

Computer Aided Ship design

-Part I. Optimal Ship Design-

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9. Heuristic Algorithm(휴리스틱 알고리즘)

9.1. Genetic Algorithm(유전 알고리즘)

9.2. Ant Algorithm(개미 알고리즘)

9.3. Simulated Annealing



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9. Heuristic Algorithm(휴리스틱 알고리즘)

9.1 Genetic Algorithm(유전 알고리즘)



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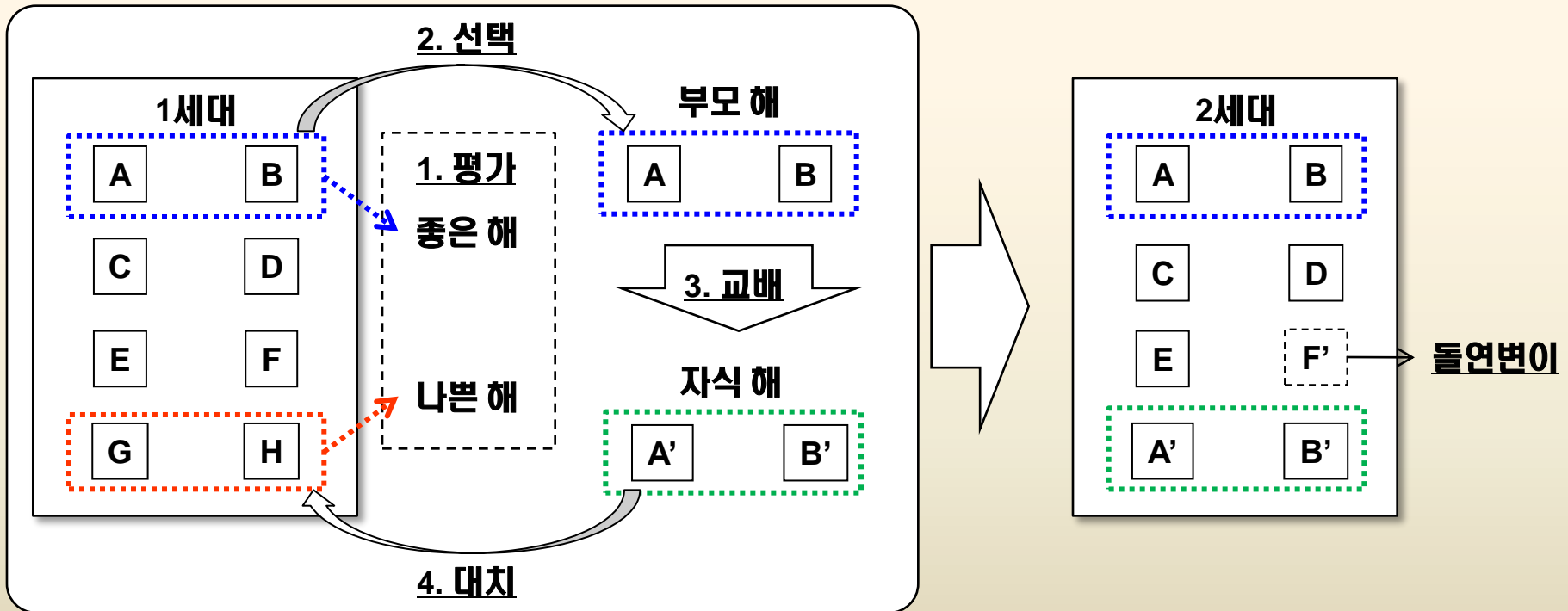


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Genetic Algorithm

- 생물의 진화 메커니즘을 이용하여 다양한 문제를 해결 하고자 하는 방법이다.
- 적자 생존과 자연 도태의 진화 원리를 도입하여 초기해로부터 선택(selection) 연산, 교배(crossover) 연산, 돌연변이(mutation) 연산과 같은 일련의 과정을 반복하여 해를 구한다.



- 평가:** 1세대의 각 해 품질을 평가한다.
- 선택(Selection):** 품질이 좋은 해를 부모 해로 선택 한다.
- 교배(Crossover):** 부모 해로부터 자식 해를 생성한다.
- 대치:** 1세대 내의 나쁜 해를 자식 해로 교체한다.

돌연변이(Mutation): 부모 해에 없는 속성을 도입하여 새로운 해를 만드는 연산이다.

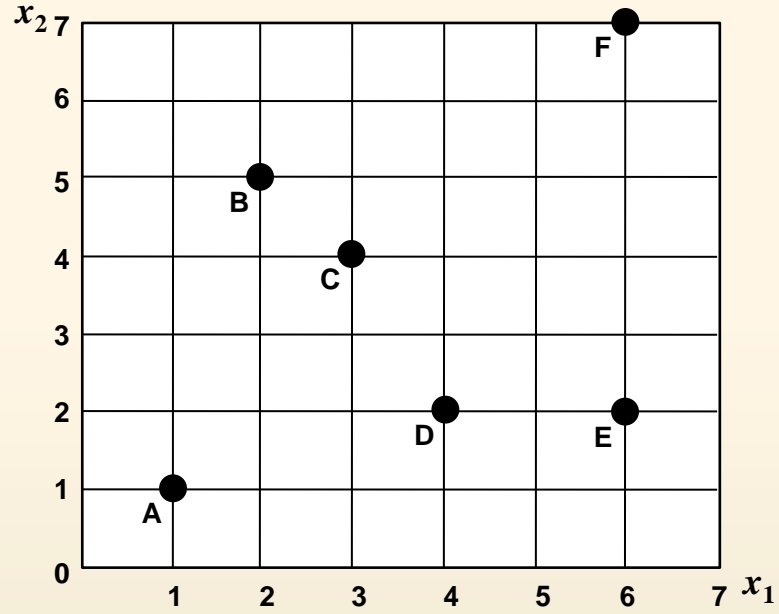
평가, 선택, 교배, 대치의 순서를 반복하며 최적 해를 찾는다.

Example of Genetic Algorithm (1/5)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 8x_1 - 8x_2$

	1세대	평가
A	(1 , 1)	-14
B	(2 , 5)	-27
C	(3 , 4)	-31
D	(4 , 2)	-28
E	(6 , 2)	-24
F	(6 , 7)	-19

부모 해
 (3 , 4)
 (4 , 2)



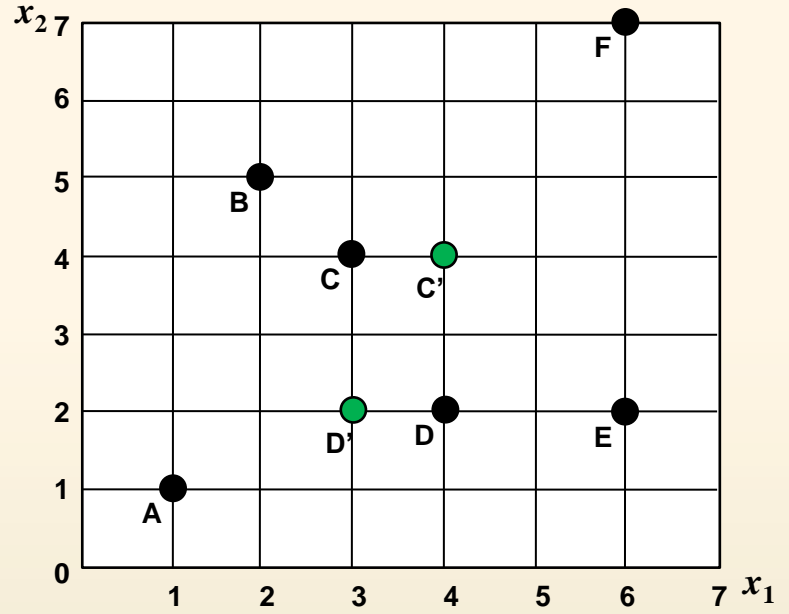
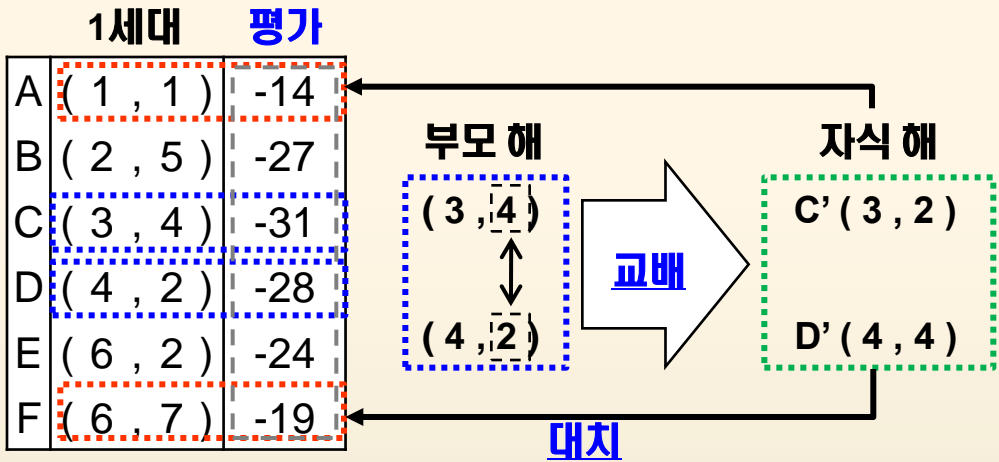
1. 평가: 1세대의 각 해 품질을 평가한다.
2. 선택(Selection): 품질이 좋은 해를 부모해로 선택 한다.
 - 본 예제에서는 품질이 가장 좋은 2개의 해를 부모 해로 선택 하였다.
 - 일반적으로는 확률적으로 해를 선택하며, 품질이 좋은 해가 선택될 확률을 높인다.

예) 품질 비례 룰렛 휠 선택

각 해의 품질에 비례하여 룰렛 휠에서 차지하는 넓이를 배정함으로써, 품질에 비례하여 선택 받을 확률을 높이는 방법

Example of Genetic Algorithm(2/5)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 8x_1 - 8x_2$



3. 교배(Crossover): 부모 해로부터 자식 해를 생성한다.

- 본 예제에서는 x_2 를 서로 교환하는 방법을 사용 하였음

일반적인 교배(Crossover) 방법 설명

4. 대치: 생성된 자식 해를 평가가 좋지 않은 해와 교환한다.

- 본 예제에서는 품질이 가장 나쁜 2개의 해를 자식 해로 대치 하였다.

해를 대치하는 다른 방법의 예

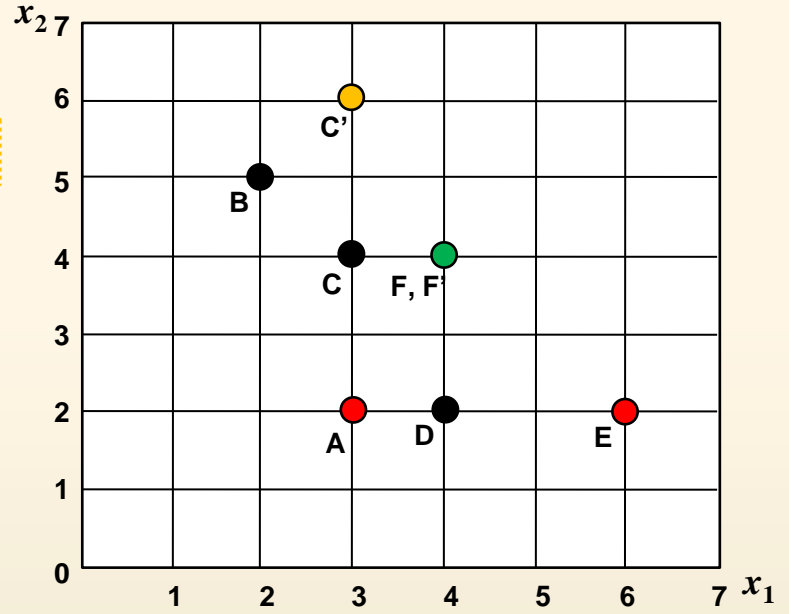
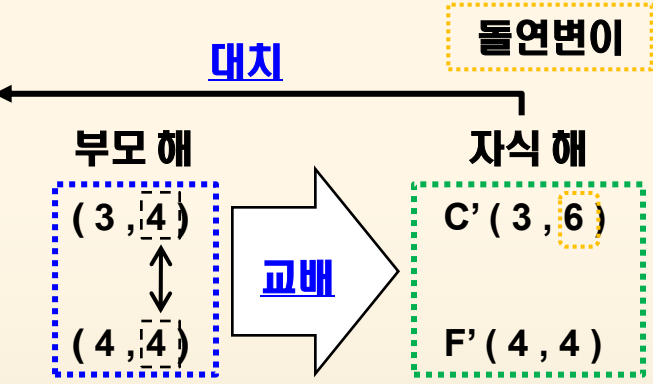
+ 예시 1) 자식 해의 품질을 즉시 평가하여, 부모 해보다 품질이 좋으면 부모 해와 대치, 그렇지 않으면 대치하지 않음

+ 예시 2) 자식 해의 품질을 즉시 평가하여, 부모 해보다 품질이 좋으면 부모 해와 대치, 그렇지 않으면 품질이 나쁜 해와 대치

Example of Genetic Algorithm(3/5)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 8x_1 - 8x_2$

	2세대	평가
A	(3 , 2)	-27
B	(2 , 5)	-27
C	(3 , 4)	-31
D	(4 , 2)	-28
E	(6 , 2)	-24
F	(4 , 4)	-32



	3세대	평가
A	(3 , 6)	
B	(2 , 5)	
C	(3 , 4)	
D	(4 , 2)	
E	(4 , 4)	
F	(4 , 4)	

다시 평가, 선택, 교배, 대치의 순서를 반복하며 최적 해를 찾는다.

돌연변이: 부모 해에 없는 속성을 도입하여 새로운 해를 만드는 연산이다.

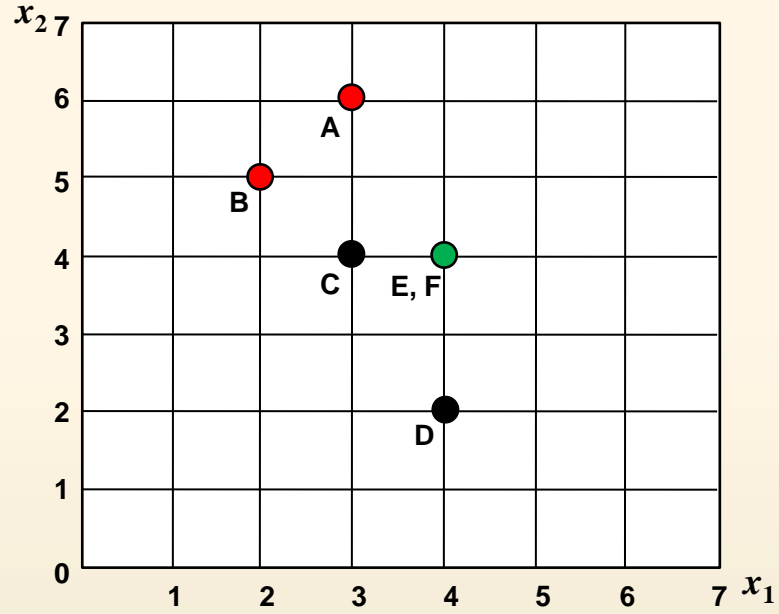
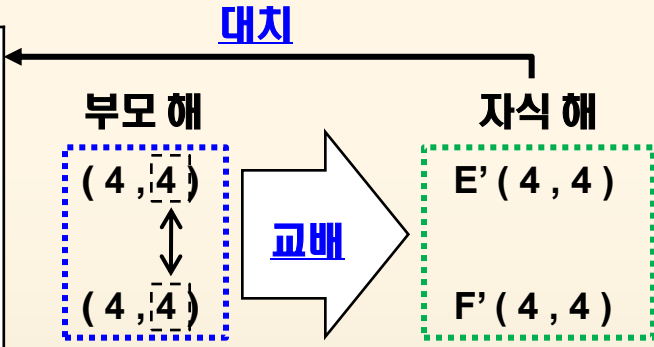
종료조건:

- 루프를 일정 횟수만큼 반복한 후 정지시키는 방법
- + 어느 정도 반복 횟수로서 해가 수렴할 것이라는 경험적 지식이 있어야 함
- 해 집합 내의 해들이 대부분(예: 70%) 똑같은지를 판단하여 종료

Example of Genetic Algorithm(4/5)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 8x_1 - 8x_2$

	3세대	평가
A	(3 , 6)	-27
B	(2 , 5)	-27
C	(3 , 4)	-31
D	(4 , 2)	-28
E	(4 , 4)	-32
F	(4 , 4)	-32



	4세대	평가
A	(4 , 4)	
B	(4 , 4)	
C	(3 , 4)	
D	(4 , 2)	
E	(4 , 4)	
F	(4 , 4)	

다시 평가, 선택, 교배, 대치의 순서를 반복하며 최적 해를 찾는다.

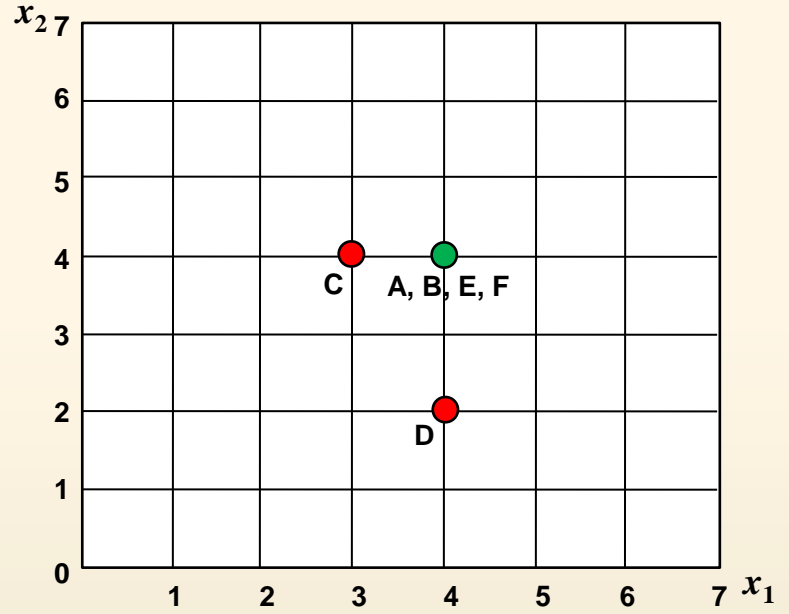
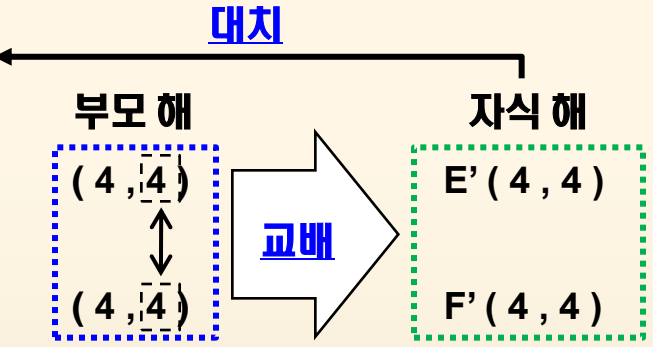
종료조건:

- 루프를 일정 횟수만큼 반복한 후 정지시키는 방법
+ 어느 정도 반복 횟수로서 해가 수렴할 것이라는 경험적 지식이 있어야 함
- 해 집합 내의 해들이 대부분(예: 70%) 똑같은지를 판단하여 종료

Example of Genetic Algorithm(5/5)

Minimize $f(\mathbf{x}) = x_1^2 + x_2^2 - 8x_1 - 8x_2$

4세대	평가
A (4 , 4)	-32
B (4 , 4)	-32
C (3 , 4)	-31
D (4 , 2)	-28
E (4 , 4)	-32
F (4 , 4)	-32



5세대	평가
A (4 , 4)	
B (4 , 4)	
C (4 , 4)	
D (4 , 4)	
E (4 , 4)	
F (4 , 4)	

다시 평가, 선택, 교배, 대치의 순서를 반복하며 최적 해를 찾는다.

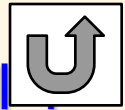
종료조건:

- 루프를 일정 횟수만큼 반복한 후 정지시키는 방법
- + 어느 정도 반복 횟수로서 해가 수렴할 것이라는 경험적 지식이 있어야 함
- 해 집합 내의 해들이 대부분(예: 70%) 똑같은지를 판단하여 종료

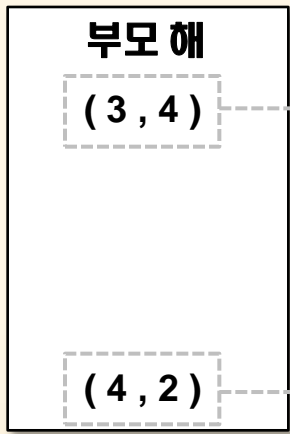
종료 → 최적점은 (4,4)



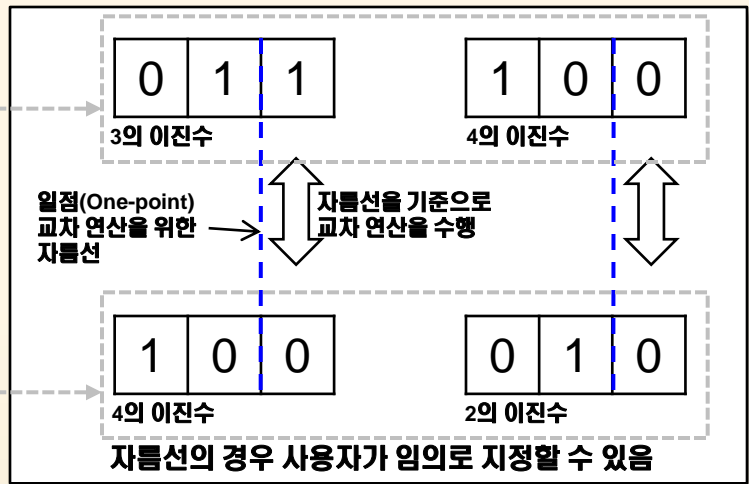
Genetic Algorithm



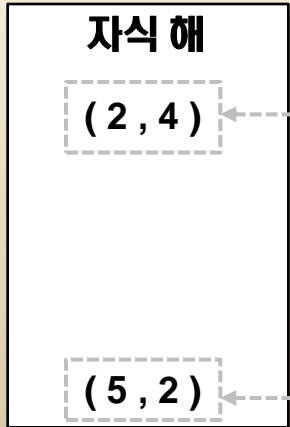
- 이진수 표현과 일점(One-point) 교차 연산을 이용한 교배(Crossover)방법의 예



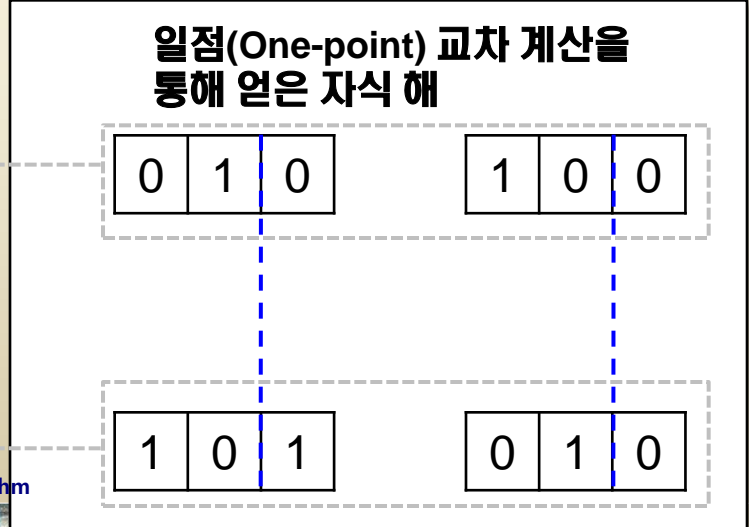
이진수 표현을 통한 Encoding



교배(Crossover)



