# 2. Wave Propagation on Transmission Lines

## *A. General Solutions of Transmission-Line Equations*

### 1) Wave solutions in the phasor domain

For uniform transmission lines with time-harmonic variation $e^{j\omega t}$,

Transmission-line equations:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \qquad \text{(8-10)}$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \qquad \text{(8-11)}$$

where $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY}$ (m$^{-1}$) (8-12)

$\alpha = Re[\sqrt{(R+j\omega L)(G+j\omega C)}]$ (Np/m) (8-12a)

$\beta = Im[\sqrt{(R+j\omega L)(G+j\omega C)}]$ (rad/m) (8-12b)

General solutions of (8-10, 11):

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z} \quad \text{(V)} \qquad \text{(8-33, 62)}$$

$$I(z) = I_o^+ \underset{\substack{\uparrow \\ \text{FTW}}}{e^{-\gamma z}} + I_o^- \underset{\substack{\uparrow \\ \text{BTW(reflected wave)}}}{e^{+\gamma z}} \quad \text{(A)} \qquad \text{(8-34, 63)}$$

where unknown amplitudes ( $V_o^+,\ V_o^-,\ I_o^+,\ I_o^-$ ) are to be determined by BCs.

Generally, $V_o^+,\ V_o^-,\ I_o^+,\ I_o^-$ are complex quantities, like $V_o^{\pm} = |V_o^{\pm}|\, e^{j\phi^{\pm}}$

(8-33) in (8-8):

$$I(z) = \frac{\gamma}{(R+j\omega L)}\left[V_o^+ e^{-\gamma z} - V_o^- e^{+\gamma z}\right] \qquad \text{(8-34)}\star$$

### 2) Characteristic impedance

Comparison of (8-34)⋆ with (8-34) leads to

$$\frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} = \frac{R+j\omega L}{\gamma} \equiv Z_o = R_o + jX_o \qquad \text{(8-35, 64)}$$

Define the Characteristic Impedance $Z_o$ of the transmission line by

$$Z_o = \frac{R+j\omega L}{\gamma} = \frac{\gamma}{G+j\omega C} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{Z}{Y}} \quad (\Omega) \qquad \text{(8-38)}$$

*Notes)*

i) $Z_o$ and $\gamma$ are independent of $z$ and the length of the line, but depends only on distributed parameters(R, L, G, C) and frequency ($\omega$).

ii) Phasor solution in terms of $Z_o$ from (8-34)⋆:

$$I(z) = \frac{V_o^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{+\gamma z} \qquad \text{(8-34)}\star\star$$

iii) Instantaneous wave solutions in the cosine-reference time domain:

$$v(z,t) = Re[V(z)e^{j\omega t}] = Re[(V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z})e^{j\omega t}]$$

$$= Re[|V_o^+|\, e^{j\omega t - (\alpha + j\beta)z + j\phi^+\}} + |V_o^-|\, e^{j\omega t + (\alpha + j\beta)z + j\phi^-\}}]$$

$$= |V_o^+| e^{-\alpha z} \cos(\underline{\omega t - \beta z} + \phi^+) + |V_o^-| e^{\alpha z} \cos(\underline{\omega t + \beta z} + \phi^-) \qquad \text{(8-33)*}$$

FTW($\omega\beta < 0$: +z direction) BTW($\omega\beta > 0$: −z direction)

attenuation of +z propagating wave attenuation of −z propagating wave

$$i(z,t) = Re\left[\left(\frac{V_o^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{+\gamma z}\right) e^{j\omega t}\right] = Re\left[\left(\frac{V_o^+}{|Z_o| e^{j\phi_{Z_o}}} e^{-\gamma z} - \frac{V_o^-}{|Z_o| e^{j\phi_{Z_o}}} e^{+\gamma z}\right) e^{j\omega t}\right]$$

$$= \frac{|V_o^+|}{|Z_o|} e^{-\alpha z} \cos(\omega t - \beta z + \phi^+ - \phi_{Z_o}) - \frac{|V_o^-|}{|Z_o|} e^{\alpha z} \cos(\omega t + \beta z + \phi^- - \phi_{Z_o}) \qquad \text{(8-34)***}$$

iV) Phase velocity: $u_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{k} = f\lambda$ (7-10, 50, 58)

## B. *Wave Characteristics on an Infinite Transmission Line*

### 1) Wave solutions

For an infinite uniform transmission line, ∃ no reflection waves (BTW).

