Naval Architecture & Ocean Engineering

Computer Aided Ship Design
Part.3 Grillage Analysis of Midship Cargo Hold

2009 Fall

Prof. Kyu-Yeul Lee

**Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering**

Seoul National Univ.

@SDAL
Advanced Ship Design Automation Lab.
http://asdal.snu.ac.kr

Summary

u: Axial Displacement G: Shear Modulus A: Sectional Area G: Shear Modulus E: Young's Modulus
w: Vertical Displacement θ: Angle of Twist l: Length J: Polar Moment of Inertia I: Moment of Inertia

Element | Differential Equation → Variational Method → •Discretization •Approximation → Finite Element Method | •Superposition of Stiffness Matrix •Coordinates Transformation **Behavior → Structure**

$$\mathbf{M\ddot{x}} = \sum \mathbf{F} \;, where\ \mathbf{\ddot{x}} = 0 \qquad\qquad \mathbf{Kd} = \mathbf{F}$$

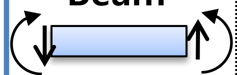
Bar

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0 \Rightarrow \delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0 \Rightarrow \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$u(x) = a_0 + a_1 x$

Tension → **Truss**, **Frame**

Beam

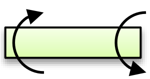


$$EI\frac{\partial^4 w(x)}{\partial x^4} + f(x) = 0 \Rightarrow \delta\int_0^l \left[\frac{EA}{2}\left(\frac{d^2w}{dx^2}\right)^2 + (f\,w)\right]dx = 0 \Rightarrow \frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{bmatrix}$$

$w(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

Bending → **Frame**, **Grillage**

Shaft



$$GJ\frac{\partial^2 \theta(x)}{\partial x^2} + f(x) = 0 \Rightarrow \delta\int_0^l \left[\frac{GJ}{2}\left(\frac{d\theta}{dx}\right)^2 - (f\,\theta)\right]dx = 0 \Rightarrow \frac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$\theta(x) = c_0 + c_1 x$

Torsion → **Grillage**

Application

Midship Cargo Hold → Grillage Modeling → Solution

Engineering Concept !

Grillage Modeling
- 3D→ 2D
- Equivalent Force & Moment
- Boundary condition

Solution
- programming
- visulaization

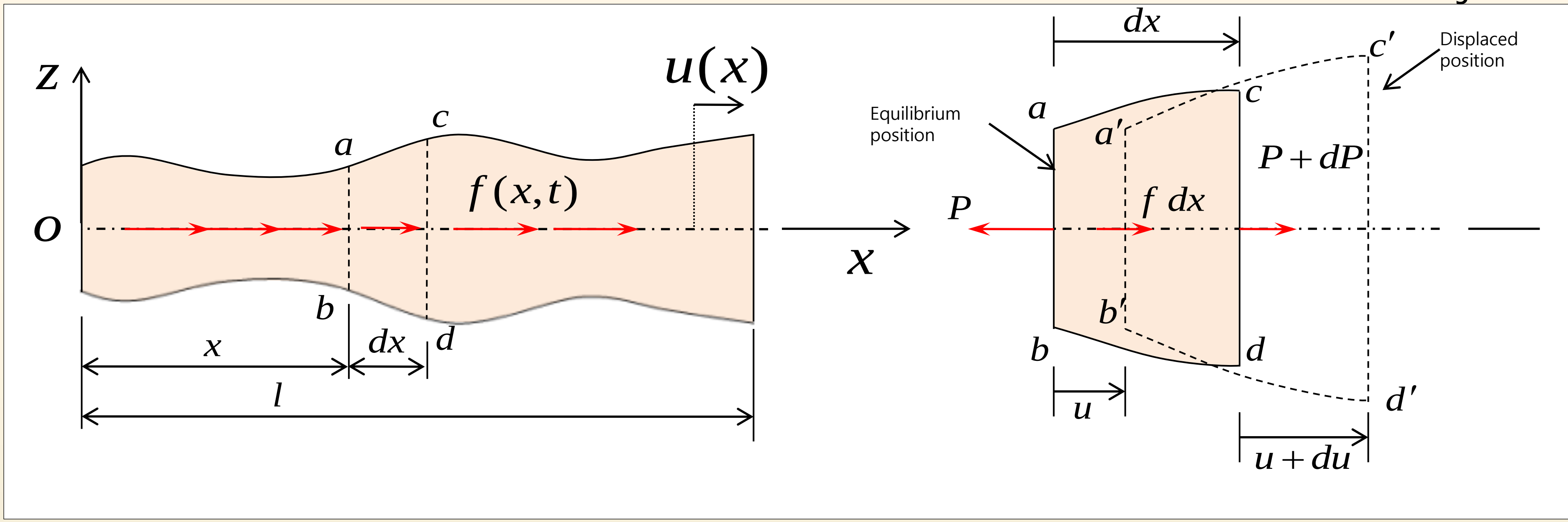
Beam Theory : Sign Convention, Deflection of Beam

Elasticity : Displacement, Strain, Stress, Force Equilibrium, Compatibility, Constitutive Equation

Chapter 1. Element : Bar

Element : Bar - Differential Eqn.

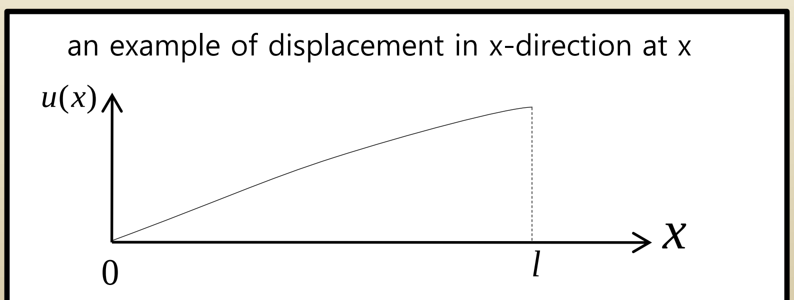
P : the axial forces acting on the cross sections of a small element of the bar of length dx

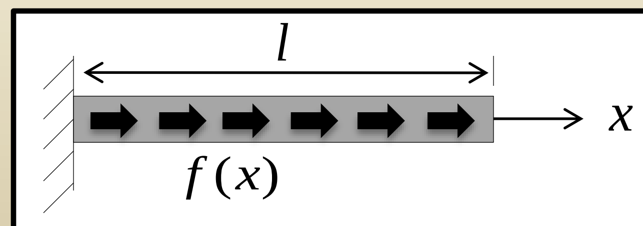


$f(x,t)$:external force per unit length

ρ :density, E : Young's Modulus, A:sectional area

if A is constant
and f is time invariant

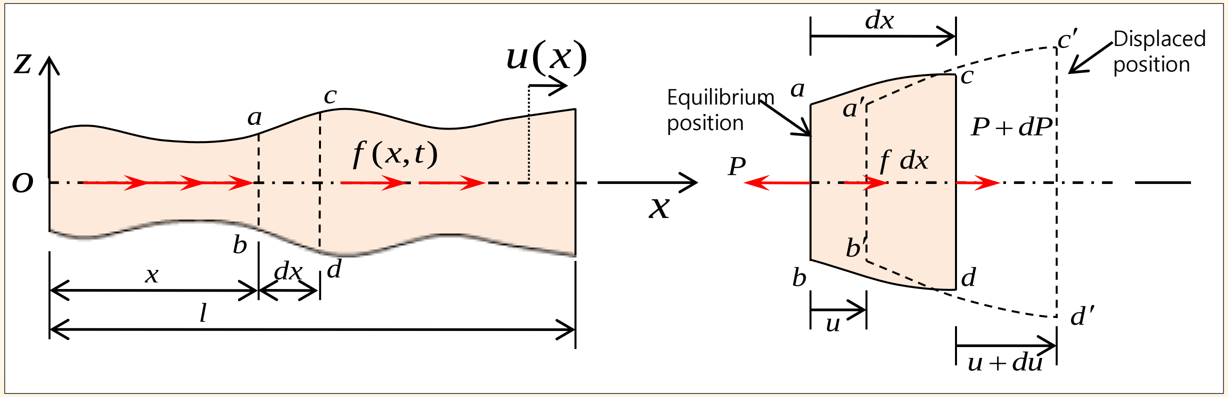




Rao,S.S.,Mechanical Vibrations, Fourth Edition, Prentice Hall, 2004,

Element : Bar - Differential Equ.

