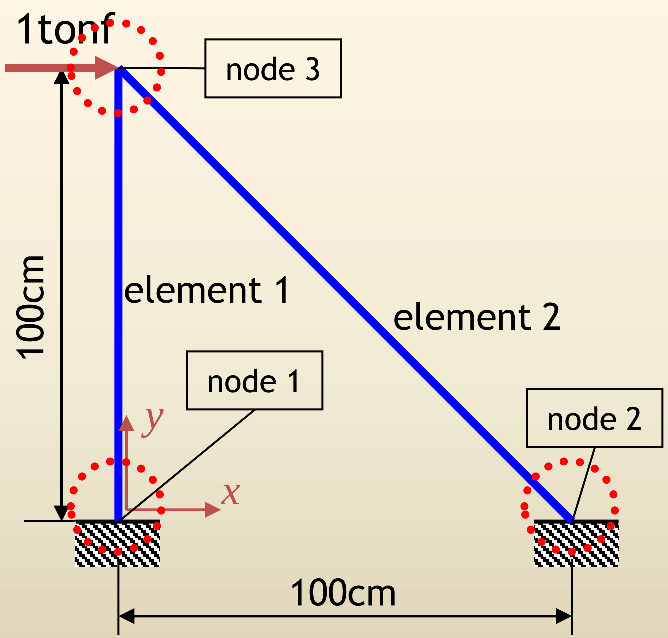
Coordinate Transformation

Node

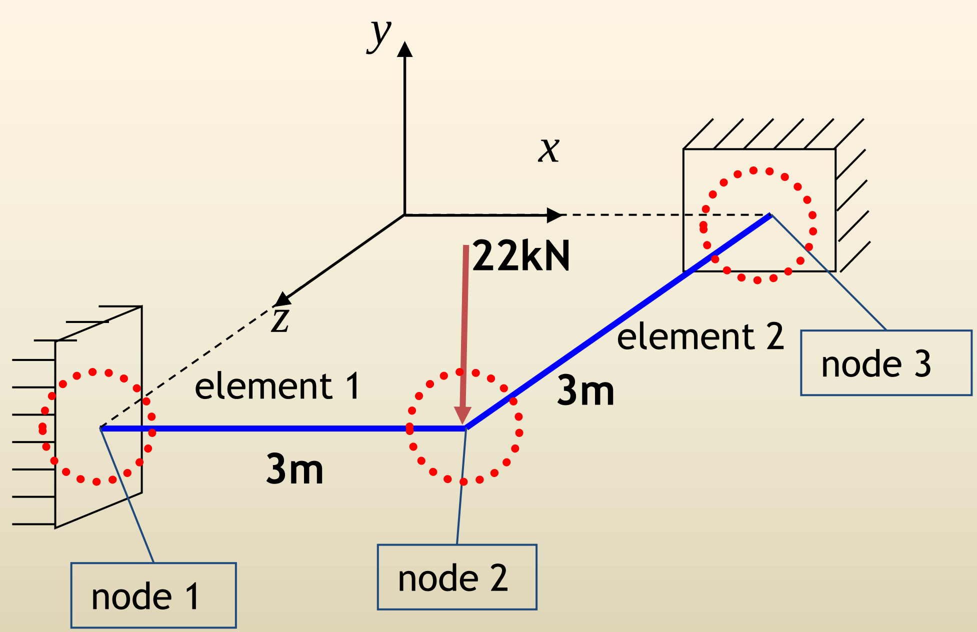


The nodes are points at which equilibrium will be enforced and displacement found. They are generally located at the ends of the elements for most common structural shapes such as bars and beams.*

ex.1)



ex.2)