Decoding



9. Heuristic Algorithm(휴리스틱 알고리즘)

9.2 Ant Algorithm(개미 알고리즘)



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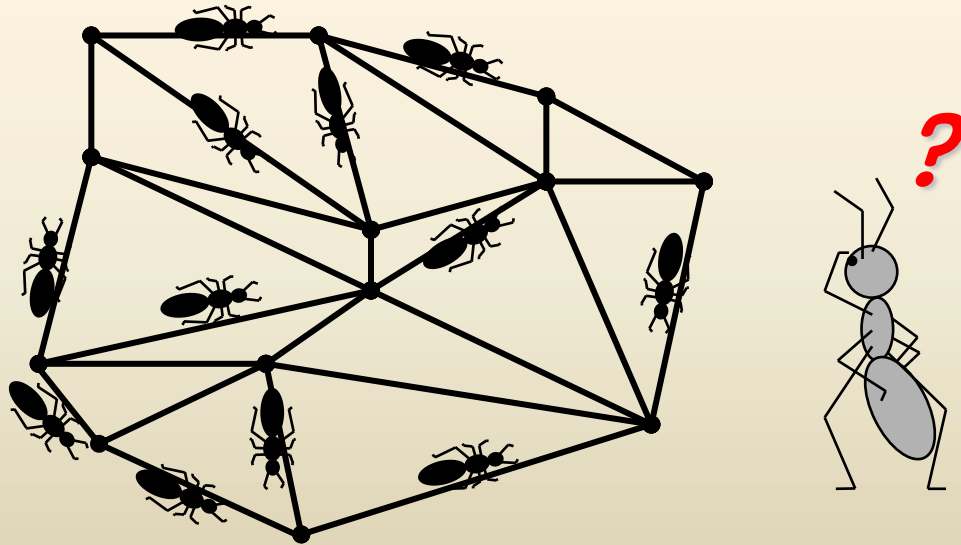


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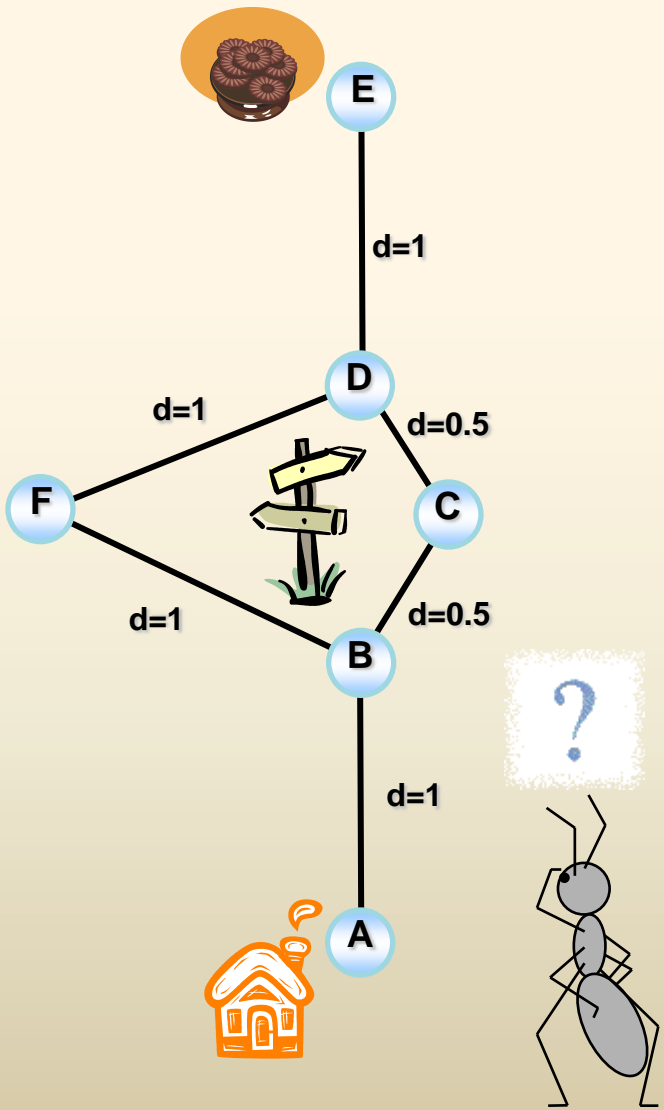


Ant System Algorithm (개미 시스템 알고리즘)

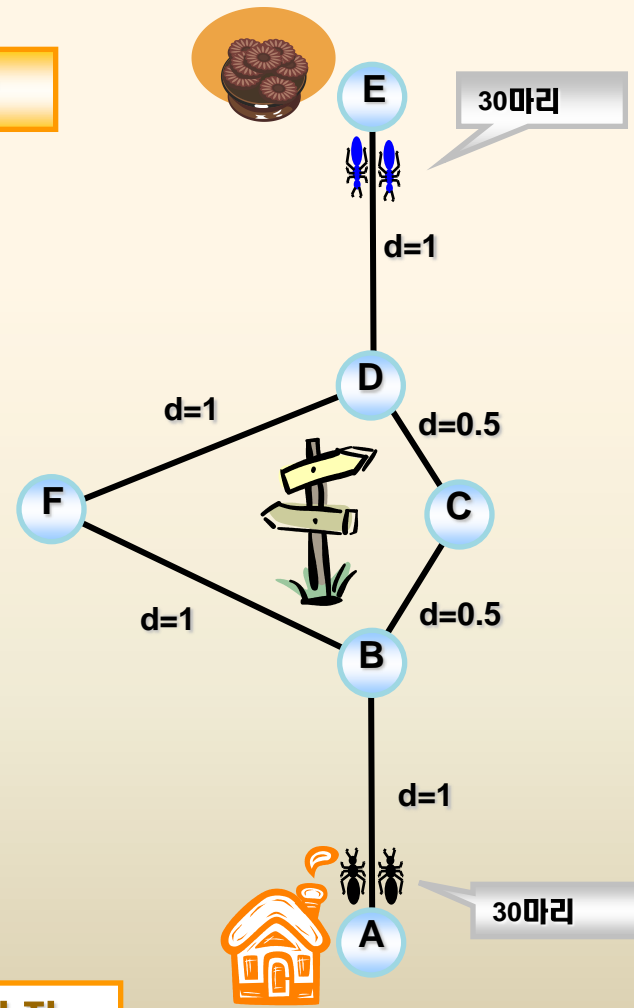
- 개미 무리들의 먹이를 옮기는 이동 행태로부터 착상
임의의 개미들이 임의의 경로로 이동 (이동경로에 페르몬 흔적 남김) → 시간 경과 후 점차적으로 페르몬 흔적이 많은 곳으로 개미들의 경로 수정 → 궁극적으로 최단 경로를 찾음



개미 시스템 알고리즘 (개미들의 이동형태)



t=0

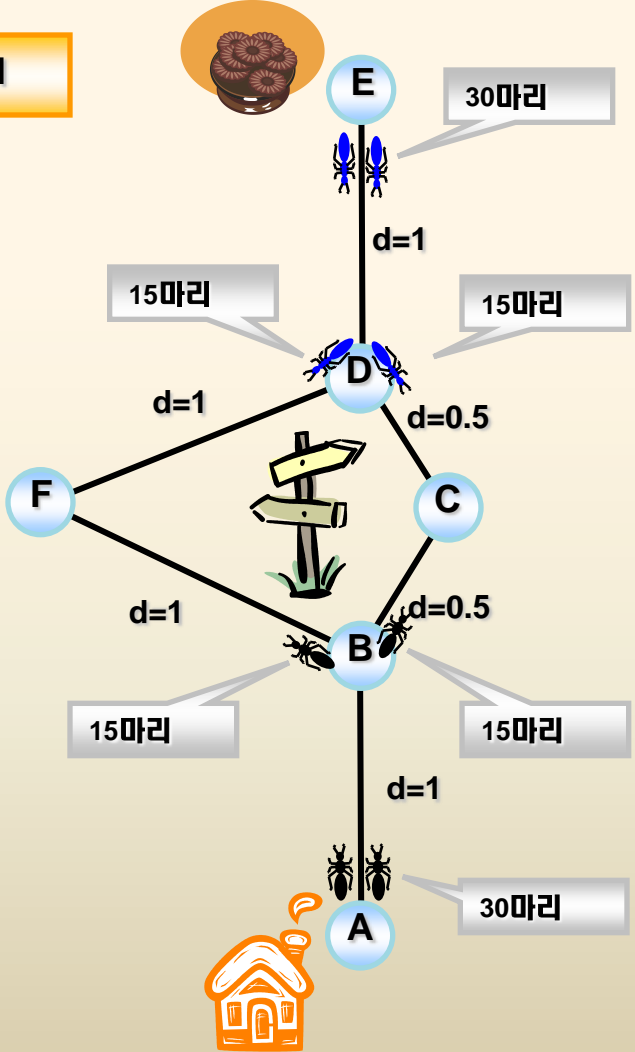


가정

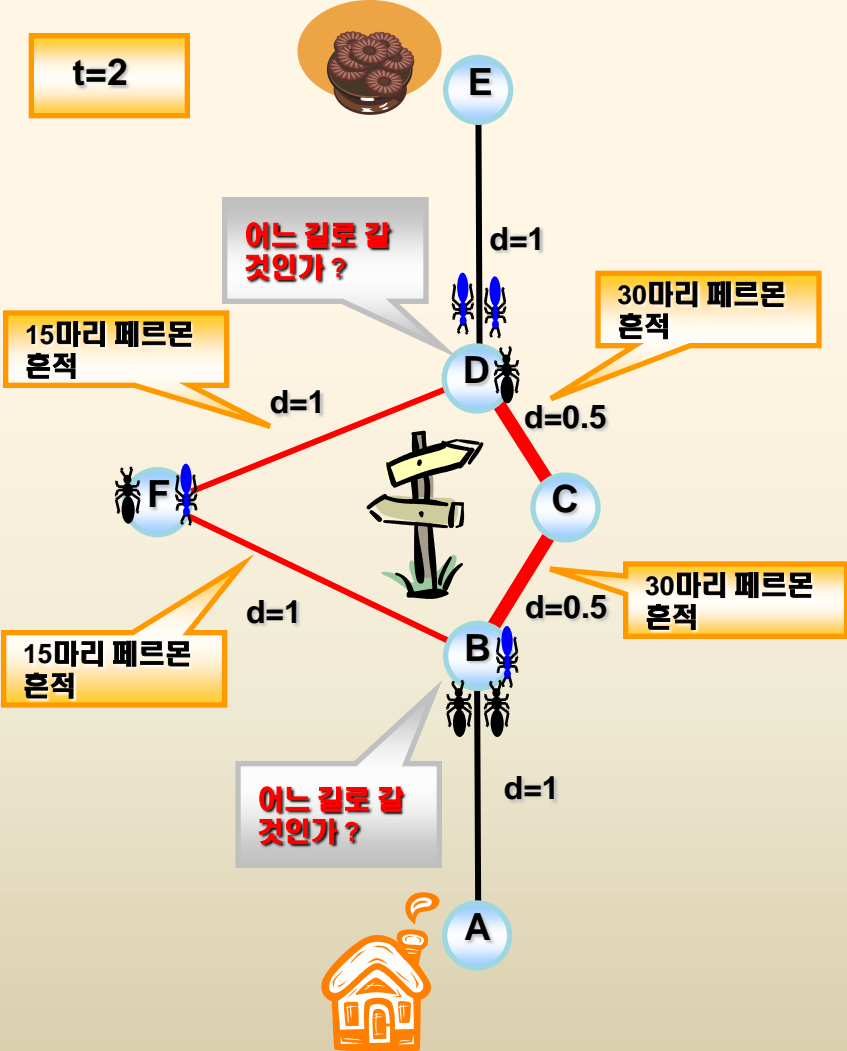
- 매 단위시간마다 A, E에서 각각 30마리씩 출발
- 모든 개미의 이동속도는 일정, 이산형 시간

개미 시스템 알고리즘 (개미들의 이동형태)

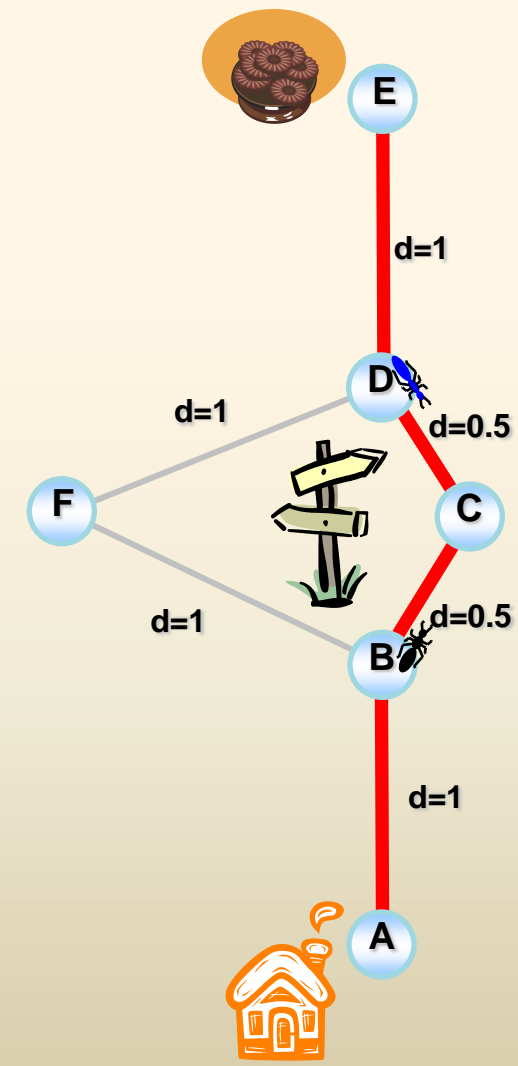
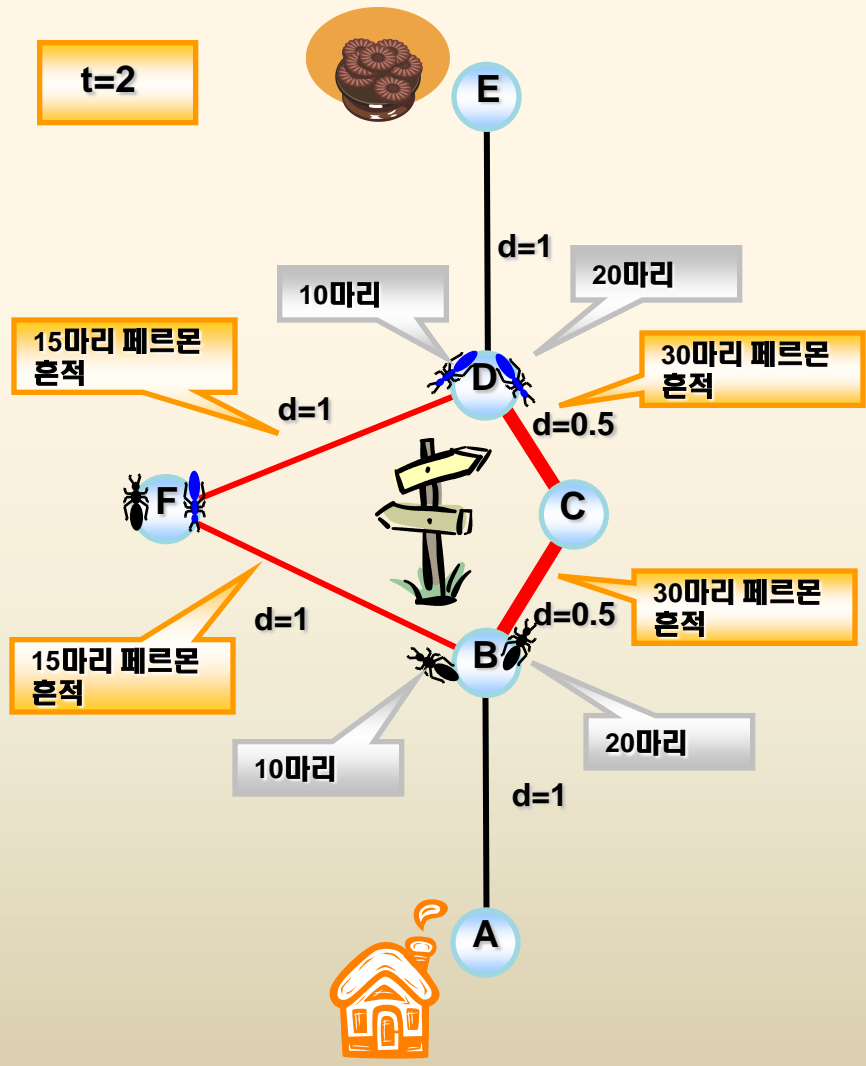
t=1



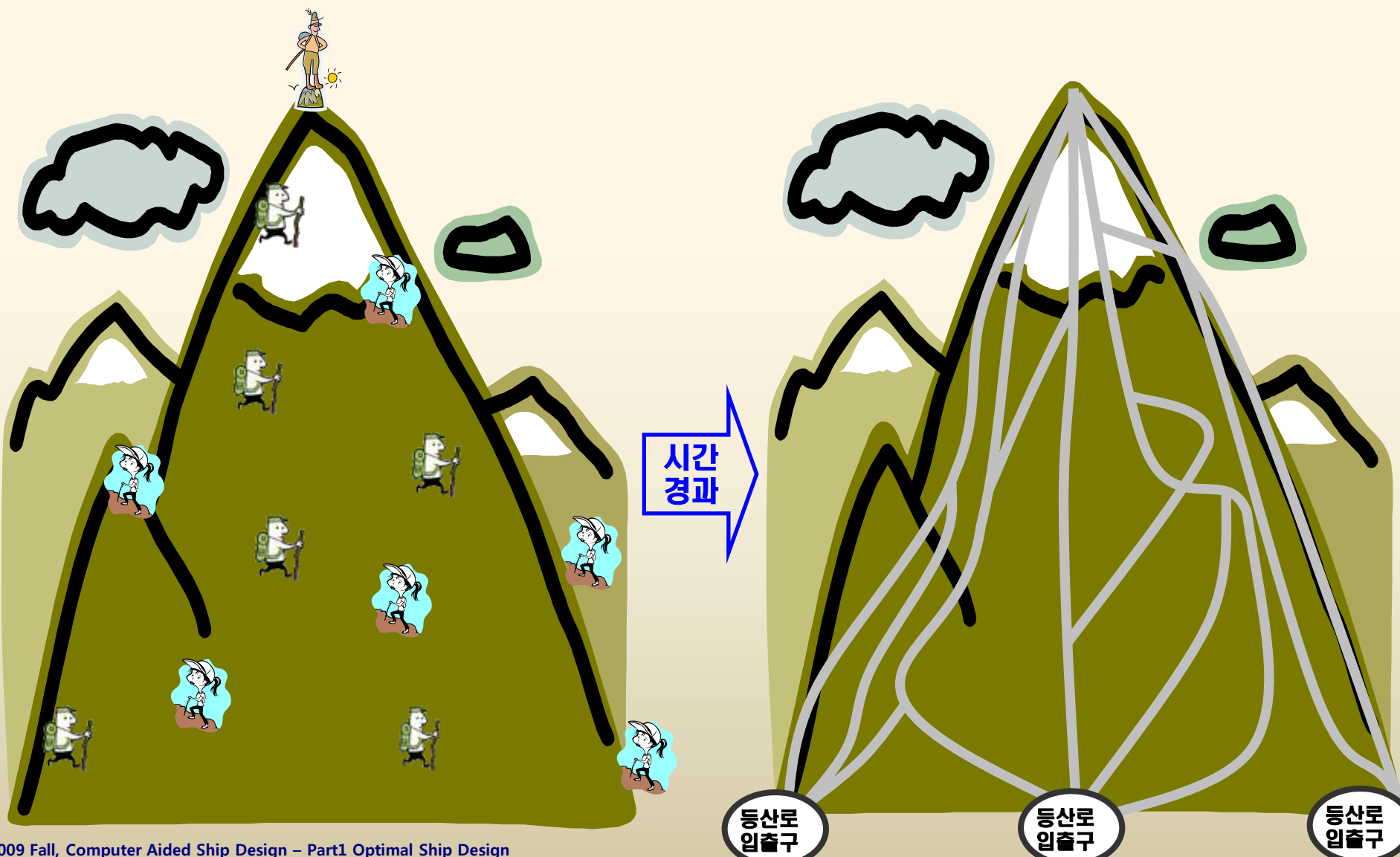
t=2



개미 시스템 알고리즘 (개미들의 이동형태)



등산로 생성과정



개미시스템의 일반적 특징

- 개미와 비슷한 에이전트(Agent)를 이용하여 모델을 구축하는데, 이것을 모의개미(artificial ant)라 한다.
- 모의개미는 실제개미처럼 페르몬을 사용하여 서로 교신하고, 이동경로 위에 놓여진 페르몬 흔적에 의한 확률적 방법으로 최단경로를 찾는다.

모의개미의 특징

1. 개미는 약간의 시력을 가짐
개미시스템에서 시인성으로 나타내며 노드간 거리에 반비례한다.
개미 k 가 도시 i 에서 도시 j 를 선택할 때, 시력은 방향성을 제시하기도 한다.
2. 개미는 약간의 기억력을 가짐
자신이 이미 방문한 노드를 기억하며, 경로탐색을 끝마칠 때까지 한번 방문한 노드에 대해 다시 방문하지 않는다.
3. 개미는 방문해야 할 모든 지점에 고르게 퍼져 존재
개미가 존재하는 각각의 노드를 출발점으로 하여 동시에 경로탐색을 가능하게 하며, 더욱 빠르고 정확한 최적해를 구할 수 있도록 한다.



개미시스템의 일반적 특징

유사점	차이점
<ul style="list-style-type: none"> ▪ 최단경로를 찾기 위해 개미간에 서로 협력한다 ▪ 페르몬을 사용한다 ▪ 페르몬은 증발한다 ▪ 확률적 방법으로 경로를 결정한다 	<p>[모의]개미는</p> <ul style="list-style-type: none"> ▪ 약간의 기억력을 가지고 있다 ▪ 약간의 시력을 가지고 있다 ▪ 시간이 불연속적인 환경(Discrete World)에서 존재한다.

개미시스템의 특징

- 점에 의한 탐색이 아닌 개체군에 의한 병렬적 탐색을 수행한다.
- 최적해는 개미수와 노드수가 동일할 때 얻을 수 있다.
- 최적해는 도해(Graph)로 정의 되어진다.
- 개미는 확률적 방법으로 다음 이동할 방향을 결정한다.
- 페르몬개선은 이동경로에 페르몬 흔적이 얼마나 많은가, 즉 개미가 얼마나 많이 통행했는가를 말한다.
- 개미시스템은 노드 수 증가에 따라 분석시간이 증가하는 특성(Time complexity)을 가지고 있다.



