Naval Architecture & Ocean Engineering

Computer Aided Ship Design Part.3 Grillage Analysis of Midship Cargo Hold

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Summary

u : Axial Displacement G : Shear Modulus A : Sectional Area G : Shear Modulus E : Young's Modulus

w : Vertical Displacement θ : Angle of Twist l : Length J : Polar Moment of Inertia I : Moment of Inertia

Element

Differential Equation → Variational Method → •Discretization •Approximation → Finite Element Method

$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F}$, where $\ddot{\mathbf{x}} = 0$

$\mathbf{Kd} = \mathbf{F}$

•Superposition of Stiffness Matrix
•Coordinates Transformation

Behavior → **Structure**

Bar

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0$$

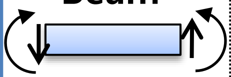
$$\delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0$$

$$u(x) = a_0 + a_1 x$$

$$\frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

Tension → **Truss**

Beam



$$EI\frac{\partial^4 w(x)}{\partial x^4} + f(x) = 0$$

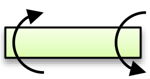
$$\delta\int_0^l \left[\frac{EA}{2}\left(\frac{d^2w}{dx^2}\right)^2 + (f\,w)\right]dx = 0$$

$$w(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$\frac{2EI}{l^3}\begin{bmatrix}6 & 3l & -6 & 3l\\ 3l & 2l^2 & -3l & l^2\\ -6 & -3l & 6 & -3l\\ 3l & l^2 & -3l & 2l^2\end{bmatrix}\begin{bmatrix}u_1\\ \theta_1\\ u_2\\ \theta_2\end{bmatrix} = \begin{bmatrix}f_1\\ M_1\\ f_2\\ M_2\end{bmatrix}$$

Bending → **Frame**

Shaft



$$GJ\frac{\partial^2\theta(x)}{\partial x^2} + f(x) = 0$$

$$\delta\int_0^l \left[\frac{GJ}{2}\left(\frac{d\theta}{dx}\right)^2 - (f\,\theta)\right]dx = 0$$

$$\theta(x) = c_0 + c_1 x$$

$$\frac{GJ}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}M_1\\ M_2\end{bmatrix}$$

Torsion → **Grillage**

Application

Midship Cargo Hold → **Grillage Modeling** → **Solution**

Engineering Concept !

Grillage Modeling
- **3D→ 2D**
- **Equivalent Force & Moment**
- **Boundary condition**

Solution
- **programming**
- **visulaization**

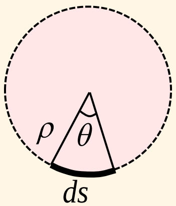
Beam Theory : Sign Convention, Deflection of Beam

Elasticity : Displacement, Strain, Stress, Force Equilibrium, Compatibility, Constitutive Equation

Chapter 2. Element : Beam

Element : Beam - Differential Eqn.

$\boldsymbol{\sigma}_x = \sigma\mathbf{i}$, $\boldsymbol{\varepsilon}_x = \varepsilon\mathbf{i}$, $\boldsymbol{\theta} = \theta\mathbf{k}$, $\mathbf{y} = y\mathbf{j}$ $\qquad \sigma = E\varepsilon$



$$\rho \cdot d\theta = ds \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho}$$

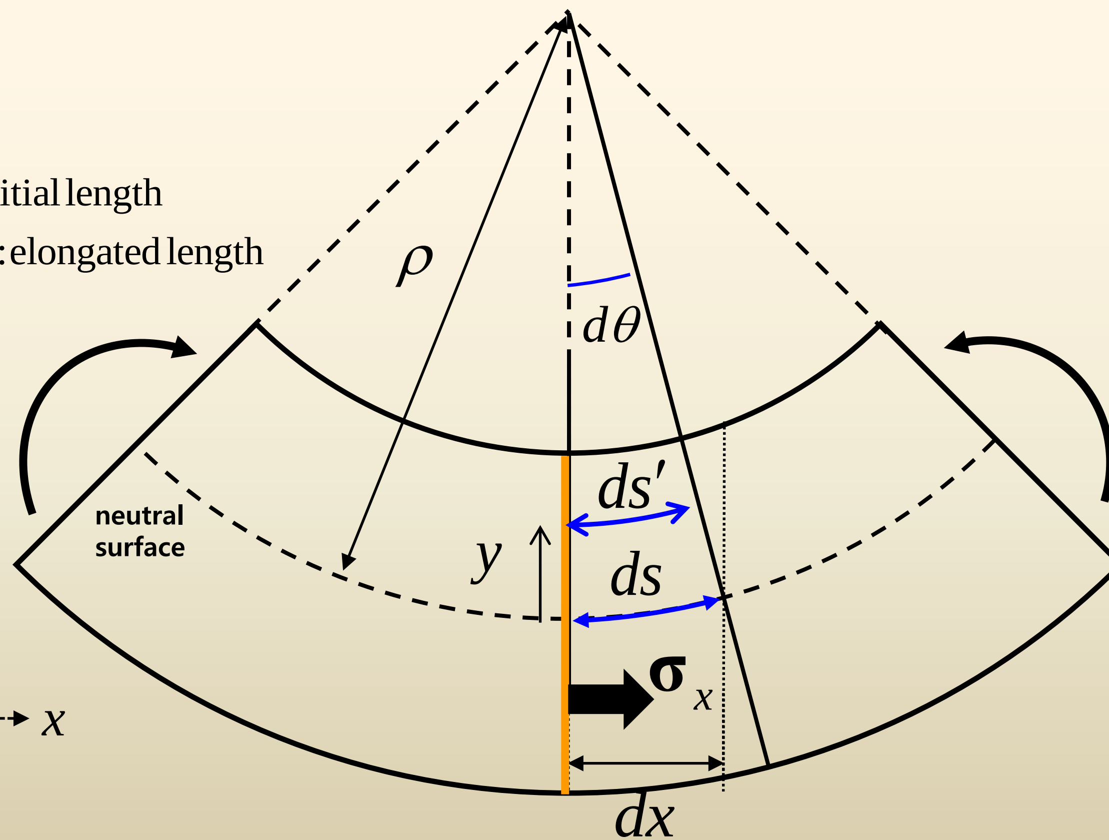
① **strain at** y **in x-direction :**

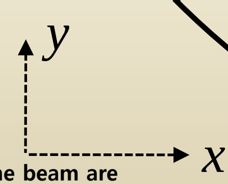
$$\boldsymbol{\varepsilon}_x = \varepsilon\mathbf{i}$$

$$\varepsilon = \frac{(\rho - y)\cdot d\theta - \rho d\theta}{ds} = -y\frac{d\theta}{ds} = -\frac{y}{\rho}$$

$, ds$: initial length

$, y \cdot d\theta$: elongated length

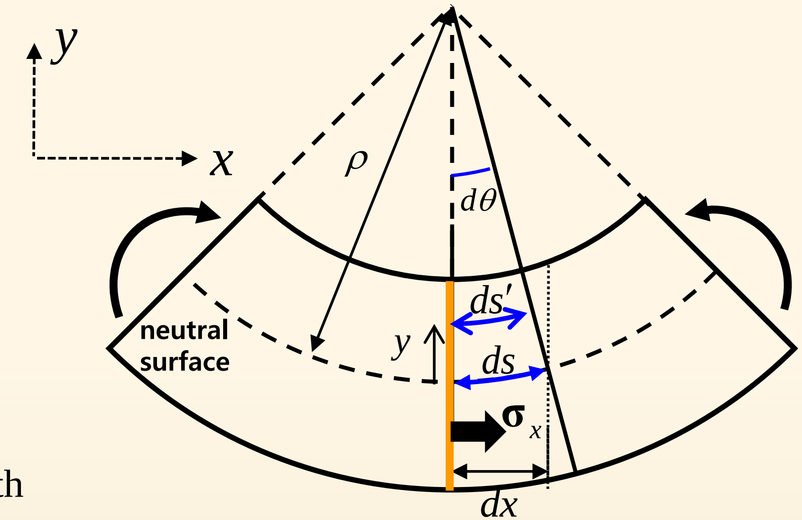


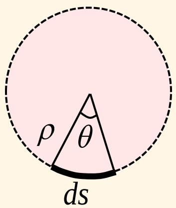


* neutral surface : Longitudinal lines on the lower part of the beam are elongated, whereas those on the upper part are shortened. Thus the lower part of the beam is in tension and the upper part is in compression. Somewhere between the top and bottom of the beam is a surface in which longitudinal lines do not change in length. This surface is called neutral surface

Element : Beam - Differential Eqn.

$\boldsymbol{\sigma}_x = \sigma \mathbf{i}$, $\boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}$, $\boldsymbol{\theta} = \theta \mathbf{k}$, $\mathbf{y} = y\mathbf{j}$ $\qquad \sigma = E\varepsilon$





$$\rho \cdot d\theta = ds \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho}$$

① **strain at y in x-direction :**

$$\boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}$$

$$\varepsilon = \frac{(\rho - y)\cdot d\theta - \rho d\theta}{ds} = -y\frac{d\theta}{ds} = -\frac{y}{\rho}$$

$, ds$: initial length

$, y \cdot d\theta$: elongated length

② **stress at y in x-direction :** $\boldsymbol{\sigma}_x = \sigma \mathbf{i} = E \cdot \varepsilon \mathbf{i}$ $, where\ \varepsilon = -\dfrac{y}{\rho}$ $\qquad \therefore \boldsymbol{\sigma}_x = \sigma \mathbf{i} = -E\dfrac{y}{\rho}\mathbf{i}$

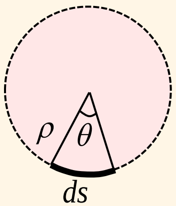
③ **force acting on dA in x-direction :** $d\mathbf{F}_x = \boldsymbol{\sigma}_x dA = (\sigma \mathbf{i})dA = \sigma dA \mathbf{i}$ $\qquad \therefore d\mathbf{F} = -E\dfrac{y}{\rho}dA\mathbf{i}$

④ **moment about z-axis :**

$$d\mathbf{M} = \mathbf{y} \times d\mathbf{F} = (y\mathbf{j}) \times (-E\frac{y}{\rho}dA\mathbf{i}) = E\frac{y^2}{\rho}dA\mathbf{k}$$

Element : Beam - Differential Eqn.