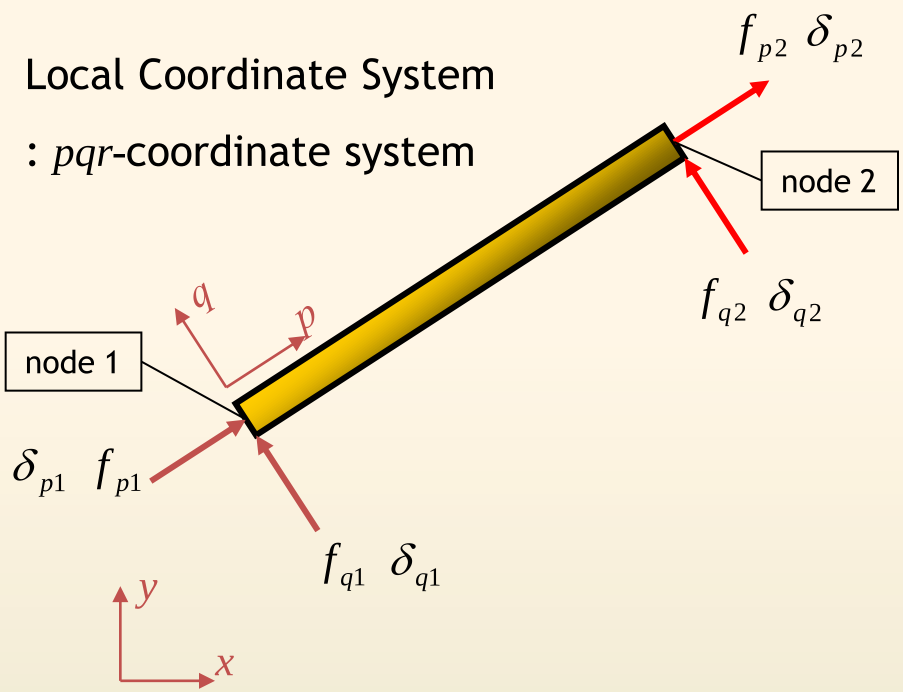


*Sennet, R.E., Matrix Analysis of Structres, Prentice-Hall, 1994, p.4

Coordinate System

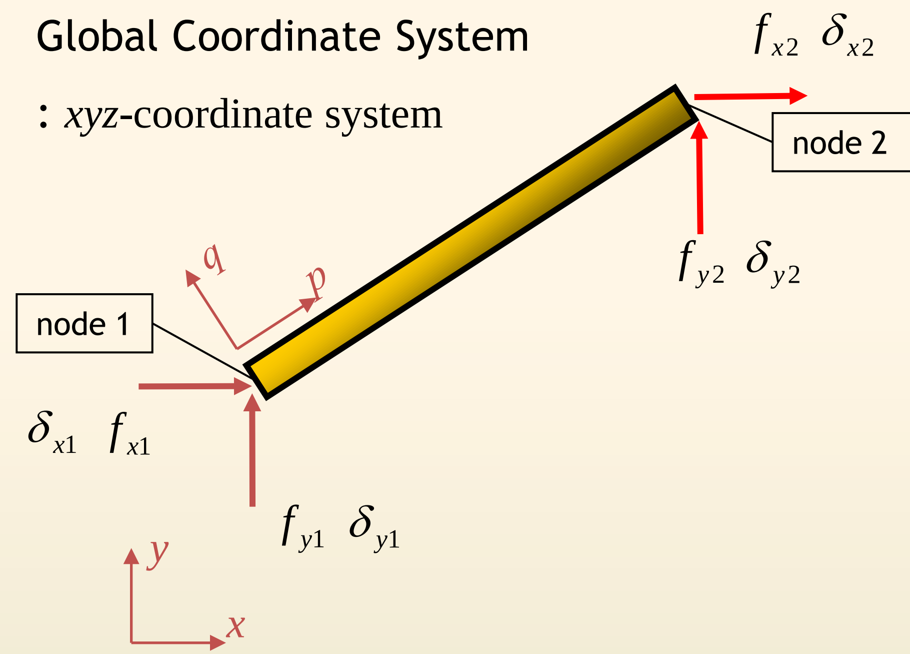
Local Coordinate System

: *pqr*-coordinate system

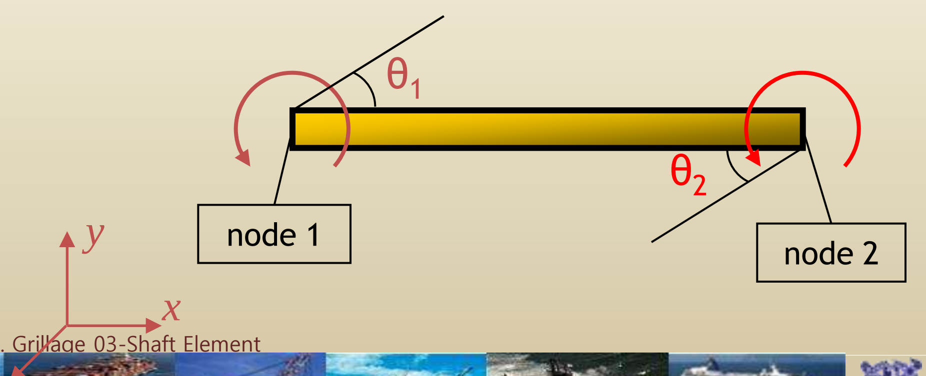


Global Coordinate System

: *xyz*-coordinate system

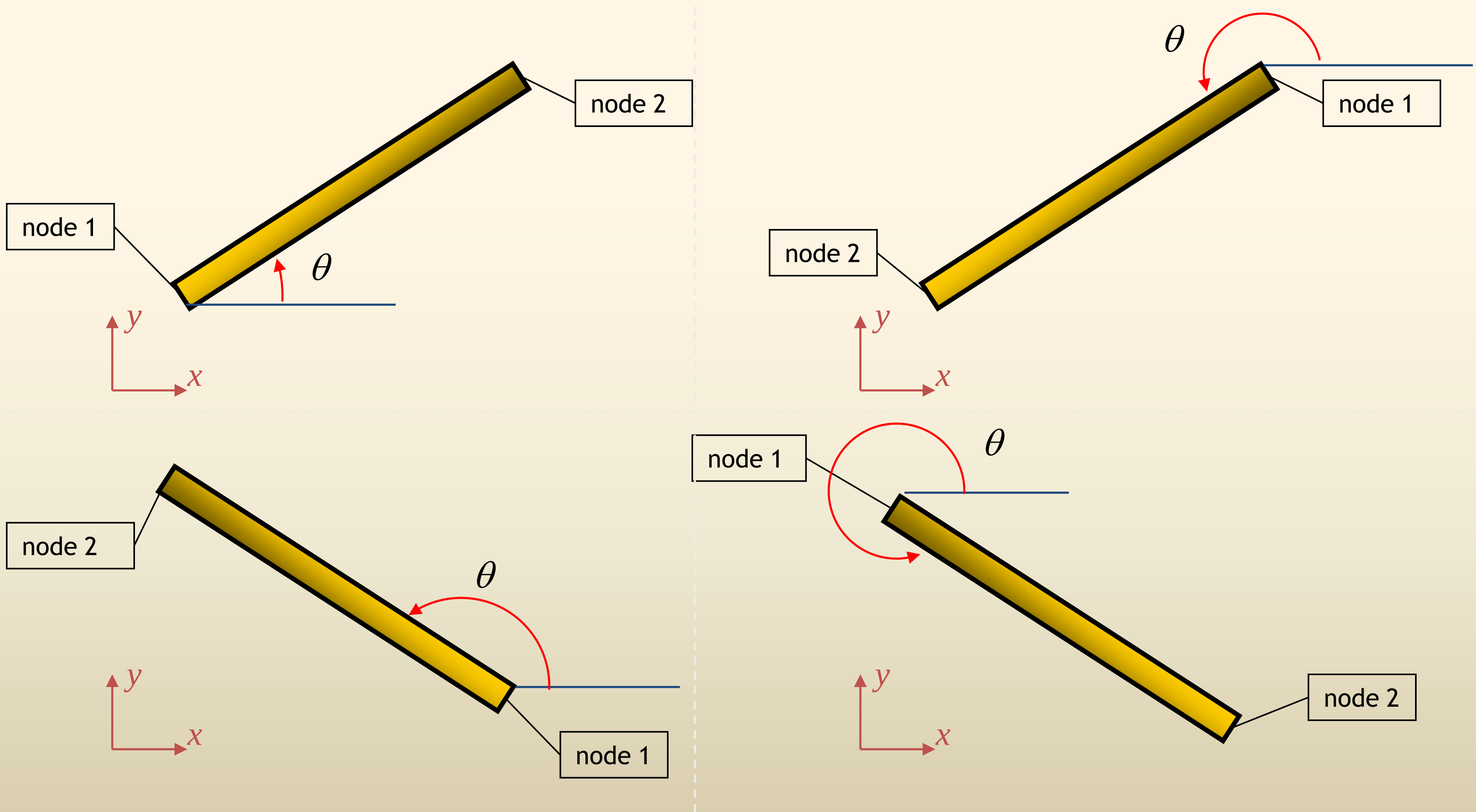


Sign Convention : positive moment



Angle

- Angles in global coordinate system : counterclockwise at the lower number of node