9. Heuristic Algorithm(휴리스틱 알고리즘)

9.3 simulated annealing



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Simulated Annealing

- A slow and controlled cooling of a heated solid (Annealing) ensures proper solidification with a highly ordered, crystalline state that corresponds to the lowest internal energy.
- Random evaluations of the objective function in such a way that transitions out of a local minimum are possible.
- It will find global optimum with a high probability even for ill-conditioned functions with numerous local minima



Simulated Annealing

(inner loop)

1. Start with initial point \mathbf{x}_0 (iteration number $i = 0$)
2. Random moves are made along each coordinate direction

3. If $\Delta f \leq 0$, accept the new point and set $\mathbf{x}_{i+1} = \mathbf{x}$
where, $\Delta f = f(\mathbf{x}_{i+1}) - f(\mathbf{x}_i)$

Otherwise, accept the new point with a probability of

$$P(\Delta f) = e^{-\Delta f / kT}$$

k : a scaling factor called Boltzmann's constant

T : a parameter called temperature

4. Repeat step 2~3 until $i \geq N_{\max}$



Simulated Annealing

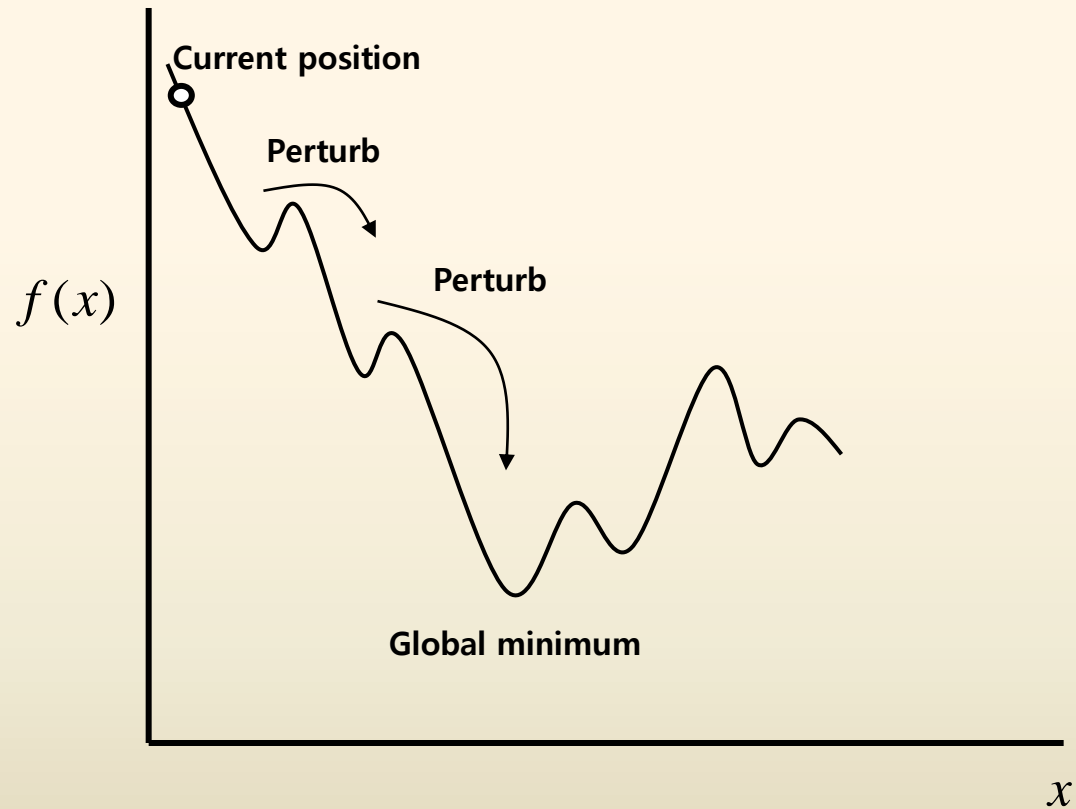
(outer loop)

5. Reduce temperature and repeat step 1~4.

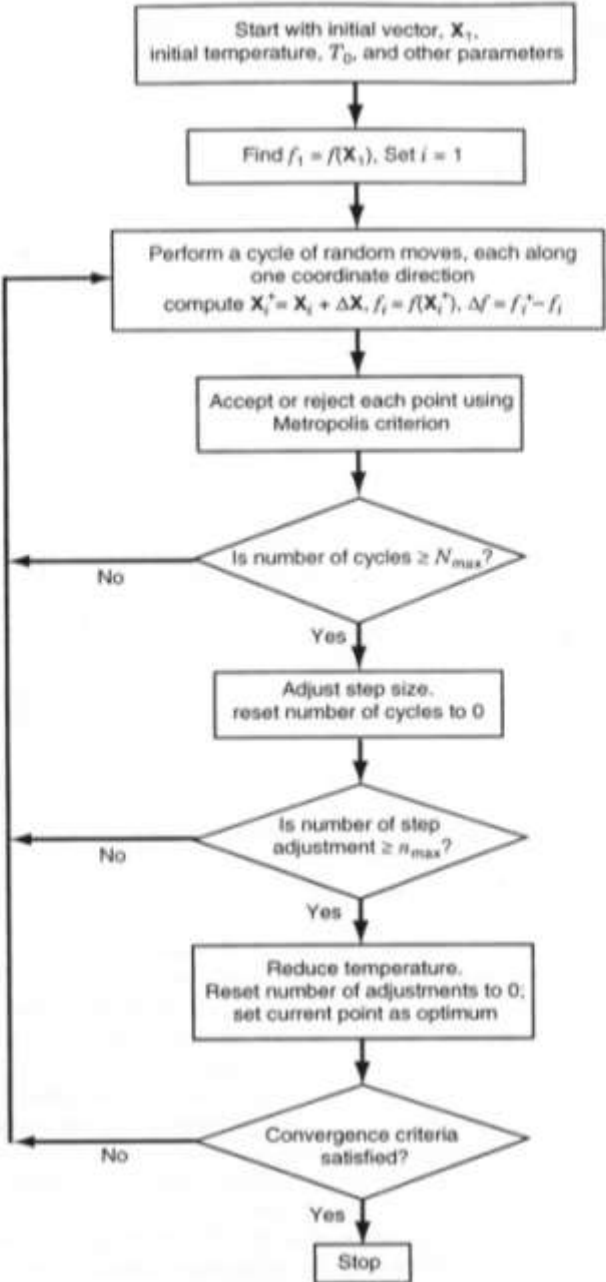
6. This process is continued until a sufficiently low temperature is reached, at which stage no more improvement in the objective function value can be expected.



Simulated Annealing



Simulated Annealing



If $\Delta f \leq 0$, accept the new point and set

$$\mathbf{x}_{i+1} = \mathbf{x}$$

Otherwise, accept the new point with a probability of

$$P(\Delta f) = e^{-\Delta f / kT}$$

k : a scaling factor called Boltzmann's constant

T : a parameter called temperature

Strengths of Simulated Annealing

- 1. Simulated annealing can deal with highly nonlinear models, chaotic and noisy data and many constraints. It is a robust and general technique.**
- 2. Its main advantages over other local search methods are its flexibility and its ability to approach global optimality.**
- 3. The algorithm is quite versatile since it does not rely on any restrictive properties of the model.**



Weakness of Simulated Annealing

1. Since SA is a metaheuristic, a lot of choices are required to turn it into an actual algorithm.
2. There is a clear tradeoff between the quality of the solutions and the time required to compute them.
3. The tailoring work required to account for different classes of constraints and to fine-tune the parameters of the algorithm can be rather delicate.
4. The precision of the numbers used in implementation is of SA can have a significant effect upon the quality of the outcome.



Ch.10 제약 비선형 최적화 프로그램 소개

10.1 제약 비선형 최적화 프로그램의 구성

10.2 제약 비선형 최적화 프로그램의 사용 방법

10.3 제약 비선형 최적화 프로그램을 이용한
최적 설계 예



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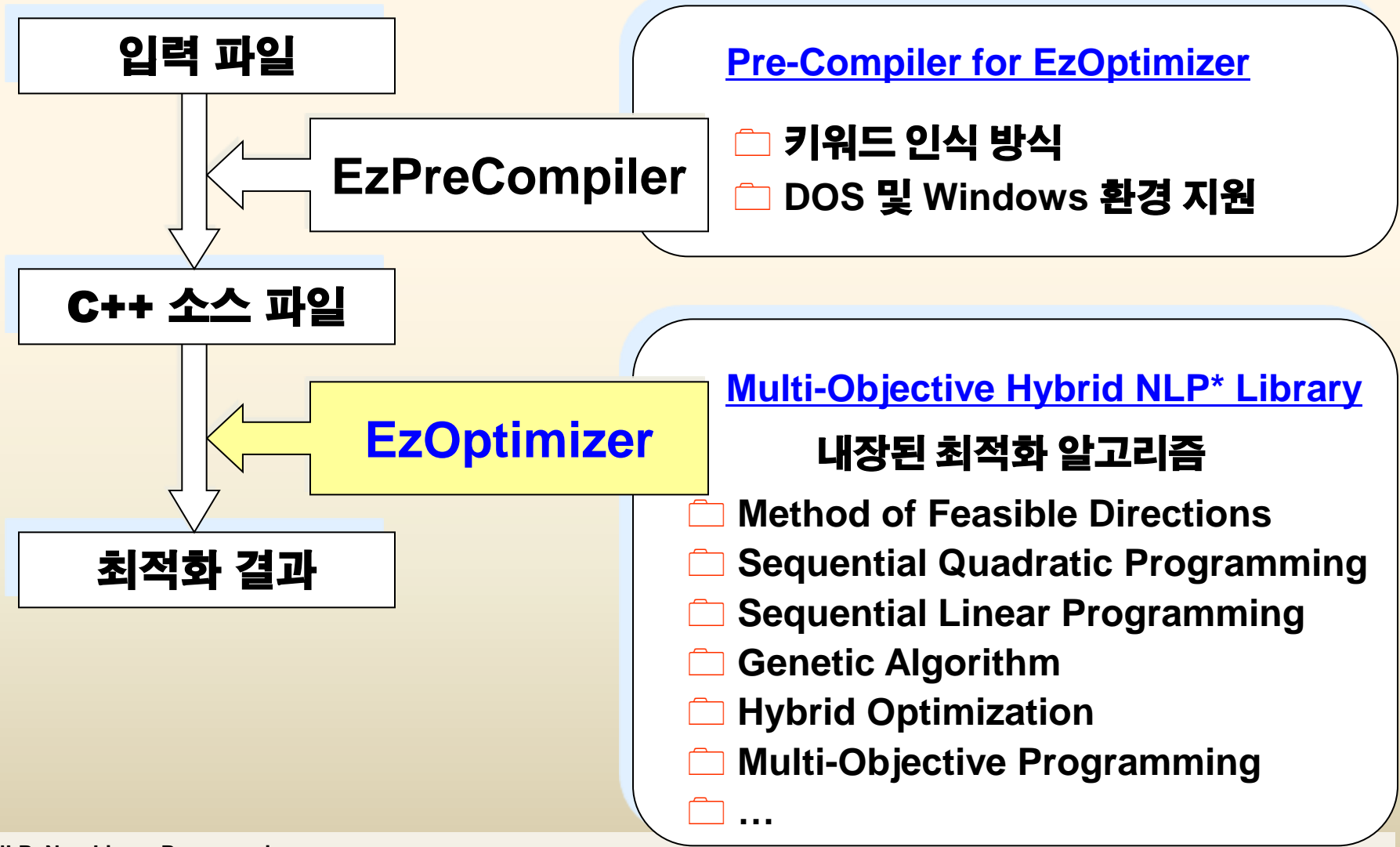


10.1 제약 비선형 최적화 프로그램의 구성(1)

- 비선형 최적화 라이브러리 '**EzOptimizer**'
 - 다양한 최적화 알고리즘을 포함
 - C++ 언어로 구현
 - 최적화 모듈을 사용자의 프로그램에 쉽게 이식 가능
 - 최적화 과정을 제어할 수 있는 다양한 옵션 지정 가능
 - 헤더 파일(EzOptimizer.h)과 라이브러리 파일(EzOptimizerLibraryD.lib/EzOptimizerLibraryR.lib)로 구성
- '**EzOptimizer**'를 위한 프리-컴파일러 '**EzPreCompiler**'
 - '**EzOptimizer**'의 보다 편리한 사용을 위해 개발
 - 키워드(Keyword) 인식 방식
 - '**EzPreCompile**'용 입력 파일을 C++ 소스 파일로 변환
 - 특정 키워드에 의한 다중 최적화 블록도 처리 가능
 - DOS 및 Windows 버전 개발



10.1 제약 비선형 최적화 프로그램의 구성(2)



* NLP: Non-Linear Programming

* 1: 노명일, 협동 최적화 방법에 의한 다분야 최적화 기법에 관한 연구, 서울대학교 조선해양공학과 석사학위논문, 2000.2

* 2: Kyu-Yeul Lee, Seon-Ho Cho, Myung-II Roh, "An Efficient Global-Local Hybrid Optimization Method Using Design Sensitivity Analysis", International Journal of Vehicle Design(SCIE), Vol. 28, No. 4, pp.300~317, 2002.7

10.2 제약 비선형 최적화 프로그램의 사용 방법

1. 사용자가 작성하는 프로그램의 디렉토리에 “EzOptimizer.h”와 “EzOptimizerD.lib”(또는 “EzOptimizerR.lib”)를 복사
2. 사용자가 작성하는 프로그램의 Source Code 상단에 다음과 같은 내용을 추가

```
#include “EzOptimizer.h”
```
3. EzOptimizer를 사용하는 프로그램 작성
4. 사용자가 작성하는 프로그램의 프로젝트 파일에 “EzOptimizerD.lib”(또는 “EzOptimizerR.lib”)를 포함
5. 사용자가 작성한 프로그램을 컴파일, 링크 후 실행



10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예

- 수학 최적화 문제

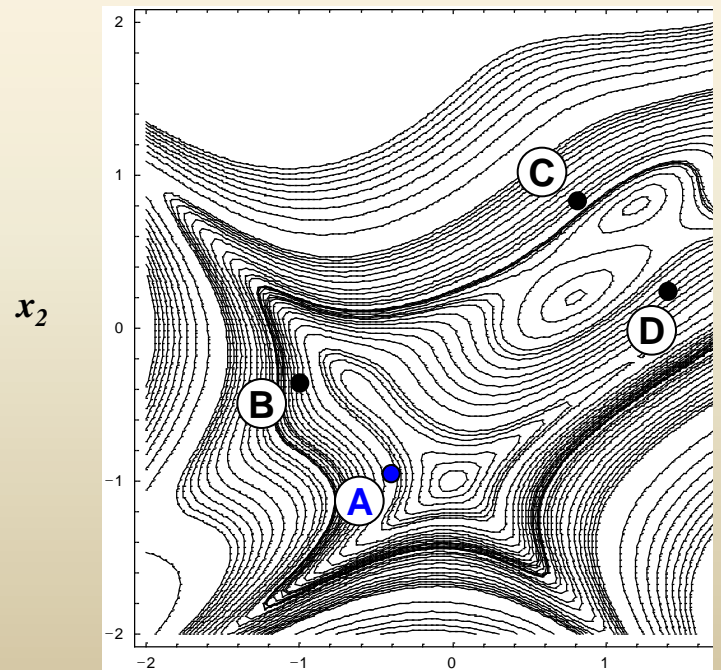
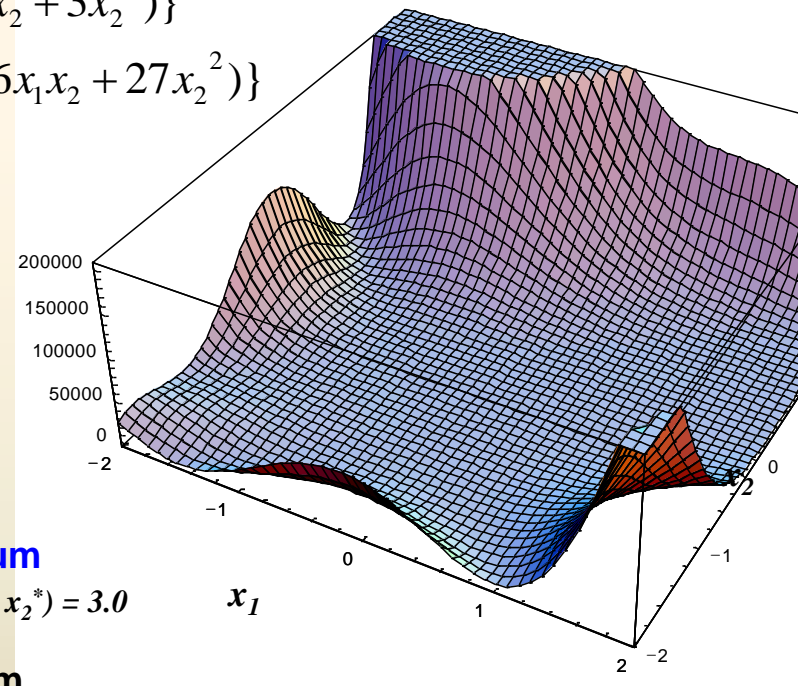
Goldstein-Price Function

Minimize

$$f(x_1, x_2) = \{1 + (x_1 + x_2 + 1)^2 \cdot (19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)\} \\ \cdot \{30 + (2x_1 - 3x_2)^2 \cdot (18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)\}$$

Subject to

$$g_1(x_1, x_2) = -2 - x_1 \leq 0, \quad g_2(x_1, x_2) = -2 - x_2 \leq 0, \\ g_3(x_1, x_2) = x_1 - 2 \leq 0, \quad g_4(x_1, x_2) = x_2 - 2 \leq 0$$



$f(x_1, x_2)$

A : Global Minimum

$$x_1^* = 0.0, x_2^* = -1.0, f(x_1^*, x_2^*) = 3.0$$

B : Local Minimum

$$x_1^* = -0.6, x_2^* = -0.4, f(x_1^*, x_2^*) = 30.0$$

C : Local Minimum

$$x_1^* = 1.2, x_2^* = 0.8, f(x_1^*, x_2^*) = 840.0$$

D : Local Minimum

$$x_1^* = 1.8, x_2^* = 0.2, f(x_1^*, x_2^*) = 84.0$$

10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예

- EzPreCompiler용 입력 파일 작성

\$\$ EzOptimizer Start

[Optimization Method]

MFD

[Print Option]

SMALL

[Design Variables]

x1, 0.0, -2.0, 2.0

◀ 제약 조건을 설계 변수에 대한 상·하한값으로 표현

x2, 0.0, -2.0, 2.0

[Objective Function]

MINIMIZE $f = (1.0 + \text{pow}(x1 + x2 + 1.0, 2.0) * (19.0 - 14.0 * x1 + 3.0 * \text{pow}(x1, 2.0) - 14.0 * x2 + 6.0 * x1 * x2 + 3.0 * \text{pow}(x2, 2.0))) * (30.0 + \text{pow}(2.0 * x1 - 3.0 * x2, 2.0) * (18.0 - 32.0 * x1 + 12.0 * \text{pow}(x1, 2.0) + 48.0 * x2 - 36.0 * x1 * x2 + 27.0 * \text{pow}(x2, 2.0)))$

\$\$ EzOptimizer End



10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예

- EzPreCompiler에 의해 자동 생성된 C++ 파일

```
EzOptimizerConfiguration MyOptimizerConfiguration00(MFD, SMALL, 2, 0, MINIMIZE);  
EzOptimizer MyOptimizer00(MyOptimizerConfiguration00);
```

```
int ResultFlag = 0;  
double* Design_Value = MyOptimizer00.GetDesignValueAddress();  
double* Lower_Value = MyOptimizer00.GetLowValueAddress();  
double* Upper_Value = MyOptimizer00.GetUpperValueAddress();  
double* Objective_Value = MyOptimizer00.GetObjectiveValueAddress();
```

```
Low_Value[ 0] = -2.000000; Design_Value[ 0] = 0.000000; Upper_Value[ 0] = 2.000000;  
Low_Value[ 1] = -2.000000; Design_Value[ 1] = 0.000000; Upper_Value[ 1] = 2.000000;
```

▶ 제약 조건을 설계 변수에 대한 상·하한값으로 표현

```
while ((ResultFlag = MyOptimizer00.Optimization()) == 1) {  
    *Objective_Value = (1.0+pow(Design_Value[0]+Design_Value[1]+1.0, 2.0)*(19.0-  
    14.0*Design_Value[0]+3.0*pow(Design_Value[0], 2.0)-  
    14.0*Design_Value[1]+6.0*Design_Value[0]*Design_Value[1] + 3.0 *pow(Design_Value[1], 2.0)))  
    * (30.0+pow(2.0*Design_Value[0]-3.0*Design_Value[1], 2.0)* (18.0-  
    32.0*Design_Value[0]+12.0*pow(Design_Value[0], 2.0)+48.0*Design_Value[1]-  
    36.0*Design_Value[0]*Design_Value[1]+27.0*pow(Design_Value[1], 2.0)));  
}  
  
if (ResultFlag == 0) { x1 = Design_Value[ 0]; x2 = Design_Value[ 1]; f = *Objective_Value; }  
else if (ResultFlag == -1) { MyOptimizer00.GetErrorMessage(); }
```



10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예

- EzOptimizer를 이용한 최적화 결과

최적화 알고리즘에 따른 결과 비교

	True Solution	MFD	MS	GA	HYBRID w/o Refine	HYBRID with Refine
x1	0.0000	-0.6000	0.0000	0.0081	0.0000	0.0000
x2	-1.0000	-0.4000	-1.0000	-1.0032	-1.0000	-1.0000
f	3.0000	30.0000	3.0000	3.0262	3.0000	3.0000
Iteration No	-	5	154	78	33	36
CPU Time(s)		0.03	0.78	1.23	0.73	0.75

Local Minimum ↑

↑
 개선된 유전 알고리즘으로부터 얻어진 근사 최적점에 대해
 local optimization을 수행함으로써 정확한 전역 최적해 도출

* MFD: Method of feasible directions, MS: Multi-start local optimization method, GA: Genetic algorithm, HYBRID: Global-local hybrid optimization method

10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예

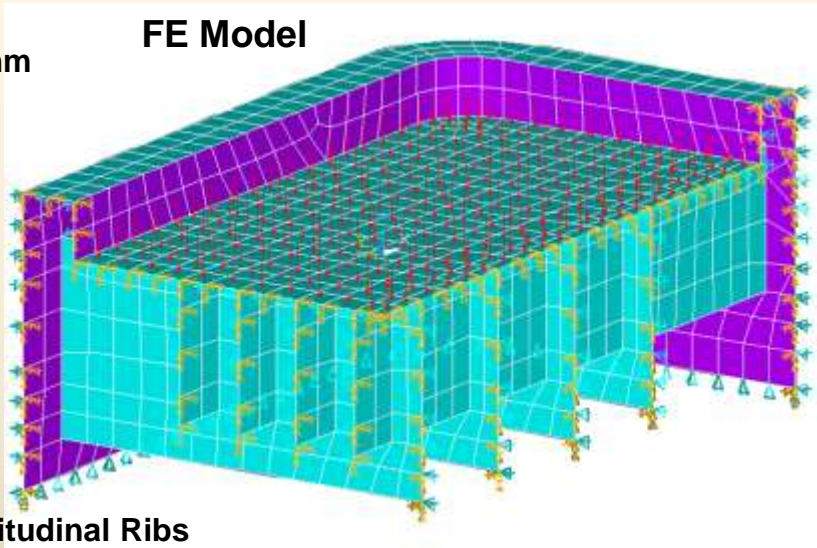
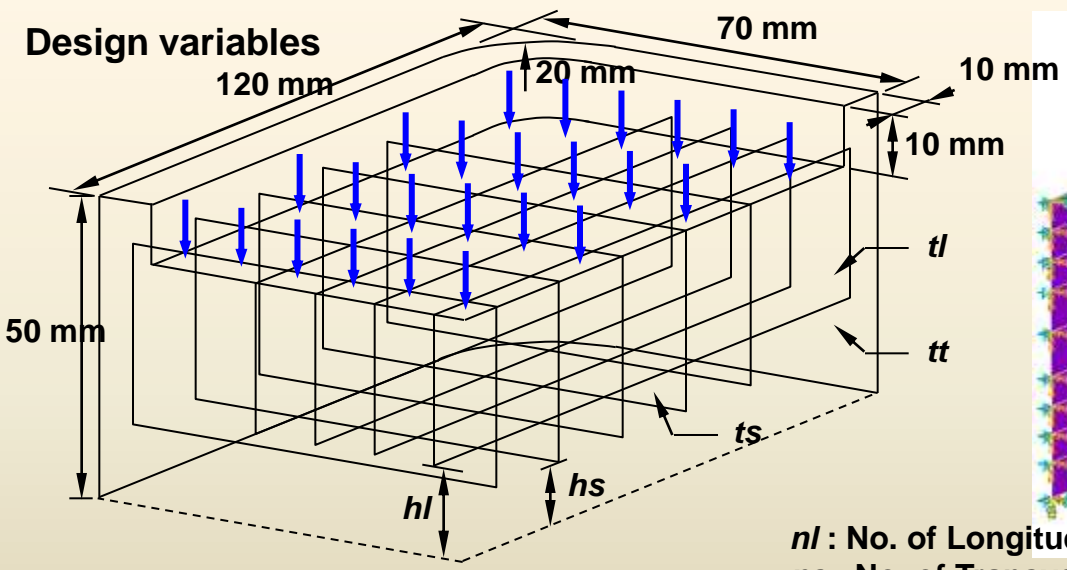
- Ribbed Tray 설계 문제