$\boldsymbol{\sigma}_x = \sigma \mathbf{i}$, $\boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}$, $\boldsymbol{\theta} = \theta \mathbf{k}$, $\mathbf{y} = y\mathbf{j}$ $\qquad \sigma = E\varepsilon$



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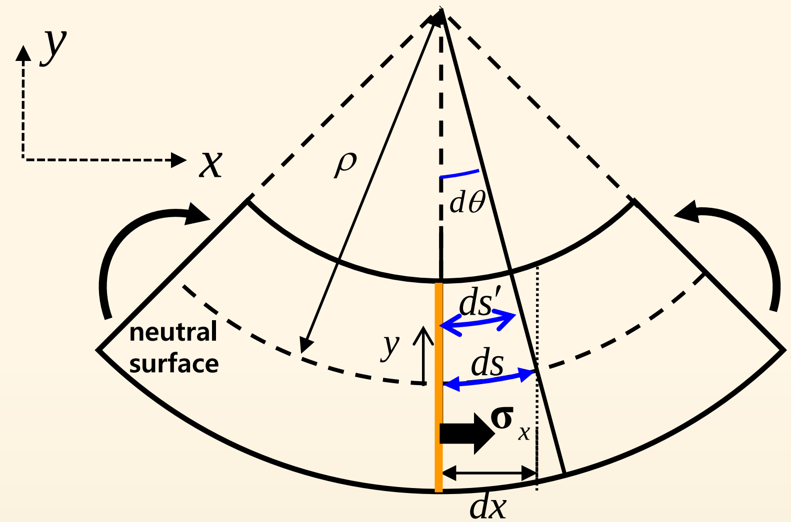


① **strain at** y **in x-direction :** $\varepsilon = \dfrac{(\rho - y)\cdot d\theta - \rho \cdot d\theta}{ds} = -y\dfrac{d\theta}{ds}$

$\boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}$ $\qquad$ $, d\theta$: initial length, $y \cdot d\theta$: elongated length

② **stress at** y **in x-direction :** $\boldsymbol{\sigma}_x = \sigma \mathbf{i} = -E\dfrac{y}{\rho}\mathbf{i}$

③ **force acting on** dA **in x-direction** : $d\mathbf{F} = -E\dfrac{y}{\rho} dA\mathbf{i}$

④ **moment about z-axis :** $d\mathbf{M} = \mathbf{y} \times d\mathbf{F} = (y\mathbf{j}) \times (-E\dfrac{y}{\rho}dA\mathbf{i}) = E\dfrac{y^2}{\rho}dA\mathbf{k}$ $\quad \therefore \mathbf{M} = \int_A d\mathbf{M} = \int_A E\dfrac{y^2}{\rho}dA\mathbf{k}$

Define $I = \int_A y^2 dA$ $\quad$ then, $\mathbf{M} = \dfrac{EI}{\rho}\mathbf{k}$, $M = \dfrac{EI}{\rho}$

$$\Rightarrow \mathbf{M} = \frac{EI}{\rho}\mathbf{k}$$

$$\Downarrow$$

$$\mathbf{M} = EI\frac{d\theta}{ds}\mathbf{k}$$

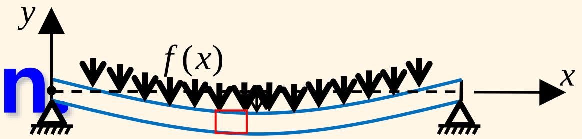
$$\Downarrow$$

$$\boxed{\mathbf{M} = EI\frac{d^2y}{dx^2}\mathbf{k} \qquad , M = EI\frac{d^2y}{dx^2}}$$

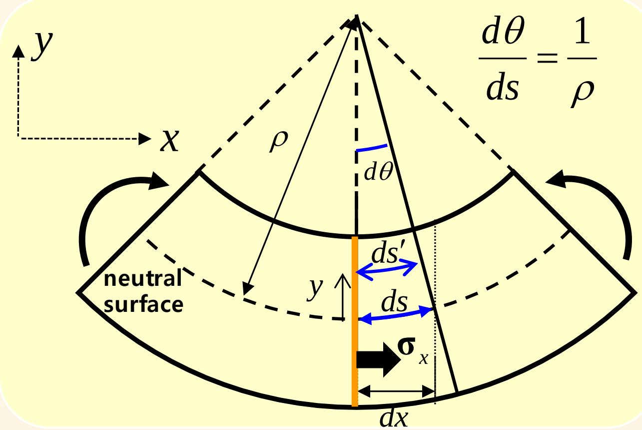
⑤ **assume** $ds \approx dx$, $\theta \approx \tan(\theta) = \dfrac{dy}{dx}$

$$\frac{d\theta}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \Rightarrow \frac{d\theta}{ds} = \frac{d^2y}{dx^2}$$

Element : Beam - Differential Eqn.



$$\boldsymbol{\sigma}_x = \sigma \mathbf{i}\ ,\ \boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}\ ,\ \boldsymbol{\theta} = \theta \mathbf{k},\ \mathbf{y} = y\mathbf{j} \qquad \sigma = E\varepsilon$$



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$\boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}$ $\quad , d\theta$: initial length, $y \cdot d\theta$: elongated length

② **stress at** y **in x-direction :** $\boldsymbol{\sigma}_x = \sigma \mathbf{i} = -E\dfrac{y}{\rho}\mathbf{i}$

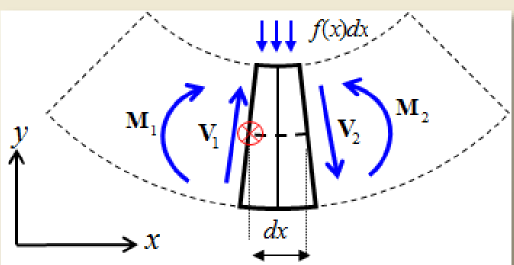
③ **force acting on** dA **in x-direction :** $d\mathbf{F} = -E\dfrac{y}{\rho}dA\mathbf{i}$

④ **moment about z-axis :**

$$d\mathbf{M} = \mathbf{y} \times d\mathbf{F} = (y\mathbf{j}) \times \left(-E\frac{y}{\rho}dA\mathbf{i}\right) = E\frac{y^2}{\rho}dA\mathbf{k} \quad \therefore \mathbf{M} = \int_A d\mathbf{M} = \int_A E\frac{y^2}{\rho}dA\mathbf{k} = \frac{EI}{\rho}\mathbf{k} \quad , I = \int_A y^2 dA$$

⑤ **assume** $ds \approx dx,\ \theta \approx \tan(\theta) = \dfrac{dy}{dx} \Rightarrow \dfrac{d\theta}{ds} = \dfrac{d^2y}{dx^2} \Rightarrow \mathbf{M} = \dfrac{EI}{\rho}\mathbf{k} = EI\dfrac{d\theta}{ds}\mathbf{k} \Rightarrow \mathbf{M} = EI\dfrac{d^2y}{dx^2}\mathbf{k}\ , M = EI\dfrac{d^2y}{dx^2}$

⑥ **relationships between loads, shear forces, and bending moments**



$$\mathbf{V}_1 = V\mathbf{j},\ \ \mathbf{V}_2 = -\left(V + \frac{\partial V}{\partial x}dx\right)\mathbf{j},\ \ \mathbf{M}_1 = -M\mathbf{k},\ \ \mathbf{M}_2 = \left(M + \frac{\partial M}{\partial x}dx\right)\mathbf{k}$$

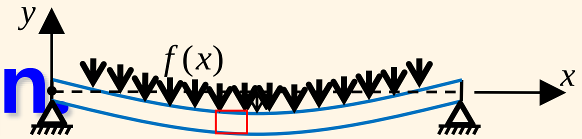
•**force equilibrium** $\sum \mathbf{F}_y = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{f}(x) = 0$

$$(V_1\mathbf{j}) + \left(-\left(V_1 + \frac{\partial V_1}{\partial x}dx\right)\mathbf{j}\right) + (-f(x)dx\mathbf{j}) = 0$$

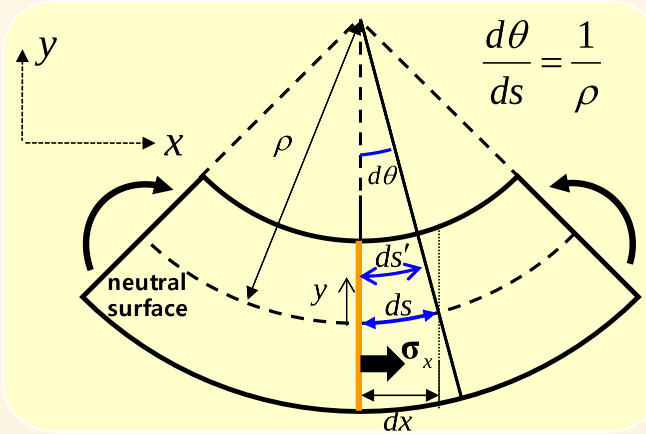
$$\left(V_1 - V_1 - \frac{\partial V_1}{\partial x}dx - f(x)dx\right)\mathbf{j} = 0$$

$$\boxed{\therefore \frac{dV}{dx} = -f(x)}$$

Element : Beam - Differential Eqn



$\boldsymbol{\sigma}_x = \sigma \mathbf{i}$, $\boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}$, $\boldsymbol{\theta} = \theta \mathbf{k}$, $\mathbf{y} = y\mathbf{j}$ $\qquad \sigma = E\varepsilon$



① strain at y in x-direction : $\varepsilon = \dfrac{(\rho - y)\cdot d\theta - \rho \cdot d\theta}{ds} = -y\dfrac{d\theta}{ds}$

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② stress at y in x-direction : $\boldsymbol{\sigma}_x = \sigma \mathbf{i} = -E\dfrac{y}{\rho}\mathbf{i}$

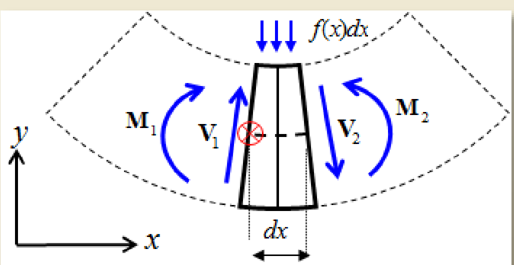
③ force acting on dA in x-direction : $d\mathbf{F} = -E\dfrac{y}{\rho}dA\mathbf{i}$

④ moment about z-axis :

$$d\mathbf{M} = \mathbf{y} \times d\mathbf{F} = (y\mathbf{j}) \times (-E\frac{y}{\rho}dA\mathbf{i}) = E\frac{y^2}{\rho}dA\mathbf{k} \quad \therefore \mathbf{M} = \int_A d\mathbf{M} = \int_A E\frac{y^2}{\rho}dA\mathbf{k} = \frac{EI}{\rho}\mathbf{k} \quad , I = \int_A y^2 dA$$

⑤ assume $ds \approx dx,\ \theta \approx \tan(\theta) = \dfrac{dy}{dx}$ ⇨ $\dfrac{d\theta}{ds} = \dfrac{d^2y}{dx^2}$ ⇨ $\mathbf{M} = \dfrac{EI}{\rho}\mathbf{k} = EI\dfrac{d\theta}{ds}\mathbf{k}$ ⇨ $\mathbf{M} = EI\dfrac{d^2y}{dx^2}\mathbf{k}$, $M = EI\dfrac{d^2y}{dx^2}$