Then, $V(z) = V^+(z) = V_o^+ e^{-\gamma z}$ (8-36)

$$I(z) = I^+(z) = I_o^+ e^{-\gamma z} = \frac{V_o^+}{Z_o} e^{-\gamma z} \qquad \text{(8-37)}$$

or $$v(z,t) = |V_o^+|\, e^{-\alpha z} \cos(\omega t - \beta z + \phi^+) \qquad \text{(8-36)*}$$

$$i(z,t) = \frac{|V_o^+|}{|Z_o|} e^{-\alpha z} \cos(\omega t - \beta z + \phi^+ - \phi_{Z_o}) \qquad \text{(8-37)*}$$

⇒ The transmission line is characterized by two fundamental properties, $\gamma$ and $Z_o$ which are specified by R, L, G, C, and $\omega$.

### 2) Characteristics in the lossless line

For lossless ($R = 0,\ G = 0$) or high frequency ($\omega L \gg R,\ \ \omega C \gg G$),

a) Propagation constant

(8-12) ⇒ $\gamma = \alpha + j\beta = j\omega\sqrt{LC}$ (8-39)

i.e., $\underline{\alpha = 0 \text{ (no attenuation)},}$ $\beta = \omega\sqrt{LC}$ (8-40, 41)

*(cf)* For lossless unbounded medium, $\gamma = jk,\ \ k = \beta = \omega\sqrt{\mu\epsilon}$ (7-4, 42)

b) Characteristic impedance

(8-38) ⇒ $Z_o = R_o + jX_o = \sqrt{L/C}$ (8-43)

i.e., $R_o = \sqrt{L/C}$ (constant) , $X_o = 0$ (8-44, 45)

*(cf)* For lossless unbounded medium, $\eta = \sqrt{\mu/\epsilon}$ (7-14)

c) Phase velocity

(8–41) in (7–50) $\Rightarrow$ $u_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$ (constant: ind. of $f$) (8–42)

$\Rightarrow$ Distortionless (or nondispersive) line

*(cf)* For lossless unbounded medium, $u_p = \dfrac{1}{\sqrt{\mu\epsilon}} = \dfrac{c}{\sqrt{\mu_r \epsilon_r}}$

### 3) Characteristics in the distortionless lossy line

For the (distortionless) condition of $\dfrac{R}{L} = \dfrac{G}{C}$ (8–46)

a) Propagation constant

(8–46) in (8–12) $\Rightarrow$ $\gamma = \alpha + j\beta = \sqrt{C/L}\,(R + j\omega L)$ (8–47)

i.e., $\underline{\alpha = R\sqrt{C/L}}$ (attenuation), $\beta = \omega\sqrt{LC}$ (8–48, 49)

b) Characteristic impedance

(8–46) in(8–38) $\Rightarrow$ $Z_o = R_o + jX_o = \sqrt{L/C}$ (8–51)

i.e., $R_o = \sqrt{L/C}$ (constant) , $X_o = 0$ (8–52, 53)

c) Phase velocity

(8–49) in (7–50) $\Rightarrow$ $u_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$ (constant: ind. of $f$) (8–42)

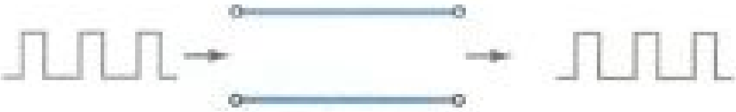
$\Rightarrow$ Distortionless (or nondispersive) lossy line

### 4) Characteristics in the lossy line

For the lossy transmission line,

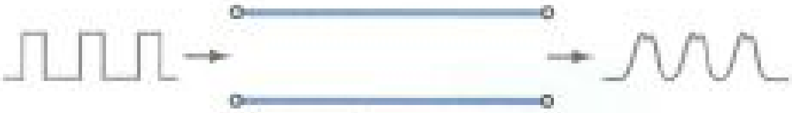
(8–12), (8–38), (7–50) $\Rightarrow$ All of $\gamma$, $\alpha$, $\beta$, $Z_o$ are functions of $f$

$\Rightarrow$ Distorted (or dispersive) lossy line



*Distortionless (or nondispersive) line*

*Distorted (or dispersive) short line*



*Distorted (or dispersive) long line*

For small losses ($\omega L \gg R, \;\; \omega C \gg G$)