Minimize $Volume = f(nl, ns, tl, ts, tt, hl, hs)$

Objective Function

Subject to $\delta_{max} \leq 1.0(mm)$ $\sigma_{max} \leq 22.5(MPa)$

Constraints



- nl : No. of Longitudinal Ribs
- ns : No. of Transversal Ribs
- tl : Thickness of Longitudinal Ribs
- ts : Thickness of Transversal Ribs
- tt : Thickness of Tray
- hl : Gap at Bottom of Longitudinal Ribs
- hs : Gap at Bottom of Transversal Ribs

$E = 3.1 \times 10^3 MPa$

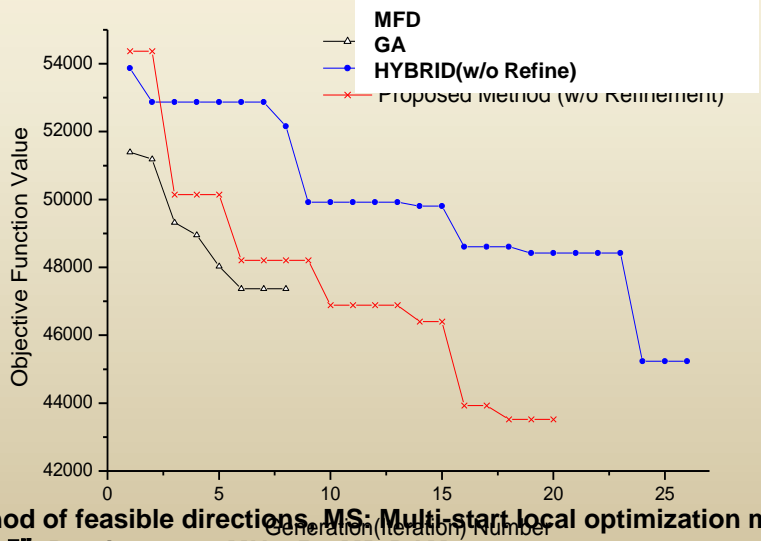
$\nu = 0.35$

Surface Load on Cavity = 0.15 MPa

* Kyu-Yeul Lee, Seon-Ho Cho, Myung-II Roh, "An Efficient Global-Local Hybrid Optimization Method Using Design Sensitivity Analysis", International Journal of Vehicle Design(SCIE), Vol. 28, No. 4, pp.300~317, 2002.7

10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예 - Ribbed Tray 문제에 대한 최적화 결과(1)

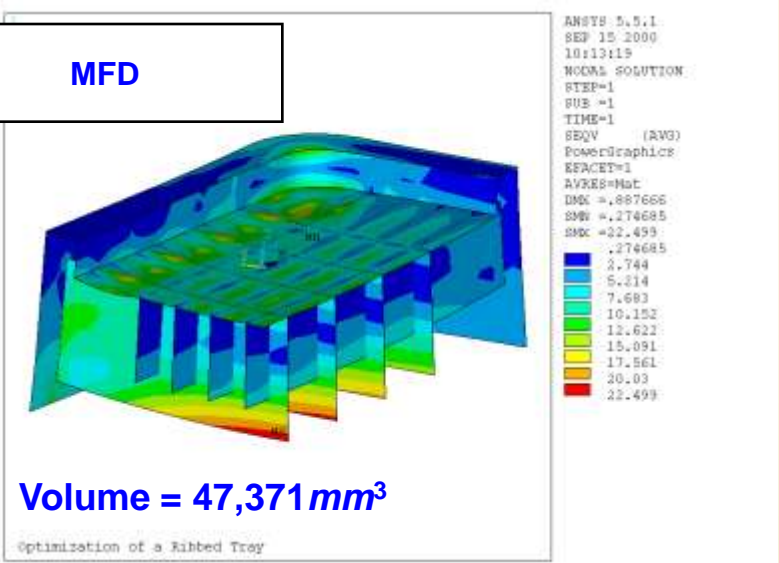
	Unit	MFD	GA	HYBRID	
				w/o Refine	with Refine
<i>Volume</i>	<i>mm³</i>	47,370.662	45,236.305	43,520.642	42,507.595
<i>nl</i>		4	3	4	4
<i>ns</i>		4	4	4	4
<i>tl</i>	<i>mm</i>	1.307021	1.199413	1.007820	0.995354
<i>ts</i>	<i>mm</i>	1.313513	1.355816	1.332356	1.327935
<i>tt</i>	<i>mm</i>	1.067905	1.156403	1.025415	1.008237
<i>hl</i>	<i>mm</i>	16.779161	16.434995	16.998045	18.129324
<i>hs</i>	<i>mm</i>	6.073858	5.619746	5.482893	5.389428
Iteration No	-	8	26	20	25
CPU Time	<i>sec</i>	2,420.38	13,091.02	19,767.95	20,973.59



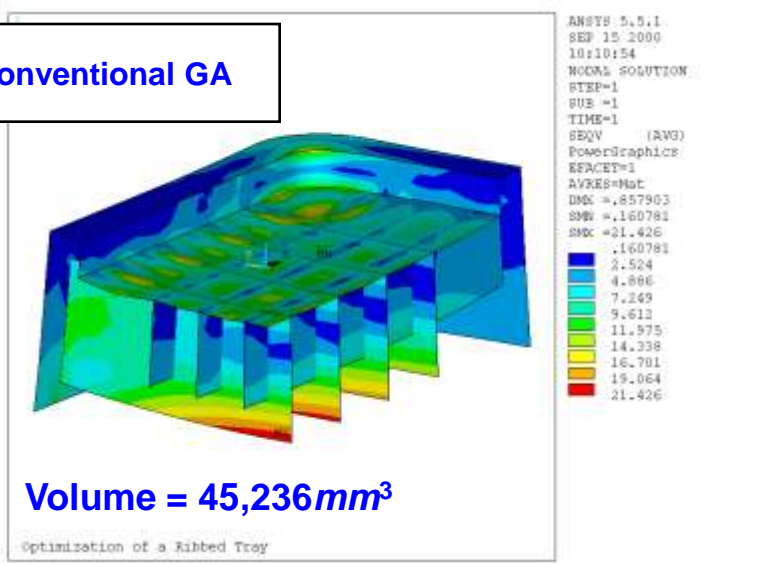
* MFD: Method of feasible directions, MS: Multi-start local optimization method, GA: Genetic algorithm, HYBRID: Global-local hybrid optimization method

10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예 - Ribbed Tray 문제에 대한 최적화 결과(2)

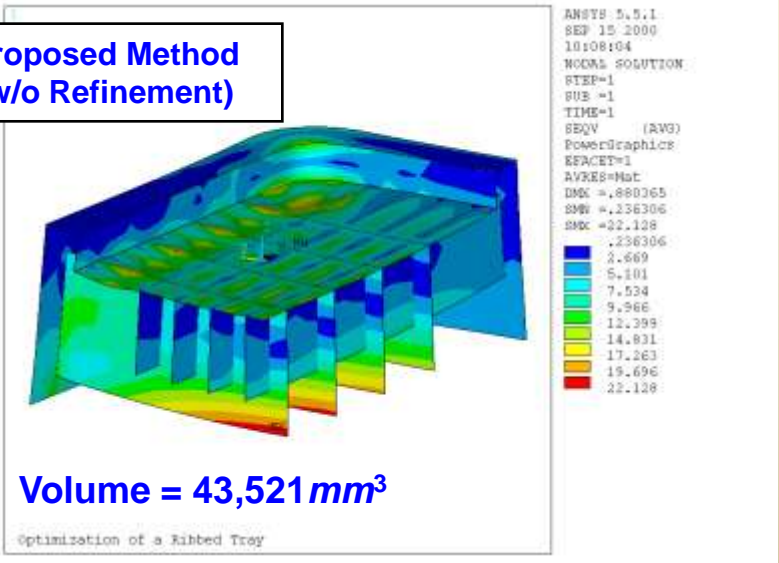
MFD



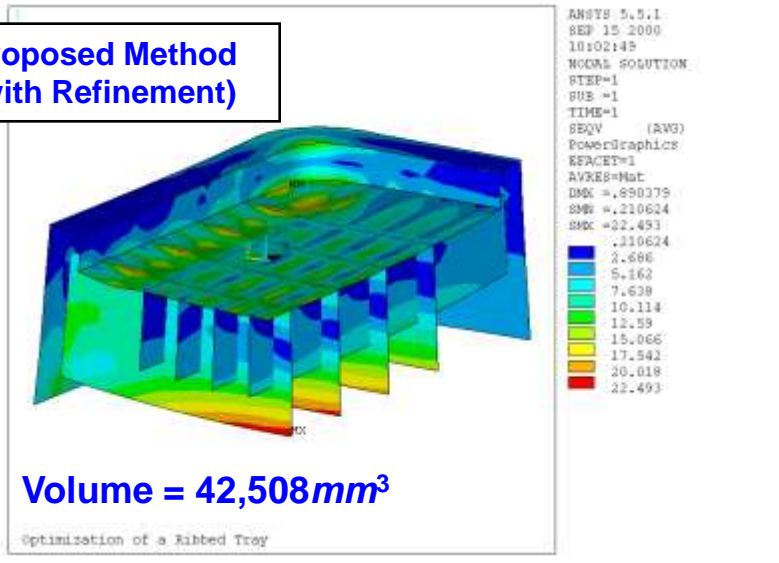
Conventional GA



Proposed Method (w/o Refinement)



Proposed Method (with Refinement)



10.3 제약 비선형 최적화 프로그램을 이용한 최적 설계 예

- 건조비를 최소화 하는 중앙 단면 설계

Find $x_i, i = 1, \dots, 16$

Minimize Building Cost

Subject to

$$t_{i,\min} - x_i \leq 0, \quad i = 6, \dots, 16$$

: minimum plate thickness

$$Z_{\min}^{deck} - Z^{deck} \leq 0$$

: minimum section modulus at deck

$$Z_{\min}^{bottom} - Z^{bottom} \leq 0$$

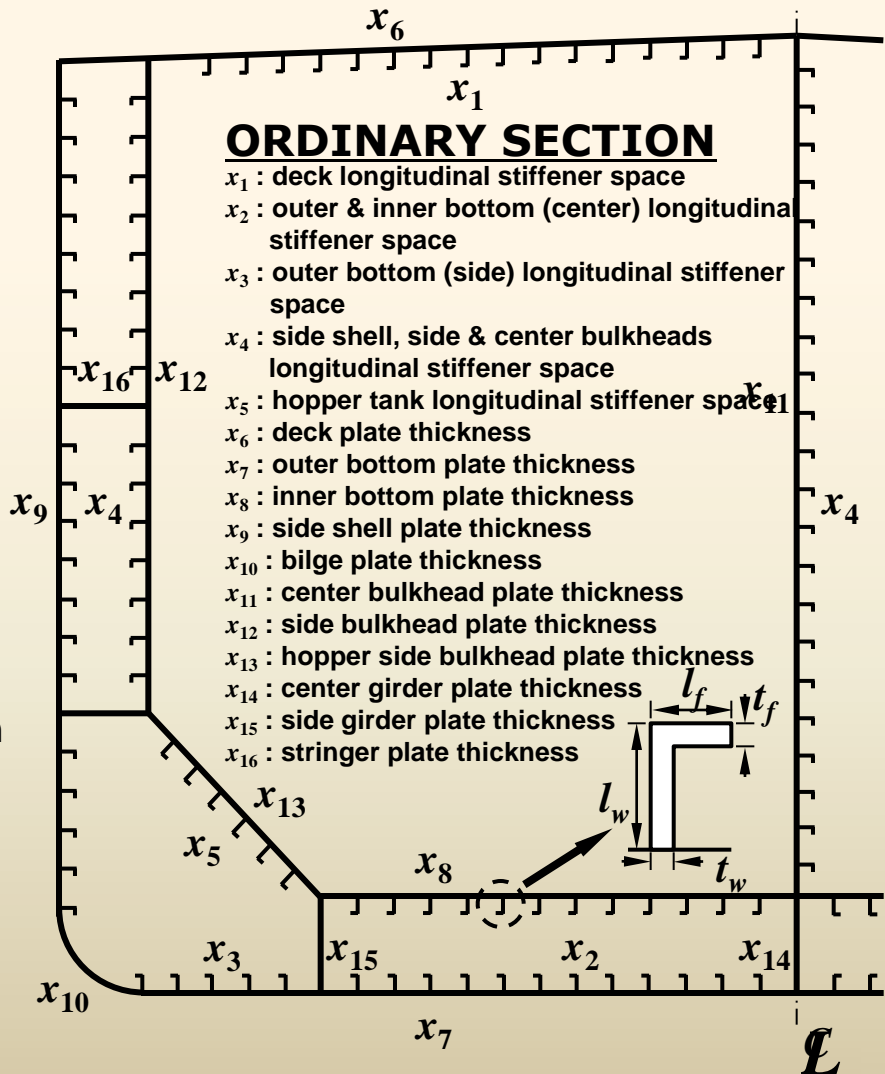
: minimum section modulus at bottom

$$\sigma^{deck} - \eta^{deck} \sigma_c^{deck} \leq 0$$

: critical buckling stress at deck

$$\sigma^{bottom} - \eta^{bottom} \sigma_c^{bottom} \leq 0$$

: critical buckling stress at bottom



* Kyu-Yeul Lee, Myung-II Roh, "An Efficient Genetic Algorithm Using Gradient Information for Ship Structural Design Optimization", Journal of Ship Technology Research, Vol. 48, No. 4, pp.161-170, 2001.

건조비를 최소화 하는 중앙 단면 설계 문제의 수학적 정식화(1)

■ 목적 함수

■ Midship Section의 단위길이당 건조비

$$\text{Building Cost} = \text{Material Cost} + \text{Labour Cost} \text{ [\$ / m]}$$

$$\text{Material Cost} = \text{Weight [ton]} \times \text{Unit Material Cost [\$ / ton]}$$

: midship section의 단위 길이당 재료비

$$\text{Labour Cost} = \text{Welding Cost} + \text{Painting Cost}$$

: midship section의 단위 길이당 인건비

$$\text{Weight} = \rho \cdot A \text{ [ton]}$$

: midship section의 단위 길이당 중량

$$\text{Welding Cost} = 2 \cdot F_w \cdot N \text{ [\$ / m]}$$

: midship section의 단위 길이당 용접비

$$\text{Painting Cost} = F_p \cdot G \text{ [\$ / m]}$$

: midship section의 단위 길이당 도장비

ρ : steel's mass density, 7.85[ton/m³]
 A : area of the midship section
 F_w : welding cost per unit length
 N : number of the plates and stiffeners
 F_p : painting cost per unit area
 G : summation of the girth length of the plates and stiffeners



■ 제약 조건

■ Minimum Plate Thickness에 관한 조건

$$t_{i,\min} - x_i \leq 0, \quad i = 6, \dots, 16$$

$$t_{i,\min} = \max \left(\frac{15.8 \cdot s_i \cdot \sqrt{p_i}}{\sqrt{\sigma_i}} + t_{k,c_i} + \frac{k_i \cdot L}{\sqrt{f_i}} + t_k \right) [mm]$$

■ Minimum Section Modulus에 관한 조건(Deck, Bottom)

$$Z_{\min}^{deck} - Z^{deck} \leq 0 \quad Z_{\min}^{bottom} - Z^{bottom} \leq 0$$

$$Z_{\min}^{deck} = \max \left(\frac{C_W \cdot L^2 \cdot B \cdot (C_B + 0.7)}{1.39}, \frac{|M_S + M_W|}{175 \cdot 1.39} \right) [cm^3]$$

$$Z_{\min}^{bottom} = \max \left(\frac{C_W \cdot L^2 \cdot B \cdot (C_B + 0.7)}{1.28}, \frac{|M_S + M_W|}{175 \cdot 1.28} \right) [cm^3]$$



건조비를 최소화 하는 중앙 단면 설계 문제의 수학적 정식화(3)

■ Critical Buckling Stress에 관한 조건(Deck, Bottom)

$$\sigma^{deck} - \eta^{deck} \sigma_c^{deck} \leq 0 \quad \sigma^{bottom} - \eta^{bottom} \sigma_c^{bottom} \leq 0$$

$$\sigma_c^{deck} = \begin{cases} \sigma_{el}^{deck} & \text{when } \sigma_{el}^{deck} < 177.5 \\ 355 \cdot \left(1 - \frac{355 \cdot \sigma_{el}^{deck}}{4} \right) & \text{when } \sigma_{el}^{deck} > 177.5 \end{cases}$$

$$\sigma_c^{bottom} = \begin{cases} \sigma_{el}^{bottom} & \text{when } \sigma_{el}^{bottom} < 117.5 \\ 235 \cdot \left(1 - \frac{355 \cdot \sigma_{el}^{bottom}}{4} \right) & \text{when } \sigma_{el}^{bottom} > 117.5 \end{cases}$$

$$\sigma^{deck} = \frac{M_S + M_W}{I_N} \cdot z_n^{deck} \cdot 10^5 \text{ [N/mm}^2\text{]}$$

$$\sigma^{bottom} = \frac{M_S + M_W}{I_N} \cdot z_n^{bottom} \cdot 10^5 \text{ [N/mm}^2\text{]}$$



중앙 단면 설계 문제에 대한 최적화 결과

	Unit	Actual Ship	MFD	MS	GA	HYBRID	
						w/o Refine	with Refine
<i>Building Cost</i>	<i>\$/m</i>	-	21,035.254748	20,637.828634	20,597.330090	20,422.478135	20,350.286893
x_1	<i>mm</i>	800.0	787.038274	811.324938	780.000000	810.000000	810.3701321
x_2	<i>mm</i>	800.0	762.891023	799.038243	750.000000	800.000000	800.1282732
x_3	<i>mm</i>	780.0	743.313979	787.034954	770.000000	790.000000	789.0923943
x_4	<i>mm</i>	835.0	814.142029	833.909455	820.000000	830.000000	834.838424
x_5	<i>mm</i>	770.0	756.434513	772.349435	790.000000	780.000000	780.002092
x_6	<i>mm</i>	16.5	16.983723	16.203495	16.000000	16.000000	16.390923
x_7	<i>mm</i>	16.0	16.829142	16.043803	16.500000	16.000000	15.989044
x_8	<i>mm</i>	15.5	16.020913	15.390394	16.000000	15.500000	15.432091
x_9	<i>mm</i>	17.0	17.329843	17.039439	16.500000	16.500000	17.139433
x_{10}	<i>mm</i>	14.5	15.001923	14.324335	15.000000	15.000000	14.780908
x_{11}	<i>mm</i>	13.5	14.192834	14.240495	14.000000	13.500000	13.5113314
x_{12}	<i>mm</i>	14.5	15.123051	15.403945	14.500000	14.500000	14.500130
x_{13}	<i>mm</i>	17.0	16.902832	16.849387	16.500000	17.000000	17.010902
x_{14}	<i>mm</i>	14.0	14.784034	14.739454	15.500000	14.500000	14.309324
x_{15}	<i>mm</i>	14.0	15.129430	14.448504	15.500000	14.500000	14.588917
x_{16}	<i>mm</i>	14.5	14.824045	14.940584	15.000000	15.000000	14.789992
Iteration No	-	-	8	912	93	64	70
CPU Time	<i>sec</i>	-	2.90	293.28	272.91	265.06	267.92

Appendix A. Engineering mathematics Review

- Directional Derivative & Gradient Vector
- Linear Algebra
- Inverse of a matrix.
Gauss-Jordan Elimination
- Quadratic Forms



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Engineering mathematics Review

- Directional Derivative & Gradient Vector



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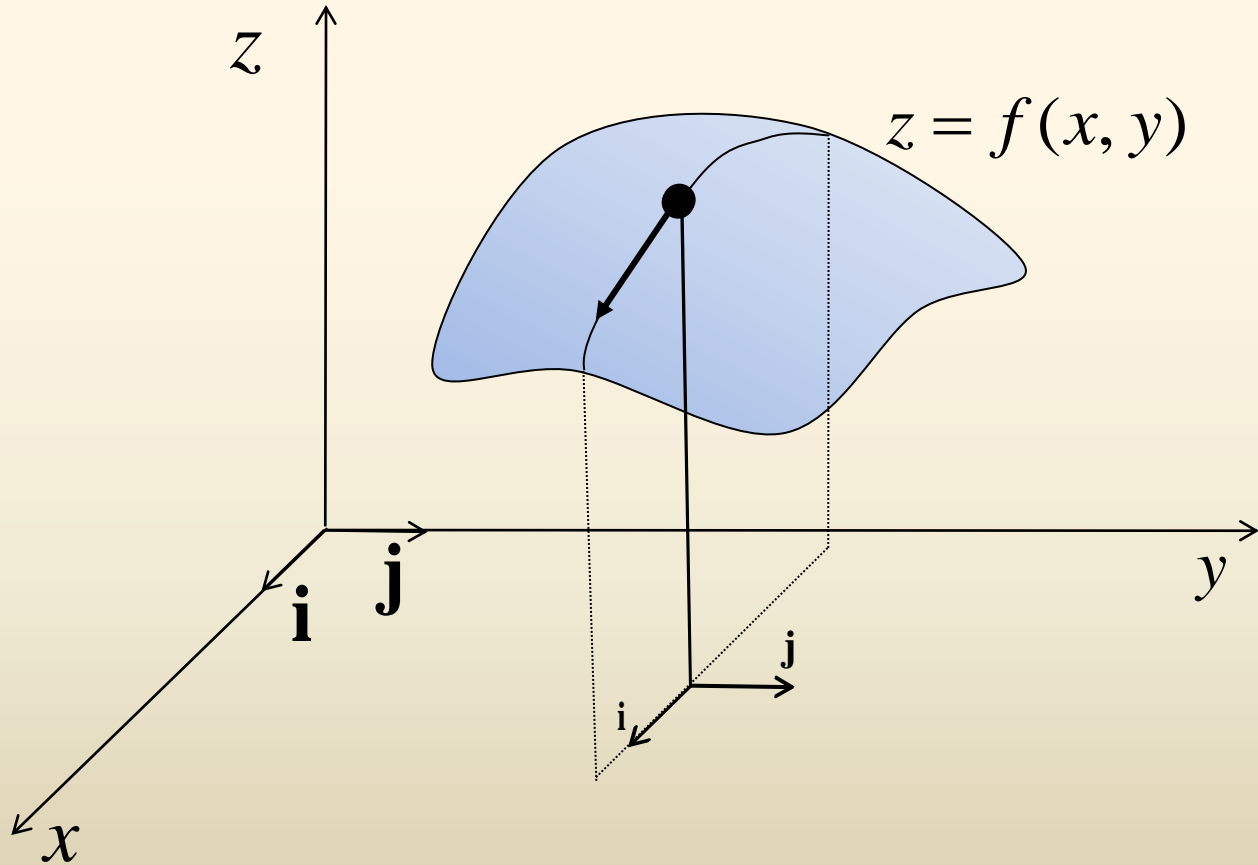