⑥ relationships between loads, shear forces, and bending moments



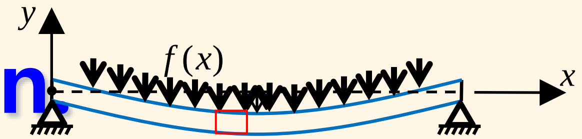
$$\mathbf{V}_1 = V\mathbf{j},\ \ \mathbf{V}_2 = -\left(V + \frac{\partial V}{\partial x}dx\right)\mathbf{j},\ \ \mathbf{M}_1 = -M\mathbf{k},\ \ \mathbf{M}_2 = \left(M + \frac{\partial M}{\partial x}dx\right)\mathbf{k}$$

•force equilibrium $\qquad \dfrac{dV}{dx} = -f(x)$

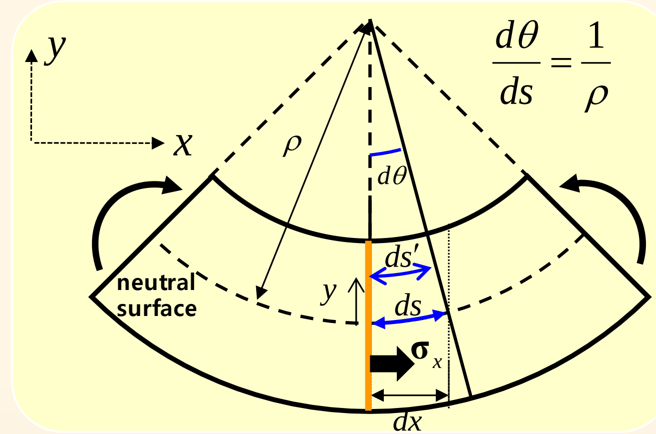
•moment equilibrium $\qquad \sum \mathbf{M}_z = \mathbf{M}_1 + \mathbf{M}_2 + d\mathbf{x} \times \mathbf{V}_2 + \dfrac{1}{2}d\mathbf{x} \times \left(\mathbf{f}(x)\cdot dx\right) = 0$

$$-M\mathbf{k} + \left(M + \frac{\partial M}{\partial x}dx\right)\mathbf{k} + (dx\mathbf{i}) \times \left(-\left(V + \frac{\partial V}{\partial x}dx\right)\mathbf{j}\right) + \left(\frac{1}{2}dx\mathbf{i}\right) \times (-f(x)dx\mathbf{j}) = 0 \qquad \boxed{\therefore \frac{dM}{dx} = V(x)}$$

Element : Beam - Differential Eqn.



$$\boldsymbol{\sigma}_x = \sigma \mathbf{i} \;,\; \boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i} \;,\; \boldsymbol{\theta} = \theta \mathbf{k},\; \mathbf{y} = y\mathbf{j} \qquad \sigma = E\varepsilon$$



$$\frac{d\theta}{ds} = \frac{1}{\rho}$$

① **strain at** y **in x-direction :** $\varepsilon = \dfrac{(\rho - y)\cdot d\theta - \rho \cdot d\theta}{ds} = -y\dfrac{d\theta}{ds}$

$\boldsymbol{\varepsilon}_x = \varepsilon \mathbf{i}$ $\quad$ $, d\theta$: initial length, $\; y\cdot d\theta$: elongated length

② **stress at** y **in x-direction :** $\boldsymbol{\sigma}_x = \sigma \mathbf{i} = -E\dfrac{y}{\rho}\mathbf{i}$

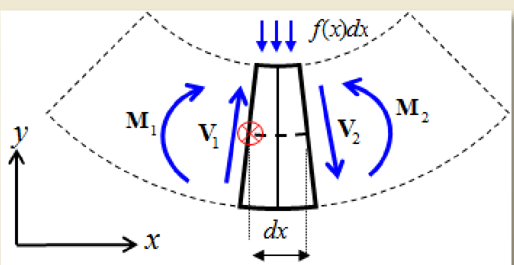
③ **force acting on** dA **in x-direction :** $d\mathbf{F} = -E\dfrac{y}{\rho}dA\mathbf{i}$

④ **moment about z-axis :**

$$d\mathbf{M} = \mathbf{y}\times d\mathbf{F} = (y\mathbf{j})\times\left(-E\frac{y}{\rho}dA\mathbf{i}\right) = E\frac{y^2}{\rho}dA\mathbf{k} \quad \therefore \mathbf{M} = \int_A d\mathbf{M} = \int_A E\frac{y^2}{\rho}dA\mathbf{k} = \frac{EI}{\rho}\mathbf{k} \;,\; I = \int_A y^2 dA$$

⑤ **assume** $ds \approx dx,\; \theta \approx \tan(\theta) = \dfrac{dy}{dx} \Rightarrow \dfrac{d\theta}{ds} = \dfrac{d^2y}{dx^2} \Rightarrow \mathbf{M} = \dfrac{EI}{\rho}\mathbf{k} = EI\dfrac{d\theta}{ds}\mathbf{k} \Rightarrow \mathbf{M} = EI\dfrac{d^2y}{dx^2}\mathbf{k} \;, M = EI\dfrac{d^2y}{dx^2}$

⑥ **relationships between loads, shear forces, and bending moments**



$$\mathbf{V}_1 = V\mathbf{j},\;\; \mathbf{V}_2 = -\left(V + \frac{\partial V}{\partial x}dx\right)\mathbf{j},\;\; \mathbf{M}_1 = -M\mathbf{k},\;\; \mathbf{M}_2 = \left(M + \frac{\partial M}{\partial x}dx\right)\mathbf{k}$$

•**force equilibrium** $\dfrac{dV}{dx} = -f(x)$ $\quad$ •**moment equilibrium** $\dfrac{dM}{dx} = V(x)$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \rightarrow \frac{d^3y}{dx^3} = \frac{1}{EI}\cdot\frac{dM}{dx} = \frac{1}{EI}\cdot V(x) \rightarrow \frac{d^4y}{dx^4} = \frac{1}{EI}\cdot\frac{dV}{dx} = -\frac{1}{EI}\cdot f(x)$$

$$\boxed{\therefore EI\frac{d^4y}{dx^4} = -f(x)}$$

Element : Beam - Variational Method

multiply by δu **and integrate**

$$\int_0^l \left(EI\frac{d^4 v}{dx^4} + f \right)\delta v\, dx = 0$$

L.H.S:

$$\int_0^l \left(EI\frac{d^4 v}{dx^4}\delta v + f\,\delta v \right) dx$$

integration by part

$$= EI\left[\frac{d^3 v}{dx^3}\delta v\right]_0^l - \int_0^l \left(EI\frac{d^3 v}{dx^3}\frac{d(\delta v)}{dx} \right) dx + \int_0^l (f\,\delta v)dx$$

$$= -\int_0^l \left(EI\frac{d^3 v}{dx^3}\frac{d(\delta v)}{dx} \right) dx + \int_0^l (f\,\delta v)dx$$

$$= -EI\left[\frac{d^2 v}{dx^2}\frac{d\delta v}{dx}\right]_0^l + \int_0^l EI\frac{d^2 v}{dx^2}\frac{d^2 \delta v}{dx^2}dx + \int_0^l (f\,\delta v)dx$$

$$= EI\int_0^l \frac{d^2 v}{dx^2}\frac{d^2 \delta v}{dx^2}dx + \int_0^l (f\,\delta v)dx$$

Differential Equation

$$EI\frac{d^4 v}{dx^4} + f(x) = 0$$

Boundary condition

$$v\big|_{x=0} = 0 \quad , v\big|_{x=l} = 0$$

$$, EI\frac{d^2 v}{dx^2}\bigg|_{x=0} = 0, EI\frac{d^2 v}{dx^2}\bigg|_{x=l} = 0$$

δ **operation**

- $f\,\delta v = \delta(fv)$
- $v\,\delta v = \delta\left(\frac{1}{2}v^2\right)$
- $\frac{\delta^2 v}{\delta x^2}\delta\frac{\delta^2 v}{\delta x^2} = \frac{1}{2}\delta\left(\frac{\delta^2 v}{\delta x^2}\right)^2$
- $\frac{d}{dx}\delta v = \delta\frac{d}{dx}v$
- $\delta\int_a^b h(x)dx = \int_a^b \delta h(x)dx$

Element : Beam - Variational Method

multiply by δu and integrate

$$\int_0^l \left(EI \frac{d^4 v}{dx^4} + f \right) \delta v \, dx = 0$$

L.H.S:

$$\int_0^l \left(EI \frac{d^4 v}{dx^4} \delta v + f \delta v \right) dx$$

integration by part

$$= \int_0^l EI \frac{d^2 v}{dx^2} \frac{d^2 \delta v}{dx^2} dx + \int_0^l (f \delta v) dx$$

$$= \int_0^l \delta \frac{1}{2} EI \frac{d^2 v}{dx^2} \frac{d^2 v}{dx^2} dx + \int_0^l (\delta f v) dx$$

$$\delta \int_0^l \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right) + (fv) \right] dx$$

Differential Equation

$$EI \frac{d^4 v}{dx^4} + f(x) = 0$$

Boundary Condition

$$v|_{x=0} = 0 \quad , v|_{x=l} = 0$$

$$, EI \left. \frac{d^2 v}{dx^2} \right|_{x=0} = 0, EI \left. \frac{d^2 v}{dx^2} \right|_{x=l} = 0$$

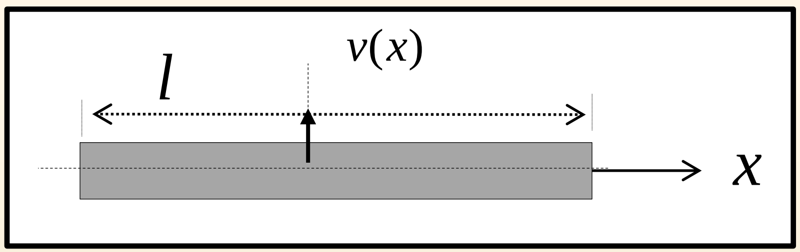
δ operation

- $f \delta v = \delta (fv)$
- $v \delta v = \delta \left(\frac{1}{2} v^2 \right)$
- $\frac{\delta^2 v}{\delta x^2} \delta \frac{\delta^2 v}{\delta x^2} = \frac{1}{2} \delta \left(\frac{\delta^2 v}{\delta x^2} \right)^2$
- $\frac{d}{dx} \delta v = \delta \frac{d}{dx} v$
- $\delta \int_a^b h(x) dx = \int_a^b \delta h(x) dx$