(8-38) : $$Z_o = \sqrt{\frac{j\omega L}{j\omega C}\left(\frac{-jR/\omega L+1}{-jG/\omega C+1}\right)} \;\cong\; \sqrt{\frac{L}{C}}\left(1-\frac{1}{2}j\frac{R}{\omega L}\right)\left(1+\frac{1}{2}j\frac{G}{\omega C}\right)$$

$$\cong \sqrt{\frac{L}{C}}\left[1+j\left(\frac{G}{2\omega C}-\frac{R}{2\omega L}\right)\right] = R_o + jX_o \; = |Z_o| \angle \phi_{Z_o} \qquad \text{(8-51)*}$$

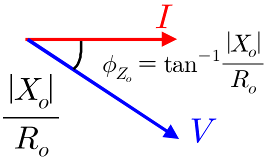
In most practical transmission lines of good conductors and very low leakage dielectrics,

$\dfrac{G}{C} < \dfrac{R}{L}$ in (8-51)* $\Rightarrow$ $X_o < 0$ (capacitive reactance)

Therefore, from (8-36) and (8-37) ,

$$V(z) = Z_o I(z) = (R_o - j|X_o|)\, I(z)$$

$$\Rightarrow V(z) \text{ lags behind } I(z) \text{ by } \phi_{Z_o} = \tan^{-1}\frac{|X_o|}{R_o}$$



## Attenuation constant from power relation

From (8-36) and (8-37) ,

$$V(z) = V_o e^{-(\alpha+j\beta)z}, \qquad I(z) = \frac{V_o}{Z_o} e^{-(\alpha+j\beta)z} \qquad \text{(8-54, 55)}$$

Time-average power along the line (like time-ave Poynting vector 7-79):

$$P(z) = \mathscr{P}_{av}(z) = \frac{1}{2}Re[V(z)I^*(z)] = \frac{V_o^2 e^{-2\alpha z}}{2} Re\left[\frac{1}{Z_o}\right]$$

$$= \frac{V_o^2 e^{-2\alpha z}}{2} Re\left[\frac{1}{R_o + jX_o}\right] = \frac{V_o^2 e^{-2\alpha z}}{2} Re\left[\frac{R_o - jX_o}{R_o^2 + X_o^2}\right] = \frac{V_o^2 R_o e^{-2\alpha z}}{2|Z_o|^2} \qquad \text{(8-56)}$$

From energy conservation law,

Decrease rate of $P(z)$ along z = Time-ave. power loss per length

$$-\frac{\partial P(z)}{\partial z} = P_L(z) \;\overset{(8-56)}{\Rightarrow}\; 2\alpha P(z) = P_L(z),$$

from which the attenuation constant can be found by

$$\alpha = \frac{P_L(z)}{2P(z)} \quad \text{(Np/m)} \qquad \text{(8-57)}$$

For lossy line, $P_L(z) = (1/2)(I^2 R + V^2 G) \overset{(8-54,55)}{=} (V_o^2/2|Z_o|^2)(R + G|Z_o|^2)\, e^{-2\alpha z}$ (8-58)

(8-58) in (8-57) : $\alpha = \dfrac{1}{2R_o}(R + G|Z_o|^2)$ (8-59)

For a low loss line with $Z_o \cong R_o = \sqrt{L/C}$,

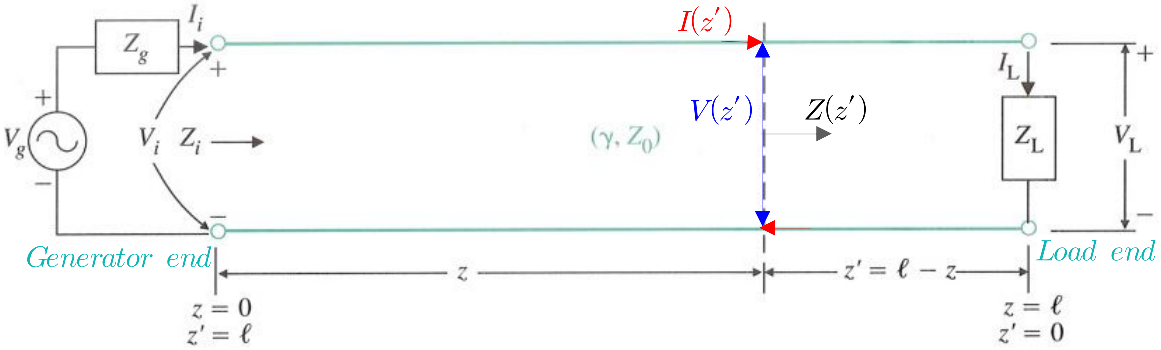
(8-59) becomes $\alpha \cong \dfrac{1}{2}(R\sqrt{C/L} + G\sqrt{L/C})$ *(cf)* (7-47) (8-60)

For a distortionless lossy line with $Z_o = R_o = \sqrt{L/C}$ using (8-46),

(8-60) yields $\alpha = R\sqrt{\dfrac{C}{L}}$ (8-61) = (8-48)

## B. Wave Characteristics on Finite (Terminated) Transmission Lines

FIGURE 8−4



### 1) General solutions

For finite uniform transmission lines, ∃ reflection waves (BTW).