Advanced Ship Design Automation Lab.
<http://asdal.snu.ac.kr>



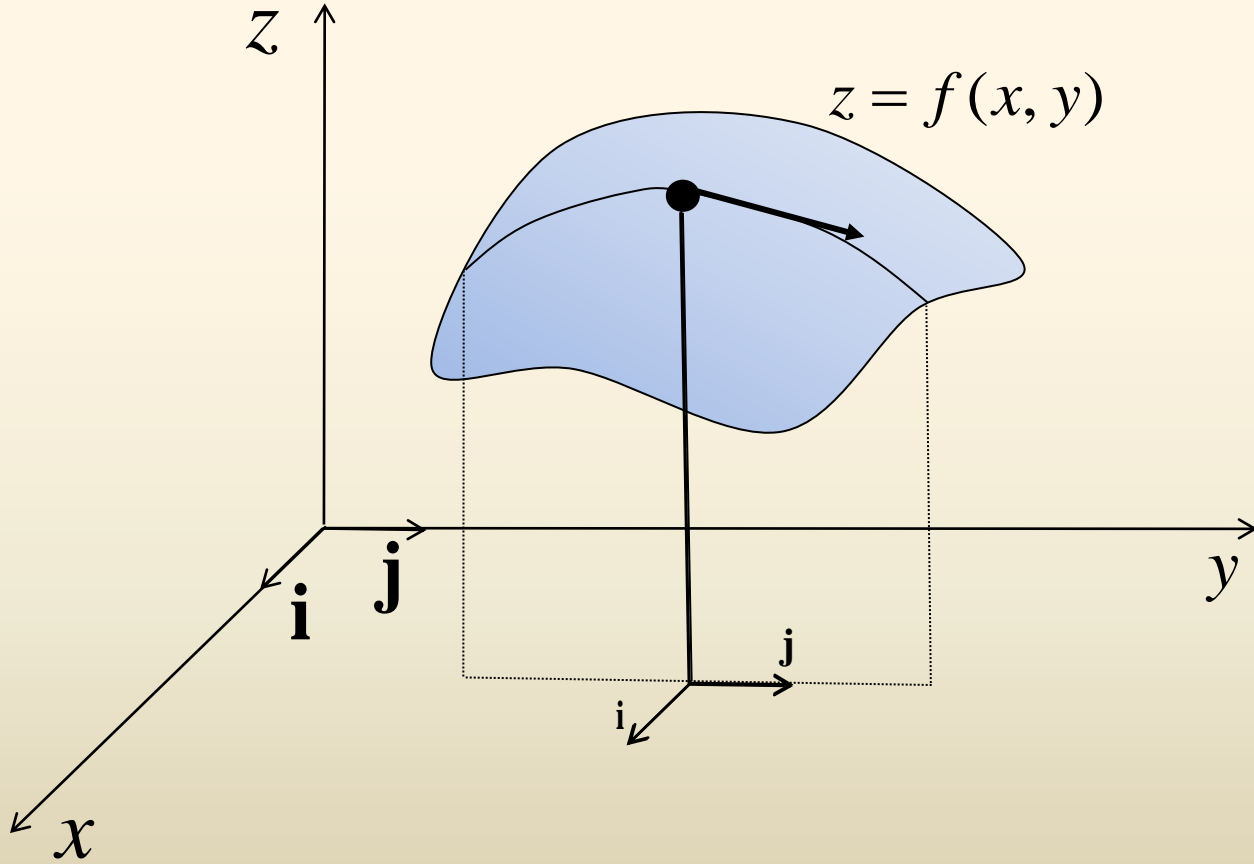
Directional Derivative

$\frac{\partial f}{\partial x}$:Rate of change of f in the \mathbf{i} -direction



Directional Derivative

$\frac{\partial f}{\partial y}$:Rate of change of f in the \mathbf{j} -direction

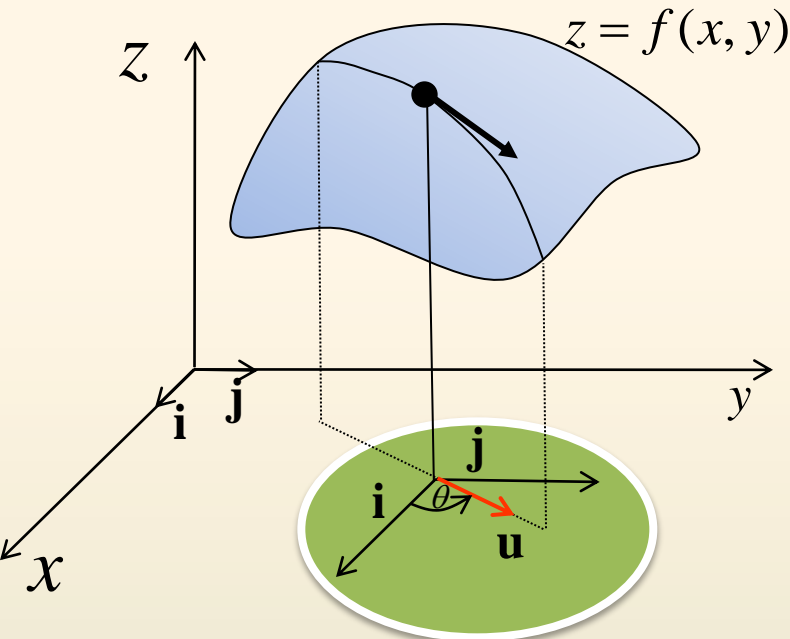


$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Directional Derivative



The rate of change of f in the direction given by the vector \mathbf{u} : $D_{\mathbf{u}}f$



The rate of change of f in the direction of \mathbf{X} : $\frac{\partial f}{\partial x}$

The component of $\frac{\partial f}{\partial x}$ in the direction of \mathbf{u} : $\frac{\partial f}{\partial x} \cos \theta$

The rate of change of f in the direction of \mathbf{Y} : $\frac{\partial f}{\partial y}$

The component of $\frac{\partial f}{\partial y}$ in the direction of \mathbf{u} : $\frac{\partial f}{\partial y} \cos(\frac{\pi}{2} - \theta)$
 $= \frac{\partial f}{\partial y} \sin \theta$

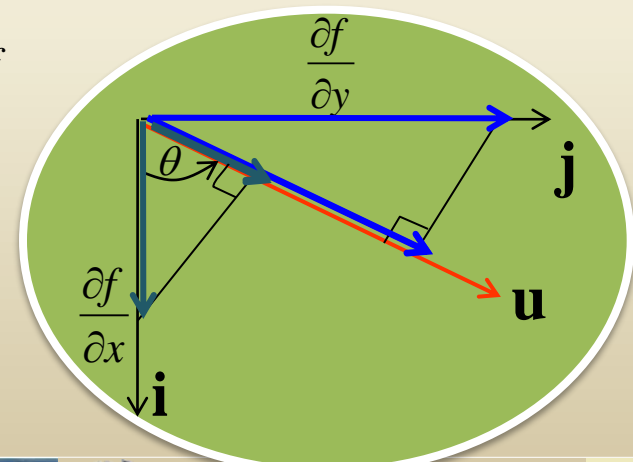
The rate of change of f in the direction given by the vector \mathbf{u} : $D_{\mathbf{u}}f$

=

+

$$= \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta$$

$$= \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \right) \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = \nabla f \cdot \mathbf{u}$$



Directional Derivative

■ Example 2

Gradient at a Point

If $F(x,y,z)=xy^2+3x^2-z^3$, find $\nabla F(x,y,z)$ at $(2,-1,4)$.

Solution)

$$\begin{aligned}\nabla F(x, y, z) &= \frac{\partial}{\partial x} (xy^2 + 3x^2 - z^3) \mathbf{i} \\ &\quad + \frac{\partial}{\partial y} (xy^2 + 3x^2 - z^3) \mathbf{j} \\ &\quad + \frac{\partial}{\partial z} (xy^2 + 3x^2 - z^3) \mathbf{k} \\ &= (y^2 + 6x) \mathbf{i} + 2xy \mathbf{j} - 3z^2 \mathbf{k}\end{aligned}$$

$$\nabla F(2,-1,4) = 13\mathbf{i} - 4\mathbf{j} - 48\mathbf{k}$$



Directional Derivative

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

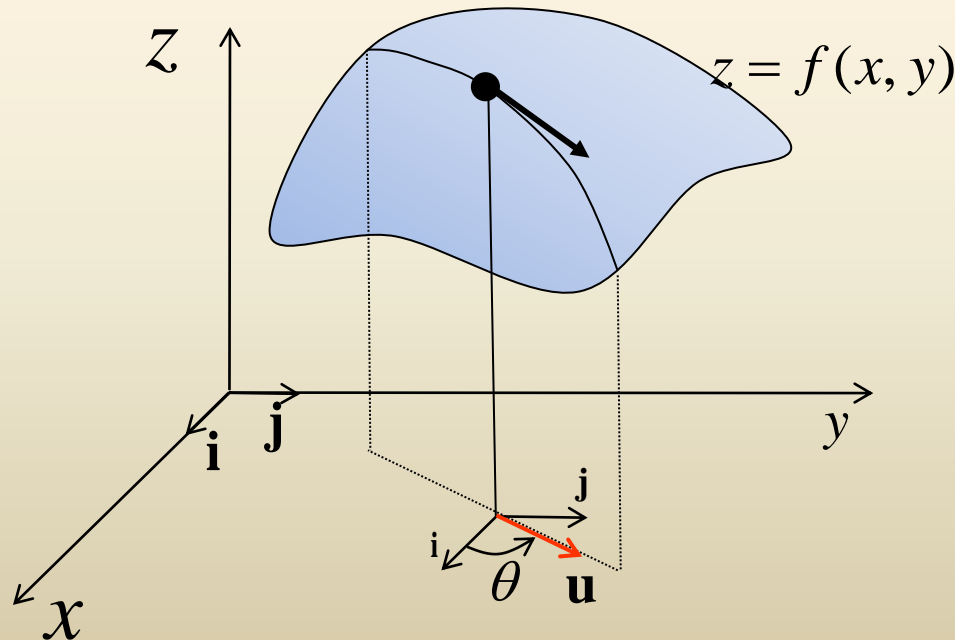
Theorem 9.6

Computing a Directional Derivative

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= [f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j}] \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \\ &= \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$



Directional Derivative

■ Example 3

Directional Derivative

Find the directional derivative of $f(x,y)=2x^2y^3+6xy$ at $(1,1)$ in the direction of a unit vector whose angle with the positive x-axis is $\pi/6$.

Solution)

$$\begin{aligned}\nabla f(x, y) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ &= (4xy^3 + 6y)\mathbf{i} + (6x^2y^2 + 6x)\mathbf{j}\end{aligned}$$

$$\nabla f(1,1) = 10\mathbf{i} + 12\mathbf{j}$$

$$\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} = \cos \frac{\pi}{6} \mathbf{i} + \sin \frac{\pi}{6} \mathbf{j}$$

$$\begin{aligned}D_{\mathbf{u}}f(1,1) &= \nabla f(1,1) \cdot \mathbf{u} \\ &= (10\mathbf{i} + 12\mathbf{j}) \cdot \left(\frac{\sqrt{3}}{2} \mathbf{i} + \frac{1}{2} \mathbf{j} \right) \\ &= 5\sqrt{3} + 6\end{aligned}$$



Directional Derivative

■ Example 4

Directional Derivative

Consider the plane that is perpendicular to the xy -plane and passes through the points $P(2,1)$ and $Q(3,2)$. What is the slope of the tangent line to the curve on intersection of this plane with the surface $f(x,y)=4x^2+y^2$ at $(2,1,17)$ in the direction of Q ?

$$\overrightarrow{PQ} = \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}}f(2,1) &= \nabla f(2,1) \bullet \mathbf{u} \\ &= (16\mathbf{i} + 2\mathbf{j}) \bullet \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \right) \\ &= 9\sqrt{2} \end{aligned}$$

Solution)

$$f(x, y) = 4x^2 + y^2$$

$$\nabla f(x, y) = 8x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f(2,1) = 16\mathbf{i} + 2\mathbf{j}$$



Directional Derivative

■ Example 5

Directional Derivative

Find the directional derivative of $F(x,y,z)=xy^2-4x^2y+z^2$ at $(1,-1,2)$ in the direction of $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

Solution)

$$\begin{aligned}f(x, y, z) &= xy^2 - 4x^2y + z^2 \\ \nabla f(x, y, z) &= \frac{\partial}{\partial x}(xy^2 - 4x^2y + z)\mathbf{i} \\ &\quad + \frac{\partial}{\partial y}(xy^2 - 4x^2y + z)\mathbf{j} \\ &\quad + \frac{\partial}{\partial z}(xy^2 - 4x^2y + z)\mathbf{k} \\ &= (y^2 - 8xy)\mathbf{i} + (2xy - 4x^2)\mathbf{j} + 2z\mathbf{k} \\ \nabla f(1, -1, 2) &= 9\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}\end{aligned}$$

$$\|6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\| = 7$$

$$\mathbf{u} = \frac{1}{7} \cdot (6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$$

$$\begin{aligned}D_{\mathbf{u}}F(1, -1, 2) &= (9\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}) \cdot \left(\frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k} \right) \\ &= \frac{54}{7}\end{aligned}$$



Directional Derivative

∇f Points in the direction of maximum increase of f at P

The rate of change of f in the direction given by the vector \mathbf{u} :

$$D_{\mathbf{u}}f(x, y) = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = \nabla f(x, y) \cdot \mathbf{u}$$

$$D_{\mathbf{u}}f = |\nabla f| |\mathbf{u}| \cos \phi = |\nabla f| \cos \phi, \quad \phi: \text{angle between } \nabla f \text{ and } \mathbf{u}$$

$$-1 \leq \cos \phi \leq 1$$

The maximum value of $D_{\mathbf{u}}f \Rightarrow D_{\mathbf{u}}f = |\nabla f|$, When $\cos \phi = 1, \phi = 0$

\mathbf{u} has the same direction of ∇f

$\Rightarrow \nabla f$ is the direction of maximum increase of f at P

$-\nabla f$ is the direction of maximum decrease of f at P



Directional Derivative

■ Example 6

Max/Min of Directional Derivative

In Example 5 the maximum value of the directional derivative at F at $(1,-1,2)$ is

$$\|\nabla F(1,-1,2)\| = \sqrt{133}.$$

The minimum value of $D_u F(1,-1,2)$ is then

$$-\sqrt{133}.$$



Directional Derivative

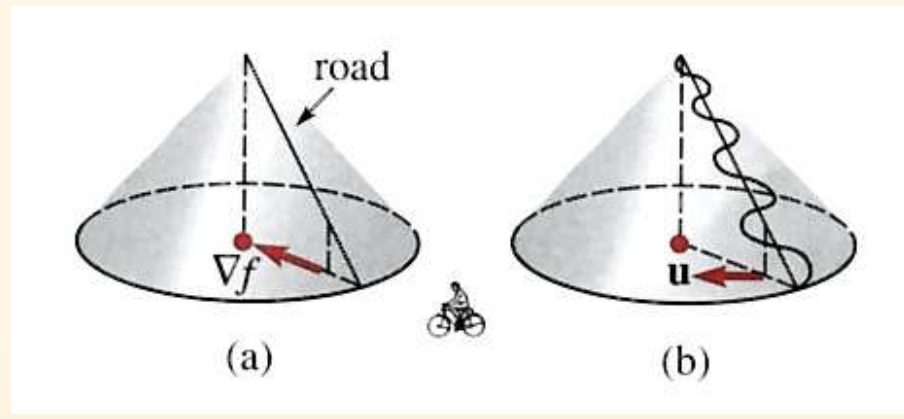
Example 7 Direction of Steepest Ascent

Each year in Los Angeles there is a bicycle race up to the top of a hill by a road known to be the steepest in the city. To understand why a bicyclist with a modicum of sanity will zigzag up the road, let us suppose the graph of

shown in Figure (a) is a mathematical model of the hill. The gradient of f is $f(x, y) = 4 - \frac{2}{3}\sqrt{x^2 + y^2}$, $0 \leq z \leq 4$,

where $\mathbf{r} = -x\mathbf{i} - y\mathbf{j}$ is a vector pointing to the center of the circular base.

$$\nabla f(x, y) = \frac{2}{3} \left[\frac{-x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{-y}{\sqrt{x^2 + y^2}} \mathbf{j} \right] = \frac{2/3}{\sqrt{x^2 + y^2}} \mathbf{r}$$



Thus the steepest ascent up the hill is a straight road whose projection in the xy -plane is a radius of the circular base. Since $D_{\mathbf{u}}f = \text{comp}_{\mathbf{u}} \nabla f$, a bicyclist will zigzag, or seek a direction \mathbf{u} other than ∇f , in order to reduce this component.



Directional Derivative

■ Example 8

Direction to Cool Off Fastest

The temperature in a rectangular box is approximated by

$$T(x, y, z) = xyz(1-x)(2-y)(3-z)$$

$$0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3.$$

If a mosquito is located at $(\frac{1}{2}, 1, 1)$, in which direction should it fly to cool off as rapidly as possible?

Solution)

$$\begin{aligned}\nabla T(x, y, z) &= \frac{\partial T(x, y, z)}{\partial x} \mathbf{i} + \frac{\partial T(x, y, z)}{\partial y} \mathbf{j} + \frac{\partial T(x, y, z)}{\partial z} \mathbf{k} \\ &= yz(2-y)(3-z)(1-2x) \mathbf{i} \\ &\quad + xz(1-x)(3-z)(2-2y) \mathbf{j} \\ &\quad + xy(1-x)(2-y)(3-2z) \mathbf{k}\end{aligned}$$

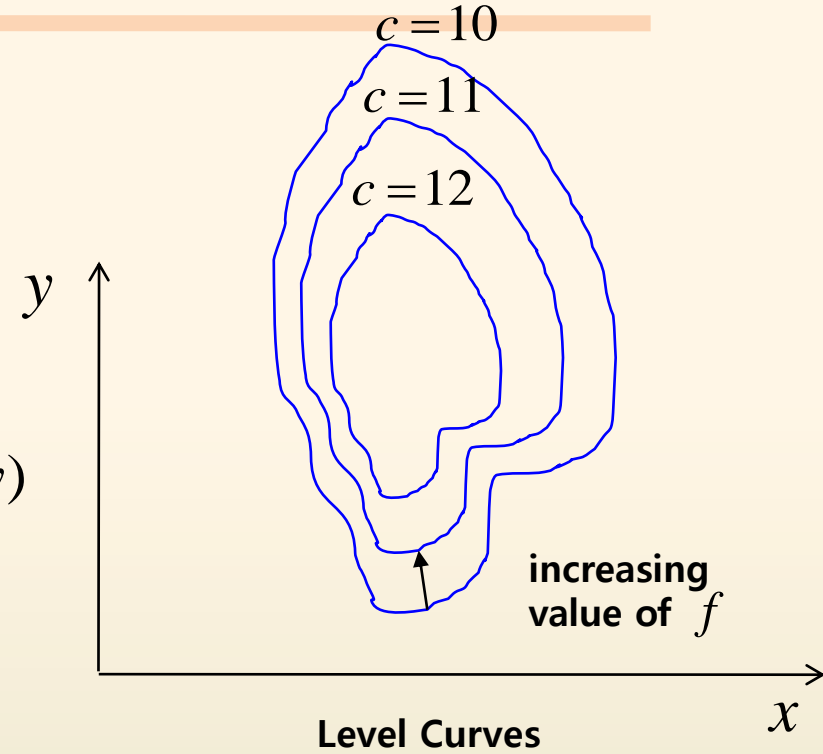
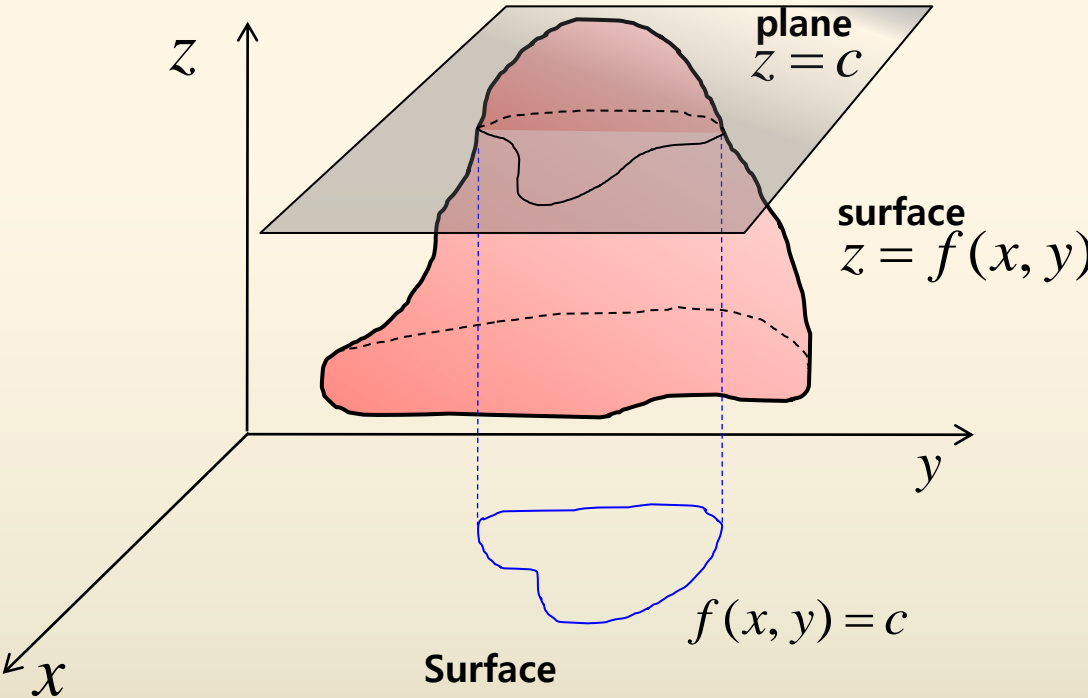
$$\nabla T\left(\frac{1}{2}, 1, 1\right) = \frac{1}{4} \mathbf{k}$$

To cool off most rapidly, the mosquito should fly in the direction of $-\frac{1}{4}\mathbf{k}$; that is, it should dive for the floor of the box, where the temperature is $T(x, y, 0) = 0$.