Element : Beam - Finite Element Method

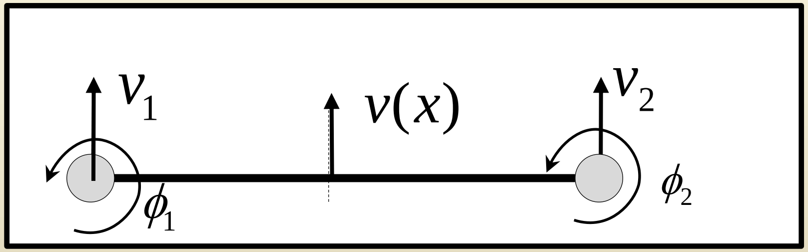
Variational Method

$$\delta \int_0^l \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 + (fv) \right] dx$$



discretization

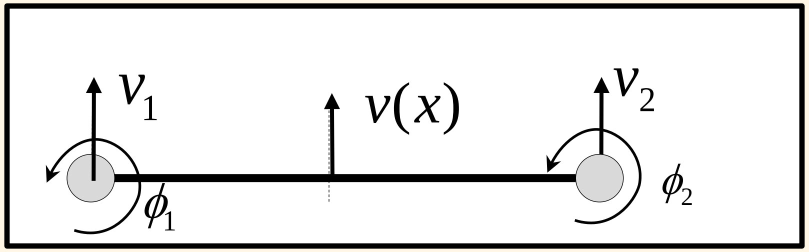
finite element method ↓ **1 element , 2 nodes**



assume: $v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, $v(0) = v_1$, $v(l) = v_2$

, $\frac{dv}{dx}(0) = \phi_1$, $\frac{dv}{dx}(l) = \phi_2$

Element : Beam - Finite Element Method



Variational Method

$$\delta \int_0^l \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 + (fv) \right] dx$$

assume: $$v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad v(0) = v_1, \quad v(l) = v_2, \quad \frac{dv}{dx}(0) = \phi_1, \quad \frac{dv}{dx}(l) = \phi_2$$

$$v(0) = c_0 \quad \Rightarrow \quad c_0 = v_1$$

$$v(l) = c_0 + c_1 l + c_2 l^2 + c_3 l^3 = v_2$$

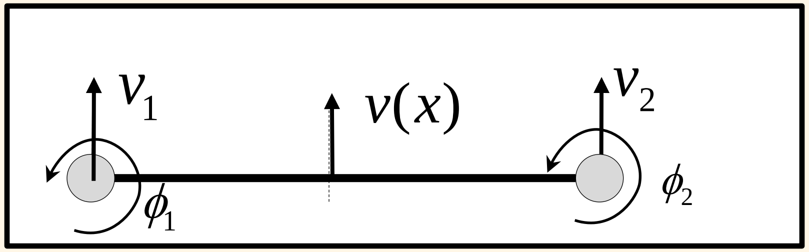
$$\frac{dv}{dx}(0) = c_1 \quad \Rightarrow \quad c_1 = \phi_1$$

$$\frac{dv}{dx}(l) = c_1 + 2c_2 l + 3c_3 l^2 = \phi_2$$

$$\Rightarrow \quad c_2 = -\frac{3}{l^2}(v_1 - v_2) - \frac{1}{l}(2\phi_1 + \phi_2)$$

$$c_3 = \frac{2}{l^3}(v_1 - v_2) + \frac{1}{l^2}(\phi_1 + \phi_2)$$

Element : Beam - Finite Element Method



Variational Method

$$\delta \int_0^l \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 + (fv) \right] dx$$

assume: $v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, $v(0) = v_1$, $v(l) = v_2$, $\dfrac{dv}{dx}(0) = \phi_1$, $\dfrac{dv}{dx}(l) = \phi_2$

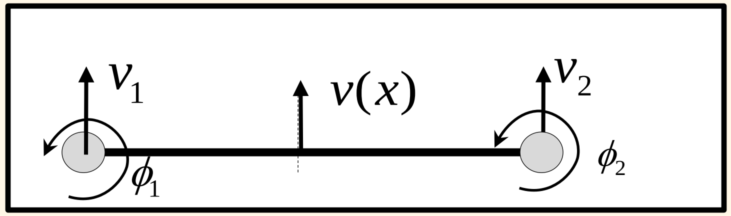
$$c_0 = v_1 \;, c_1 = \phi_1 \;, c_2 = -\frac{3}{l^2}(v_1 - v_2) - \frac{1}{l}(2\phi_1 + \phi_2)$$

$$c_3 = \frac{2}{l^3}(v_1 - v_2) + \frac{1}{l^2}(\phi_1 + \phi_2)$$

$$v(x) = v_1 + \phi_1 x + \left[-\frac{3}{l^2}(v_1 - v_2) - \frac{1}{l}(2\phi_1 + \phi_2) \right] x^2 + \left[\frac{2}{l^3}(v_1 - v_2) + \frac{1}{l^2}(\phi_1 + \phi_2) \right] x^3$$

$$\text{or } v(x) = \frac{1}{l^3}(2x^3 - 3x^2 l + l^3) v_1 + \frac{1}{l^3}(x^3 l - 2x^2 l^2 + x l^3)\phi_1 + \frac{1}{l^3}(-2x^3 + 3x^2 l) v_2 + \frac{1}{l^3}(x^3 l - x^2 l^2)\phi_2$$

Element : Beam - Finite Element Method



Variational Method

$$\delta \int_0^l \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 + (fv) \right] dx$$

$$v(x) = \frac{1}{l^3}(2x^3 - 3x^2 l + l^3)v_1 + \frac{1}{l^3}(x^3 l - 2x^2 l^2 + x l^3)\phi_1 + \frac{1}{l^3}(-2x^3 + 3x^2 l)v_2 + \frac{1}{l^3}(x^3 l - x^2 l^2)\phi_2$$

$$\Downarrow$$

$$v(x) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{bmatrix}$$

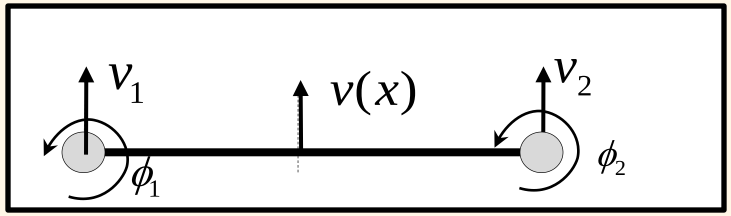
$$N_1 = \frac{1}{l^3}(2x^3 - 3x^2 l + l^3)$$

$$N_2 = \frac{1}{l^3}(x^3 l - 2x^2 l^2 + x l^3)$$

$$N_3 = \frac{1}{l^3}(-2x^3 + 3x^2 l)$$

$$N_4 = \frac{1}{l^3}(x^3 l - x^2 l^2)$$

Element : Beam - Finite Element Method



Variational Method

$$\delta \int_0^l \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 + (fv) \right] dx$$

$$v(x) = \frac{1}{l^3}(2x^3 - 3x^2 l + l^3)v_1 + \frac{1}{l^3}(x^3 l - 2x^2 l^2 + x l^3)\phi_1 + \frac{1}{l^3}(-2x^3 + 3x^2 l)v_2 + \frac{1}{l^3}(x^3 l - x^2 l^2)\phi_2$$

$$\Downarrow$$

$$v(x) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{bmatrix}$$

$$N_1 = \frac{1}{l^3}(2x^3 - 3x^2 l + l^3) \qquad N_2 = \frac{1}{l^3}(x^3 l - 2x^2 l^2 + x l^3)$$

$$N_3 = \frac{1}{l^3}(-2x^3 + 3x^2 l) \qquad N_4 = \frac{1}{l^3}(x^3 l - x^2 l^2)$$

$\Downarrow$ **differentiation with respect to** x **twice**

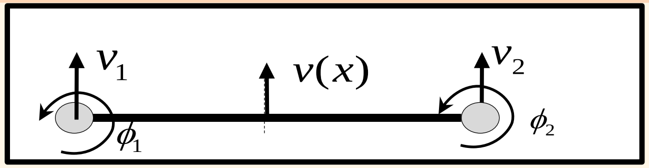
$$\frac{d^2 v(x)}{dx^2} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{bmatrix}$$

$$B_1 = \frac{1}{l^3}(12x - 6l) \qquad B_2 = \frac{1}{l^3}(6xl - 4l^2)$$

$$B_3 = \frac{1}{l^3}(-12x + 6l) \qquad B_4 = \frac{1}{l^3}(6xl - 2l^2)$$

$$\therefore v(x) = \mathbf{N}\mathbf{d}, \quad \frac{d^2 v(x)}{dx^2} = \mathbf{B}\mathbf{d}$$

Element : Beam - Finite Element Method



Variational Method

$$\delta\int_0^l\left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2+(fv)\right]dx$$

$$v(x)=\mathbf{N}\,\mathbf{d}$$

$$\frac{d^2v(x)}{dx^2}=\mathbf{B}\,\mathbf{d}$$

$$\delta\int_0^l\left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2+(fv)\right]dx=0$$

$$\text{where } \mathbf{N}=\left[N_1\ \ N_2\ \ N_3\ \ N_4\right],\ \ \mathbf{B}=\left[B_1\ \ B_2\ \ B_3\ \ B_4\right],\ \ \mathbf{d}=\begin{bmatrix}v_1\\ \phi_1\\ v_2\\ \phi_2\end{bmatrix}$$

$$\Rightarrow\ \delta\left\{\frac{EI}{2}\int_0^l\left(\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}\right)dx-\int_0^l(f\ \mathbf{N}\mathbf{d})dx\right\}=0$$

derivation

$$\mathbf{B}^T\mathbf{B}=\begin{bmatrix}\frac{1}{l^3}(12x-6l)\\ \frac{1}{l^3}(6xl-4l^2)\\ \frac{1}{l^3}(-12x+6l)\\ \frac{1}{l^3}(6xl-2l^2)\end{bmatrix}\begin{bmatrix}\frac{1}{l^3}(12x-6l) & \frac{1}{l^3}(6xl-4l^2) & \frac{1}{l^3}(-12x+6l) & \frac{1}{l^3}(6xl-2l^2)\end{bmatrix}$$