Rotational Transformation : Point

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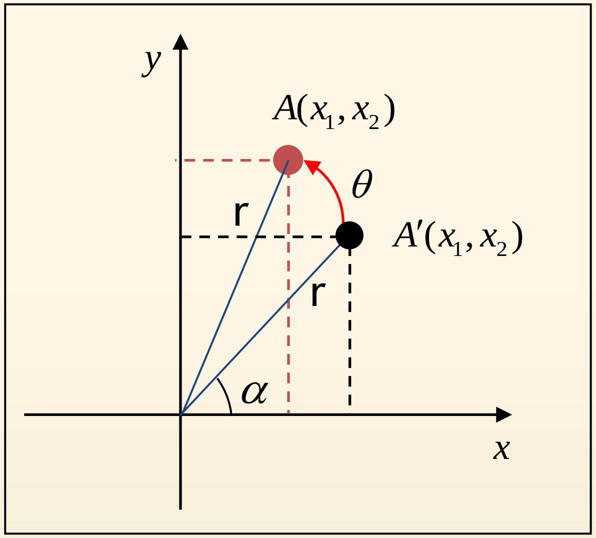

① trigonometric identities : angle sum

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\cos\alpha\cos\beta$$

② components of point A

$$x_2 = r\cos(\alpha+\theta)$$

$$y_2 = r\sin(\alpha+\theta)$$

③ by using the angle sum identities

$$\begin{aligned} x_2 &= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ &= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta \\ &= x_1\cos\theta - y_1\sin\theta \end{aligned}$$

$$\begin{aligned} y_2 &= r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \\ &= (r\sin\alpha)\cos\theta + (r\cos\alpha)\sin\theta \\ &= y_1\cos\theta + x_1\sin\theta \end{aligned}$$

④ in matrix form

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Rotational Transformation : Coordinate System

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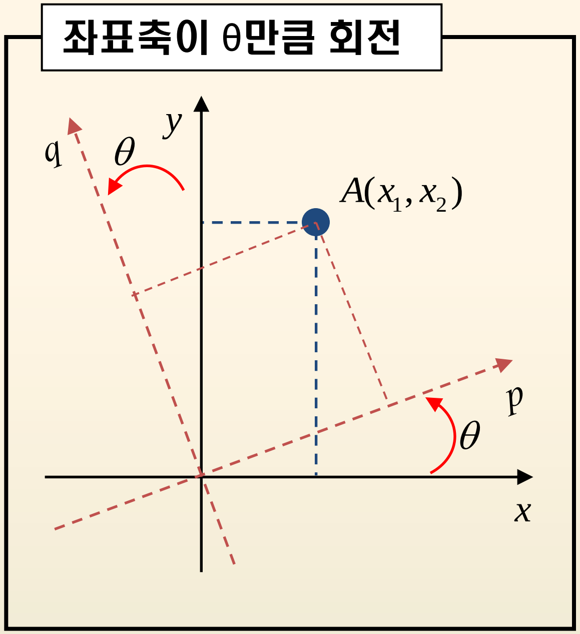
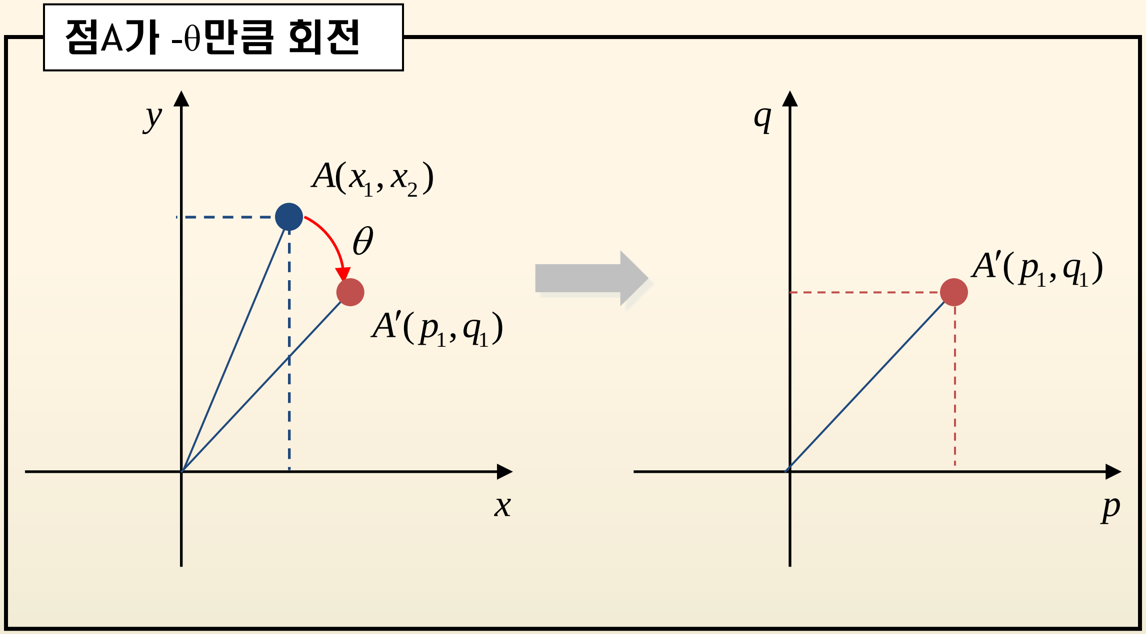




※ Rotational Transformation : Point

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Rotation of point by $-\theta$

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Rotational Transformation : Coordinate System

- 2 Dimension $(xy \rightarrow pq)$

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

- 3 Dimension $(xyz \rightarrow pqr)$

$$\begin{bmatrix} p_1 \\ q_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

: rotation along with z axis

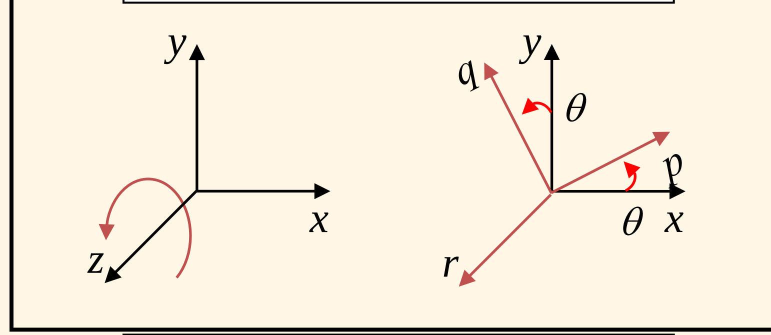
$$\begin{bmatrix} p_1 \\ q_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