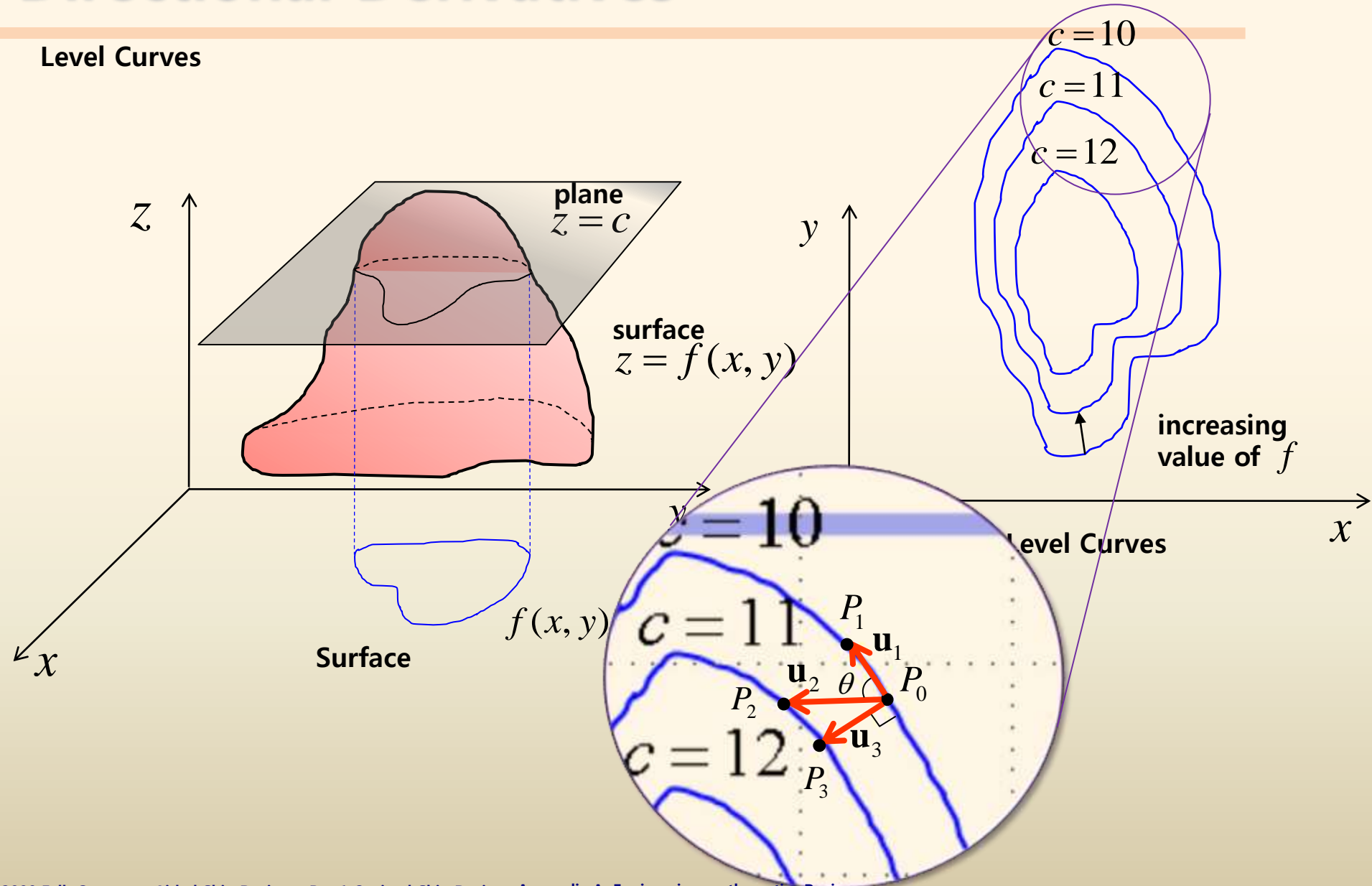
Directional Derivatives

Level Curves



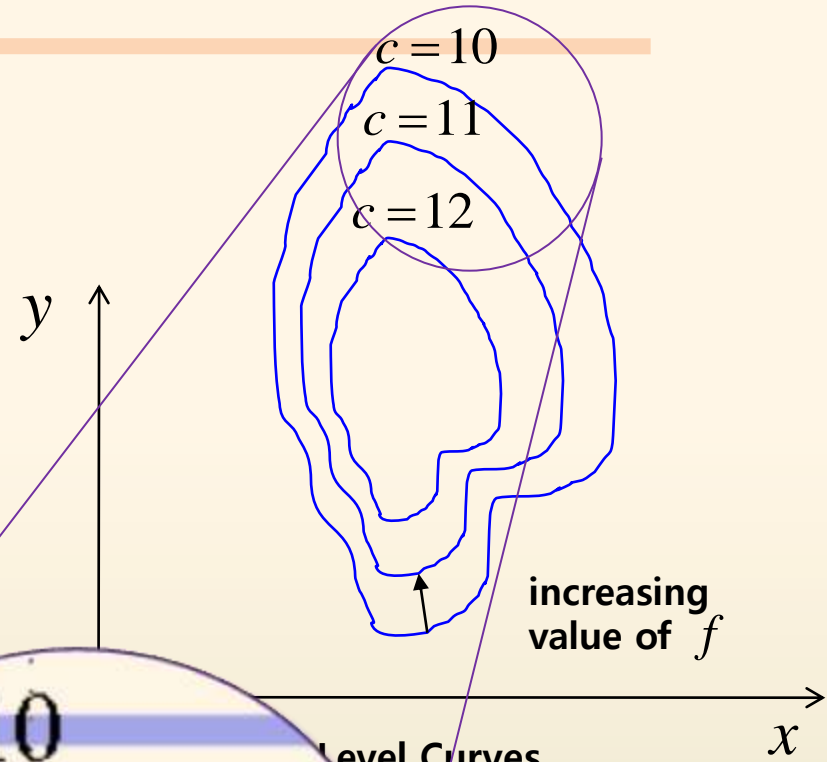
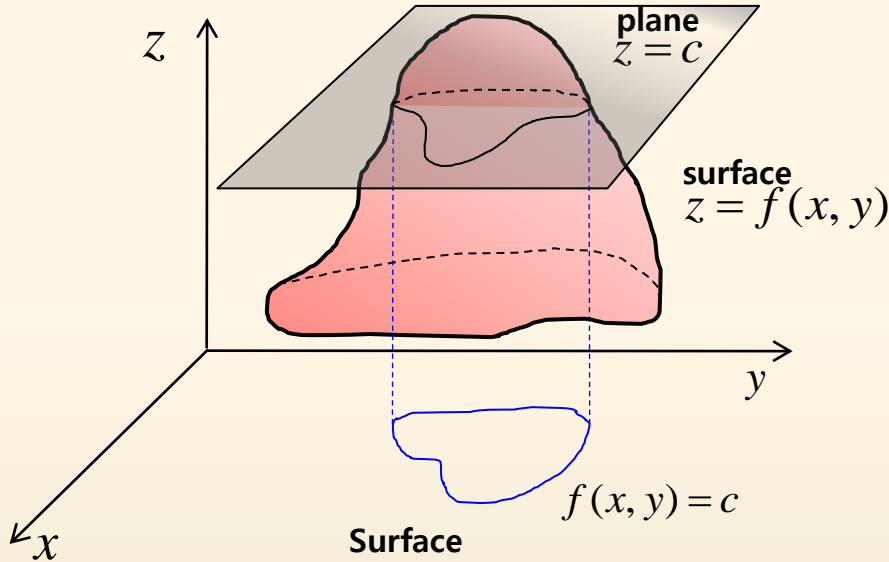
Directional Derivatives

Level Curves



Directional Derivatives

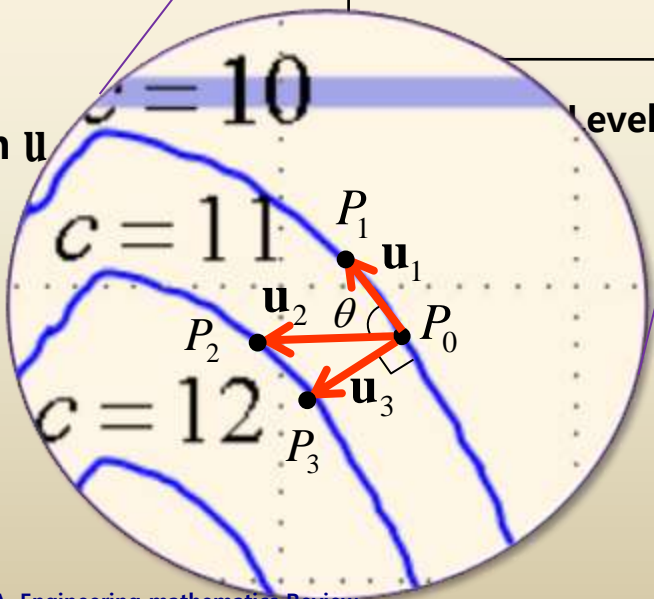
Level Curves



The rate of change of f in the direction \mathbf{u} given by the vector: $D_{\mathbf{u}}f(x, y)$

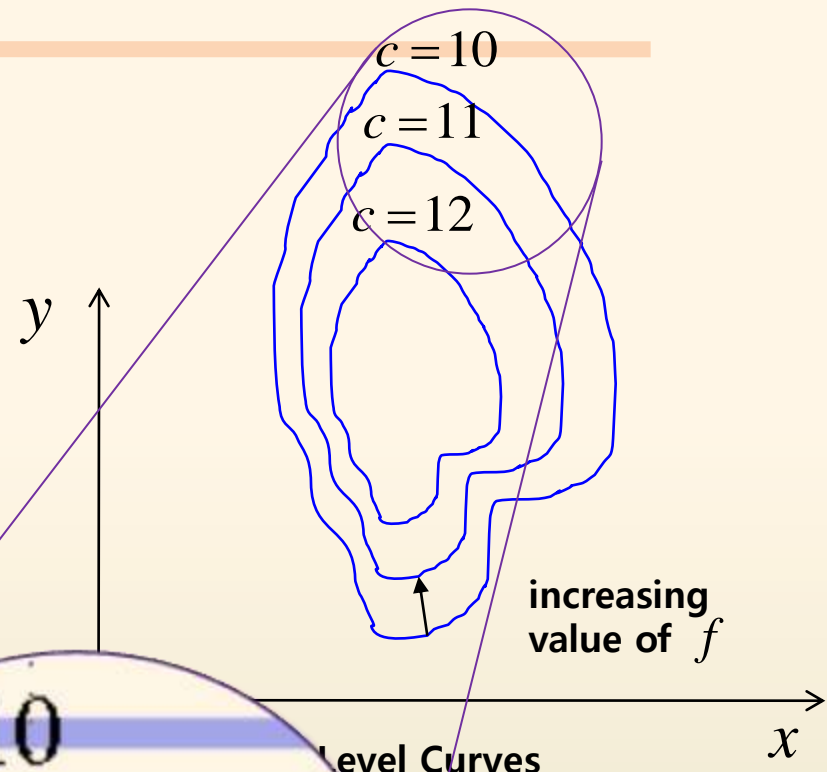
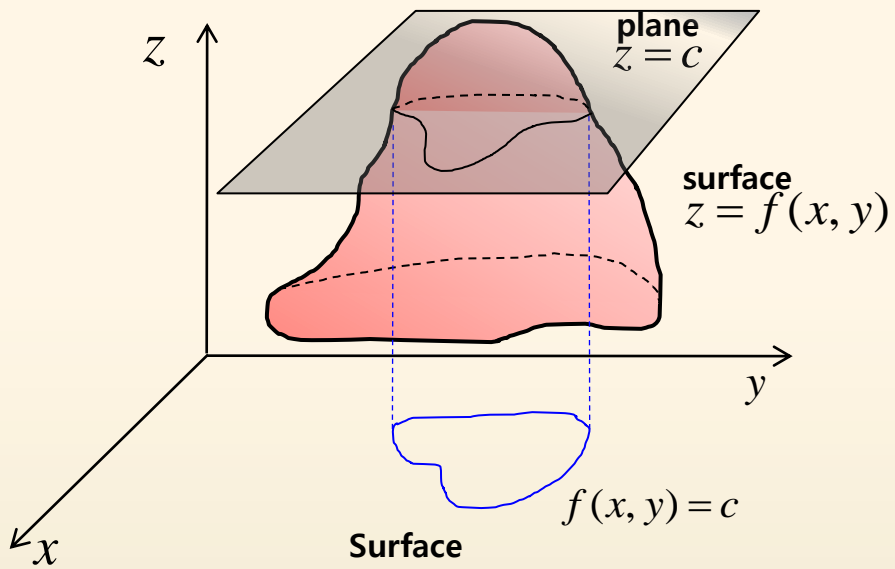
? $D_{\mathbf{u}_1}f(x, y) = 0$

$$\therefore \frac{f(P_1) - f(P_0)}{P_1P_0} = \frac{10 - 10}{P_1P_0} = 0$$



Directional Derivatives

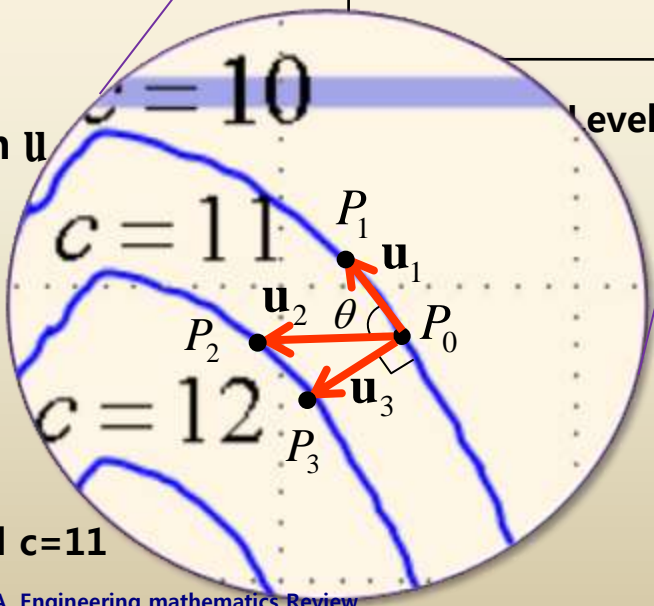
Level Curves



The rate of change of f in the direction \mathbf{u} given by the vector: $D_{\mathbf{u}}f(x, y)$

$D_{\mathbf{u}_2}f(x, y)$ Vs. $D_{\mathbf{u}_3}f(x, y)$

$$\frac{f(P_2) - f(P_0)}{\overline{P_2P_0}} < \frac{f(P_3) - f(P_0)}{\overline{P_3P_0}}$$

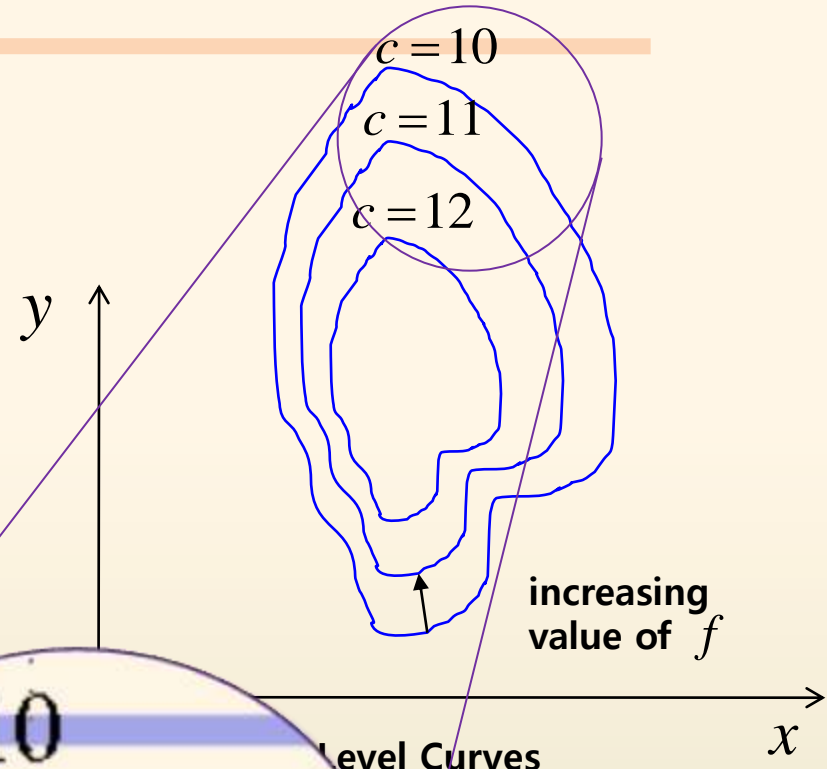
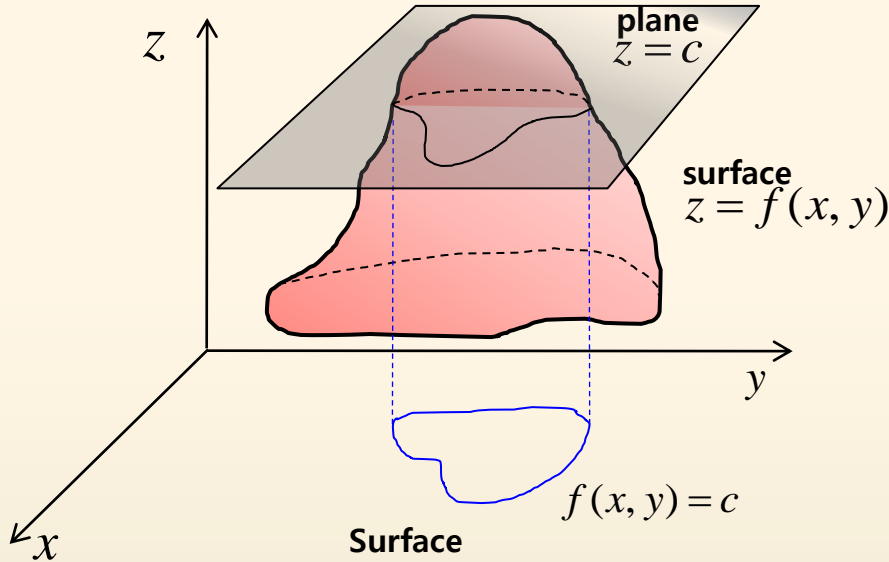


$\therefore \overline{P_3P_0}$ is the shortest path between $c=10$ and $c=11$



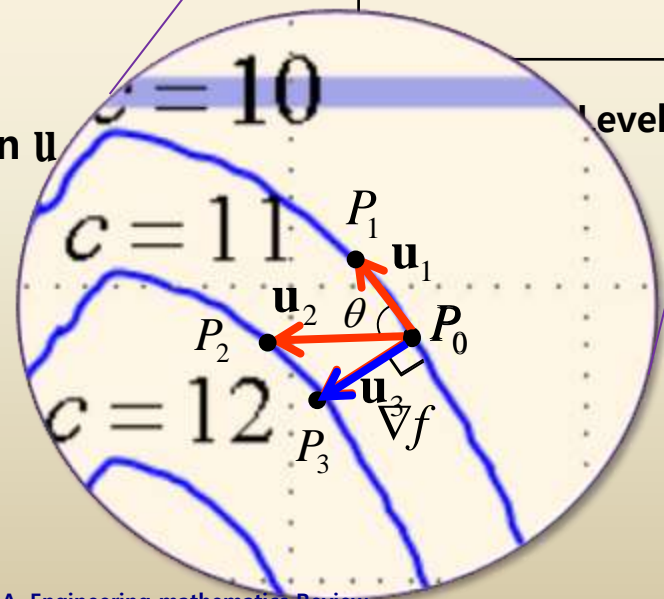
Directional Derivatives

Level Curves



The rate of change of f in the direction \mathbf{u} given by the vector: $D_{\mathbf{u}}f(x, y)$

∇f is the direction of maximum increase of f at P_0



Engineering mathematics Review

- Linear Algebra



Seoul
National
Univ.



Advanced Ship Design Automation Lab.
<http://asdal.snu.ac.kr>



Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

row2 + row1x(-3)

row2 + row1x(-3)

$$\begin{aligned} &3x_1 - x_2 - x_3 = 2 \\ +) &-3x_1 - 6x_2 - 3x_3 = -3 \\ \hline &-7x_2 - 4x_3 = -1 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$



Linear Systems Vs Matrices

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$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

row3 + row1x(-2)

row3 + row1x(-2)

$$\begin{aligned} & 2x_1 + 3x_2 - x_3 = -3 \\ +) & -2x_1 - 4x_2 - 2x_3 = -2 \\ \hline & -x_2 - 3x_3 = -5 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \\ 0 \cdot x_1 - x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$



Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \\ 0 \cdot x_1 - x_2 - 3x_3 &= -5 \end{aligned}$$

row 2 ↔ row 3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - x_2 - 3x_3 &= -5 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \end{aligned}$$

row 2x(-1)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$



Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

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$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

row 3 + row 2 x 7

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 - 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

The last equations and matrix are equal to given equations.



Linear Independence

Definition 3.1

Linear Dependence / Independence

A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is said to be '**linearly dependent**' on an interval I if there exist constant c_1, c_2, \dots, c_n , not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for every x in the interval.

If the set of functions is not linearly dependent on the interval, it is said to be '**linearly independent**'

In other words, a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \dots = c_n = 0$

"two functions are **linearly independent** when neither is a constant multiple of the other"

$$\begin{cases} f_1(x) = \sin 2x \\ f_2(x) = \sin x \cos x \end{cases} \quad \text{on } (-\infty, \infty)$$

$$f_2(x) = \sin x \cos x = \frac{1}{2} \sin 2x$$

$$\therefore f_2(x) = \frac{1}{2} f_1(x)$$

Linearly Dependent

$$\begin{cases} f_1(t) = e^t \\ f_2(t) = e^{2t} \end{cases} \quad \text{on } (-\infty, \infty)$$

$$c_1 e^t + c_2 e^{2t} = 0$$

Satisfied only when $c_1 = c_2 = 0$ on the interval

Linearly Independent



Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} f_1 : x_1 + 2x_2 + x_3 - 1 &= 0 \\ f_2 : 0 \cdot x_1 + x_2 + 3x_3 - 5 &= 0 \\ f_3 : 0 \cdot x_1 + 0 \cdot x_2 + 17x_3 - 34 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 + 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

No. of equations which are linearly independent ?

$$c_1(f_1) + c_2(f_2) + c_3(f_3) = 0$$

$$c_1(x_1 + 2x_2 + x_3 - 1) + c_2(0 \cdot x_1 + x_2 + 3 \cdot x_3 - 5) + c_3(0 \cdot x_1 + 0 \cdot x_2 + 17x_3 - 34) = 0$$

$$\underbrace{(c_1 + 0 \cdot c_2 + 0 \cdot c_3)}_{c_1 = 0} x_1 + \underbrace{(2c_1 + c_2 + 0 \cdot c_3)}_{c_2 = 0} x_2 + \underbrace{(c_1 + 3c_2 + 17c_3)}_{c_3 = 0} x_3 + (-c_1 - 5c_2 - 34c_3) = 0$$

$\therefore f_1, f_2, f_3$: linearly independent.

→ rank : 3



Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

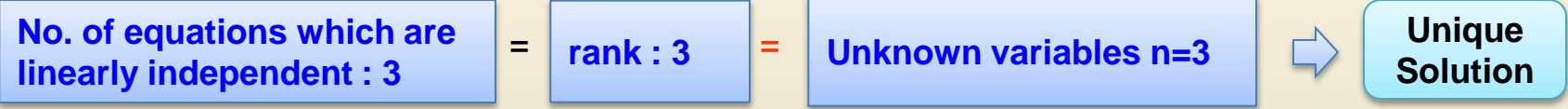
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 + 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

No. of equations which are linearly independent : 3

rank : 3

Unknown variables x_1, x_2, x_3 n=3



$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 2x_2 + 6x_3 &= 10 \end{aligned}$$



Linear Systems Vs Matrices



$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 2x_2 + 6x_3 &= 10 \end{aligned}$$

0.5*row 3

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 &= 0 \end{aligned}$$

$$\begin{aligned} f_1 : x_1 + 2x_2 + x_3 - 1 &= 0 \\ f_2 : 0 \cdot x_1 + x_2 + 3x_3 - 5 &= 0 \\ f_3 : 0 \cdot x_1 + 2x_2 + 6x_3 - 10 &= 0 \end{aligned}$$

No. of equations which are linearly independent ?

$$c_1(f_1) + c_2(f_2) + c_3(f_3) = 0$$

$$c_1(x_1 + 2x_2 + x_3 - 1) + c_2(0 \cdot x_1 + x_2 + 3 \cdot x_3 - 5) + c_3(0 \cdot x_1 + 2x_2 + 6x_3 + 10) = 0$$

$$\underbrace{(c_1 + 0 \cdot c_2 + 0 \cdot c_3)}_{c_1} x_1 + \underbrace{(2c_1 + c_2 + 2c_3)}_{c_2} x_2 + \underbrace{(c_1 + 3c_2 + 6c_3)}_{c_3} x_3 + \underbrace{(-c_1 - 5c_2 + 10c_3)}_{c_3} = 0$$

$c_1 = 0$ $c_2 = -2c_3$ $c_3 = \text{arbitrary number}$

$\therefore f_1, f_2, f_3$: linearly dependent.



Linear Systems Vs Matrices



$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 2x_2 + 6x_3 &= 10 \end{aligned}$$

0.5*row 3

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \end{aligned}$$



$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 &= 0 \end{aligned}$$

rank : 2

$$\begin{aligned} f_1 : x_1 + 2x_2 + x_3 - 1 &= 0 \\ f_2 : 0 \cdot x_1 + x_2 + 3x_3 - 5 &= 0 \\ f_3 : 0 \cdot x_1 + 2x_2 + 6x_3 - 10 &= 0 \end{aligned}$$

No. of equations which are linearly independent ?

$$c_1(f_1) + c_2(f_2) = 0$$

$$c_1(x_1 + 2x_2 + x_3 - 1) + c_2(0 \cdot x_1 + x_2 + 3 \cdot x_3 - 5) = 0$$

$$\underbrace{(c_1 + 0 \cdot c_2)}_{c_1 = 0} x_1 + \underbrace{(2c_1 + c_2)}_{c_2 = 0} x_2 + (c_1 + 3c_2)x_3 + (-c_1 - 5c_2) = 0$$

$\therefore f_1, f_2$: linearly independent.



Linear Systems Vs Matrices



$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 2x_2 + 6x_3 &= 10 \end{aligned}$$



$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$



Solution ?

$$\mathbf{Ax} = \lambda \mathbf{x} \longrightarrow \mathbf{x} = \mathbf{0} \text{ Trivial Solution}$$



Linear Systems Vs Matrices

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Rank : 3
Trivial x

$$(A - \lambda I)x = 0$$

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = (5-\lambda)(\lambda+3)^2 = 0$$

when $\lambda = 5$
 $A - \lambda I = A - 5I$

$$= \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

Row reduction

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank : 2

infinite no. of x

? Solution ?



$$Ax = \lambda x \longrightarrow x = 0 \text{ Trivial Solution}$$

$$(A - \lambda I)x = 0$$

x: Infinite many solutions

to have a solution x except x=0

$$rank(A - \lambda I) < n$$

$$\det(A - \lambda I) = 0$$

Zero row

λ : eigenvalues, x: eigenvectors

x: Infinite many solutions

Optimization Problem

Objective function

Solution determined



Engineering mathematics Review

- Inverse of a matrix.
Gauss-Jordan Elimination



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Notation of inverse matrix

In this inverse section, only *square matrices* are considered exclusively.