(derivation)

$$\delta\int_0^l\left[\frac{EA}{2}\left(\frac{d^2v}{dx^2}\right)^2+(fv)\right]dx \qquad f\,\mathbf{Nd}=(\mathbf{Nd})^{\mathrm{T}}f=\mathbf{d}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}f \quad \because \mathbf{Nd}: scalar$$

$$=\delta\left\{\frac{EI}{2}\int_0^l\left(\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}\right)dx+\int_0^l(f\,\mathbf{N}\,\mathbf{d})dx\right\}$$

$$=\delta\left\{\frac{EI}{2}\int_0^l\left(\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}\right)dx+\int_0^l(\mathbf{d}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\,f\,)dx\right\}$$

$$=\delta\left\{\frac{1}{2}\mathbf{d}^{\mathrm{T}}\boxed{\left[\int_0^l EI\left(\mathbf{B}^T\mathbf{B}\right)dx\right]}\mathbf{d}+\mathbf{d}^{\mathrm{T}}\boxed{\left[\int_0^l(\mathbf{N}^{\mathrm{T}}f\,)dx\right]}\right\}$$

$$=\delta\left\{\frac{1}{2}\mathbf{d}^{\mathrm{T}}\boxed{\mathbf{K}}\mathbf{d}-\mathbf{d}^{\mathrm{T}}\boxed{\mathbf{F}}\right\}$$

$$=\frac{1}{2}(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{Kd}+\frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{K}\delta\mathbf{d}-(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{F}$$

$$=\boxed{\frac{1}{2}(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{Kd}+\frac{1}{2}(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{Kd}}-(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{F} \quad \because(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{Kd}=\mathbf{d}^{\mathrm{T}}\mathbf{K}\delta\mathbf{d}$$

$$=\boxed{(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{Kd}}-(\delta\mathbf{d})^{\mathrm{T}}\,\mathbf{F}$$

$$=(\delta\mathbf{d})^{\mathrm{T}}\left(\mathbf{Kd}-\mathbf{F}\right)$$

$$\mathbf{K}=EI\int_0^l\left(\mathbf{B}^T\mathbf{B}\right)dx$$

$$=\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}$$

⇨ $\mathbf{K}=\mathbf{K}^{\mathrm{T}}$

symmetry

$$\mathbf{F}=-\int_0^l(\mathbf{N}^{\mathrm{T}}f\,)dx$$

$$d=[v_1\ \phi_1\ v_2\ \phi_2]^T$$

(derivation)

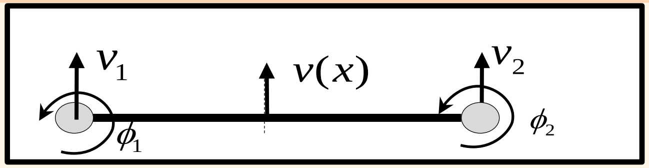
$$\mathbf{K} = EI\int_0^l \left(\mathbf{B}^T\mathbf{B}\right)dx$$

$$= EI\int_0^l \begin{bmatrix} \frac{1}{l^3}(12x-6l)\frac{1}{l^3}(12x-6l) & \frac{1}{l^3}(12x-6l)\frac{1}{l^3}(6xl-4l^2) & \frac{1}{l^3}(12x-6l)\frac{1}{l^3}(-12x+6l) & \frac{1}{l^3}(12x-6l)\frac{1}{l^3}(6xl-2l^2) \\ \frac{1}{l^3}(6xl-4l^2)\frac{1}{l^3}(12x-6l) & \frac{1}{l^3}(6xl-4l^2)\frac{1}{l^3}(6xl-4l^2) & \frac{1}{l^3}(6xl-4l^2)\frac{1}{l^3}(-12x+6l) & \frac{1}{l^3}(6xl-4l^2)\frac{1}{l^3}(6xl-2l^2) \\ \frac{1}{l^3}(-12x+6l)\frac{1}{l^3}(12x-6l) & \frac{1}{l^3}(-12x+6l)\frac{1}{l^3}(6xl-4l^2) & \frac{1}{l^3}(-12x+6l)\frac{1}{l^3}(-12x+6l) & \frac{1}{l^3}(-12x+6l)\frac{1}{l^3}(6xl-2l^2) \\ \frac{1}{l^3}(6xl-2l^2)\frac{1}{l^3}(12x-6l) & \frac{1}{l^3}(6xl-2l^2)\frac{1}{l^3}(6xl-4l^2) & \frac{1}{l^3}(6xl-2l^2)\frac{1}{l^3}(-12x+6l) & \frac{1}{l^3}(6xl-2l^2)\frac{1}{l^3}(6xl-2l^2) \end{bmatrix} dx$$

$$= EI\int_0^l \begin{bmatrix} \frac{1}{l^6}(144x^2-144xl+36l^2) & \frac{1}{l^6}(72x^2l-84xl^2+24l^3) & \frac{1}{l^6}(-144x^2+144xl-36l^2) & \frac{1}{l^6}(72x^2l-60xl^2+12l^3) \\ \frac{1}{l^6}(72x^2l-84xl^2+24l^3) & \frac{1}{l^6}(36x^2l^2-48xl^3+16l^4) & \frac{1}{l^6}(-72x^2l+84xl^2-24l^3) & \frac{1}{l^6}(36x^2l^2-36xl^3+8l^4) \\ \frac{1}{l^6}(-144x^2+144xl-36l^2) & \frac{1}{l^6}(-72x^2l+84xl^2-24l^3) & \frac{1}{l^6}(144x^2-144xl+36l^2) & \frac{1}{l^6}(-72x^2l+60xl^2-12l^3) \\ \frac{1}{l^6}(72x^2l-60xl^2+12l^3) & \frac{1}{l^6}(36x^2l^2-36xl^3+8l^4) & \frac{1}{l^6}(-72x^2l+60xl^2-12l^3) & \frac{1}{l^3}(36x^2l^2-24xl^3+4l^4) \end{bmatrix} dx$$

$$= \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Element : Beam - Finite Element Method



Variational Method

$$\delta\int_0^l \left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 + (fv) \right] dx$$

$$v(x) = \mathbf{N}\,\mathbf{d}$$

$$\frac{d^2v(x)}{dx^2} = \mathbf{B}\,\mathbf{d}$$

$$\boxed{\delta\int_0^l \left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 + (fv) \right] dx = 0}$$

$$\textit{where } \mathbf{N} = [N_1\ N_2\ N_3\ N_4],\ \ \mathbf{B} = [B_1\ B_2\ B_3\ B_4],\ \ \mathbf{d} = \begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{bmatrix}$$

$$\Rightarrow\ \delta\left\{ \frac{EI}{2}\int_0^l \left(\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}\right)dx - \int_0^l (f\ \mathbf{N}\mathbf{d})dx \right\} = 0$$

derivation

$$\Rightarrow\ (\delta\mathbf{d})^{\mathrm{T}}(\mathbf{K}\mathbf{d} - \mathbf{F}) = 0$$

$$\therefore \mathbf{K}\mathbf{d} = \mathbf{F} \qquad \textit{where, } \mathbf{K} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix},\ \mathbf{F} = -\int_0^l (\mathbf{N}^{\mathrm{T}} f)dx$$

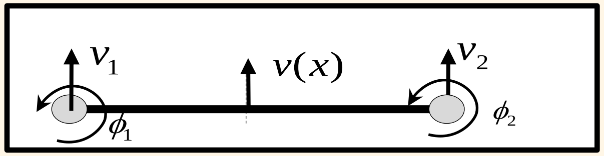
Element : Beam - Finite Element Metho

Variational Method

$$\delta\int_0^l\left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2+(fv)\right]dx$$

equivalent nodal forces

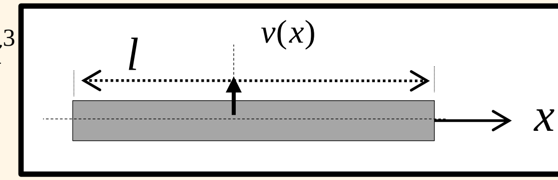
$$\mathbf{Kd}=\mathbf{F}$$



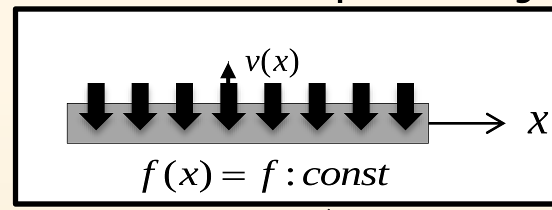
assume: $v(x)=c_0+c_1x+c_2x^2+c_3x^3$

$, v(0)=v_1,\ v(l)=v_2,$

$v'(0)=\phi_1,\ v'(l)=\phi_2$



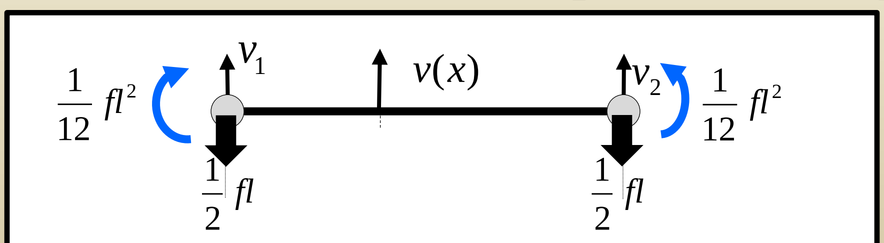
constant external force per unit length

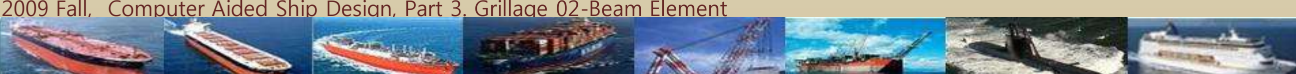


$$\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{bmatrix}v_1\\ \phi_1\\ v_2\\ \phi_2\end{bmatrix}=\mathbf{F}\quad ,\mathbf{F}=-\int_0^l(\mathbf{N}^{\mathrm{T}}f)dx$$

equivalent nodal forces

$$\mathbf{F}=\begin{bmatrix}f_1\\ m_1\\ f_2\\ m_2\end{bmatrix}=-\int_0^l(\mathbf{N}^{\mathrm{T}}f)dx=-\int_0^l\begin{bmatrix}\frac{1}{l^3}(2x^3-3x^2l+l^3)\\ \frac{1}{l^3}(x^3l-2x^2l^2+xl^3)\\ \frac{1}{l^3}(-2x^3+3x^2l)\\ \frac{1}{l^3}(x^3l-x^2l^2)\end{bmatrix}f\,dx=-\frac{1}{l^3}\left[\begin{bmatrix}\frac{x^4}{2}-x^3l+xl^3\\ \frac{x^4}{4}l-\frac{2}{3}x^3l^2+\frac{x^2}{2}l^3\\ -\frac{x^4}{2}+x^3l\\ \frac{x^4}{4}l-\frac{x^3}{3}l^2\end{bmatrix}f\right]_0^l=\begin{bmatrix}-\frac{l}{2}f\\ -\frac{l^2}{12}f\\ -\frac{l}{2}f\\ \frac{l^2}{12}f\end{bmatrix}$$