A Bar in Axial Vibration



$f(x,t)$:external force per unit length

ρ :density, E : Young's Modulus, A:sectional area

$$\sum F = ma$$

$$(P+dP)+f\,dx-P=\rho A(x)\,dx\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$dP + f\,dx=\rho A(x)\,dx\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$P=\sigma A(x)=EA(x)\varepsilon=EA(x)\frac{\partial u(x,t)}{\partial x}$$ 'constitutive equation'

$$dP=\frac{\partial P}{\partial x}dx=\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u(x,t)}{\partial x}\right)dx$$

$$\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u(x,t)}{\partial x}\right)dx+f(x,t)\,dx=\rho A(x)\,dx\frac{\partial^2 u(x,t)}{\partial t^2}$$

$if \quad A(x)=A \quad : const.$

$$EA\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t)=\rho A\frac{\partial^2 u(x,t)}{\partial t^2}$$

dynamics (vibration)

subjected to

$$\begin{cases} u(x,t=0)=u_0(x), & 0\le x\le l \\ \dfrac{\partial u}{\partial t}(x,t=0)=\dot{u}_0(x), & 0\le x\le l \end{cases}$$ ⇨ **I.V.P**

$$\begin{cases} u(0,t)=0, & t>0 \\ AE\dfrac{\partial u}{\partial x}(l,t)=0 \text{ or } \dfrac{\partial u}{\partial x}(l,t)=0, & t>0 \end{cases}$$ ⇨ **B.V.P**

at the free end , axial force

$$\frac{\partial u}{\partial t}=0$$

$$EA\frac{d^2 u(x)}{dx^2}+f(x)=0$$ ⇨ **statics**

"Longitudinal Vibration of a Bar or Rod : Rao,S.S.,Mechanical Vibrations, Fourth Edition, Prentice Hall, 2004, pp597-560

Element : Bar - Variational Method

multiply by δu and integrate

$$\int_0^l \left(EA\frac{d^2u}{dx^2} + f \right)\delta u\, dx = 0$$

L.H.S:

$$\int_0^l \left(EA\frac{d^2u}{dx^2}\delta u + f\,\delta u \right) dx$$

integration by part

$$= EA\left[\frac{du}{dx}\delta u\right]_0^l - \int_0^l \left(EA\frac{du}{dx}\frac{d(\delta u)}{dx} - f\,\delta u \right) dx$$

$$= -\int_0^l \left(EA\frac{du}{dx}\delta\frac{du}{dx} - f\,\delta u \right) dx$$

$$= -\int_0^l \left[\delta\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - \delta(f\,u) \right] dx$$

$$\therefore \delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$

Differential Equation

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0$$

Boundary condition

$$u\big|_{x=0} = 0 \quad , \quad EA\frac{du}{dx}\bigg|_{x=l} = 0$$

δ operation

- $f\,\delta u = \delta(fu)$
- $u\,\delta u = \delta\left(\frac{1}{2}u^2\right)$
- $\frac{\delta u}{\delta x}\delta\frac{\delta u}{\delta x} = \frac{1}{2}\delta\left(\frac{\delta u}{\delta x}\right)^2$
- $\frac{d}{dx}\delta u = \delta\frac{d}{dx}u$
- $\delta\int_a^b h(x)dx = \int_a^b \delta h(x)dx$

Element : Bar - Rayleigh-Ritz method

$$\delta\int_0^l\left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2-(f\,u)\right]dx=0$$

Rayleigh-Ritz method

assume, $u=\sum_{k=1}^{n}a_k x^k$

$$u=a_1x+a_2x^2$$

$$\delta\int_0^l\left[\frac{EA}{2}\left(a_1+2a_2x\right)^2-f\,a_1x-f\,a_2x^2\right]dx=0$$

▷ solution

Differential Equation

$$EA\frac{d^2u(x)}{dx^2}+f(x)=0$$

Boundary condition

$$u\big|_{x=0}=0\ ,\quad EA\frac{du}{dx}\bigg|_{x=l}=0$$

Variational Method

$$\delta\int_0^l\left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2-(f\,u)\right]dx=0$$

(solution)

$$\delta\int_0^l\left[\frac{EA}{2}\left(a_1+2a_2x\right)^2-f\,a_1x-f\,a_2x^2\right]dx$$

$$=\delta\int_0^l\left[\frac{EA}{2}\left(a_1^2+4a_1a_2x+4a_2^2x^2\right)-f\,a_1x-f\,a_2x^2\right]dx$$

$$=\delta\left[\frac{EA}{2}\left(a_1^2x+2a_1a_2x^2+a_2^2\frac{4x^3}{3}\right)-f\,a_1\frac{x^2}{2}-f\,a_2\frac{x^3}{3}\right]_0^l$$

$$=\delta\left(\frac{EA}{2}\left(l\,a_1^2+2l^2a_1a_2+\frac{4l^3}{3}a_2^2\right)-\frac{f\cdot l^2}{2}a_1-\frac{f\cdot l^3}{3}a_2\right)$$

$$=\left(\frac{EA}{2}\left(2l\,a_1\delta a_1+2l^2\delta a_1a_2+2l^2a_1\delta a_2+\frac{8l^3}{3}a_2\delta a_2\right)-\frac{f\cdot l^2}{2}\delta a_1-\frac{f\cdot l^3}{3}\delta a_2\right)$$

$$=\left(EAl\,a_1\delta a_1+EAl^2\delta a_1a_2+EAl^2a_1\delta a_2+EA\frac{4l^3}{3}a_2\delta a_2-\frac{f\cdot l^2}{2}\delta a_1-\frac{f\cdot l^3}{3}\delta a_2\right)$$

$$=\left(EAl\,a_1+EAl^2a_2-\frac{f\cdot l^2}{2}\right)\delta a_1+\left(EAl^2a_1+EA\frac{4l^3}{3}a_2-\frac{f\cdot l^3}{3}\right)\delta a_2$$

(solution)

$$\delta\int_0^l\left[\frac{EA}{2}\left(a_1+2a_2x\right)^2-f\,a_1x-f\,a_2x^2\right]dx$$

$$=\left(EAl\,a_1+EAl^2a_2-\frac{f\cdot l^2}{2}\right)\delta a_1+\left(EAl^2a_1+EA\frac{4l^3}{3}a_2-\frac{f\cdot l^3}{3}\right)\delta a_2$$

since

$$\delta\int_0^l\left[\frac{EA}{2}\left(a_1+2a_2x\right)^2-f\,a_1x-f\,a_2x^2\right]dx=0$$

$$\therefore\begin{cases}EAl\,a_1+EAl^2a_2-\dfrac{f\cdot l^2}{2}=0\\[2ex] EAl^2a_1+EA\dfrac{4l^3}{3}a_2-\dfrac{f\cdot l^3}{3}=0\end{cases}$$

$$EA\begin{bmatrix} l & l^2 \\ l^2 & \dfrac{4l^3}{3}\end{bmatrix}\begin{bmatrix} a_1 \\ a_2\end{bmatrix}=\begin{bmatrix}\dfrac{f\cdot l^2}{2}\\[2ex]\dfrac{f\cdot l^3}{3}\end{bmatrix}$$

(solution)

$$EA\begin{bmatrix} l & l^2 \\ l^2 & \dfrac{4l^3}{3} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{f\cdot l^2}{2} \\ \dfrac{f\cdot l^3}{3} \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{3}{EAl^4}\begin{bmatrix} \dfrac{4l^3}{3} & -l^2 \\ -l^2 & l \end{bmatrix}\begin{bmatrix} \dfrac{f\cdot l^2}{2} \\ \dfrac{f\cdot l^3}{3} \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{3}{EAl^4}\begin{bmatrix} \dfrac{2f\cdot l^5}{3} - \dfrac{f\cdot l^5}{3} \\ -\dfrac{f\cdot l^4}{2} + \dfrac{f\cdot l^4}{3} \end{bmatrix} = \frac{3}{EAl^4}\begin{bmatrix} \dfrac{f\cdot l^5}{3} \\ -\dfrac{f\cdot l^4}{6} \end{bmatrix} = \begin{bmatrix} \dfrac{f\cdot l}{EA} \\ -\dfrac{f}{2EA} \end{bmatrix} \qquad \therefore \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{f\cdot l}{EA} \\ -\dfrac{f}{2EA} \end{bmatrix}$$

Element : Bar - Rayleigh-Ritz method

$$\delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$

Rayleigh-Ritz method

assume, $u = \sum_{k=1}^{n} a_k x^k$

$$u = a_1 x + a_2 x^2$$

$$\delta\int_0^l \left[\frac{EA}{2}\left(a_1 + 2a_2 x\right)^2 - f\,a_1 x - f\,a_2 x^2 \right] dx = 0$$

$$\therefore \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{f \cdot l}{EA} \\ -\dfrac{f}{2EA} \end{bmatrix}$$

$$\therefore u(x) = \frac{f \cdot l}{EA} x - \frac{f}{2EA} x^2$$

Differential Equation

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0$$

Boundary condition

$$u\big|_{x=0} = 0 \quad , \quad EA\frac{du}{dx}\bigg|_{x=l} = 0$$

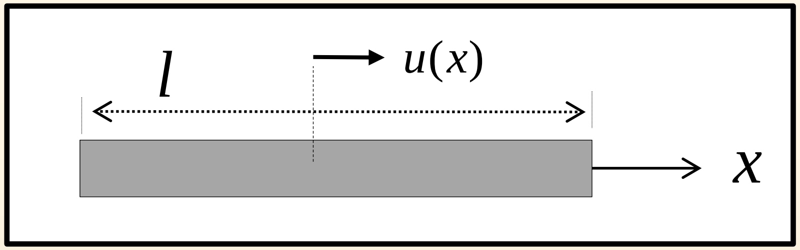
Variational Method

$$\delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$

Element : Bar - Finite Element Method

Variational Method

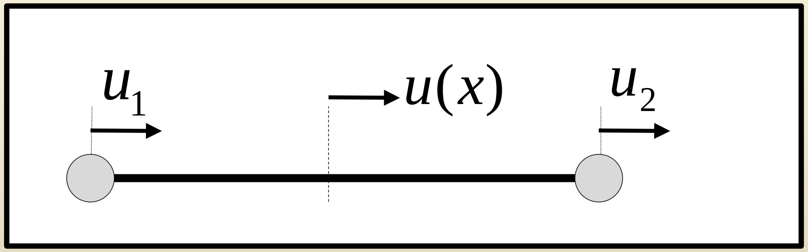
$$\delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$