General solutions of (8−10, 11):

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z} \qquad (8\text{−}33,\ 62)$$

$$I(z) = \underset{\substack{\uparrow \\ \text{FTW}}}{I_o^+ e^{-\gamma z}} + \underset{\substack{\uparrow \\ \text{BTW (reflected wave)}}}{I_o^- e^{+\gamma z}} = Z_o^{-1}[V_o^+ e^{-\gamma z} - V_o^- e^{+\gamma z}] \qquad (8\text{−}34,\ 63)$$

where $\frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} \equiv Z_o = R_o + jX_o$ : characteristic impedance (8−64)

BCs : (8−62, 63) at the load end ($z = l$) using (8−64),

$$V_L = V_o^+ e^{-\gamma l} + V_o^- e^{+\gamma l}\ , \qquad I_L = \frac{V_o^+}{Z_o} e^{-\gamma l} - \frac{V_o^-}{Z_o} e^{+\gamma l} \qquad (8\text{−}66,\ 67)$$

Solution of (8−66, 67) in (8−62, 63) with change of variable $z' = l - z$,

$$V(z') = \frac{I_L}{2}[(Z_L + Z_o)\overset{FTW}{e^{\gamma z'}} + (Z_L - Z_o)\overset{BTW}{e^{-\gamma z'}}] = \frac{I_L}{2}(Z_L + Z_o)e^{\gamma z'}[1 + \Gamma e^{-2\gamma z'}] \qquad (8\text{−}72,\ 87)$$

$$= I_L(Z_L \cosh\gamma z' + Z_o \sinh\gamma z') \qquad (8\text{−}74)$$

$$I(z') = \frac{I_L}{2Z_o}[(Z_L + Z_o)e^{\gamma z'} - (Z_L - Z_o)e^{-\gamma z'}] = \frac{I_L}{2Z_o}(Z_L + Z_o)e^{\gamma z'}[1 - \Gamma e^{-2\gamma z'}] \qquad (8\text{−}73,\ 89)$$

$$= \frac{I_L}{Z_o}(Z_L \sinh\gamma z' + Z_o \cosh\gamma z') \qquad (8\text{−}75)$$

where $\Gamma \equiv \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_o}{Z_L + Z_o} = |\Gamma| e^{j\theta_\Gamma}$ = (voltage) reflection coeff. of $Z_L$ (8−88)

: complex value with $|\Gamma| \le 1$

*Notes)*

i) Current reflection coeff. $\equiv \frac{I_o^-}{I_o^+} \overset{(8-63,\ 88)}{=} -\Gamma$ (out of phase)

ii) For $Z_L = Z_o$, $\Gamma = 0$ and $V_o^- = 0$ (no reflection wave)

⇒ The transmission line is said to be matched to the load.

iii) For an open-circuit line ($Z_L \to \infty$), $\Gamma = 1$ and $V_o^- = V_o^+$ (in phase)

iv) For a short-circuit line ($Z_L = 0$), $\Gamma = -1$ and $V_o^- = -V_o^+$ (out of phase)

v) For $Z_L \neq Z_o$, $\exists$ standing voltage and current waves along the line,

standing-wave ratio (SWR): $S = \dfrac{|V_{\max}|}{|V_{\min}|} = \dfrac{1+|\Gamma|}{1-|\Gamma|}$ or $20\log_{10} S$ in (dB) (8-90)

$(8-87)$

$$\Rightarrow \quad |\Gamma| = \frac{S-1}{S+1} \qquad (8\text{-}91)$$

$|\Gamma| = 0 : matched$ $\quad$ $|\Gamma| = 1 : o.c.$ or $s.c.$

$$1 \leq S < \infty$$

vi) For a lossless($\alpha = 0$, $X_o = 0$ ; $\gamma = j\beta$, $Z_o = R_o$) line, (8-87, 89) become

$$V(z') = \frac{I_L}{2}(Z_L + R_o)\, e^{j\beta z'}\left[1 + |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right] \qquad (8\text{-}92)$$

$$I(z') = \frac{I_L}{2R_o}(Z_L + R_o)\, e^{j\beta z'}\left[1 - |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right] \qquad (8\text{-}93)$$