: rotation along with y axis

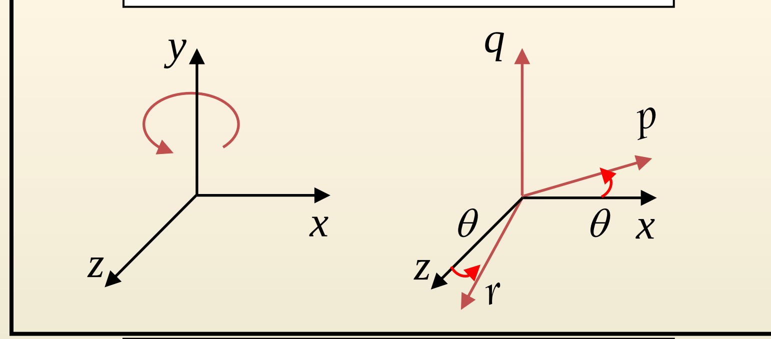
$$\begin{bmatrix} p_1 \\ q_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

: rotation along with x axis

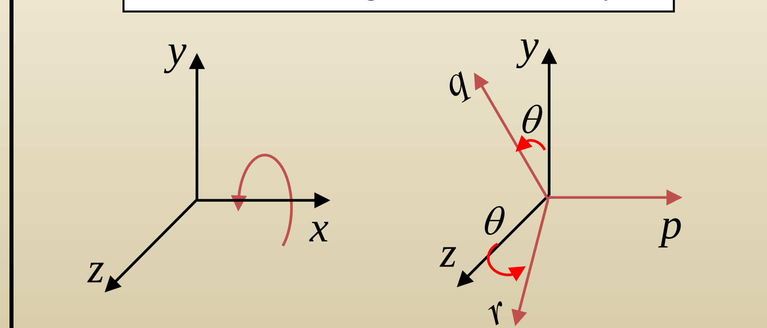
rotation along with z axis by θ



rotation along with y axis by θ



rotation along with x axis by θ



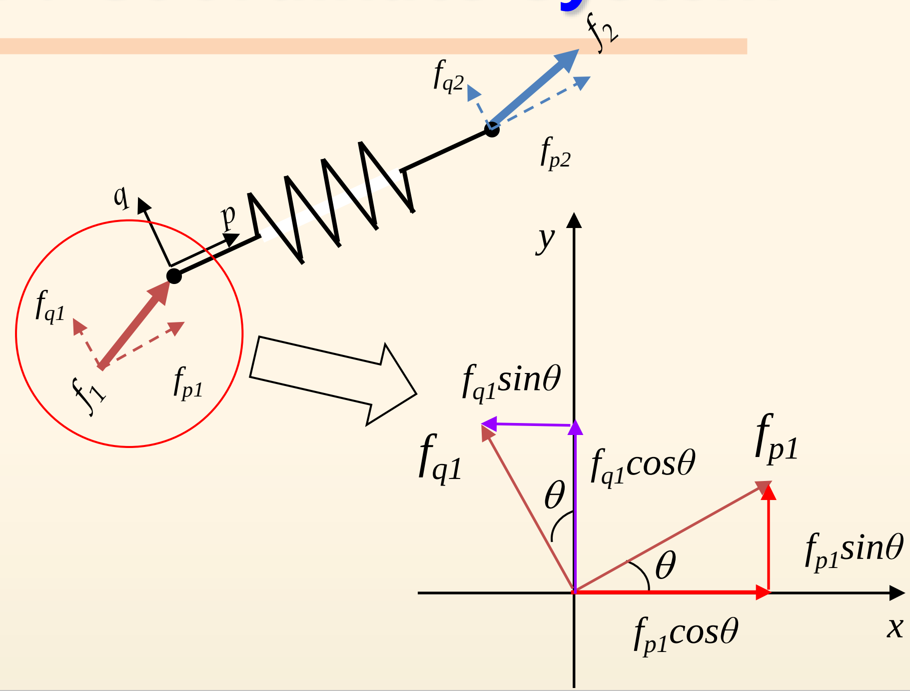
Rotational Transformation : Coordinate System

- 2 Dimension $(xy \rightarrow pq)$

rotation of coordinate system of node 1 by θ

$$\begin{bmatrix} f_{p1} \\ f_{q1} \\ f_{p2} \\ f_{q2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix}$$

rotation of coordinate system of node 2 by θ



- 3 Dimension $(xyz \rightarrow pqr)$

rotation along with z axis by θ

$$\begin{bmatrix} f_{p1} \\ f_{q1} \\ M_{z1} \\ f_{p2} \\ f_{q2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{x1} \\ f_{y1} \\ M_{z1} \\ f_{x2} \\ f_{y2} \\ M_{z2} \end{bmatrix}$$

Element : Bar

E : Young's Modulus, *A* : sectional area

Element — Differential Equation ⇨ Variational Method ⇨ •Discretization •Approximation ⇨ Finite Element Method

$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F}$, *where* $\ddot{\mathbf{x}} = 0$

$\mathbf{Kd} = \mathbf{F}$

Bar

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0 \;\Rightarrow\; \delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$

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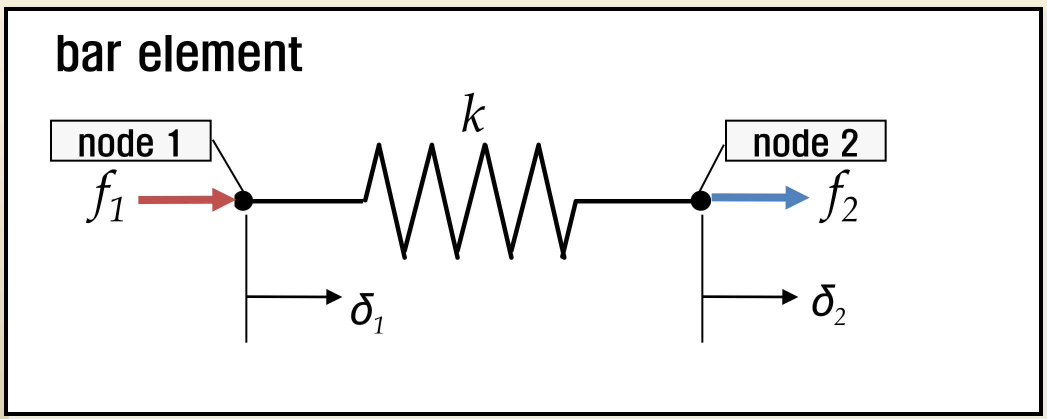
$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