discretization

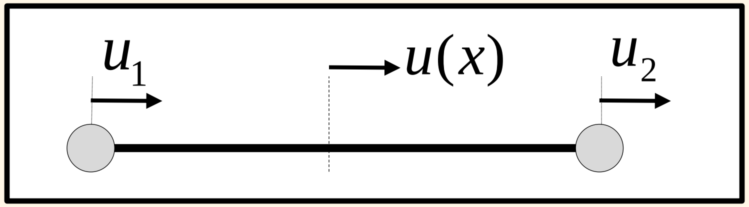
finite element method

1 element , 2 nodes



assume: $u(x) = c_1 + c_2 x \;\;,\; u(0) = u_1,\;\; u(l) = u_2$

Element : Bar - Finite Element Method



Variational Method

$$\delta \int_0^l \left[\frac{EA}{2} \left(\frac{du}{dx} \right)^2 - (f\, u) \right] dx = 0$$

assume: $u(x) = c_1 + c_2 x \;,\; u(0) = u_1,\;\; u(l) = u_2$

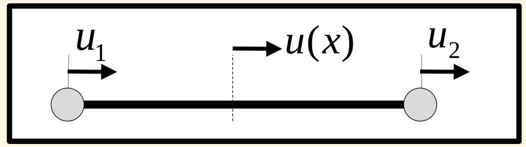
$$u(0) = c_1 \;\Rightarrow\; c_1 = u_1$$

$$u(l) = c_1 + c_2 l \;\Rightarrow\; c_2 = \frac{u_2 - u_1}{l}$$

$$\therefore u(x) = u_1 + \left(\frac{u_2 - u_1}{l} \right) x$$

$$or,\; u(x) = \left(1 - \frac{x}{l} \right) u_1 + \frac{x}{l}\, u_2$$

Element : Bar - Finite Element Method



Variational Method

$$\delta \int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$

$$u(x) = \left(1 - \frac{x}{l}\right) u_1 + \frac{x}{l} u_2$$

$$u(x) = \begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

differentiation with respect to x

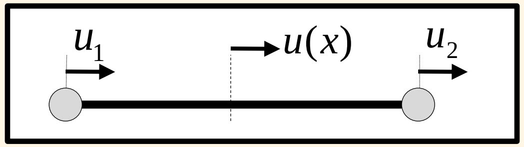
$$\frac{du(x)}{dx} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\therefore u(x) = \mathbf{N}\mathbf{d}, \quad \frac{du(x)}{dx} = \mathbf{B}\,\mathbf{d}$$

$$where\ \mathbf{N} = \begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element : Bar - Finite Element Method

$$u(x)=\left(1-\frac{x}{l}\right)u_1+\frac{x}{l}u_2$$



Variational Method

$$\delta\int_0^l\left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2-(f\,u)\right]dx=0$$

$$u(x)=\mathbf{N}\mathbf{d}$$

$$\frac{du(x)}{dx}=\mathbf{B}\mathbf{d}$$

$$\boxed{\delta\int_0^l\left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2-(f\,u)\right]dx=0}$$

$$\textit{where}\ \mathbf{N}=\begin{bmatrix}1-\frac{x}{l} & \frac{x}{l}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix},\quad \mathbf{d}=\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\Rightarrow\ \delta\left\{\frac{EA}{2}\int_0^l\left(\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}\right)dx-\int_0^l(f\,\mathbf{N}\mathbf{d})dx\right\}=0$$

▷ **derivation**

(derivation)

$$\delta\int_0^l\left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2-(f\,u)\right]dx \qquad f\,\mathbf{Nd}=(\mathbf{Nd})^{\mathrm{T}}f=\mathbf{d}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}f \quad \because \mathbf{Nd}: scalar$$

$$=\delta\left\{\frac{EA}{2}\int_0^l\left(\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}\right)dx-\int_0^l(f\,\mathbf{N}\,\mathbf{d})dx\right\}$$

$$=\delta\left\{\frac{EA}{2}\int_0^l\left(\mathbf{d}^T\mathbf{B}^T\mathbf{B}\,\mathbf{d}\right)dx-\int_0^l(\mathbf{d}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\,f\,)dx\right\}$$

$$=\delta\left\{\frac{1}{2}\mathbf{d}^{\mathrm{T}}\boxed{\left[\int_0^l EA\left(\mathbf{B}^T\mathbf{B}\,\right)dx\right]}\mathbf{d}-\mathbf{d}^{\mathrm{T}}\boxed{\left[\int_0^l(\mathbf{N}^{\mathrm{T}}f\,)dx\right]}\right\}$$

$$=\delta\left\{\frac{1}{2}\mathbf{d}^{\mathrm{T}}\boxed{\mathbf{K}}\mathbf{d}-\mathbf{d}^{\mathrm{T}}\boxed{\mathbf{F}}\right\}$$

$$=\frac{1}{2}(\delta\mathbf{d})^{\mathrm{T}}\mathbf{Kd}+\frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{K}\delta\mathbf{d}-(\delta\mathbf{d})^{\mathrm{T}}\mathbf{F}$$

$$=\boxed{\frac{1}{2}(\delta\mathbf{d})^{\mathrm{T}}\mathbf{Kd}+\frac{1}{2}(\delta\mathbf{d})^{\mathrm{T}}\mathbf{Kd}}-(\delta\mathbf{d})^{\mathrm{T}}\mathbf{F} \quad \because(\delta\mathbf{d})^{\mathrm{T}}\mathbf{Kd}=\mathbf{d}^{\mathrm{T}}\mathbf{K}\delta\mathbf{d}$$

$$=\boxed{(\delta\mathbf{d})^{\mathrm{T}}\mathbf{Kd}}-(\delta\mathbf{d})^{\mathrm{T}}\mathbf{F}$$

$$=(\delta\mathbf{d})^{\mathrm{T}}(\mathbf{Kd}-\mathbf{F}\,)$$

$$\mathbf{N}=\begin{bmatrix}1-\frac{x}{l} & \frac{x}{l}\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}$$

$$\mathbf{K}=EA\int_0^l\left(\mathbf{B}^T\mathbf{B}\,\right)dx$$

$$=EA\int_0^l\begin{bmatrix}-\frac{1}{l}\\ \frac{1}{l}\end{bmatrix}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx$$

$$=\frac{EA}{l^2}\int_0^l\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx$$

$$=\frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \Rightarrow \mathbf{K}=\mathbf{K}^{\mathrm{T}}$$

symmetry

$$\mathbf{F}=\int_0^l(\mathbf{N}^{\mathrm{T}}f\,)dx$$

$$\mathbf{d}=\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

(cf. Taylor Series for a Function of Two Variables)

Taylor Series for a Function $f(x_1, x_2)$ at (x_1^*, x_2^*)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}^* = \begin{bmatrix} x^*{}_1 \\ x^*{}_2 \end{bmatrix}, \quad \mathbf{d} = \mathbf{x} - \mathbf{x}^*$$

$$f(x_1, x_2) = f(x_1^*, x_2^*)$$
$$+ \boxed{\frac{\partial f(x_1^*, x_2^*)}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}(x_2 - x_2^*)}$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right) + R$$

$$\boxed{\frac{\partial f(x_1^*, x_2^*)}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}(x_2 - x_2^*)} = \begin{bmatrix} \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} & \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix} = \begin{bmatrix} \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} \\ \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$
$$= \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*)$$

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \boxed{\nabla f(\mathbf{x}^*)^T \mathbf{d}} + \cdots$$

(cf. Taylor Series for a Function of Two Variables)

Taylor Series for a Function $f(x_1, x_2)$ at (x_1^*, x_2^*)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}^* = \begin{bmatrix} x^*_1 \\ x^*_2 \end{bmatrix}, \quad \mathbf{d} = \mathbf{x} - \mathbf{x}^*$$

$$f(x_1, x_2) = f(x_1^*, x_2^*)$$
$$+ \frac{\partial f(x_1^*, x_2^*)}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}(x_2 - x_2^*)$$
$$+ \boxed{\frac{1}{2}\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right)} + R$$

$$\boxed{\frac{1}{2}\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right)}$$

$$= \frac{1}{2}\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right)$$

$$= \frac{1}{2}\left(\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_2 - x_2^*) \right)(x_1 - x_1^*) + \left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*) \right)(x_2 - x_2^*) \right)$$

$$= \frac{1}{2}\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_2 - x_2^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*) \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

(cf. Taylor Series for a Function of Two Variables)

Taylor Series for a Function $f(x_1, x_2)$ at (x_1^*, x_2^*)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}^* = \begin{bmatrix} x^*_1 \\ x^*_2 \end{bmatrix}, \quad \mathbf{d} = \mathbf{x} - \mathbf{x}^*$$

$$\begin{aligned} f(x_1, x_2) &= f(x_1^*, x_2^*) \\ &+ \frac{\partial f(x_1^*, x_2^*)}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}(x_2 - x_2^*) \\ &+ \frac{1}{2}\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right) + R \end{aligned}$$

$$\frac{1}{2}\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right)$$

$$= \frac{1}{2}\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_2 - x_2^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*) \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} x_1 - x_1^* & x_2 - x_2^* \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$

(cf. Taylor Series for a Function of Two Variables)

Taylor Series for a Function $f(x_1, x_2)$ at (x_1^*, x_2^*)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}^* = \begin{bmatrix} x^*_1 \\ x^*_2 \end{bmatrix}, \quad \mathbf{d} = \mathbf{x} - \mathbf{x}^*$$