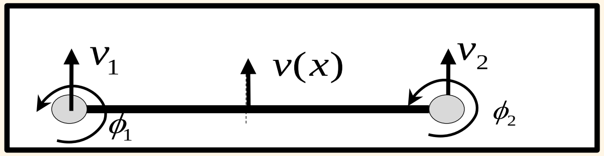
Element : Beam - Finite Element Method

Variational Method

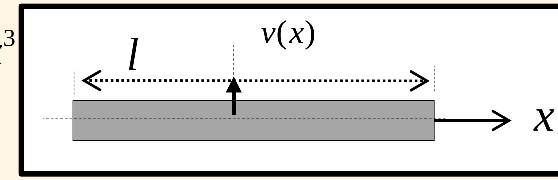
$$\delta\int_0^l\left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2+(fv)\right]dx$$

equivalent nodal forces

$$\mathbf{Kd}=\mathbf{F}$$



assume: $v(x)=c_0+c_1x+c_2x^2+c_3x^3$

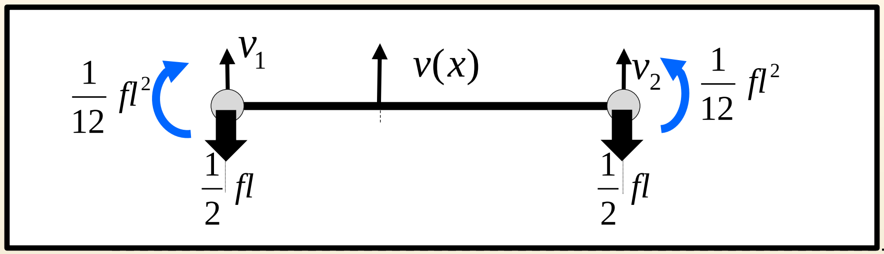


$, v(0)=v_1,\ v(l)=v_2,$

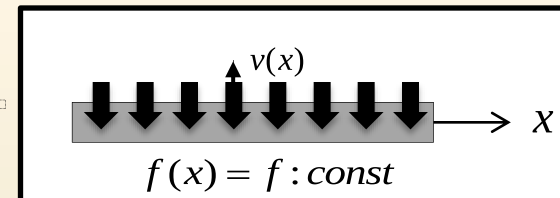
$v'(0)=\phi_1,\ v'(l)=\phi_2$

$$\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{bmatrix}v_1\\ \phi_1\\ v_2\\ \phi_2\end{bmatrix}=\mathbf{F} \qquad ,\mathbf{F}=-\int_0^l(\mathbf{N}^{\mathrm{T}}f)dx$$

constant external force per unit length



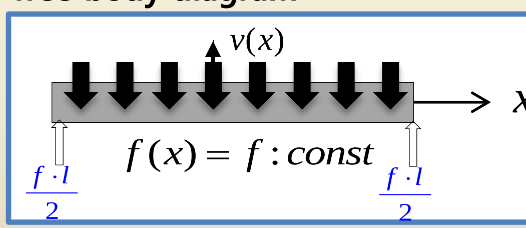
$$\mathbf{F}=\begin{bmatrix}f_1\\ m_1\\ f_2\\ m_2\end{bmatrix}=\begin{bmatrix}-\frac{l}{2}f\\ -\frac{l^2}{12}f\\ -\frac{l}{2}f\\ \frac{l^2}{12}f\end{bmatrix}$$



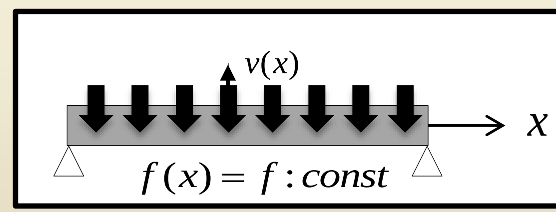
equivalent nodal forces

$$\therefore\mathbf{F}=\begin{bmatrix}-\frac{l}{2}f\\ -\frac{l^2}{12}f\\ -\frac{l}{2}f\\ \frac{l^2}{12}f\end{bmatrix}+\begin{bmatrix}\frac{l}{2}f\\ 0\\ \frac{l}{2}f\\ 0\end{bmatrix}=\begin{bmatrix}0\\ -\frac{l^2}{12}f\\ 0\\ \frac{l^2}{12}f\end{bmatrix}$$

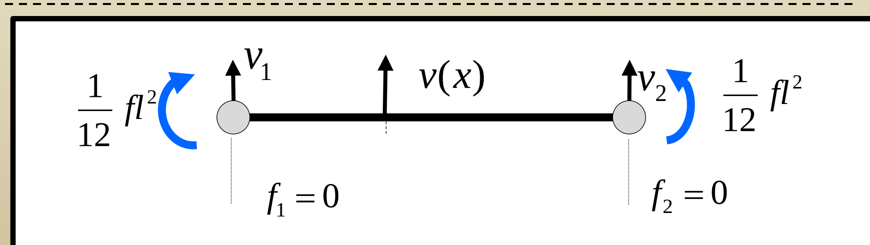
free body diagram



boundary condition



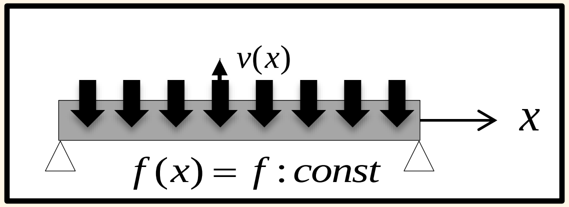
$$\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{bmatrix}0\\ \phi_1\\ 0\\ \phi_2\end{bmatrix}=\begin{bmatrix}0\\ -\frac{l^2}{12}f\\ 0\\ \frac{l^2}{12}f\end{bmatrix}$$

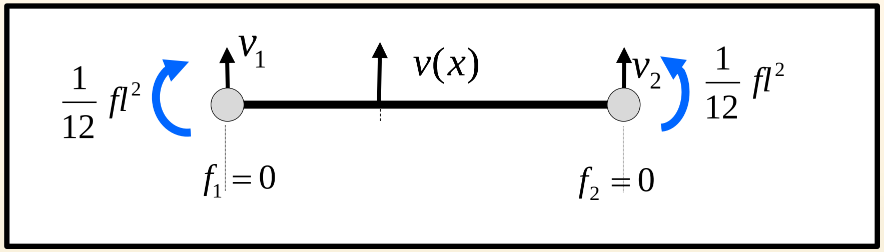


Element : Beam - Finite Element Method

Variational Method
$$\delta\int_0^l\left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2+(fv)\right]dx$$

equivalent nodal forces





$$\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{bmatrix}0\\ \phi_1\\ 0\\ \phi_2\end{bmatrix}=\begin{bmatrix}0\\ -\frac{l^2}{12}f\\ 0\\ \frac{l^2}{12}f\end{bmatrix}$$

$$\underset{\text{find}}{\mathbf{Kd}}=\underset{\text{given}}{\mathbf{F}}$$

$$v_1=0,\ \phi_1=-\frac{fl^3}{24EI},\ v_2=0,\ \phi_2=\frac{fl^3}{24EI}$$

$$v(x)=\frac{f}{24EI}(x^3l-2x^2l^2+xl^3)\phi_1+\frac{1}{l^3}(x^3l-x^2l^2)\phi_2\ ,0\le x\le l$$

displacement

given : x

find : $v(x)$

$$v(x)=\frac{f}{24EI}(2x^2l^2-xl^3)$$

$ref.)\ \int_0^1 (-u'v' + uv - xv)\,dx = 0$

Galerkin's Residual Method

Differential Equation

$$\boxed{EI\frac{d^4v(x)}{dx^4} + f = 0}$$

residual

$$EA\frac{d^4 v(x)}{dx^4} + f \neq 0 = R$$

since it is approximated solution

$$u(x) = \mathbf{N}\,\mathbf{d}$$

$$where\ \mathbf{N} = [N_1 \quad N_2 \quad N_3 \quad N_4], \quad \mathbf{d} = \begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{bmatrix}$$

basis function

Thus substituting the approximated solution into the differential equation results in a residual over the whole region of the problem as follows

$$\iiint_V R\,dV$$

In the residual method, we require that a weighted value of the residual be a minimum over the whole region. The weighting functions allow the weighted integral of residuals to go to zero

$$\iiint_V R\,W\,dV = 0$$

weighting function or test function

Galerkin Method

the basis functions N_i are chosen to play the role of the weighting functions *W*

$$\iiint_V R\,N_i\,dV = 0 \qquad ,(i = 1, 2)$$

Element : Beam - Galerkin's Residual Method

$$ref.)\quad \int_0^1 (-u'v' + uv - xv)\,dx = 0$$

Beam - Galerkin's Residual Method

$$\int_0^l \left[EI\frac{d^4 v(x)}{dx^4} + f \right] N_i\,dx = 0 \quad ,(i=1,2,3,4)$$

$$where,\quad v(x) = \mathbf{N}\,\mathbf{d}$$

$$N_1 = \frac{1}{l^3}(2x^3 - 3x^2 l + l^3)$$

$$N_2 = \frac{1}{l^3}(x^3 l - 2x^2 l^2 + x l^3)$$

$$N_3 = \frac{1}{l^3}(-2x^3 + 3x^2 l)$$

$$N_4 = \frac{1}{l^3}(x^3 l - x^2 l^2)$$

Galerkin Method

the test functions N_i are chosen to play the role of the weighting functions *W*

$$\iiint_V R\,N_i\,dV = 0 \qquad ,(i=1,2)$$

N_i → weighting function

R → residual (test function $\mathbf{N}$ used)

integration by parts

$$\left[N_i EI \frac{d^3 v}{dx^3} \right]_0^l - \int_0^l EI\frac{d^3 v}{dx^3}\frac{dN_i}{dx}\,dx + \int_0^l f\,N_i\,dx = 0 \quad ,(i=1,2,3,4)$$

Differential Equation

$$EI\frac{d^4 v(x)}{dx^4} + f = 0$$

integration by parts again

$$EI\int_0^l \frac{d^2 N_i}{dx^2}\frac{d^2 v}{dx^2}\,dx + EI\left[N_i\frac{d^3 v}{dx^3} - \frac{dN_i}{dx}\frac{d^2 v}{dx^2} \right]_0^l + \int_0^l f\,N_i\,dx = 0 \quad ,(i=1,2,3,4)$$

Recall,

$$\frac{d^3 v}{dx^3} = V(x),\quad \frac{d^2 v}{dx^2} = m(x)$$

$$EI\int_0^l \frac{d^2 N_i}{dx^2}\mathbf{B}\,dx\,\mathbf{d} + EI\left[N_i V - \frac{dN_i}{dx}m \right]_0^l + \int_0^l f\,N_i\,dx = 0 \quad ,(i=1,2,3,4)$$