## 2) Input impedance

Impedance $Z(z')$ looking toward the load end at $z'$ from the load:

$$Z(z') \equiv \frac{V(z')}{I(z')} = Z_o \frac{Z_L + Z_o \tanh\gamma z'}{Z_o + Z_L \tanh\gamma z'} = Z_o \frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma e^{-2\gamma z'}} \quad (\Omega) \qquad (8\text{-}77)$$

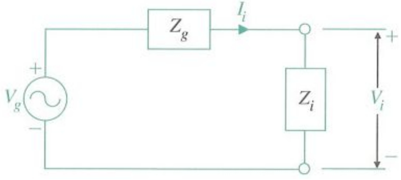
$\dfrac{(8-74)}{(8-75)}$ $\quad$ $\dfrac{(8-87)}{(8-89)}$

Input impedance $Z_i$ looking into the line from the source at $z' = l$:

$$Z_i \equiv \frac{V_i}{I_i} = (Z)_{\substack{z'=l \\ z=0}}$$

$$= Z_o \frac{Z_L + Z_o \tanh\gamma l}{Z_o + Z_L \tanh\gamma l} = Z_o \frac{1 + \Gamma e^{-2\gamma l}}{1 - \Gamma e^{-2\gamma l}} \qquad (8\text{-}78)$$

*FIGURE* $8-5$



*Note)* When $Z_L = Z_o$, $Z_i = Z_o$ irrespective of the length $l$

$\Rightarrow$ The transmission line is matched

For a lossless($\alpha = 0$, $X_o = 0$ ; $\gamma = j\beta$, $Z_o = R_o$) line, (8-78) become

$$Z_i = R_o \frac{Z_L + jR_o \tan\beta l}{R_o + jZ_L \tan\beta l} \qquad (8\text{-}79)$$

From the standpoint of the generator circuit,

$$V_i = Z_i I_i = \frac{Z_i V_g}{Z_g + Z_i} = V_g - I_i Z_g \qquad (8\text{-}94)$$

If $Z_L \neq Z_o$ $but$ $Z_g = Z_o$, reflected at the load and ending at the generator

If $Z_L \neq Z_o$ $but$ $Z_g \neq Z_o$, reflected at both the load and generator

repeating indefinitely

## 3) Standing waves

For lossless lines $(\gamma = j\beta)$, (8-62, 63) with (8-88) becomes

$$V(z') = V_o^+(e^{j\beta z'} + \Gamma e^{-j\beta z'}) \qquad \text{(8-62)*}$$

$$I(z') = \frac{V_o^+}{Z_o}(e^{j\beta z'} - \Gamma e^{-j\beta z'}) \qquad \text{(8-63)*}$$

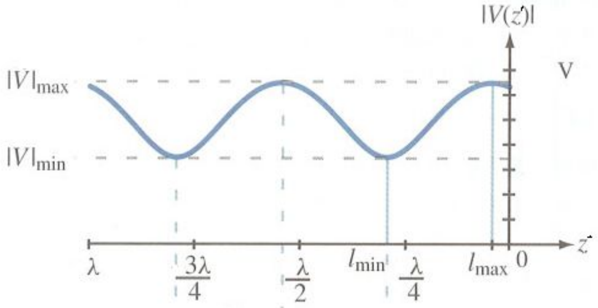
Polar expression of (8-88) in (8-62)* using $|V(z')| = [V(z')\,V^*(z')]^{1/2}$ gives

$$|V(z')| = \{[V_o^+(e^{j\beta z'} + |\Gamma| e^{j\theta_\Gamma} e^{-j\beta z'})] \cdot [(V_o^+)^* e^{-j\beta z'} + |\Gamma| e^{-j\theta_\Gamma} e^{j\beta z'})]\}^{1/2}$$