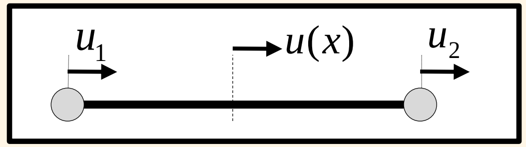
$$f(x_1, x_2) = f(x_1^*, x_2^*)$$
$$+ \boxed{\frac{\partial f(x_1^*, x_2^*)}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}(x_2 - x_2^*)}$$
$$+ \boxed{\frac{1}{2}\left(\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2}(x_2 - x_2^*)^2 \right)} + R$$

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \boxed{\nabla f(\mathbf{x}^*)^T \mathbf{d}} + \boxed{\frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d}} + R$$

$$\boxed{\begin{bmatrix} \dfrac{\partial f(x_1^*, x_2^*)}{\partial x_1} & \dfrac{\partial f(x_1^*, x_2^*)}{\partial x_2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}} \qquad \boxed{\frac{1}{2}\begin{bmatrix} x_1 - x_1^* & x_2 - x_2^* \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}}$$

Element : Bar - Finite Element Method

$$u(x) = \left(1 - \frac{x}{l}\right) u_1 + \frac{x}{l} u_2$$



Variational Method

$$\delta \int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$

$$u(x) = \mathbf{N}\,\mathbf{d}$$

$$\frac{du(x)}{dx} = \mathbf{B}\,\mathbf{d}$$

$$\delta \int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0 \quad \text{where } \mathbf{N} = \begin{bmatrix} 1-\frac{x}{l} & \frac{x}{l} \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix},\ \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow \delta \left\{ \frac{EA}{2} \int_0^l \left(\mathbf{d}^T \mathbf{B}^T \mathbf{B}\,\mathbf{d}\right) dx - \int_0^l (f\,\mathbf{N}\mathbf{d})\,dx \right\} = 0$$

derivation

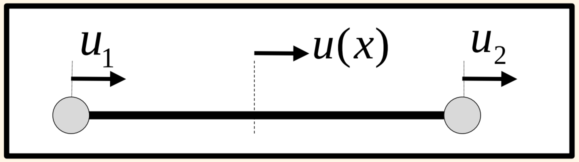
$$\Rightarrow (\delta \mathbf{d})^{\mathrm{T}} (\mathbf{K}\mathbf{d} - \mathbf{F}) = 0$$

$$\therefore \mathbf{K}\mathbf{d} = \mathbf{F} \quad \text{where}\ , \mathbf{K} = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ \mathbf{F} = \int_0^l (\mathbf{N}^{\mathrm{T}} f\,)\,dx$$

Element : Bar - Finite Element Method

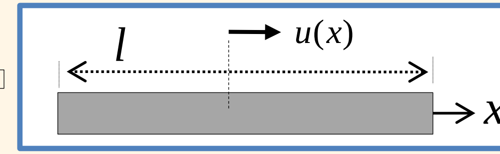
equivalent nodal forces

$$\mathbf{Kd} = \mathbf{F}$$



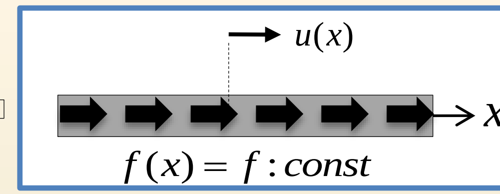
assume: $u(x) = c_1 + c_2 x$

$, \; u(0) = u_1, \; u(l) = u_2$



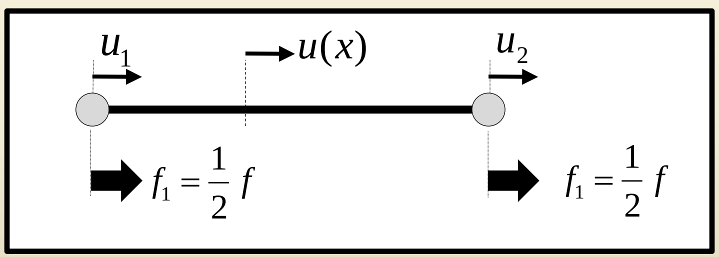
$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{F} \quad , \mathbf{F} = \int_0^l (\mathbf{N}^{\mathrm{T}} f\,)dx$$

constant **external force** per unit length



equivalent nodal forces

$$\mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \int_0^l (\mathbf{N}^{\mathrm{T}} f\,)dx = \int_0^l \begin{bmatrix} 1 - \dfrac{x}{l} \\ \dfrac{x}{l} \end{bmatrix} f\,dx = \left[\begin{bmatrix} x - \dfrac{x^2}{2l} \\ \dfrac{x^2}{2l} \end{bmatrix} f \right]_0^l = \begin{bmatrix} \dfrac{1}{2} f \cdot l \\ \dfrac{1}{2} f \cdot l \end{bmatrix}$$



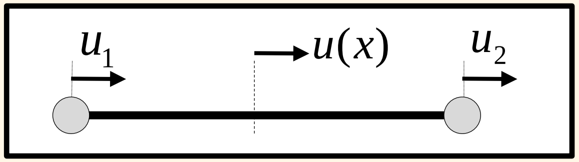
Element : Bar - Finite Element Method

Variational Method

$$\delta \int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$

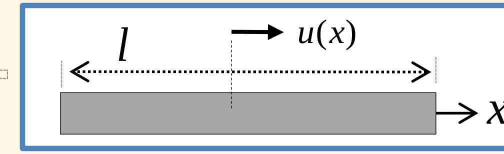
equivalent nodal forces

$$\mathbf{Kd} = \mathbf{F}$$



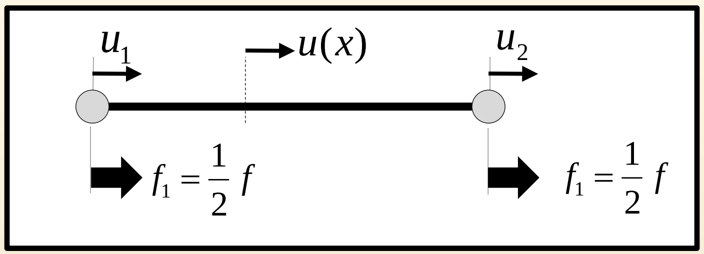
assume: $u(x) = c_1 + c_2 x$

$, u(0) = u_1,\ u(l) = u_2$

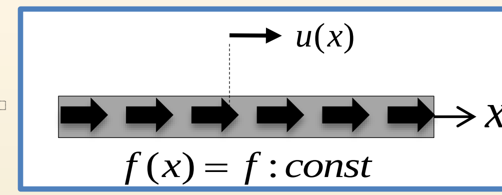


$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{F} \quad , \mathbf{F} = \int_0^l (\mathbf{N}^{\mathrm{T}} f\,)dx$$

constant external force per unit length



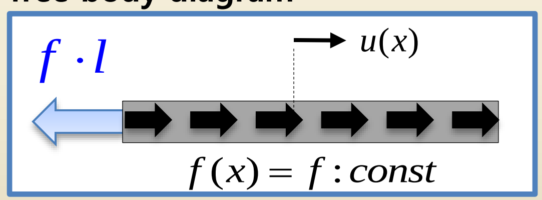
$$\mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} f \cdot l \\ \frac{1}{2} f \cdot l \end{bmatrix}$$



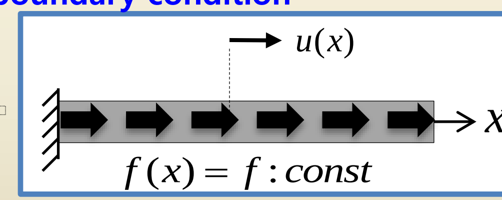
equivalent nodal forces

$$\therefore \mathbf{F} = \begin{bmatrix} f_1 - f \cdot l \\ f_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} f \cdot l \\ \frac{1}{2} f \cdot l \end{bmatrix}$$

free body diagram

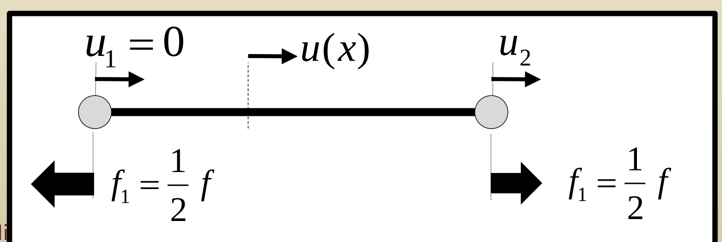


boundary condition



and $u_1 = 0$

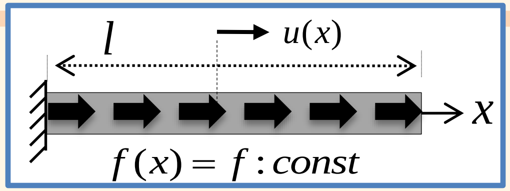
$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} f \cdot l \\ \frac{1}{2} f \cdot l \end{bmatrix}$$