In matrix form,

$$EI\int_0^l \mathbf{B}^T\mathbf{B}\,dx\,\mathbf{d} = EI\left[\frac{d\mathbf{N}^T}{dx}m - \mathbf{N}^T V \right]_0^l - \int_0^l \mathbf{N}^T f\;dx = 0$$

Element : Beam - Galerkin's Residual Method

$ref.)\ \int_0^1 (-u'v' + uv - xv)\,dx = 0$

Beam - Galerkin's Residual Method

$$\int_0^l \left[EI \frac{d^4 v(x)}{dx^4} + f \right] N_i \, dx = 0 \quad ,(i=1,2)$$

$$where, \quad v(x) = \mathbf{N}\mathbf{d}$$

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$$N_2 = \frac{1}{l^3}(x^3 l - 2x^2 l^2 + x l^3)$$
$$N_3 = \frac{1}{l^3}(-2x^3 + 3x^2 l)$$
$$N_4 = \frac{1}{l^3}(x^3 l - x^2 l^2)$$

Galerkin Method

the test functions N_i are chosen to play the role of the weighting functions *W*

$$\iiint_V R\, N_i \, dV = 0 \qquad ,(i=1,2)$$

weighting function

residual (test function $\mathbf{N}$ used)

integration by parts

$$\underbrace{\boxed{EI\int_0^l \mathbf{B}^T \mathbf{B}\, dx}}_{\mathbf{K}}\mathbf{d} = EI\left[\frac{d\mathbf{N}^T}{dx} m - \mathbf{N}^T V \right]_0^l - \int_0^l \mathbf{N}^T f \; dx = 0$$

$$\mathbf{N}(x) = \frac{1}{l^3}\begin{bmatrix} 2x^3 - 3x^2 l + l^3 & x^3 l - 2x^2 l^2 + x l^3 & -2x^3 + 3x^2 l & x^3 l - x^2 l^2 \end{bmatrix}$$

$$\frac{d\mathbf{N}(x)}{dx} = \frac{1}{l^3}\begin{bmatrix} 6x^2 - 6xl & 3x^2 l - 4x l^2 + l^3 & -6x^2 + 6xl & 3x^2 l - 2x l^2 \end{bmatrix}$$

$$\mathbf{N}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad , \mathbf{N}(l) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$, \left.\frac{d\mathbf{N}(x)}{dx}\right|_{x=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \; , \left.\frac{d\mathbf{N}(x)}{dx}\right|_{x=l} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

R.H.S

$$EI\left[\frac{d\mathbf{N}^T}{dx} m - \mathbf{N}^T V \right]_0^l - \int_0^l \mathbf{N}^T f \, dx = m(l)\frac{d\mathbf{N}}{dx}(l) - V(l)\mathbf{N}(l) - m(0)\frac{d\mathbf{N}^T}{dx}(0) + V(0)\mathbf{N}^T(0) - \int_0^l \mathbf{N}^T f \, dx$$

Differential Equation

$$EI\frac{d^4 v(x)}{dx^4} + f = 0$$

$$\mathbf{K} = EI\int_0^l \left(\mathbf{B}^T \mathbf{B} \right) dx$$

$$= \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Element : Beam - Galerkin's Residual Method

$$ref.)\quad \int_0^1 (-u'v' + uv - xv)\,dx = 0$$

Beam - Galerkin's Residual Method

$$\int_0^l \left[EI\frac{d^4 v(x)}{dx^4} + f \right] N_i\,dx = 0 \quad ,(i=1,2)$$

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$$N_1 = \frac{1}{l^3}(2x^3 - 3x^2 l + l^3)$$

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$$N_3 = \frac{1}{l^3}(-2x^3 + 3x^2 l)$$

$$N_4 = \frac{1}{l^3}(x^3 l - x^2 l^2)$$

Galerkin Method

the test functions N_i are chosen to play the role of the weighting functions W

$$\iiint_V R\, N_i\, dV = 0 \qquad ,(i=1,2)$$

N_i: weighting function

R: residual (test function $\mathbf{N}$ used)

integration by parts

$$\underbrace{EI\int_0^l \mathbf{B}^T\mathbf{B}\,dx}_{\mathbf{K}}\,\mathbf{d} = EI\left[\frac{d\mathbf{N}}{dx}m - \mathbf{N}V\right]_0^l - \int_0^l f\,\mathbf{N}\,dx$$

R.H.S

$$EI\left[\frac{d\mathbf{N}^T}{dx}m - \mathbf{N}^T V\right]_0^l - \int_0^l \mathbf{N}^T f\,dx = m(l)\frac{d\mathbf{N}}{dx}(l) - V(l)\mathbf{N}(l) - m(0)\frac{d\mathbf{N}^T}{dx}(0) + V(0)\mathbf{N}^T(0) - \int_0^l \mathbf{N}^T f\,dx$$

$$= m(l)\begin{bmatrix}0\\0\\0\\1\end{bmatrix} - V(l)\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - m(0)\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + V(0)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-\frac{l}{2}f\\ -\frac{l^2}{12}f\\ -\frac{l}{2}f\\ \frac{l^2}{12}f\end{bmatrix} = \underbrace{\begin{bmatrix}V(0)-\frac{l}{2}f\\ -m(0)-\frac{l^2}{12}f\\ -V(l)-\frac{l}{2}f\\ m(l)+\frac{l^2}{12}f\end{bmatrix}}_{\mathbf{F}}$$

Differential Equation

$$EI\frac{d^4 v(x)}{dx^4} + f = 0$$

$$\mathbf{K} = EI\int_0^l \left(\mathbf{B}^T\mathbf{B}\right)dx$$

$$= \frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}$$

$$\mathbf{N}(0) = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} \quad ,\mathbf{N}(l) = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}$$

$$,\left.\frac{d\mathbf{N}(x)}{dx}\right|_{x=0} = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix},\ \left.\frac{d\mathbf{N}(x)}{dx}\right|_{x=l} = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$$

$$\therefore \mathbf{Kd} = \mathbf{F} \qquad where,\ \mathbf{K} = \frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix},\ \mathbf{F} = \begin{bmatrix}V(0)-\frac{l}{2}f\\ -m(0)-\frac{l^2}{12}f\\ -V(l)-\frac{l}{2}f\\ m(l)+\frac{l^2}{12}f\end{bmatrix}$$

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$$\int_0^l \left[EI\frac{d^4 v(x)}{dx^4} + f \right] N_i\,dx = 0 \quad ,(i=1,2)$$

$$where,\quad v(x) = \mathbf{N}\mathbf{d}$$

Galerkin Method

the test functions N_i are chosen to play the role of the weighting functions *W*

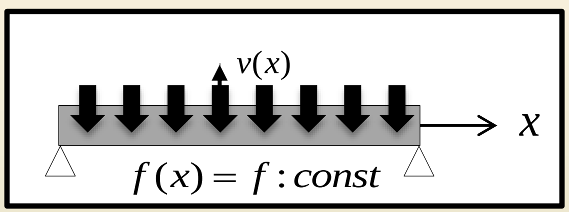
$$\iiint_V \underbrace{R}_{\text{residual (test function } \mathbf{N} \text{ used)}}\ \underbrace{N_i}_{\text{weighting function}}\,dV = 0 \qquad ,(i=1,2)$$

$$\therefore \mathbf{Kd} = \mathbf{F} \quad where,\ \mathbf{K} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix},\ \mathbf{F} = [f_1\ \ f_2\ \ f_3\ \ f_4]^T$$

Differential Equation

$$EI\frac{d^4 v(x)}{dx^4} + f = 0$$

For simple support beam,

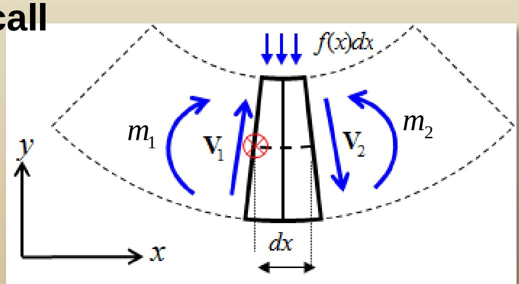


$$f_1 = V(0) - \frac{l}{2}f,\quad f_2 = -m(0) - \frac{l^2}{12}f$$

$$f_3 = -V(l) - \frac{l}{2}f,\quad f_4 = m(l) + \frac{l^2}{12}f$$

$$V(0) = \frac{l}{2}f, m(0) = 0, V(l) = -\frac{l}{2}f, m(l) = 0$$

recall



$$\mathbf{F} = [f_1\ \ f_2\ \ f_3\ \ f_4]^T = \begin{bmatrix} 0 \\ -\dfrac{l^2}{12}f \\ 0 \\ \dfrac{l^2}{12}f \end{bmatrix}$$

Element : Beam - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

the total potential energy Π is *defined as the* sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

Element : Beam - Potential Energy Approach

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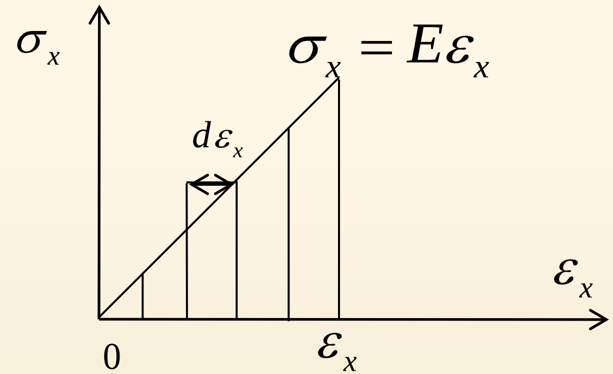
To evaluate the strain energy for a bar,
we consider only the work done by the internal forces during deformation.



Linear-elastic (Hooke's law)material

$$d\Pi_{in} = \int_0^{\varepsilon_x} \sigma \, d\varepsilon_x \, dxdydz$$

$$= \int_0^{\varepsilon_x} E\varepsilon_x \, d\varepsilon_x \, dxdydz \quad = \frac{1}{2} E\left(\varepsilon_x\right)^2 d\varepsilon_x dxdydz$$

$$= \frac{1}{2} \sigma \varepsilon_x dxdydz$$

$$\Pi_{in} = \iiint_V d\Pi_{in} = \frac{1}{2} \iiint_V \sigma_x \varepsilon_x dV$$

the strain energy for one-dimensional stress.