! Notation

$$k = \frac{EA}{l}$$



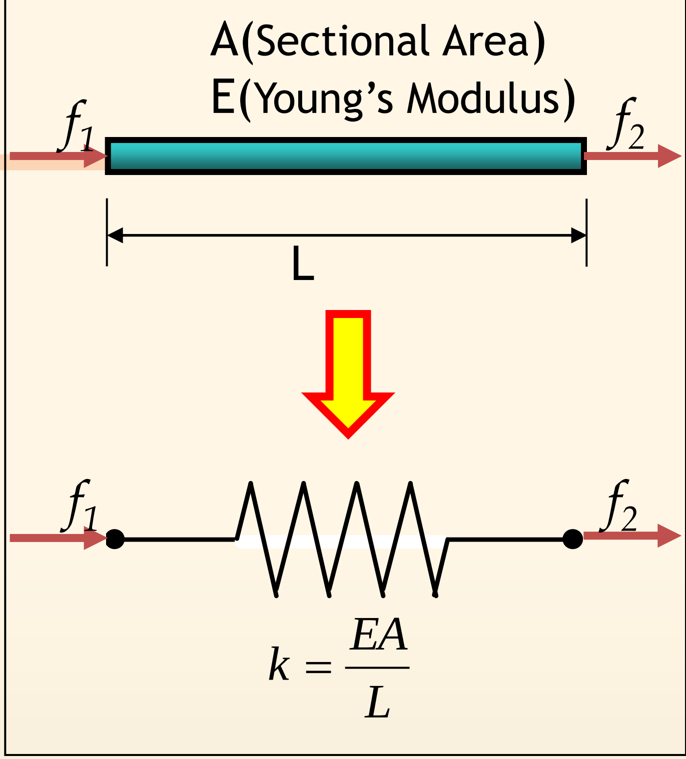
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

→ stiffness matrix

$$[f] = [K][\delta]$$

stiffness equation

Element : Bar - Linearity

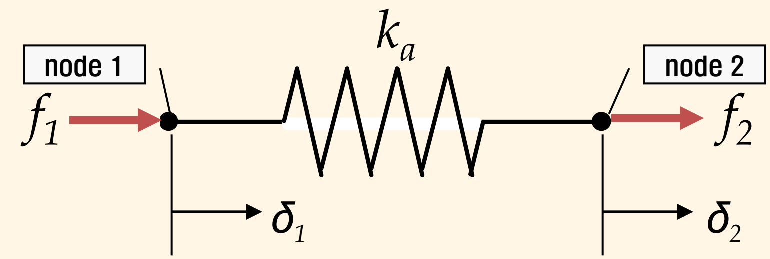


Linearity (α : Scalar)
$L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$ $L(\alpha\, \mathbf{v}_1) = \alpha \cdot L(\mathbf{v}_1)$ ⇒ Definition of Linearity

- Bar - Linearity (α : Scalar)

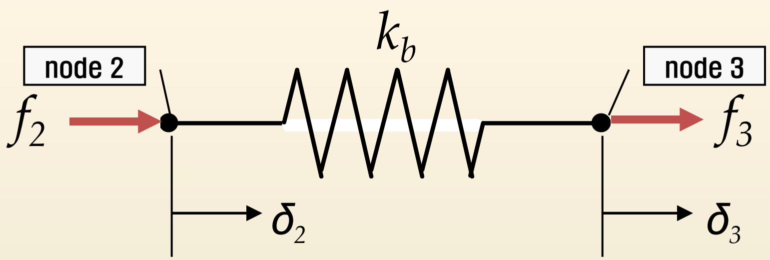
$f(\delta_1) = k\delta_1 \quad , \quad f(\delta_2) = k\delta_2$ $f(\delta_1) + f(\delta_2) = k\delta_1 + k\delta_2 = k(\delta_1 + \delta_2)$ $f(\delta_1 + \delta_2) = k(\delta_1 + \delta_2)$ $\therefore f(\delta_1 + \delta_2) = f(\delta_1) + f(\delta_2)$	$f(\delta_1) = k\delta_1$ $f(\alpha \cdot \delta_1) = k(\alpha\delta_1) = k\alpha\delta_1$ $= \alpha(k\delta_1) = \alpha \cdot f(\delta_1)$ $\therefore f(\alpha \cdot \delta_1) = \alpha \cdot f(\delta_1)$

Element : Bar - Superposition



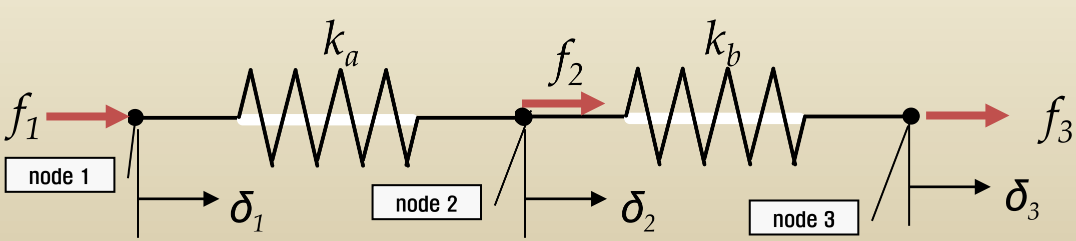
$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

+



$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

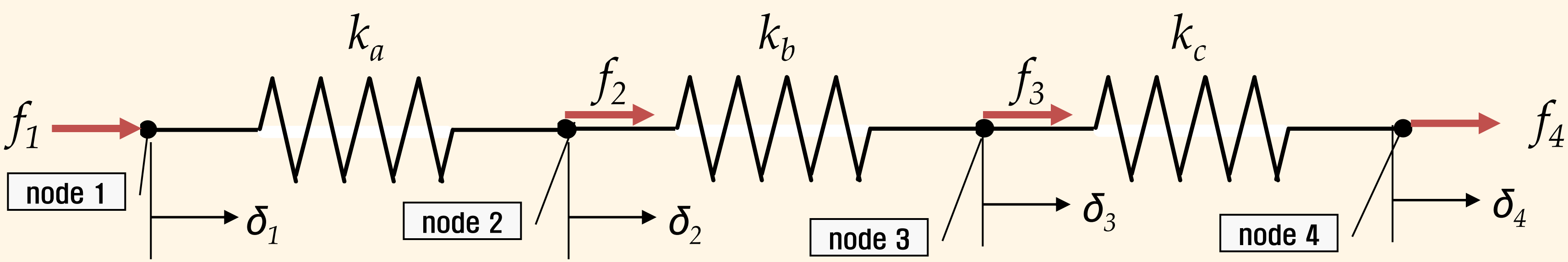
=



$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

Element : Bar - Superposition

ex.) Find a stiffness equation of the following system:

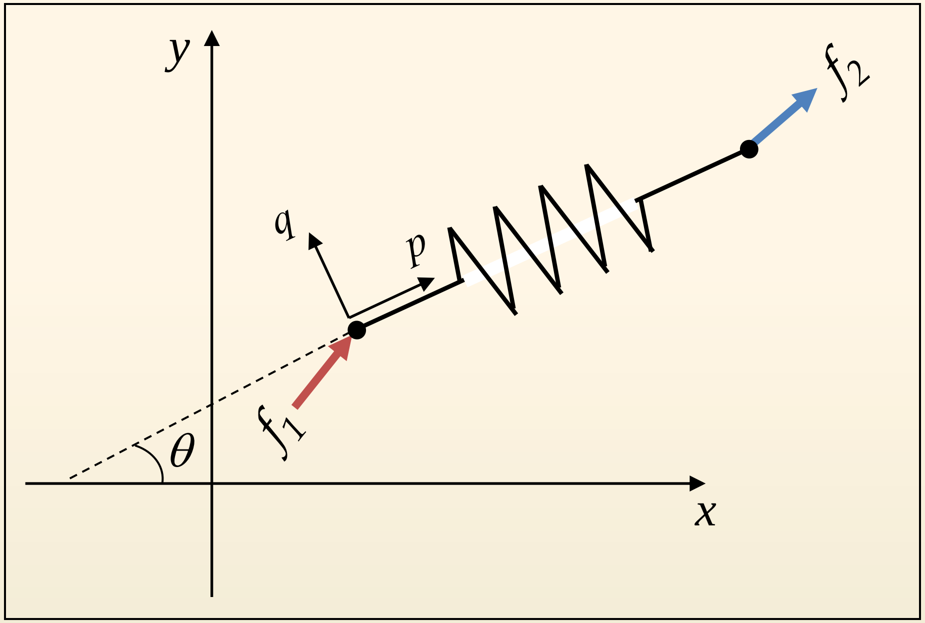


$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} k_a & -k_a & 0 & 0 \\ -k_a & k_a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_b & -k_b & 0 \\ 0 & -k_b & k_b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_c & -k_c \\ 0 & 0 & -k_c & k_c \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

$$= \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} k_a & -k_a & 0 & 0 \\ -k_a & k_a + k_b & -k_b & 0 \\ 0 & -k_b & k_b + k_c & -k_c \\ 0 & 0 & -k_c & k_c \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

Element : 2-Dimensional Bar

- Solution of 2-Dimensional Bar



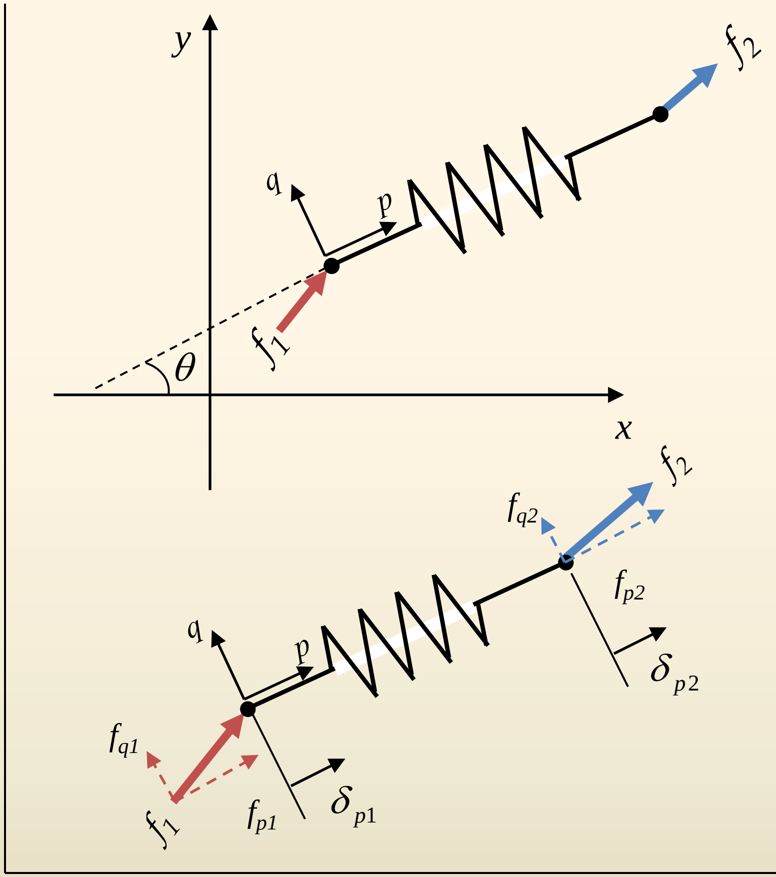
Step1. Find stiffness matrix in the local coordinate system (***pq-coordinate system)***

Step2. Find transformation matrix between the local and the global coordinate system

Step3. Find stiffness matrix in the global coordinate system (***xy-coordinate system)***

Element : 2-Dimensional Bar

Step1. Find stiffness matrix in the local coordinate system (***pq-coordinate system)***



Notation

δ_{pi} : displacement perpendicular to the p *axis* at node i

δ_{qi} : displacement perpendicular to the q *axis* at node i

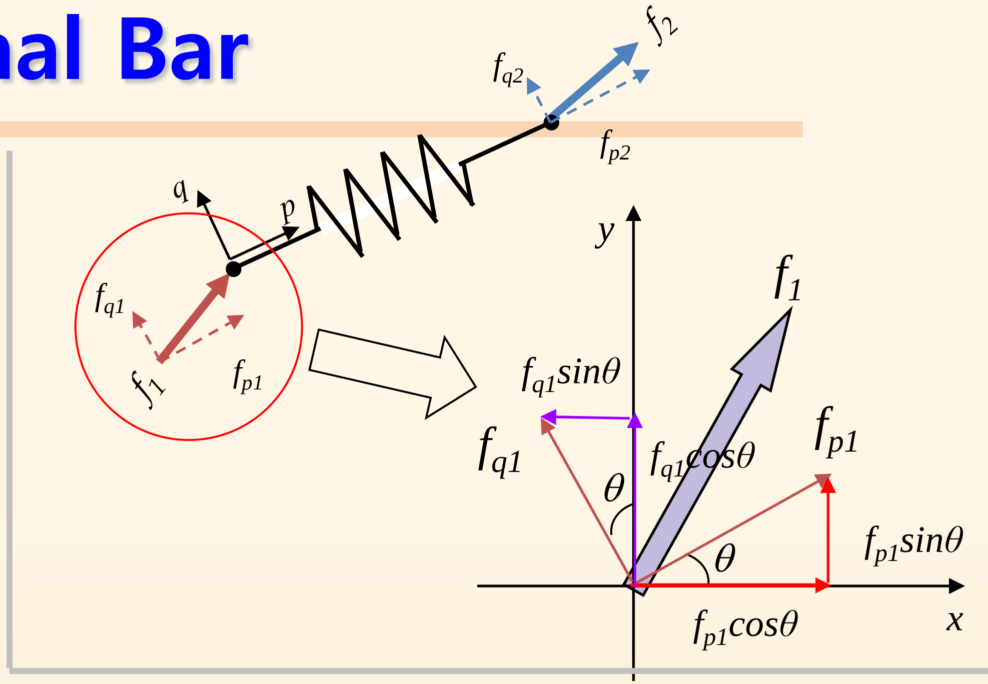
f_{pi} : force perpendicular to the p *axis* at node i

f_{qi} : force perpendicular to the q *axis* at node i

$$\begin{bmatrix} f_{p1} \\ f_{q1} \\ f_{p2} \\ f_{q2} \end{bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{p1} \\ \delta_{q1} \\ \delta_{p2} \\ \delta_{q2} \end{bmatrix}$$