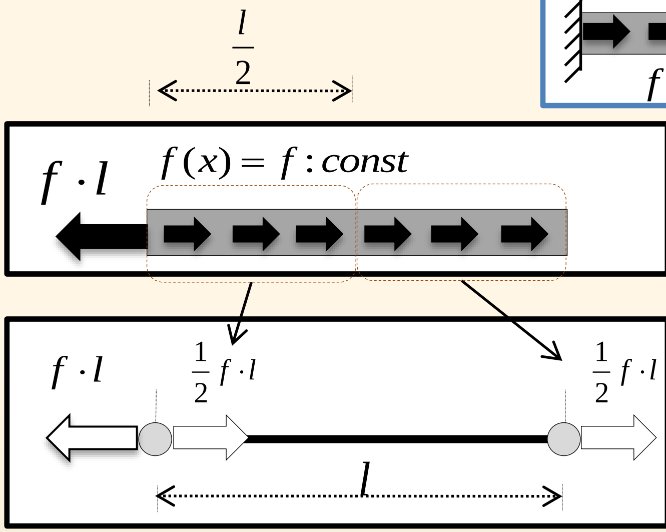


Element : Bar - Finite Element Method

Variational Method

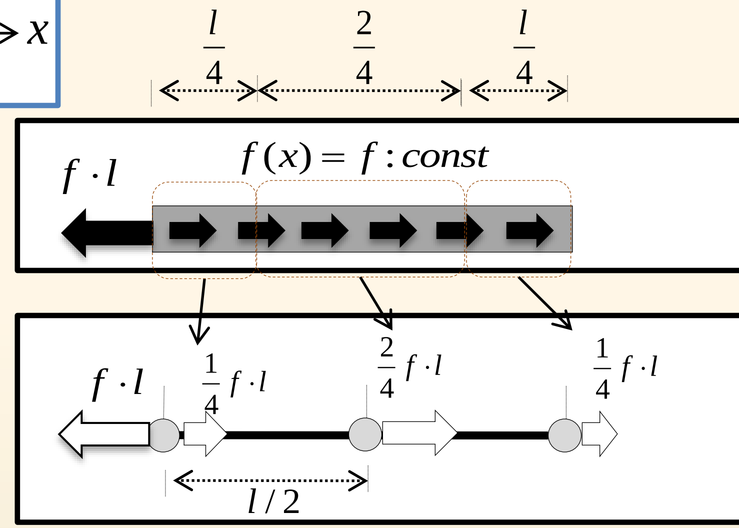
$$\delta \int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$





free body diagram

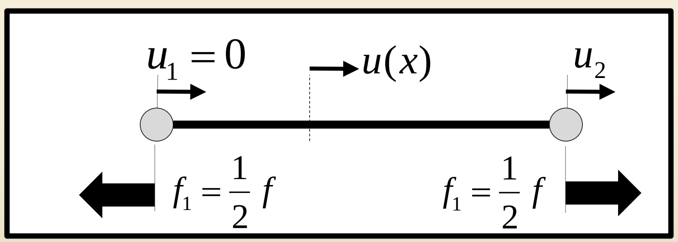
equivalent nodal forces

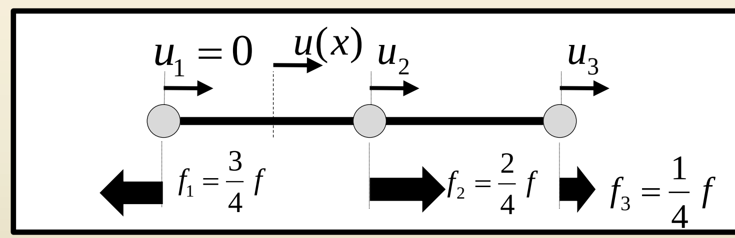


1 element , 2 nodes

2 element , 3 nodes

F.E.M





$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} f \cdot l \\ \frac{1}{2} f \cdot l \end{bmatrix}$$

$$\mathbf{Kd} = \mathbf{F}$$

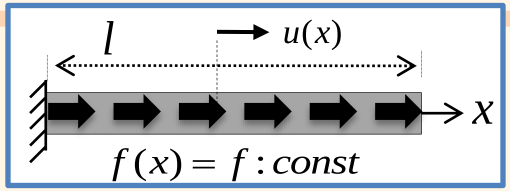
$$\frac{EA}{l/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & \boxed{1+1} & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} f \cdot l \\ \frac{2}{4} f \cdot l \\ \frac{1}{4} f \cdot l \end{bmatrix}$$

superposition of stiffness matrix

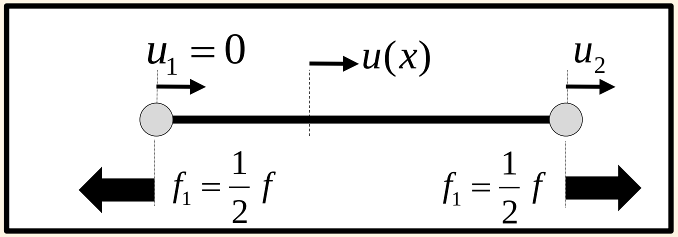
Element : Bar - Finite Element Method

Variational Method

$$\delta \int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u) \right] dx = 0$$



1 element , 2 nodes



$$\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} f \cdot l \\ \frac{1}{2} f \cdot l \end{bmatrix}$$

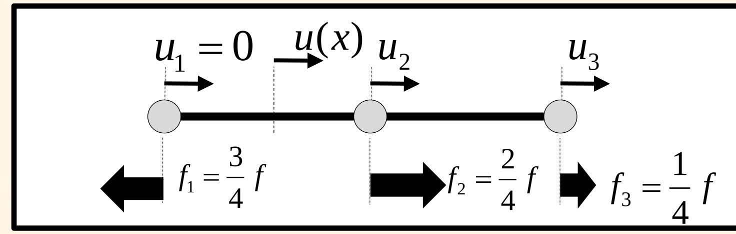
sol) $u_1 = 0, \quad u_2 = \frac{1}{2}\frac{f \cdot l^2}{EA}$

F.E.M

$$\mathbf{Kd} = \mathbf{F}$$

find (K) given (F)

2 element , 3 nodes



superposition of stiffness matrix

$$\frac{EA}{l/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & \boxed{1+1} & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} f \cdot l \\ \frac{2}{4} f \cdot l \\ \frac{1}{4} f \cdot l \end{bmatrix}$$

sol) $u_1 = 0, \quad u_2 = \frac{3}{8}\frac{f \cdot l^2}{EA}, \quad u_3 = \frac{4}{8}\frac{f \cdot l^2}{EA}$

---

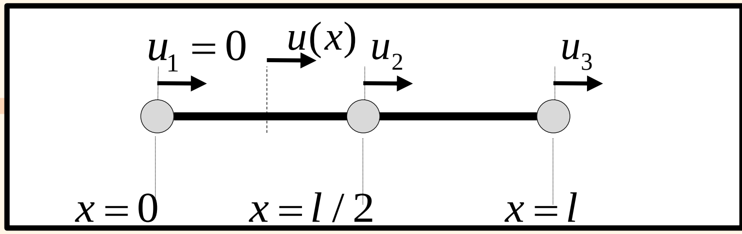
displacement

$$u(x) = \left(1 - \frac{x}{l}\right) u_1 + \frac{x}{l}\, u_2 \quad , 0 \le x \le l$$

▷ **solution**

(solution)

2 element , 3 nodes



$i)\ 0 \le x \le \dfrac{l}{2}$

$$u(x)=c_1^1+c_2^1 x$$

$$,\ u(0)=u_1,\ \ u(l/2)=u_2$$

$$u(0)=c_1^1 \quad \Rightarrow \quad c_1^1=u_1$$

$$u(l/2)=c_1+c_2 l/2 \quad \Rightarrow \quad c_2=\frac{u_2-u_1}{l/2}$$

$$\therefore u(x)=u_1+\left(\frac{u_2-u_1}{l/2}\right)x$$

$$or,\ u(x)=\left(1-\frac{x}{l/2}\right)u_1+\frac{x}{l/2}\ u_2$$

$ii)\ \dfrac{l}{2} \le x \le l$

$$u(x)=c_1^2+c_2^2 x$$

$$,\ u(0)=u_2,\ \ u(l)=u_3$$

$$u(l/2)=c_1^2+c_2^2 l/2=u_2 \quad \Rightarrow \quad c_1^2=2u_2-u_3$$

$$u(l)=c_1^2+c_2^2 l=u_3 \quad \Rightarrow \quad c_2^2=\frac{u_3-u_2}{l/2}$$

$$\therefore u(x)=2u_2-u_3+\left(\frac{u_3-u_2}{l/2}\right)x$$

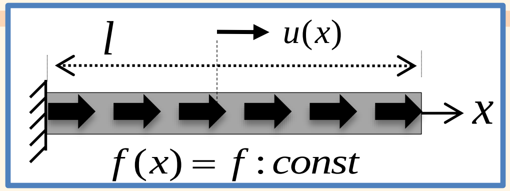
$$or,\ u(x)=\left(2-\frac{x}{l/2}\right)u_2+\left(-1+\frac{x}{l/2}\right)\ u_3$$

$$\therefore u(x)=\begin{cases}\left(1-\dfrac{x}{l/2}\right)u_1+\dfrac{x}{l/2}\ u_2 & ,0\le x\le \dfrac{l}{2}\\[2ex] \left(2-\dfrac{x}{l/2}\right)u_2+\left(-1+\dfrac{x}{l/2}\right)u_3 & ,\dfrac{l}{2}\le x\le l\end{cases}$$

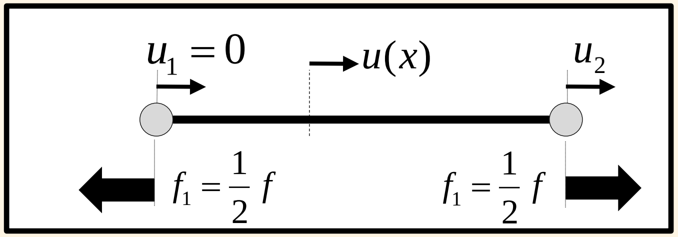
Element : Bar - Finite Element Method

Variational Method

$$\delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0$$



1 element , 2 nodes



$$\frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ u_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} f\cdot l\\ \frac{1}{2} f\cdot l\end{bmatrix}$$