Element : Beam - Potential Energy Approach

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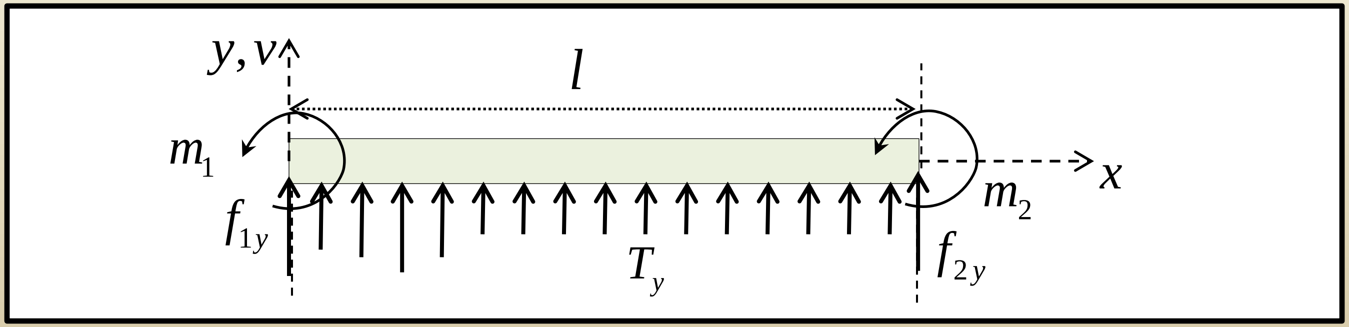
sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

The potential energy of the external forces, being opposite in sign from the extenal work expression because the potential energy of external forces is lost when the work is done by the external forces, is given by

$$\Pi_{ext} = -\iint_{S_1} T_y v_s dS - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

- **body forces X_b typically from the self-weight of the bar (in units of force per unit volume) moving through displacement function v**
- **surface loading or traction T_y typically from distributed loading acting along the surface of the element (in units of force per unit surface area) moving through displacements v_s where v_s are the displacements occurring over surface S_1**
- **nodal concentrated force f_{iy} moving through nodal displacements v_i**



Element : Beam - Potential Energy Approach

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$$\Pi = \Pi_{in} + \Pi_{ext}$$

$$\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iint_{S_1} T_y v_s dS - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

1. Formulate an expression for the total potential energy.
2. Assume the displacement pattern to vary with a finite set of undetermined parameters (here these are the nodal displacements v_i), which are substituted into the expression for total potential energy.
3. Obtain a set of simultaneous equations minimizing the total potential energy with respect to these nodal parameters. These resulting equations represent the element equations.

Element : Beam - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

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$$\Pi = \Pi_{in} + \Pi_{ext} \qquad \Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iint_{S_1} T_y v_s dS - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

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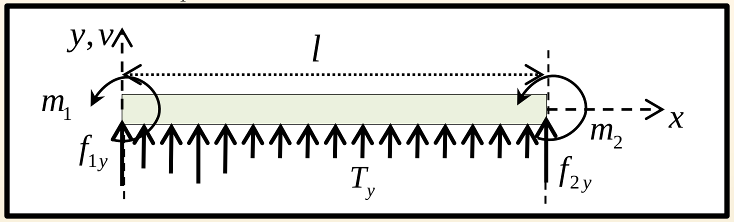
assume that there is no surface traction and body force and the sectional area A is constant



Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

The differential volume for the beam element

$$dV = dAdx$$

The differential area over which the surface loading acts is

$$dS = bdx \quad , b: \text{ Width of beam}$$

$$\therefore \Pi = \Pi_{in} + \Pi_{ext}$$

$$= \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV - \iint_{S_1} T_y v_s dS - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

$$= \frac{1}{2}\iiint_{x\ A} \sigma_x \varepsilon_x dAdx - \int_0^l bT_y v_s dx - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

Element : Beam - Potential Energy Approach

the principle of minimum potential energy

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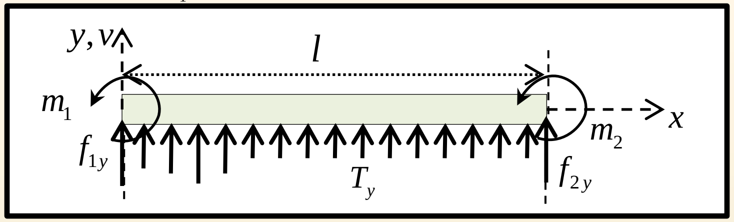
---

assume that there is no surface traction and body force and the sectional area A is constant

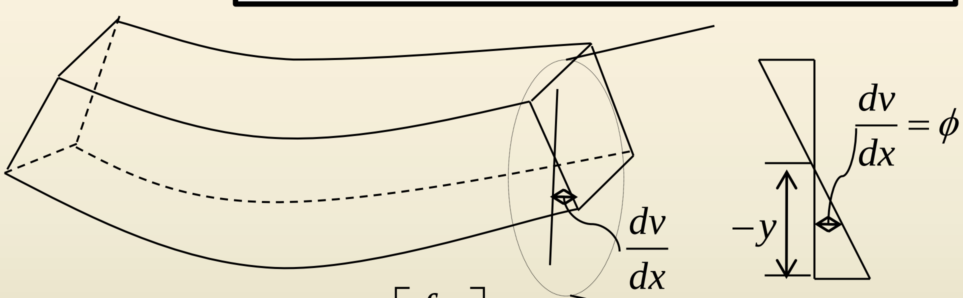


$$\therefore \Pi = \frac{1}{2}\iiint_{x\ A} \sigma_x \varepsilon_x dA dx - \int_0^l b T_y v_s dx - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

$$\varepsilon_x = \frac{du}{dx} = -y\frac{d^2 v}{dx^2}$$

$$= -y\mathbf{B}\mathbf{d}$$

$$\sigma_x = E\,\varepsilon_x = -yE\mathbf{B}\mathbf{d}$$



$$u = -y\frac{dv}{dx}$$

$$bT_y = f_y \quad , \int_0^l b T_y v_s dx = \int b T_y \mathbf{d}^T \mathbf{N}^T dx$$

$$-\int_0^l f_y \mathbf{d}^T \mathbf{N}^T dx - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i = -\mathbf{d}^T \int_0^l \mathbf{N}^T f_y dx - [v_1\ \phi_1\ v_2\ \phi_2]\begin{bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{bmatrix} = -\mathbf{d}^T \mathbf{F}$$

$$\boxed{\Pi = \frac{EI}{2}\int_0^l \mathbf{d}^T \mathbf{B}^T \mathbf{B}\,\mathbf{d}\,dx - \mathbf{d}^T \mathbf{F}}$$

Element : Beam - Potential Energy Approach

the principle of minimum potential energy

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$$\Pi = \Pi_{in} + \Pi_{ext}$$

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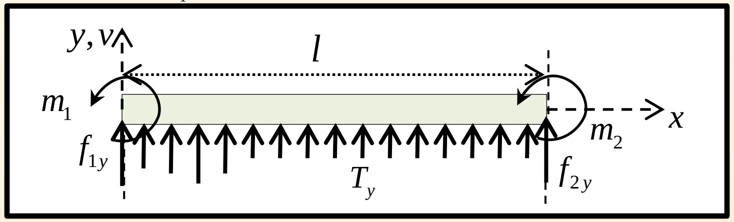
---

assume that there is no surface traction and body force and the sectional area A is constant

$$\Pi = \frac{1}{2}\iiint_{x\,A} \sigma_x \varepsilon_x dA dx - \int_0^l b T_y v_s dx - \sum_{i=1}^{2} f_{iy} v_i - \sum_{i=1}^{2} m_i \phi_i$$

$$\boxed{\Pi = \frac{EI}{2}\int_0^l \mathbf{d}^{\mathrm{T}} \mathbf{B}^{\mathrm{T}} \mathbf{B} \mathbf{d} dx - \mathbf{d}^{\mathrm{T}} \mathbf{F}}$$



3. Obtain a set of simultaneous equations minimizing the total potential energy with respect to these nodal parameters. These resulting equations represent the element equations.

The minimization of Π with respect to each nodal displacement requires that

$$\frac{\partial \Pi}{\partial v_1} = 0 \;, \frac{\partial \Pi}{\partial \phi_1} = 0 \;, \frac{\partial \Pi}{\partial v_2} = 0 \quad \text{and} \quad , \frac{\partial \Pi}{\partial \phi_2} = 0$$

Element : Beam - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

the total potential energy Π is *defined as the sum* of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

$$\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

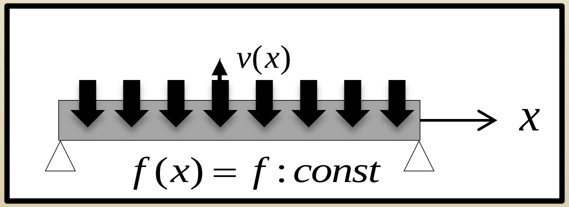
$$\Pi = \frac{EI}{2}\int_0^l \boxed{\mathbf{d}^T\mathbf{B}^T\mathbf{B}\mathbf{d}}\, dx - \mathbf{d}^T\mathbf{F}$$

$$\boxed{\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}} = \frac{EI}{l^3}\begin{bmatrix} v_1 & \phi_1 & v_2 & u_2 \end{bmatrix}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{bmatrix}$$

$$d\Pi = 0$$

$$EI\int_0^l \mathbf{B}^T\mathbf{B}dx\,\mathbf{d} - \mathbf{F} = 0 \quad , where\ \mathbf{F} = -\int_0^l \mathbf{N} f_y dx - \begin{bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{bmatrix}$$

$$\therefore \mathbf{Kd} = \mathbf{F} \qquad where,\ \mathbf{K} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad , \mathbf{F} = \int_0^l \mathbf{N} f_y dx + \begin{bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{bmatrix}$$



Element : Beam - Potential Energy Approach

the principle of minimum potential energy

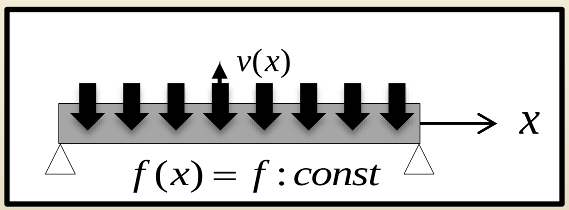
Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

the total potential energy Π is *defined as the sum* of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext} \qquad \Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

$$\mathbf{Kd} = \mathbf{F} \qquad where,\ \mathbf{K} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad , \mathbf{F} = \int_0^l \mathbf{N} f_y dx + \begin{bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{bmatrix}$$

For simple support beam with uniform load f



$$f_y = -f, \quad f_{1y} = \frac{f}{2}, \quad m_1 = 0, \quad f_{2y} = \frac{f}{2}, \quad m_2 = 0$$

$$, \mathbf{F} = -\int_0^l \mathbf{N} f dx + \begin{bmatrix} \frac{f}{2} \\ 0 \\ \frac{f}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{l}{2} f + \frac{l}{2} f \\ -\frac{l^2}{12} f + 0 \\ -\frac{l}{2} f + \frac{l}{2} f \\ \frac{l^2}{12} f + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{l^2}{12} f \\ 0 \\ \frac{l^2}{12} f \end{bmatrix}$$