Notation of inverse of an $n \times n$ matrix $A = [a_{jk}] : A^{-1}$

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad , \text{ where } \mathbf{I} \text{ is the } n \times n \text{ unit matrix.}$$

Nonsingular matrix : A matrix that has an inverse.

(If a matrix has an inverse, the inverse is unique)

Singular matrix : A matrix that has **no inverse**.

Proof of uniqueness of inverse matrix

If B and C are inverses of A ($\mathbf{AB} = \mathbf{I}$ & $\mathbf{CA} = \mathbf{I}$),

$$\text{We obtain } \mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}$$

(the uniqueness of inverse)



Inverse by the Gauss-Jordan Method

For Practical determination of the inverse A^{-1} of a nonsingular $n \times n$ matrix A , Gauss elimination can be used.

: This method is called Gauss-Jordan elimination

Step 1. Make **augmented matrix**. $\tilde{A} = [A \ I]$

Step 2. Make Multiplication of $AX=I$ by A^{-1}
(by applying Gauss elimination to $\tilde{A} = [A \ I]$)

→ This gives a matrix of the form $[U \ H]$

Step 3. **Reduce U** by further elementary row operations to **diagonal form**.
(Eliminate the entries of U above the main diagonal and making the diagonal entries all 1 by multiplication. See the example next page.)



Inverse of a Matrix. Gauss-Jordan elimination.

Determine the inverse \mathbf{A}^{-1} of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Step 1. Make augmented matrix.

$$[\mathbf{A} \ \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array} \right]$$

Step 2. Make Multiplication of $\mathbf{AX}=\mathbf{I}$ by \mathbf{A}^{-1} by applying Gauss elimination to



Inverse of a Matrix. Gauss-Jordan elimination

$$[A \ I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & -0.6 & -0.4 & 0.4 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array} \right]$$

Row1 +0.4Row3
Row2 +1.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array} \right]$$

Row2 + 3Row1
Row3 - 3Row1

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0.7 & -0.2 & -0.3 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array} \right]$$

diagonal matrix
Row1 -0.5Row2

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array} \right]$$

Row3 - Row2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right]$$

-Row1
0.5Row2
-0.2Row3



Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right] \mathbf{A}^{-1}$$

Check the result.

$$\text{Let } \mathbf{A}\mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \left[\begin{array}{ccc|ccc} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{array} \right]$$

$$b_{11} = (-1) \times (-0.7) + 1 \times (-1.3) + 2 \times 0.8 = 1$$

$$b_{12} = (-1) \times (0.2) + 1 \times (-0.2) + 2 \times 0.2 = 0$$

$$b_{13} = (-1) \times (0.3) + 1 \times (0.7) + 2 \times (-0.2) = 0$$

$$b_{21} = (3) \times (-0.7) + (-1) \times (-1.3) + 1 \times (0.8) = 0$$

$$b_{22} = (3) \times (0.2) + (-1) \times (-0.2) + 1 \times (0.2) = 1$$

$$b_{23} = (3) \times (0.3) + (-1) \times (0.7) + 1 \times (-0.2) = 0$$

$$b_{31} = (-1) \times (-0.7) + (3) \times (-1.3) + 4 \times (0.8) = 0$$

$$b_{32} = (-1) \times (0.2) + (3) \times (-0.2) + 4 \times (0.2) = 0$$

$$b_{33} = (-1) \times (0.3) + (3) \times (0.7) + 4 \times (-0.2) = 1$$

$$\therefore \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

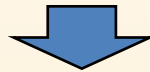


Inverse of a Matrix. Gauss-Jordan elimination.

$$-x_1 + x_2 + 2x_3 = 7$$

$$3x_1 - x_2 + x_3 = 4$$

$$-x_1 + 3x_2 + 4x_3 = 17$$



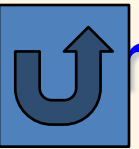
$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 17 \end{bmatrix}$$



$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 7 \\ 3 & -1 & 1 & 4 \\ -1 & 3 & 4 & 17 \end{array} \right]$$



Inverse of a Matrix. Gauss-Jordan elimination



$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 7 \\ 3 & -1 & 1 & 4 \\ -1 & 3 & 4 & 17 \end{array} \right]$$



$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 7 \\ 0 & 2 & 7 & 25 \\ 0 & 2 & 2 & 10 \end{array} \right]$$

Row2 + 3Row1

Row3 - Row1

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 7 \\ 0 & 2 & 7 & 25 \\ 0 & 0 & -5 & -15 \end{array} \right]$$

Row3 - Row2

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -5 & -15 \end{array} \right]$$

Row1 + 0.4Row3

Row2 + 1.4Row3



$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -5 & -15 \end{array} \right]$$

Row1 - 0.5Row2

diagonal matrix



$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

-Row1

0.5Row2

-0.2Row3

모든 변수를 하나의 행에만 남기고
다른 행에서는 모두 소거 하였음
→ 바로 해를 구할 수 있음



Engineering mathematics Review - Quadratic Forms



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Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T \mathbf{C} \mathbf{x} \quad \mathbf{C} = [c_{jk}], \quad c_{jk} = \frac{1}{2} (a_{jk} + a_{kj})$$



Principal Axes Theorem

(Quadratic Forms)
$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{j=1}^n \sum_{k=1}^n a_{jk} x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix \mathbf{A} has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \dots, \mathbf{x}_n$. Let \mathbf{X} be

$$\mathbf{X} = [\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n]$$

\mathbf{X} is *orthogonal*, so that $\mathbf{X}^{-1} = \mathbf{X}^T$, we obtain

$$\mathbf{D} = \mathbf{X}^{-1} \mathbf{A} \mathbf{X}$$

$$\therefore \mathbf{A} = \mathbf{X} \mathbf{D} \mathbf{X}^{-1} = \mathbf{X} \mathbf{D} \mathbf{X}^T$$

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T (\mathbf{X} \mathbf{D} \mathbf{X}^T) \mathbf{x}$$

Symmetric: $\mathbf{A}^T = \mathbf{A}$

Skew-symmetric: $\mathbf{A}^T = -\mathbf{A}$

Orthogonal: $\mathbf{A}^T = \mathbf{A}^{-1}$



Principal Axes Theorem

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T \mathbf{X} \mathbf{D} \mathbf{X}^T \mathbf{x}$$

If we set $\mathbf{X}^T \mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^T = \mathbf{X}^{-1}$, we get

$$\mathbf{x} = (\mathbf{X}^T)^{-1} \mathbf{y} = \mathbf{X} \mathbf{y}$$

Furthermore, we have

$$\mathbf{x}^T \mathbf{X} = (\mathbf{X}^T \mathbf{x})^T = \mathbf{y}^T$$

So Q becomes simply

$$\begin{aligned} Q &= \mathbf{x}^T \mathbf{X} \mathbf{D} \mathbf{X}^T \mathbf{x} = \mathbf{y}^T \mathbf{D} \mathbf{y} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$



Principal Axes Theorem

$$\begin{aligned}
 Q &= \mathbf{x}^T \mathbf{A} \mathbf{x} \longleftarrow \mathbf{D} = \mathbf{X}^{-1} \mathbf{A} \mathbf{X} \\
 &= \mathbf{x}^T \mathbf{X} \mathbf{D} \mathbf{X}^T \mathbf{x} \longleftarrow \mathbf{y} = \mathbf{X}^{-1} \mathbf{x} = \mathbf{X}^T \mathbf{x} \\
 &= \mathbf{y}^T \mathbf{D} \mathbf{y} \longleftarrow \mathbf{y}^T = (\mathbf{X}^T \mathbf{x})^T = \mathbf{x} \mathbf{X}^T
 \end{aligned}$$

Theorem 8.4.5 Principal Axes Theorem

The substitution $\mathbf{x} = \mathbf{X} \mathbf{y}$ transforms a **quadratic form**

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{j=1}^n \sum_{k=1}^n a_{jk} x_j x_k \quad (a_{kj} = a_{jk})$$

to the **principal axes form** or **canonical form**

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1, \dots, \lambda_n$ are the (not necessarily distinct) eigenvalues of the **symmetric matrix** \mathbf{A} , and \mathbf{X} is an **orthogonal matrix** with corresponding eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$, respectively, as column vectors.



$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T \mathbf{X} \mathbf{D} \mathbf{X}^T \mathbf{x}$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

Find out what type of conic section the following quadratic form represents and transform it to principal axes. (Ex 8.2-1)

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = \mathbf{x}^T \mathbf{A} \mathbf{x}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

Characteristic Equation:

$$\begin{vmatrix} 17 - \lambda & -15 \\ -15 & 17 - \lambda \end{vmatrix} = 0$$

$$(17 - \lambda)^2 - 15^2 = 0 \quad \therefore \lambda_1 = 2, \lambda_2 = 32$$

Eigenvectors $\lambda_1 = 2, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\lambda_2 = 32, \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T \mathbf{X} \mathbf{D} \mathbf{X}^T \mathbf{x} = \mathbf{y}^T \mathbf{D} \mathbf{y}$$

$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2y_1^2 + 32y_2^2 = 128$$

$$\therefore \frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$



Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector \mathbf{x}_1 .

$$\mathbf{x}_1 = \left[\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right]^T$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T \mathbf{A} \mathbf{x}$$

$$\lambda_1 = 2, \lambda_2 = 32$$

2) $\lambda = \lambda_2 = 32$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$-15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector \mathbf{x}_2 .

$$\mathbf{x}_2 = \left[-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right]^T$$



Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

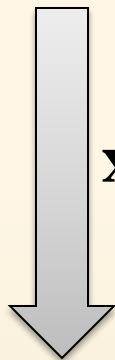
$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

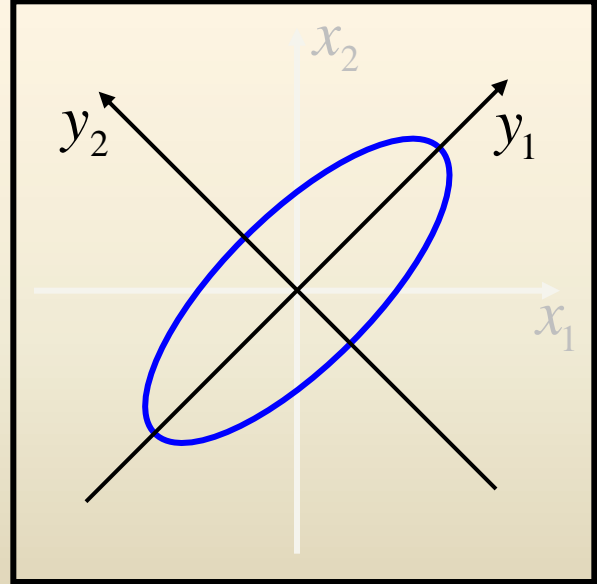
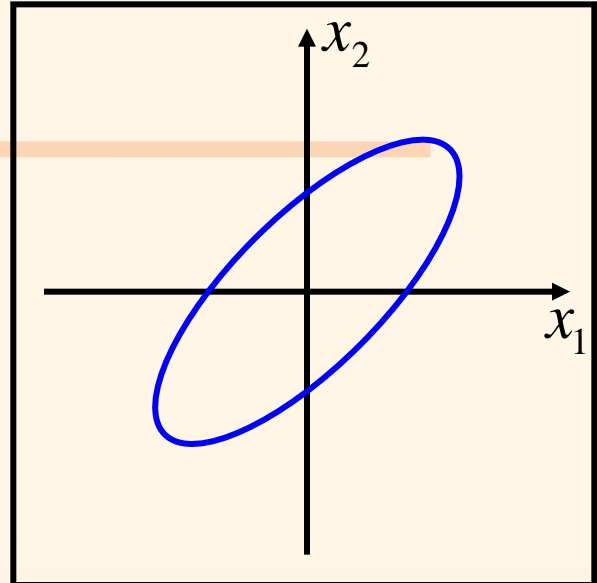


$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$

$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



→ This means a 45° rotation (of principal axes. (See example 8.2-1.)



Quadratic form(Definiteness)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ and its (symmetric!) matrix \mathbf{A} are called

(a) **positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) **negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) **indefinite** if $Q(\mathbf{x})$ takes both positive and negative values.



Quadratic form(Definiteness)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ and its (symmetric!) matrix \mathbf{A} are called

(a) **positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) **negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) **indefinite** if $Q(\mathbf{x})$ takes both positive and negative values.

A necessary and sufficient condition for positive definiteness is that all the “**principal minors**” are positive, that is,

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, \det \mathbf{A} > 0$$

Show that the form in Prob. 23 is positive definite, whereas that in Prob. 19 is indefinite.

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the “**principal minors**” are positive, that is,

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - (\sqrt{3})^2 = 5 > 0$$

→ positive definite

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

$$a_{11} = 1 > 0$$

$$\begin{vmatrix} 1 & 12 \\ 12 & -6 \end{vmatrix} = -6 - 12^2 = -150 < 0$$

→ indefinite





the eigenvalues of A are

(a) positive definite: all positive

(b) negative definite: all negative

(c) indefinite: both positive and negative

$$\begin{aligned} Q &= \mathbf{x}^T \mathbf{A} \mathbf{x}, \quad \mathbf{x} = \mathbf{X} \mathbf{y} \\ &= \mathbf{y}^T \mathbf{A} \mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad (1) \end{aligned}$$

Because $\mathbf{y} = \mathbf{X}^{-1} \mathbf{x}$, if $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{y} \neq \mathbf{0}$.

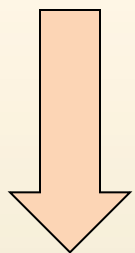
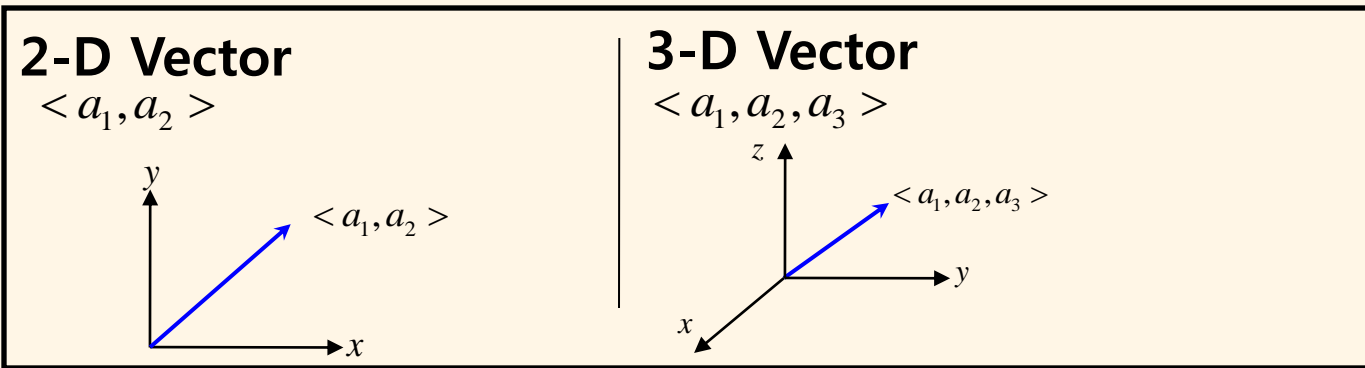
From equation (1),

If all eigenvalues are positive, $Q(\mathbf{x})$ is positive.

If all eigenvalues are negative, $Q(\mathbf{x})$ is negative.



기하학적 Vector 개념의 확장



Arthur Cayley, James Joseph Sylvester, William Rowan Hamilton
 - Concepts for point and vector could be **generalized**.
 - A realization developed that vectors could be defined by **analytic** rather than **geometric properties**.


Vector in n-space

$\langle a_1, a_2, \dots, a_n \rangle$

n차원 공간에서는 유향성분으로 보일 수 없음.


Arthur Cayley

- 1821. 8.16. ~ 1895. 1. 26.
- British mathematician
- Projective geometry, Group theory




James Joseph Sylvester

- 1814. 9.3. ~ 1897. 3. 15.
- British mathematician
- Sylvester's sequence



William Rowan Hamilton

- 1805. 8.4. ~ 1865. 9. 2.
- Irish mathematician, physicist, astronomer
- Quaternions, Hamiltonians.



출처 : Dennis G. Zill, Michael R. Cullen, Advanced Engineering Mathematics 3rd Edition, pp. 321-322, Johns and Bartlett, 2006



Appendix B. Calculus of Variation

- D'Alembert's Principle
- Virtual Work
- Equation of Euler-Lagrange
- Hamilton's principle



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Calculus of Variation

- Mechanical System의 운동 방정식 유도

Newton's 2nd law

$$\mathbf{F} = m\ddot{\mathbf{r}}$$



D'Alembert's Principle
Virtual work



보존력이 작용하는 Mechanical System의 Kinetic energy(T)와 Potential energy(V)의 차이(L)를 Equation of Euler-Lagrange에 대입후 정리하면 Mechanical System의 운동 방정식을 유도할 수 있음

$$\delta W = \sum_{i=1}^N (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$$



Hamilton's Principle



- Mechanical System에 가해지는 힘은 보존력임
- System의 초기 상태와 최종 상태는 정의되어 있음

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad \text{define : } L = T - V \quad (\text{T: Kinetic energy, V: Potential energy})$$



Equation of Euler-Lagrange



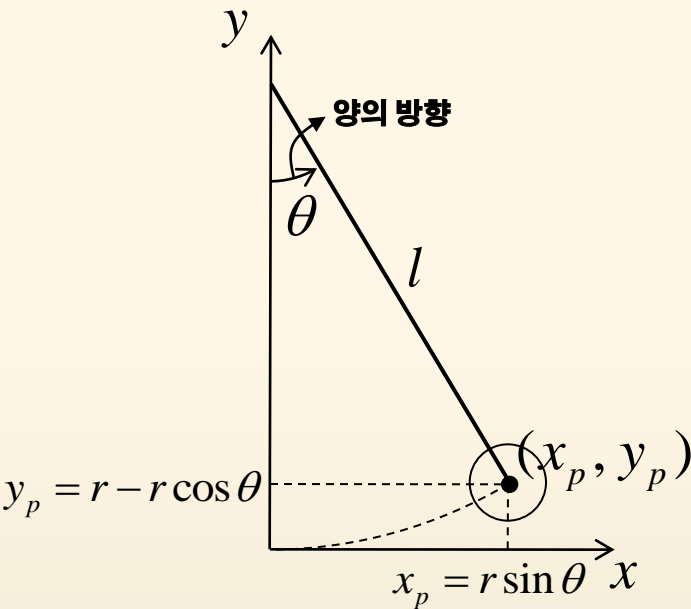
$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$



진자(Pendulum)의 운동 방정식 유도

- 예시

보존력이 작용하는 Mechanical System의 Kinetic energy(T)와 Potential energy(V)의 차이(L)를 Equation of Euler-Lagrange에 대입후 정리하면 Mechanical System의 운동 방정식을 유도할 수 있음



define : $L = T - V$

$$T = \frac{1}{2} m \dot{\mathbf{r}}^2 = \frac{1}{2} m r^2 \dot{\theta}^2, \quad V = mg(l - l \cos \theta)$$

$$\dot{x}_p = l \cos \theta \cdot \dot{\theta}$$

$$\dot{y}_p = l \sin \theta \cdot \dot{\theta}$$

$$\dot{\mathbf{r}}^2 = \dot{x}_p^2 + \dot{y}_p^2 = l^2 \dot{\theta}^2 \cos^2 \theta + l^2 \dot{\theta}^2 \sin^2 \theta = l^2 \dot{\theta}^2$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mg(l - l \cos \theta) \text{----- ①}$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \text{----- ②}$$

(단, $q = \theta$)

식①은 ②를 만족함

$$-mgl \sin \theta - \frac{d}{dt} ml^2 \dot{\theta} = 0$$

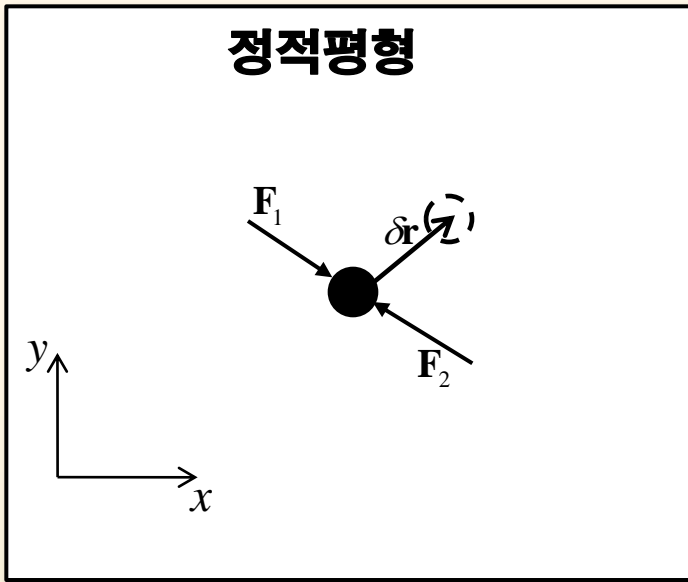
$$-mgl \sin \theta - ml^2 \ddot{\theta} = 0$$

$$-g \sin \theta - l \ddot{\theta} = 0$$

$$\boxed{-\frac{g}{l} \sin \theta = \ddot{\theta}}$$

Virtual work and D'Alembert's Principle

정적평형



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$$

힘의 합력이 0이므로 물체는 정지해 있음
“정적 평형 상태”

물체를 미소 변위 $\delta \mathbf{r}$ 만큼 움직인다고 가정 하면
질점에 작용하는 모든 힘이 한 일은 다음과 같다.

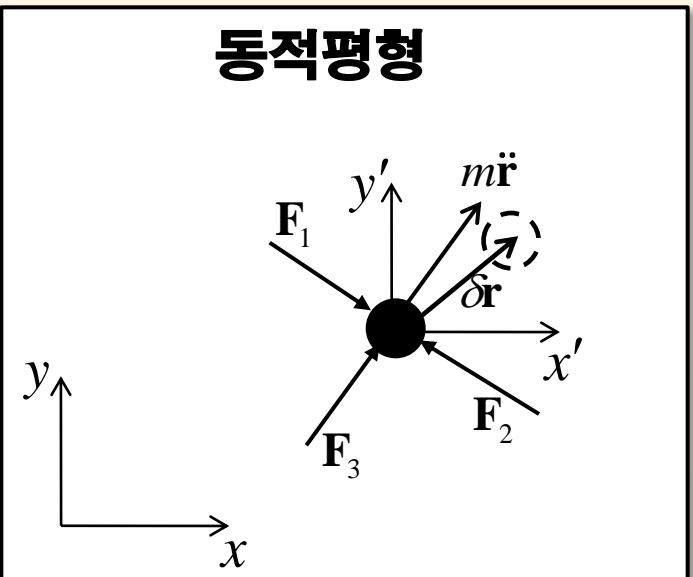
$$\begin{aligned}\delta W &= \mathbf{F} \cdot \delta \mathbf{r} \\ &= \mathbf{F}_1 \cdot \delta \mathbf{r} + \mathbf{F}_2 \cdot \delta \mathbf{r} \\ &= 0\end{aligned}$$

가상일의 원리



Virtual work and D'Alembert's Principle

동적평형



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = m\ddot{\mathbf{r}}$$

해석

- 질량과 가속도의 곱은 물체에 가해지는 힘과 같다.
- 물체에 F라는 힘이 가해져서 m의 질량을 가진 물체가 가속도 운동을 한다.

만일 물체에 고정되어 있는 좌표계(x',y')에서 물체를 바라본다면?