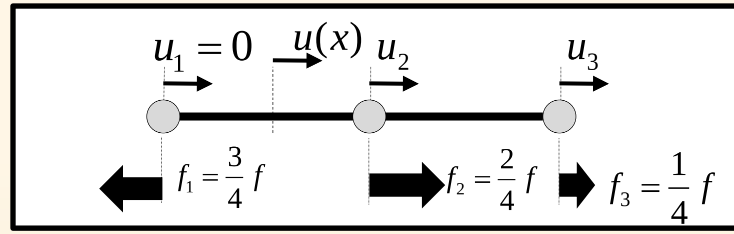
sol) $u_1 = 0,\ \ u_2 = \frac{1}{2}\frac{f\cdot l^2}{EA}$

F.E.M

$$\mathbf{K}\mathbf{d} = \mathbf{F}$$

find (K d) given (F)

2 element , 3 nodes



superposition of stiffness matrix

$$\frac{EA}{l/2}\begin{bmatrix}1 & -1 & 0\\ -1 & \boxed{1 + 1} & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}0\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}-\frac{3}{4} f\cdot l\\ \frac{2}{4} f\cdot l\\ \frac{1}{4} f\cdot l\end{bmatrix}$$

sol) $u_1 = 0,\ \ u_2 = \frac{3}{8}\frac{f\cdot l^2}{EA},\ u_3 = \frac{4}{8}\frac{f\cdot l^2}{EA}$

---

displacement

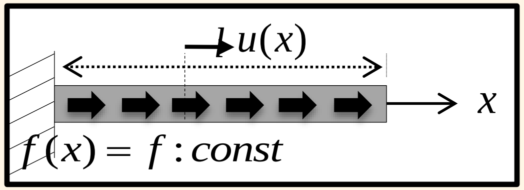
given : x

find : $u(x)$

$$u(x) = \left(1 - \frac{x}{l}\right)u_1 + \frac{x}{l}u_2 \quad ,0 \le x \le l$$

$$u(x) = \begin{cases}\left(1 - \frac{x}{l/2}\right)u_1 + \frac{x}{l/2}u_2 & ,0 \le x \le \frac{l}{2}\\ \left(2 - \frac{x}{l/2}\right)u_2 + \left(-1 + \frac{x}{l/2}\right)u_3 & ,\frac{l}{2} \le x \le l\end{cases}$$

Element : Bar - Comparison



mathematical modeling
by using Newton's second law

Differential Equation

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0$$

Boundary Condition

$$u\big|_{x=0} = 0 \quad , \quad EA\frac{du}{dx}\bigg|_{x=l} = 0$$

Variational Method

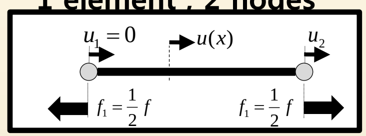
$$\delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0$$

Finite Element Method

Rayleigh-Ritz Method

$$u(x) = \frac{f\cdot l}{EA}x - \frac{f}{2EA}x^2$$

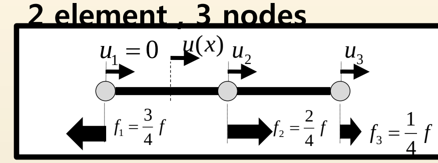
1 element , 2 nodes



$$u(x) = \left(1 - \frac{x}{l}\right)u_1 + \frac{x}{l}u_2$$

$$u_1 = 0, \quad u_2 = \frac{1}{2}\frac{f\cdot l^2}{EA}$$

2 element , 3 nodes



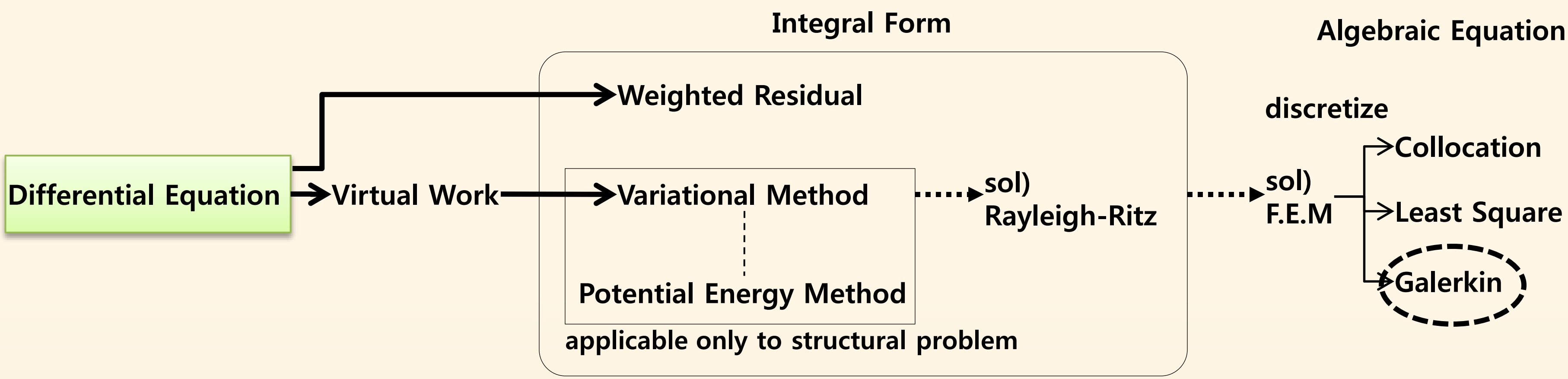
$$u(x) = \begin{cases} \left(1 - \dfrac{x}{l/2}\right)u_1 + \dfrac{x}{l/2}u_2 & , 0 \le x \le \dfrac{l}{2} \\ \left(2 - \dfrac{x}{l/2}\right)u_2 + \left(-1 + \dfrac{x}{l/2}\right)u_3 & , \dfrac{l}{2} \le x \le l \end{cases}$$

$$u_1 = 0, \quad u_2 = \frac{3}{8}\frac{f\cdot l^2}{EA}, u_3 = \frac{4}{8}\frac{f\cdot l^2}{EA}$$

for example,

$u\left(\frac{l}{2}\right) = \frac{f\cdot l}{EA}\cdot\frac{l}{2} - \frac{f}{2EA}\cdot\left(\frac{l}{2}\right)^2 = \frac{3}{8}\frac{f\cdot l^2}{EA}$	$u\left(\frac{l}{2}\right) = \frac{l/2}{l}\,\frac{1}{2}\frac{f\cdot l^2}{EA} = \frac{1}{4}\frac{f\cdot l^2}{EA}$	$u\left(\frac{l}{2}\right) = u_2 = \frac{3}{8}\frac{f\cdot l^2}{EA}$
$u(l) = \frac{f\cdot l}{EA}\cdot l - \frac{f}{2EA}\cdot(l)^2 = \frac{1}{2}\frac{f\cdot l^2}{EA}$	$u(l) = u_2 = \frac{1}{2}\frac{f\cdot l^2}{EA}$	$u(l) = u_3 = \frac{4}{8}\frac{f\cdot l^2}{EA} = \frac{1}{2}\frac{f\cdot l^2}{EA}$

Classification



ref. Logan D.L., A First Course in the Finite Element Method, Third edition, Brooks/Cole, p.116

We developed the bar finite element equations by the direct method in Section 3.1 and by the potential energy method (one or a number of variational methods) in Section 3.10.

In fields other than structural/solid mechanics, it is quite probable that a variational principle, analogous to the principle or minimum potential energy, for instance, may not be known or even exist. In some flow problems in fluid mechanics and in mass transport problems (Chapter 13), we often have only the differential equation and boundary conditions available. However, the finite element method can still be applied.
The methods or weighted residuals applied directly to the differential equation can be used to develop the finite element equations. In this section, we describe Galerkin's residual method in general and then apply it to the bar element. This development provides the basis for later applications of Galerkin's method to the nonstructural heat-transfer element (specifically, the one-dimensional combined conduction, convection, and mass transport element described in Chapter 13). Because of the mass transport phenomena, the variational formulation is not known (or certainly is difficult to obtain), so Galerkin's method is necessarily applied to develop the finite element equations.