① $$[\mathbf{F}_{pq}] = [\mathbf{K}_{pq}][\delta_{pq}]$$

Element : 2-Dimensional Bar



Step2. Find transformation matrix between the local and the global coordinate system

(1) the forces with respect to the global coordinate system

- f_{p1}
  x : $f_{p1}\cos\theta$
  y : $f_{q1}\sin\theta$
- f_{q1}
  x : $-f_{q1}\sin\theta$
  y : $f_{q1}\cos\theta$

- Total force in the global coordinate system

x : $f_{x1} = f_{p1}\cos\theta - f_{q1}\sin\theta$

y : $f_{y1} = f_{p1}\sin\theta + f_{q1}\cos\theta$

- In the same way

x : $f_{x2} = f_{p2}\cos\theta - f_{q2}\sin\theta$

y : $f_{y2} = f_{p2}\sin\theta + f_{q2}\cos\theta$

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} f_{p1} \\ f_{q1} \\ f_{p2} \\ f_{q2} \end{bmatrix}$$

inverse

$$\begin{bmatrix} f_{p1} \\ f_{q1} \\ f_{p2} \\ f_{q2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix}$$

transformation matrix

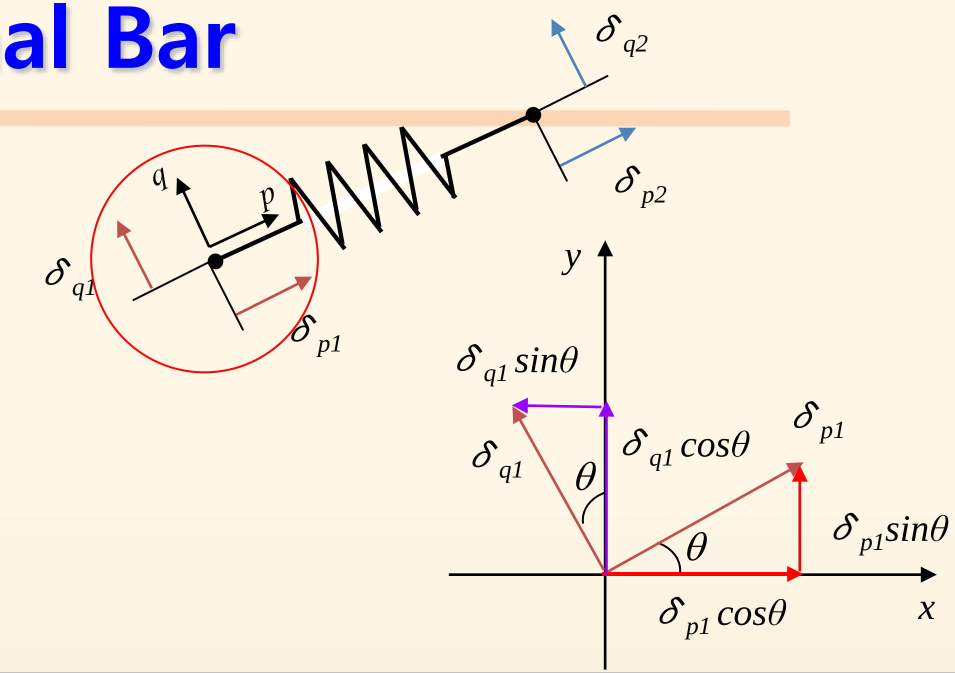
②

$$[\mathbf{F}_{pq}] = [\mathbf{T}][\mathbf{F}_{xy}]$$

Element : 2-Dimensional Bar

Step2. Find transformation matrix between the local and the global coordinate system

(2) the displacements with respect to the global coordinate system



- δ_{p1}

x : $\delta_{p1}\cos\theta$

y : $\delta_{p1}\sin\theta$

- δ_{q1}

x : $-\delta_{q1}\sin\theta$

y : $\delta_{q1}\cos\theta$

- displacements in Global Coordinate

x : $\delta_{x1} = \delta_{p1}\cos\theta - \delta_{q1}\sin\theta$

y : $\delta_{y1} = \delta_{p1}\sin\theta + \delta_{q1}\cos\theta$

- In the same way

x : $\delta_{x2} = \delta_{p2}\cos\theta - \delta_{q2}\sin\theta$

y : $\delta_{y2} = \delta_{p2}\sin\theta + \delta_{q2}\cos\theta$

$$\begin{bmatrix}\delta_{x1}\\ \delta_{y1}\\ \delta_{x2}\\ \delta_{y2}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & \cos\theta & -\sin\theta\\ 0 & 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\delta_{p1}\\ \delta_{q1}\\ \delta_{p2}\\ \delta_{q2}\end{bmatrix}$$

inverse

$$\begin{bmatrix}\delta_{p1}\\ \delta_{q1}\\ \delta_{p2}\\ \delta_{q2}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0 & 0\\ -\sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & \cos\theta & \sin\theta\\ 0 & 0 & -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\delta_{x1}\\ \delta_{y1}\\ \delta_{x2}\\ \delta_{y2}\end{bmatrix}$$

transformation matrix

③ $[\delta_{pq}] = [\mathbf{T}][\delta_{xy}]$

Element : 2-Dimensional Bar

Step3. Find stiffness matrix in the global coordinate system (*xy-coordinate system)*

① $[\mathbf{F}_{pq}] = [\mathbf{K}_{pq}][\delta_{pq}]$ → $[\mathbf{T}][\mathbf{F}_{xy}] = [\mathbf{K}_{pq}][\mathbf{T}][\delta_{xy}]$