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 - m\ddot{\mathbf{r}} = \mathbf{0}$$

해석

물체에 외력 이외에 **관성력**($-m\ddot{\mathbf{r}}$)이 가해져서 힘의 합력은 0이고, 움직이지 않는다.
 “동적 평형 상태”

D'Alembert's Principle

동적 평형 상태에 가상일의 원리를 적용

$$\begin{aligned} \delta W &= (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 - m\ddot{\mathbf{r}}) \delta \mathbf{r} \\ &= (\mathbf{F} - m\ddot{\mathbf{r}}) \cdot \delta \mathbf{r} = 0 \end{aligned}$$

외력과 관성력이 한 가상 일(Virtual work)의 총 합은 0이다.

$$\delta W = (\mathbf{F} - m\ddot{\mathbf{r}}) \delta \mathbf{r} = 0$$



Hamilton's Principle (1/3)

부분 적분

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

$$= (\text{적분})(\text{그대로}) - \int (\text{적분})(\text{미분})dx$$

$$\delta W = \sum_{i=1}^N (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0 \quad \leftarrow \text{From D'Alembert's principle}$$

양변을 t_1 부터 t_2 까지 적분하면

$$\int_{t_1}^{t_2} \delta W dt = \int_{t_1}^{t_2} \sum_{i=1}^N (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i dt$$

$$= \int_{t_1}^{t_2} \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i dt - \int_{t_1}^{t_2} \sum_{i=1}^N m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i dt$$

$$\int_{t_1}^{t_2} \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i dt = \int_{t_1}^{t_2} \delta U dt = - \int_{t_1}^{t_2} \delta V dt$$

작용하는 외력(F)이 보존력이라면 $U = -V$ 가 성립한다.
 U: Work function, V: Potential energy ▶

$$\int_{t_1}^{t_2} \sum_{i=1}^N m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i dt = \left[\sum_{i=1}^N m\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{i=1}^N m\dot{\mathbf{r}}_i \cdot \frac{d}{dt}(\delta \mathbf{r}_i) dt \quad (\text{부분 적분 적용})$$

$\because \delta \mathbf{r}_i(t_1) = \delta \mathbf{r}_i(t_2) = 0$

시스템의 초기 상태와 마지막 상태는 정의되어 있으며, 변분에 의해 변화하지 않는다.



Hamilton's Principle (2/3)

$$\delta W = \sum_{i=1}^N (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$$

양변을 t1 부터 t2까지 적분하면

$$\begin{aligned} \int_{t_1}^{t_2} \delta W dt &= \int_{t_1}^{t_2} \sum_{i=1}^N (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i dt \\ &= \int_{t_1}^{t_2} \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i dt - \int_{t_1}^{t_2} \sum_{i=1}^N m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i dt \\ &= -\delta \int_{t_1}^{t_2} V dt + \delta \int_{t_1}^{t_2} T dt \end{aligned}$$

$$\int_{t_1}^{t_2} \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i dt = \int_{t_1}^{t_2} \delta U dt = -\int_{t_1}^{t_2} \delta V dt$$

작용하는 외력(F)이 보존력이라면 U = -V가 성립한다.
U: Work function, V: Potential energy

$$\begin{aligned} \int_{t_1}^{t_2} \sum_{i=1}^N m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i dt &= -\int_{t_1}^{t_2} \sum_{i=1}^N m\dot{\mathbf{r}}_i \cdot \frac{d}{dt}(\delta \mathbf{r}_i) dt = -\int_{t_1}^{t_2} \sum_{i=1}^N m\dot{\mathbf{r}}_i \cdot \delta \dot{\mathbf{r}}_i dt = -\int_{t_1}^{t_2} \sum_{i=1}^N m \frac{1}{2} \delta(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) dt \\ &= -\delta \int_{t_1}^{t_2} \sum_{i=1}^N \frac{1}{2} m\dot{\mathbf{r}}_i^2 dt = -\delta \int_{t_1}^{t_2} T dt \end{aligned}$$

$\because \delta(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) = (\delta \dot{\mathbf{r}}_i) \cdot \dot{\mathbf{r}}_i + \dot{\mathbf{r}}_i \cdot (\delta \dot{\mathbf{r}}_i) = 2\dot{\mathbf{r}}_i \cdot (\delta \dot{\mathbf{r}}_i)$

T: Kinetic energy

Hamilton's Principle (3/3)

$$\delta W = \sum_{i=1}^N (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$$

양변을 t1 부터 t2까지 적분하면

$$\begin{aligned} \int_{t_1}^{t_2} \delta W dt &= \int_{t_1}^{t_2} \sum_{i=1}^N (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i dt \\ &= \int_{t_1}^{t_2} \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i dt - \int_{t_1}^{t_2} \sum_{i=1}^N m\ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i dt \\ &= -\delta \int_{t_1}^{t_2} V dt + \delta \int_{t_1}^{t_2} T dt \\ &= \delta \int_{t_1}^{t_2} T - V dt = 0 \\ &= \delta \int_{t_1}^{t_2} L dt = 0 \end{aligned}$$

$$L = T - V \quad \text{라고 정의 하면}$$

V: Potential energy
T: Kinetic energy

초기 상태와 최종 상태가 정의되어 있는 Mechanical System의 Potential energy와 Kinetic energy의 차이를 L이라 정의하면 L의 정적분값은 stationary value가 된다. (L의 정적분값의 변화율은 0이 된다.)

참고(Equation of Euler-Lagrange)

- Give $I = \int_a^b F(y, y', x) dx$
- n: $f(a) = \alpha, f(b) = \beta$
- Find: 적분 I가 stationary value를 갖도록 하는 $y = f(x)$

• F의 정적분값의 변화율(δI)이 0이 되도록 하는 $y = f(x)$ 를 찾아야 함

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0$$

The Calculus of Variation

- Equation of Euler-Lagrange (1/6)

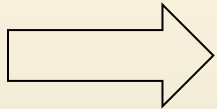
$$\therefore \delta \int_a^b F(x) dx = \int_a^b \delta F(x) dx$$

The stationary value of a definite integral treated by the calculus of variation

•Given:
$$I = \int_a^b F(y, y', x) dx$$

where $y = f(x)$, $f(a) = \alpha$, $f(b) = \beta$

•Find: 적분 I 가 stationary value를 갖도록 하는 $y = f(x)$



• $y = f(x)$ 가 I 가 stationary value를 갖도록 하는 함수라 가정한다.

• y 의 변분(variation) δy 를 고려한다.

• y 가 $y + \delta y$ 로 이동했을 때, 적분 I 의 변분 δI 를 살펴본다.

• 함수 F 의 변분 δF 의 값을 살펴본다.

• δI 의 계산식을 구한다.

• 적분 I 의 변화율($\delta I / \varepsilon$)이 0이 되도록 하는 조건을 만족하는 $y = f(x)$ 가 적분 I 가 stationary value를 갖도록 한다.



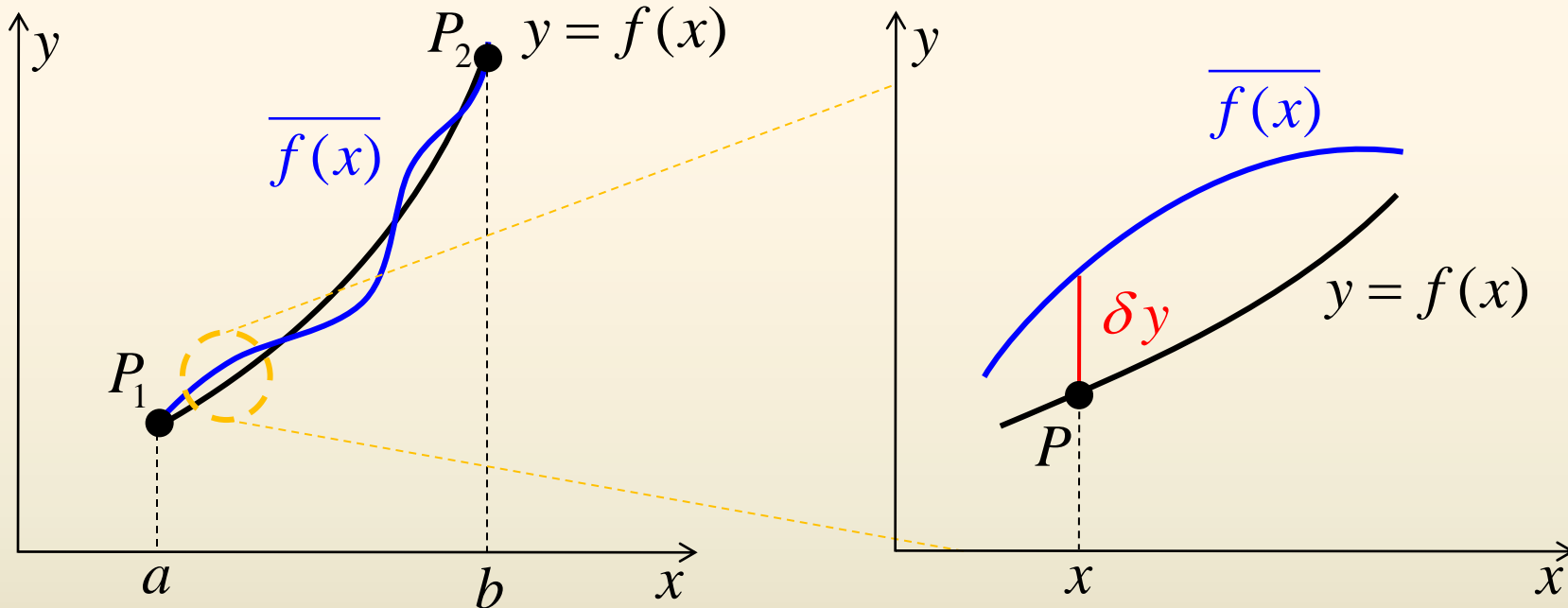
The Calculus of Variation

- Equation of Euler-Lagrange (2/6)

• Given: $I = \int_a^b F(y, y', x) dx$
 $f(a) = \alpha, f(b) = \beta$

• Find: 적분 I 가 stationary value를 갖도록 하는 $y = f(x)$

- $y = f(x)$ 가 I 가 stationary value를 갖도록 하는 함수라 가정한다.
- y 의 변분(variation) δy 를 고려한다.



The given function: $y = f(x)$

The modified function: $\overline{f(x)} = f(x) + \varepsilon\phi(x)$

- $\Phi(x)$: arbitrary new function, continuous and differentiable.

$$\phi(a) = \phi(b) = 0$$

변분 δy :

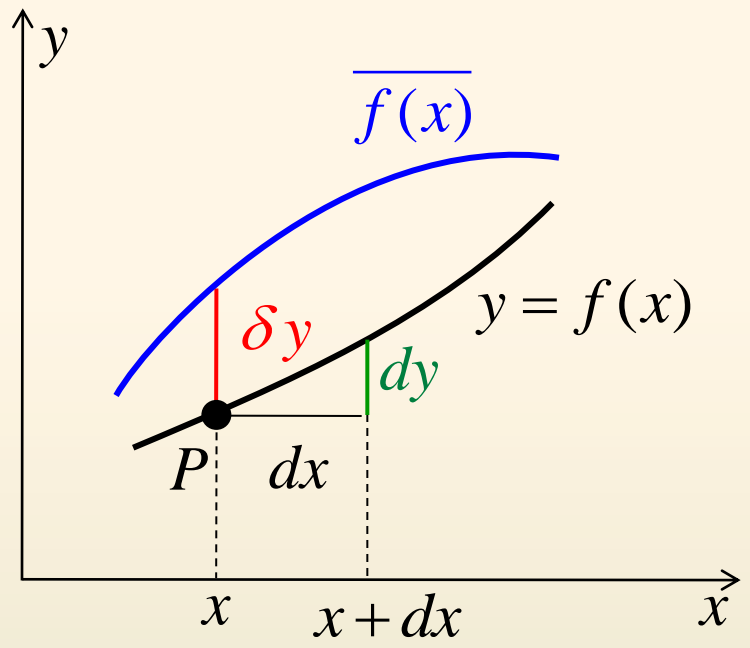
$$\delta y = \overline{f(x)} - f(x) = \varepsilon\phi(x)$$

- parameter ε 이 감소함에 따라 0으로 접근(infinitesimal change), 임의의 방향으로 변경(virtual change)

The Calculus of Variation

- Equation of Euler-Lagrange (3/6)

•Give $I = \int_a^b F(y, y', x) dx$
 n: $f(a) = \alpha, f(b) = \beta$
 •Find: 적분 I 가 stationary value를 갖도록 하는 $y = f(x)$



•dy 와 δy

- dy : 주어진 함수 $y = f(x)$ 의 독립변수 x 가 미소변위 dx 만큼 이동하여 생기는, 함수 $f(x)$ 의 변화량
- δy : 주어진 함수 $y = f(x)$ 에 새로운 함수를 더함으로써 생긴 변화량



The Calculus of Variation

- Equation of Euler-Lagrange (4/6)

$$\left. \begin{array}{l} y = f(x) \\ \overline{f(x)} = f(x) + \varepsilon\phi(x) \end{array} \right\} \begin{array}{l} \delta y = \varepsilon\phi(x) \\ \phi(a) = \phi(b) = 0 \end{array}$$

•Give $I = \int_a^b F(y, y', x)dx$
n: $f(a) = \alpha, f(b) = \beta$

•Find: 적분 I 가 stationary value를 갖도록 하는 $y = f(x)$

- y 가 $y + \delta y$ 로 이동했을 때, 적분 I 의 변분 δI 를 살펴본다.
- 함수 F 의 변분 δF 의 값을 살펴본다.
- δI 의 계산식을 구한다.

•함수 F 의 변분 δF

$$\delta F(y, y', x) = F(y + \varepsilon\phi, y' + \varepsilon\phi', x) - F(y, y', x)$$

↓ Taylor series expansion 이용

$$= \frac{\partial F}{\partial y} \varepsilon\phi + \frac{\partial F}{\partial y'} \varepsilon\phi' + \frac{1}{2} \left(\frac{\partial^2 F}{\partial y^2} \varepsilon\phi + 2 \frac{\partial^2 F}{\partial y \partial y'} \varepsilon\phi \varepsilon\phi' + \frac{\partial^2 F}{\partial y'^2} \varepsilon\phi' \right) + \dots$$

ε 이 작으므로, 고차항은 무시함

$$= \varepsilon \left(\frac{\partial F}{\partial y} \phi + \frac{\partial F}{\partial y'} \phi' \right)$$

The Calculus of Variation

- Equation of Euler-Lagrange (5/6)

•Give $I = \int_a^b F(y, y', x) dx$
n: $f(a) = \alpha, f(b) = \beta$

•Find: 적분 I 가 stationary value를 갖도록 하는 $y = f(x)$

•함수 F 의 변분 δF :
$$\delta F(y, y', x) = \varepsilon \left(\frac{\partial F}{\partial y} \phi + \frac{\partial F}{\partial y'} \phi' \right)$$

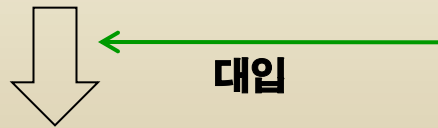
- y 가 $y + \delta y$ 로 이동했을 때, 적분 I 의 변분 δI 를 살펴본다.
- 함수 F 의 변분 δF 의 값을 살펴본다.
- δI 의 계산식을 구한다.

•적분 I 의 변분 δI

$$\delta I = \delta \int_a^b F dx \stackrel{\square}{=} \int_a^b \delta F dx = \varepsilon \int_a^b \left(\frac{\partial F}{\partial y} \phi + \frac{\partial F}{\partial y'} \phi' \right) dx$$

•적분 I 의 변화율(양변을 ε 으로 나눔)

$$\frac{\delta I}{\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial y} \phi + \frac{\partial F}{\partial y'} \phi' \right) dx$$



$$\therefore \frac{\delta I}{\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \phi dx$$

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

=(적분)(그대로) - ∫ (적분)(미분)dx

부분 적분 \square 를 이용하여 구함.

$$\int_a^b \frac{\partial F}{\partial y'} \phi' dx = \left[\frac{\partial F}{\partial y'} \phi \right]_a^b - \int_a^b \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \phi dx$$

0 ($\because \phi(a) = \phi(b) = 0$)

The Calculus of Variation

- Equation of Euler-Lagrange (6/6)

- Give $I = \int_a^b F(y, y', x) dx$
- n: $f(a) = \alpha, f(b) = \beta$
- Find: 적분 I 가 stationary value를 갖도록 하는 $y = f(x)$

• 적분 I 의 변화율:
$$\frac{\delta I}{\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \phi dx$$

• 적분 I 의 변화율(δI)이 0이 되도록 하는 $y = f(x)$ 를 찾아야 함

$$\therefore \frac{\delta I}{\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \phi dx = 0$$

• 함수 $\phi(x)$ 는 임의의 함수이므로, 위 식이 항상 0이 되기 위해서는 괄호 안의 식이 0이 되어야 한다.

$$\therefore \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0$$

적분 I 가 stationary value를 갖게 하는 필요충분조건

• 따라서 적분 I 가 stationary value를 갖도록 하는 $y = f(x)$ 는 위 미분방정식을 만족하는 함수이다.



(참고) 곱의 미분

$$\frac{df}{dx} = f'(x), \quad \frac{dg}{dx} = g'(x)$$

$$\begin{aligned} \frac{d}{dx}(f(x)g(x)) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\{f(x+h)g(x+h) - f(x)g(x+h)\} + \{f(x)g(x+h) - f(x)g(x)\}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \rightarrow 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h \rightarrow 0} f(x) \cdot \frac{g(x+h) - g(x)}{h} \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$



(참고) 부분 적분

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Integral with respect to x

$$\int (f(x)g(x))' dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$f(x)g(x) - \int f(x)g'(x)dx = \int f'(x)g(x)dx$$

$$\begin{aligned} \int f'(x)g(x)dx &= f(x)g(x) - \int f(x)g'(x)dx \\ &= (\text{적분})(\text{그대로}) - \int (\text{적분})(\text{미분})dx \end{aligned}$$

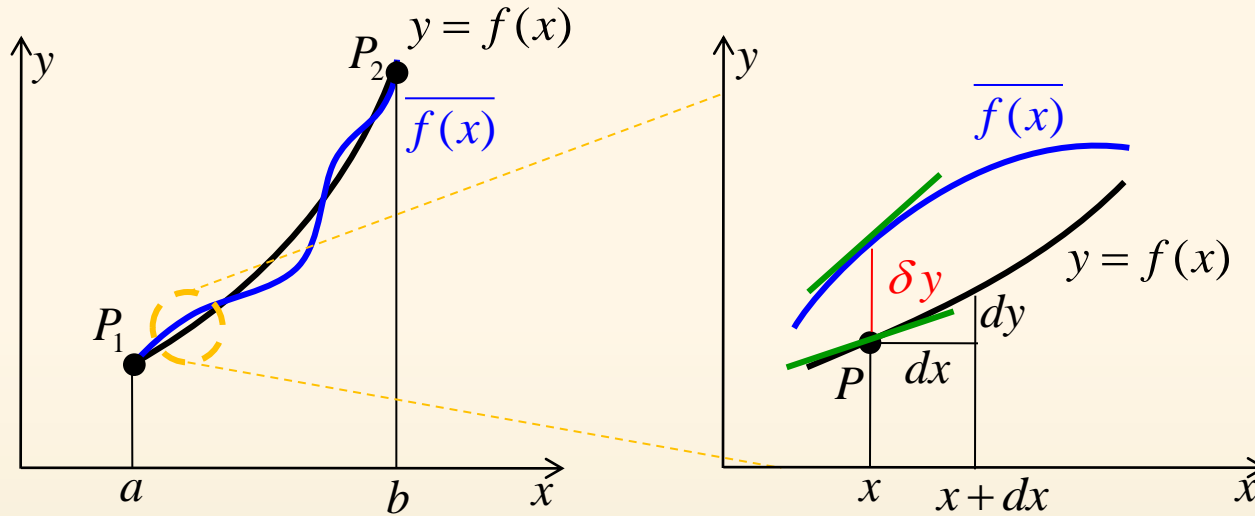
$$\int u'v dx = uv - \int uv' dx, \quad (\text{where } u = f(x), v = g(x))$$

u' 으로 가정하는 순서 : 지수함수, 삼각함수, 다항함수, 로그함수

The commutative properties of the δ -process (1)

$$\overline{f(x)} = f(x) + \varepsilon\phi(x)$$

$$\delta y = \overline{f(x)} - f(x) = \varepsilon\phi(x)$$



•The derivative of the variation

$$\frac{d}{dx} \delta y = \frac{d}{dx} \left[\overline{f(x)} - f(x) \right] = \frac{d}{dx} \left[\varepsilon\phi(x) \right] = \varepsilon\phi'(x)$$

•The variation of the derivative

$$\delta \frac{d}{dx} y = \left[\overline{f'(x)} - f'(x) \right] = (y' + \varepsilon\phi') - y' = \varepsilon\phi'(x)$$

$f(x)$ 와 $\overline{f(x)}$ 의
차이의 변화율
(기울기)

$$\therefore \frac{d}{dx} \delta y = \delta \frac{d}{dx} y$$

$f(x)$ 와 $\overline{f(x)}$ 의
변화율(기울기)
의 차이

The commutative properties of the δ -process (2)

$$y = f(x) \quad \overline{f(x)} = f(x) + \varepsilon\phi(x)$$

$$\therefore \frac{d}{dx} \delta y = \delta \frac{d}{dx} y$$

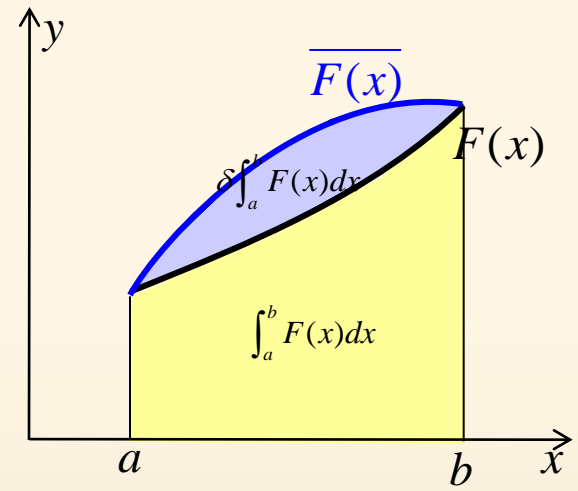
The variation of a definite integral

- The given integrand: $F(x)$
- The modified integrand: $\overline{F(x)} = F(x) + \delta F(x)$
- The variation of a definite integral

$$\delta \int_a^b F(x) dx = \int_a^b \overline{F(x)} dx - \int_a^b F(x) dx$$

$$= \int_a^b [\overline{F(x)} - F(x)] dx = \int_a^b \delta F(x) dx$$

$$\therefore \delta \int_a^b F(x) dx = \int_a^b \delta F(x) dx$$



- The δ -process reveals two characteristic properties:
- Variation and differentiation are permutable processes.
 - Variation and integration are permutable processes.



Appendix C. Solution of Systems of equation by optimization method



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Solution of Systems of equation by optimization method

다음과 같은 연립 방정식이 있다.

변수	: x_1, x_2
Given 식	: $f_1(x_1, x_2)=0$ $f_2(x_1, x_2)=0$
Find	두 식을 만족하는 x_1^*, x_2^*

따라서 다음과 같은 목적함수를 최소화 하는 점 (x_1^*, x_2^*) 가 연립 방정식의 해가 된다.

minimize

$$G = \left(f_1(x_1, x_2) \right)^2 + \left(f_2(x_1, x_2) \right)^2$$

⇒ 최적화 방법으로 풀이

[참고] 다음을 통해 위의 최적화 문제가 연립 방정식과 같은 문제임을 보일 수 있다.

$(f_1(x_1, x_2))^2 \geq 0, (f_2(x_1, x_2))^2 \geq 0$ 이기 때문에
목적함수 G 를 최소화 하는 곳은
 $f_1(x_1, x_2) = 0, f_2(x_1, x_2) = 0$ 인 점이다.

함수 G 가 최소 값을 가질 필요 조건

$$\frac{\partial G}{\partial f_1} = 2f_1 = 0, \frac{\partial G}{\partial f_2} = 2f_2 = 0$$


최적화 기법을 이용하여 위의 문제를 풀 수 있을까?

식 $f_1(x_1, x_2)=0$ 와 $f_2(x_1, x_2)=0$ 는 점 (x_1^*, x_2^*) 에서만 만족한다.

즉 점 (x_1^*, x_2^*) 가 아닌 곳에서는 $f_1(x_1, x_2) \neq 0, f_2(x_1, x_2) \neq 0$ 이다.