Galerkin's Residual Method

Differential Equation

$$EA\frac{d^2u(x)}{dx^2} = 0$$

residual

$$EA\frac{d^2\boxed{u(x)}}{dx^2} \neq 0 = R$$

since it is approximated solution

$$\boxed{u(x)} = \mathbf{N}\,\mathbf{d}$$

$$where\ \boxed{\mathbf{N} = \begin{bmatrix} 1-\frac{x}{l} & \frac{x}{l} \end{bmatrix}},\quad \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

basis function

$$\mathbf{N} = \begin{bmatrix} \underbrace{1-\frac{x}{l}}_{N_1} & \underbrace{\frac{x}{l}}_{N_2} \end{bmatrix}$$

Thus substituting the approximated solution into the differential equation results in a residual over the whole region of the problem as follows

$$\iiint_V R\,dV$$

In the residual method, we require that a weighted value of the residual be a minimum over the whole region. The weighting functions allow the weighted integral of residuals to go to zero

$$\iiint_V R\,W\,dV = 0$$

weighting function or test function

Galerkin Method

the basis functions N_i are chosen to play the role of the weighting functions *W*

$$\iiint_V R\,N_i\,dV = 0 \qquad ,(i = 1, 2)$$

Element : Bar - Galerkin's Residual Method

$$ref.)\quad \int_0^1 (-u'v' + uv - xv)\,dx = 0$$

Bar - Galerkin's Residual Method

$$\int_0^l AE\frac{d^2u(x)}{dx^2}N_i\,dx = 0 \quad ,(i=1,2)$$

$$where,\ u(x) = \mathbf{N}\mathbf{d} = N_1u_1 + N_2u_2 \quad ,N_1 = 1-\frac{x}{l},\ \ N_2 = \frac{x}{l}$$

Galerkin Method

the test functions N_i are chosen to play the role of the weighting functions W

$$\iiint_V \boxed{R}\, N_i\, dV = 0 \qquad ,(i=1,2)$$

weighting function

residual (test function $\mathbf{N}$ used)

integration by parts

$$\left[N_iAE\frac{du}{dx}\right]_0^l - \int_0^l AE\frac{du}{dx}\frac{dN_i}{dx}dx = 0$$

since $$\frac{du}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 = \begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

Differential Equation

$$EA\frac{d^2u(x)}{dx^2} = 0$$

$$AE\int_0^l \frac{dN_i}{dx}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = \left[N_iAE\frac{du}{dx}\right]_0^l \quad ,(i=1,2)$$

$$i=1:\quad AE\int_0^l \frac{dN_1}{dx}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = \boxed{\left[N_1AE\frac{du}{dx}\right]_0^l} \Rightarrow \cancel{N_1}AE\frac{du}{dx}\bigg|_{x=l} - N_1AE\frac{du}{dx}\bigg|_{x=0} \Rightarrow -N_1f\,\big|_{x=0} \Rightarrow -f_1$$

$$i=2:\quad AE\int_0^l \frac{dN_2}{dx}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = \boxed{\left[N_2AE\frac{du}{dx}\right]_0^l} \Rightarrow N_2AE\frac{du}{dx}\bigg|_{x=l} - \cancel{N_2}AE\frac{du}{dx}\bigg|_{x=0} \Rightarrow N_2f\,\big|_{x=l} \Rightarrow f_2$$

since $N_1(0)=1,\ N_2(l)=0$

$N_2(0)=0,\ N_2(l)=1$

since

$$AE\frac{du}{dx} = AE\varepsilon = A\sigma = f$$

Element : Bar - Galerkin's Residual Method

$$ref.)\quad \int_0^1 (-u'v' + uv - xv)\,dx = 0$$

Bar - Galerkin's Residual Method

$$\int_0^l AE\frac{d^2u(x)}{dx^2}N_i\,dx = 0 \quad ,(i=1,2)$$

$$where,\ u(x) = \mathbf{N}\mathbf{d} = N_1u_1 + N_2u_2 \quad ,N_1 = 1-\frac{x}{l},\ N_2 = \frac{x}{l}$$

Galerkin Method

the test functions N_i are chosen to play the role of the weighting functions *W*

$$\iiint_V R\,N_i\,dV = 0 \qquad ,(i=1,2)$$

N_i → weighting function

R → residual (test function $\mathbf{N}$ used)

integration by parts

$$\begin{cases} AE\int_0^l \frac{dN_1}{dx}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = -f_1 \\ AE\int_0^l \frac{dN_2}{dx}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = f_2 \end{cases}$$

Differential Equation

$$EA\frac{d^2u(x)}{dx^2} = 0$$

$$\Rightarrow \begin{cases} AE\int_0^l \begin{bmatrix}-\frac{1}{l}\end{bmatrix}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = AE\frac{1}{l^2}\int_0^l\begin{bmatrix}1 & -1\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = AE\frac{1}{l^2}\begin{bmatrix}l & -l\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \\ AE\int_0^l \begin{bmatrix}\frac{1}{l}\end{bmatrix}\begin{bmatrix}-\frac{1}{l} & \frac{1}{l}\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = AE\frac{1}{l^2}\int_0^l\begin{bmatrix}-1 & 1\end{bmatrix}dx\begin{bmatrix}u_1\\u_2\end{bmatrix} = AE\frac{1}{l^2}\begin{bmatrix}-l & l\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \frac{AE}{l}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \end{cases}$$

$$\therefore \begin{cases} \frac{AE}{l}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = f_1 \\ \frac{AE}{l}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = f_2 \end{cases}$$

$$\Rightarrow \therefore \mathbf{Kd} = \mathbf{F} \qquad where\ ,\mathbf{K} = \frac{EA}{l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix},\mathbf{d} = \begin{bmatrix}u_1\\u_2\end{bmatrix},\mathbf{F} = \begin{bmatrix}f_1\\f_2\end{bmatrix}$$

Element : Bar - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume,
the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

the total potential energy Π is *defined as the* sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

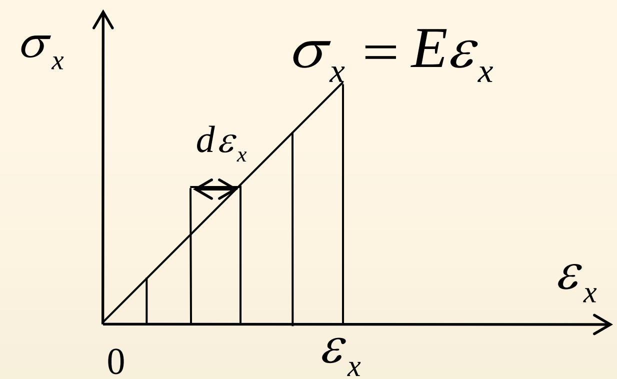
$$\Pi = \Pi_{in} + \Pi_{ext}$$

Element : Bar - Potential Energy Approach

the total potential energy Π is *defined* as the
sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

To evaluate the strain energy for a bar,
we consider only the work done by the internal forces during deformation.



Linear-elastic
(Hooke's law)material

$$d\Pi_{in} = \int_0^{\varepsilon_x} \sigma \, d\varepsilon_x \, dxdydz$$
$$= \int_0^{\varepsilon_x} E\varepsilon_x \, d\varepsilon_x \, dxdydz \;\; = \frac{1}{2} E\left(\varepsilon_x\right)^2 d\varepsilon_x dxdydz$$
$$= \frac{1}{2}\sigma\varepsilon_x dxdydz$$

$$\Pi_{in} = \iiint_V d\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV$$

the strain energy for one-dimensional stress.

Element : Bar - Potential Energy Approach

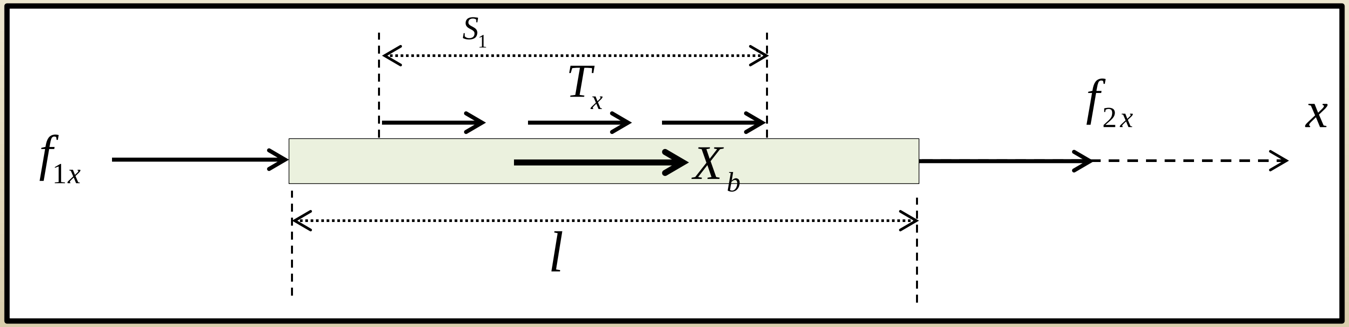
the total potential energy Π is *defined* *as the*
sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

The potential energy of the external forces, being opposite in sign from the extenal work expression because the potential energy of external forces is lost when the work is done by the external forces, is given by

$$\Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

- **body forces X_b typically from the self-weight of the bar (in units of force per unit volume) moving through displacement function u**
- **surface loading or traction T_x typically from distributed loading acting along the surface of the element (in units of force per unit surface area) moving through displacements u_s where u_s are the displacements occurring over surface S_1**
- **nodal concentrated force f_{ix} moving through nodal displacements u_i**



Element : Bar - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume,
the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

the total potential energy Π is *defined as the* sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

$$\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

1. **Formulate an expression for the total potential energy.**
2. **Assume the displacement pattern to vary with a finite set of undetermined parameters (here these are the nodal displacements u_i), which are substituted into the expression for total potential energy.**
3. **Obtain a set of simultaneous equations minimizing the total potential energy with respect to these nodal parameters. These resulting equations represent the element equations.**

Element : Bar - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

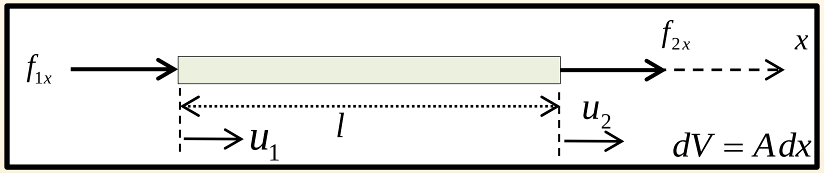
the total potential energy Π is *defined as the* sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

$$\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

---

assume that there is no surface traction and body force and the sectional area A is constant



Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

1. Formulate an expression for the total potential energy.

$$\Pi = \frac{A}{2}\int_0^l \sigma_x \varepsilon_x dx - f_{1x}u_1 - f_{2x}u_2$$

2. Assume the displacement pattern to vary with a finite set of undetermined parameters (here these are the nodal displacements u_i) ,which are substituted into the expression for total potential energy.

we have the axial displacement function expressed in terms of the shape functions and nodal displacements by

$$u(x) = \mathbf{N}\mathbf{d} \quad where,\ \mathbf{N} = \begin{bmatrix} 1-\frac{x}{l} & \frac{x}{l} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element : Bar - Potential Energy Approach

the principle of minimum potential energy

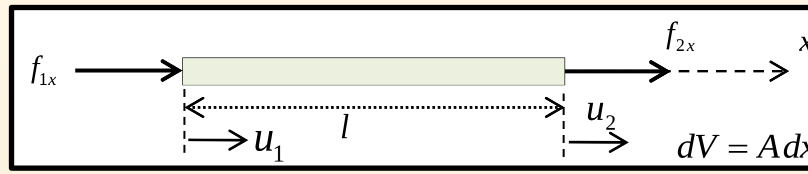
**Of all the geometrically possible shapes that a body can assume,
the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified
by a minimum value of the total potential energy**

the total potential energy Π is *defined as the* sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

$$\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

assume that there is no surface traction and body force and the sectional area A is constant



$$\Pi = \frac{A}{2}\int_0^l \sigma_x \varepsilon_x dx - f_{1x}u_1 - f_{2x}u_2$$

$$u(x) = \mathbf{N}\mathbf{d} \quad where,\ \mathbf{N} = \begin{bmatrix} 1-\frac{x}{l} & \frac{x}{l} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\varepsilon_x = \frac{du}{dx} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{B}\mathbf{d}$$

$$\sigma_x = E\varepsilon_x = E\mathbf{B}\mathbf{d}$$

$$-f_{1x}u_1 - f_{2x}u_2 = [u_1 \quad u_2]\begin{bmatrix} f_{1x} \\ f_{2x} \end{bmatrix} = \mathbf{d}^{\mathrm{T}}\mathbf{F}$$

$$\Pi = \frac{A}{2}\int_0^l (E\mathbf{B}\mathbf{d})^{\mathrm{T}}\mathbf{B}\mathbf{d}dx - \mathbf{d}^{\mathrm{T}}\mathbf{F} \Rightarrow \boxed{\Pi = \frac{EA}{2}\int_0^L \mathbf{d}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{d}dx - \mathbf{d}^{\mathrm{T}}\mathbf{F}}$$

Element : Bar - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

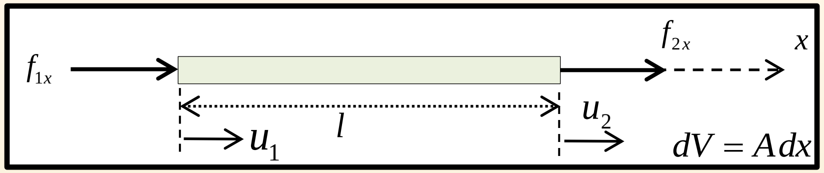
the total potential energy Π is *defined as the sum* of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

$$\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

---

assume that there is no surface traction and body force and the sectional area A is constant



$$\Pi = \frac{A}{2}\int_0^l \sigma_x \varepsilon_x dx - f_{1x}u_1 - f_{2x}u_2$$

$$\boxed{\Pi = \frac{EA}{2}\int_0^l \mathbf{d}^T\mathbf{B}^T\mathbf{B}\mathbf{d}dx - \mathbf{d}^T\mathbf{F}}$$

$$u(x) = \mathbf{N}\mathbf{d} \quad where,\ \mathbf{N} = \begin{bmatrix} 1-\frac{x}{l} & \frac{x}{l} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\varepsilon_x = \frac{du}{dx} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{B}\mathbf{d}$$

$$\sigma_x = E\,\varepsilon_x = E\mathbf{B}\mathbf{d}$$

$$-f_{1x}u_1 - f_{2x}u_2 = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} f_{1x} \\ f_{2x} \end{bmatrix} = \mathbf{d}^T\mathbf{F}$$

3. Obtain a set of simultaneous equations minimizing the total potential energy with respect to these nodal parameters. These resulting equations represent the element equations.

The minimization of Π with respect to each nodal displacement requires that

$$\frac{\partial \Pi}{\partial u_1} = 0 \quad \textbf{and} \quad \frac{\partial \Pi}{\partial u_2} = 0$$

Element : Bar - Potential Energy Approach

the principle of minimum potential energy

Of all the geometrically possible shapes that a body can assume,
the true one, corresponding to the safisfaction of stable equilibrium of the body, is identified
by a minimum value of the total potential energy

the total potential energy Π is *defined as the* sum of the internal strain energy Π_{in} *and the potential energy of the external forces* Π_{ext}

$$\Pi = \Pi_{in} + \Pi_{ext}$$

$$\Pi_{in} = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x dV \quad , \Pi_{ext} = -\iiint_V X_b u dV - \iint_{S_1} T_x u_s dS - \sum_{i=1}^{M} f_{ix} u_i$$

$$\Pi = \frac{EA}{2}\int_0^l \boxed{\mathbf{d}^T\mathbf{B}^T\mathbf{B}\mathbf{d}}\, dx - \mathbf{d}^T\mathbf{F}$$

$$\boxed{\mathbf{d}^T\mathbf{B}^T\mathbf{D}^T\mathbf{B}\,\mathbf{d}} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{bmatrix}\begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{l^2}\begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \frac{1}{l^2}(u_1^2 - 2u_1u_2 + u_2^2)$$

$$\therefore \Pi = \frac{EA}{2l^2}\int_0^l (u_1^2 - 2u_1u_2 + u_2^2)dx - f_{1x}u_1 - f_{2x}u_2$$

The minimization of Π with respect to each nodal displacement requires that $\frac{\partial \Pi}{\partial u_1} = 0$ and $\frac{\partial \Pi}{\partial u_2} = 0$

$$\begin{cases} \dfrac{\partial \Pi}{\partial u_1} = \dfrac{EA}{2l^2}\displaystyle\int_0^l (2u_1 - 2u_2)dx - f_{1x} = \dfrac{EA}{l}(u_1 - u_2) - f_{1x} \\ \dfrac{\partial \Pi}{\partial u_2} = \dfrac{EA}{2l^2}\displaystyle\int_0^l (-2u_1 + 2u_2)dx - f_{2x} = \dfrac{EA}{l}(-u_1 + u_2) - f_{2x} \end{cases}$$

$$\mathbf{B}\,\mathbf{d} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{d}^T\mathbf{F} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} f_{1x} \\ f_{2x} \end{bmatrix} = -f_{1x}u_1 - f_{2x}u_2$$

$$\Rightarrow \begin{cases} \dfrac{EA}{l}(u_1 - u_2) - f_{1x} = 0 \\ \dfrac{EA}{l}(-u_1 + u_2) - f_{2x} = 0 \end{cases}$$

In matrix form, we express

$$\Rightarrow \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{2x} \end{bmatrix}$$

$$\boxed{\therefore \mathbf{Kd} = \mathbf{F} \quad where\ , \mathbf{K} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} f_{1x} \\ f_{2x} \end{bmatrix}}$$

E: Young's Modulus, A: sectional area

Element : Bar

Element

Differential Equation → Galerkin's Weighted Residual / Variational Method / Energy Method → •Discretization •Approximation → Finite Element Method

$\mathbf{M\ddot{x}} = \sum \mathbf{F}$, where $\mathbf{\ddot{x}} = 0$

$\mathbf{Kd} = \mathbf{F}$

from now on, we will use this!

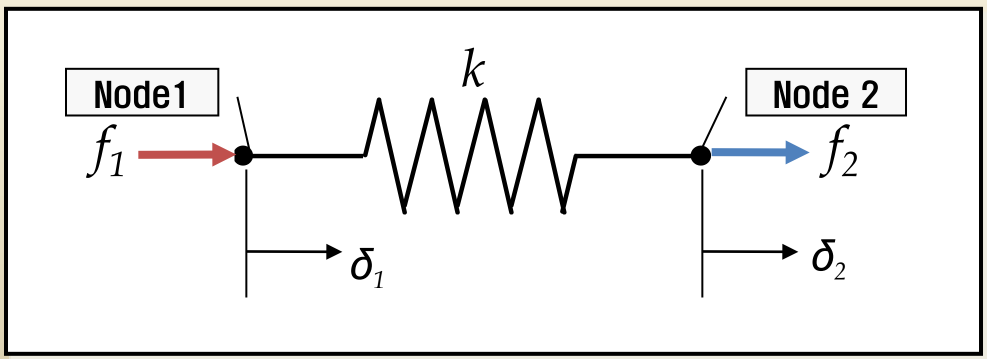
Bar

$$EA\frac{d^2u(x)}{dx^2} + f(x) = 0 \Rightarrow \delta\int_0^l \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - (f\,u)\right]dx = 0 \Rightarrow \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$u(x) = a_0 + a_1 x$$

! Notation

$$k = \frac{EA}{l}$$



$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

stiffness matrix

$$[f] = [K][\delta]$$