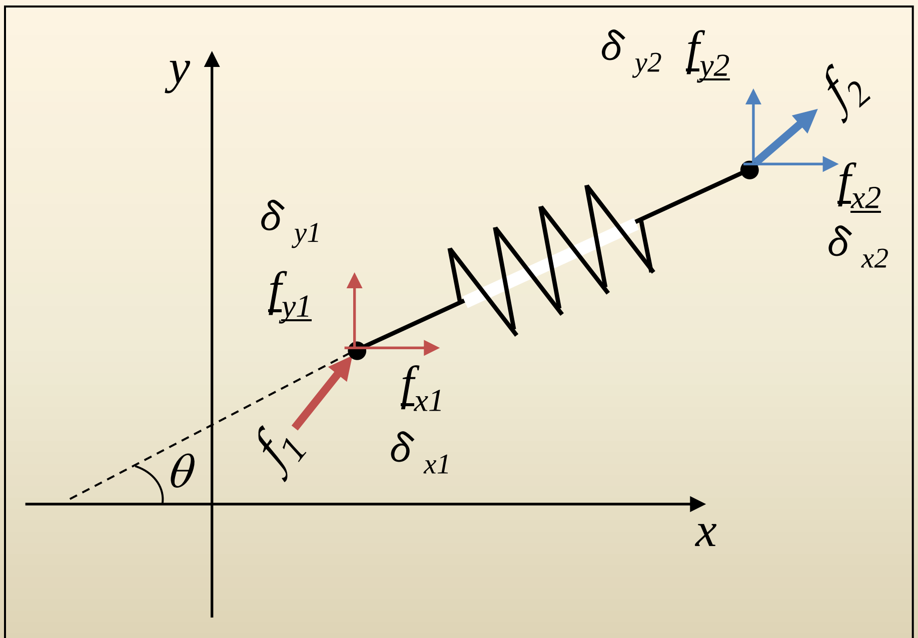
② $[\mathbf{F}_{pq}] = [\mathbf{T}][\mathbf{F}_{xy}]$

③ $[\delta_{pq}] = [\mathbf{T}][\delta_{xy}]$

multiply $[\mathbf{T}]^{-1} = [\mathbf{T}]^T$

$$[\mathbf{F}_{xy}] = [\mathbf{T}]^T[\mathbf{K}_{pq}][\mathbf{T}][\delta_{xy}]$$

C:cos
S:sin



$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = k \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{x2} \\ \delta_{y2} \end{bmatrix}$$

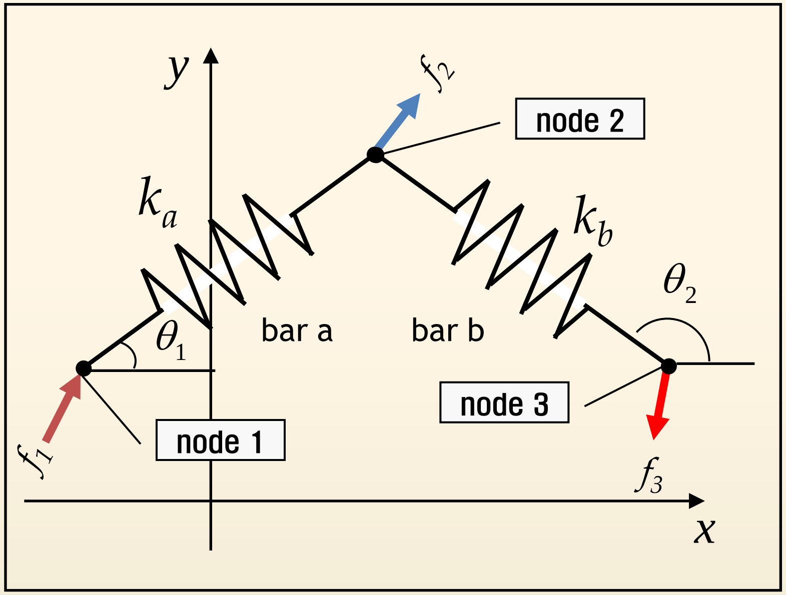
$$[\mathbf{F}_{xy}] = [\mathbf{K}_{xy}][\delta_{xy}]$$

stiffness equation

Element : 2-Dimensional Bar

C:cos , S:sin

ex.) Find a stiffness equation of the following system:



(1) Stiffness equation of bar a

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} = k_a \begin{bmatrix} C^2\theta_1 & C\theta_1 S\theta_1 & -C^2\theta_1 & -C\theta_1 S\theta_1 \\ C\theta_1 S\theta_1 & S^2\theta_1 & -C\theta_1 S\theta_1 & -S^2\theta_1 \\ -C^2\theta_1 & -C\theta_1 S\theta_1 & C^2\theta_1 & C\theta_1 S\theta_1 \\ -C\theta_1 S\theta_1 & -S^2\theta_1 & C\theta_1 S\theta_1 & S^2\theta_1 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{x2} \\ \delta_{y2} \end{bmatrix}$$

(2) Stiffness equation of bar b

$$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = k_b \begin{bmatrix} C^2\theta_2 & C\theta_2 S\theta_2 & -C^2\theta_2 & -C\theta_2 S\theta_2 \\ C\theta_2 S\theta_2 & S^2\theta_2 & -C\theta_2 S\theta_2 & -S^2\theta_2 \\ -C^2\theta_2 & -C\theta_2 S\theta_2 & C^2\theta_2 & C\theta_2 S\theta_2 \\ -C\theta_2 S\theta_2 & -S^2\theta_2 & C\theta_2 S\theta_2 & S^2\theta_2 \end{bmatrix} \begin{bmatrix} \delta_{x2} \\ \delta_{y2} \\ \delta_{x3} \\ \delta_{y3} \end{bmatrix}$$

(3) Superposition

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} k_a C^2\theta_1 & k_a C\theta_1 S\theta_1 & -k_a C^2\theta_1 & -k_a C\theta_1 S\theta_1 & 0 & 0 \\ k_a C\theta_1 S\theta_1 & k_a S^2\theta_1 & -k_a C\theta_1 S\theta_1 & -k_a S^2\theta_1 & 0 & 0 \\ -k_a C^2\theta_1 & -k_a C\theta\ S\theta_1 & k_a C^2\theta_1 + k_b C^2\theta_2 & k_a C\theta_1 S\theta_1 + k_b C\theta_2 S\theta_2 & -k_b C^2\theta_2 & -k_b C\theta_2 S\theta_2 \\ -k_a C\theta_1 S\theta_1 & -k_a S^2\theta_1 & k_a C\theta_1 S\theta_1 + k_b C\theta_2 S\theta_2 & k_a S^2\theta_1 + k_b S^2\theta_2 & -k_b C\theta\ S\theta & -k_b S^2\theta_2 \\ 0 & 0 & -k_b C^2\theta_2 & -k_b C\theta_2 S\theta_2 & k_b C^2\theta_2 & k_b C\theta_2 S\theta_2 \\ 0 & 0 & -k_b C\theta_2 S\theta_2 & -k_b S^2\theta_2 & k_b C\theta_2 S\theta_2 & k_b S^2\theta_2 \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{x2} \\ \delta_{y2} \\ \delta_{x3} \\ \delta_{y3} \end{bmatrix}$$