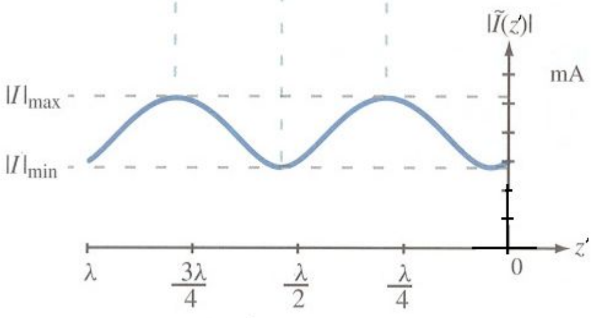
$$= |V_o^+|[1 + |\Gamma|^2 + |\Gamma|(e^{j(2\beta z' - \theta_\Gamma)} + e^{-j(2\beta z' - \theta_\Gamma)})]^{1/2}$$

$$= |V_o^+|[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z' - \theta_\Gamma)]^{1/2} \qquad \text{(8-62)**}$$

⇒ Standing-wave pattern resulted from interference of incid. and reflec. waves



(a) $|V(z')|$ *versus* $z'$



(b) $|I(z')|$ *versus* $z'$

$$|V|_{\max} = |V_o^+|(1 + |\Gamma|) \qquad (3)$$

when $\cos(2\beta z'_{\max} - \theta_\Gamma) = 1 \quad \Rightarrow \quad 2\beta z'_{\max} - \theta_\Gamma = 2n\pi$

$$\Rightarrow \quad z'_{\max} = (\theta_\Gamma + 2n\pi)/2\beta = \left(\frac{\theta_\Gamma}{4\pi} + \frac{n}{2}\right)\lambda \ \text{ for} \begin{cases} n = 1, 2, \ldots.. & \text{for } \theta_\Gamma < 0 \\ n = 0, 1, 2, \ldots.. & \text{for } \theta_\Gamma > 0 \end{cases} \qquad (4)$$

$-\pi \le \theta_\Gamma \le \pi, \quad \beta = 2\pi/\lambda$

$$|V|_{\min} = |V_o^+|(1 - |\Gamma|) \qquad (5)$$

when $\cos(2\beta z'_{\min} - \theta_\Gamma) = -1 \quad \Rightarrow \quad 2\beta z'_{\min} - \theta_\Gamma = (2n+1)\pi$

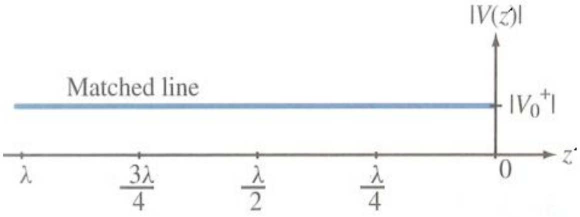
$$\Rightarrow \quad z'_{\min} = \left(\frac{\theta_\Gamma}{4\pi} + \frac{2n+1}{4}\right)\lambda \qquad n = 0,\ 1,\ 2,\ \ldots \qquad (6)$$

1st minimum position $(n=0)$: $\quad l_{\min} = \left(\frac{\theta_\Gamma}{\pi} + 1\right)\frac{\lambda}{4} \qquad (7)$

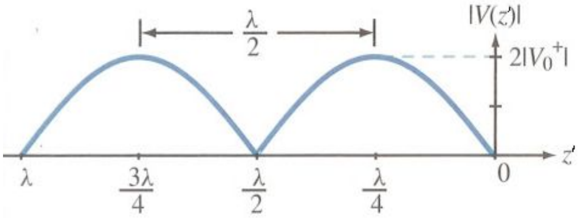
1st maximum position $(n = 0 \text{ or } 1)$: $\quad l_{\max} = \frac{\theta_\Gamma \lambda}{4\pi}$ or $\frac{\theta_\Gamma \lambda}{4\pi} + \frac{\lambda}{2} \qquad (8)$

Using $|z'_{\min} - z'_{\max}| = \lambda/4$,

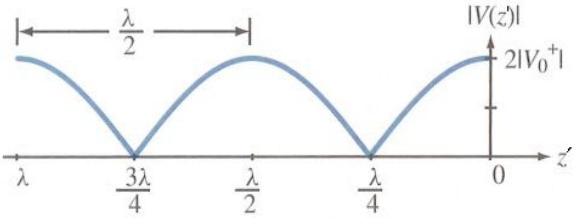
$$l_{\min} = \begin{cases} l_{\max} + \lambda/4 & \text{for } l_{\max} < \lambda/4 \\ l_{\max} - \lambda/4 & \text{for } l_{\max} \ge \lambda/4 \end{cases} \qquad (9)$$



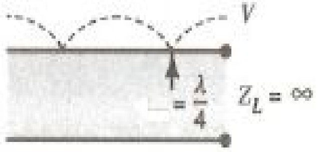
(a) $Z_L = Z_0$ (*matched line*) : $\Gamma = 0$



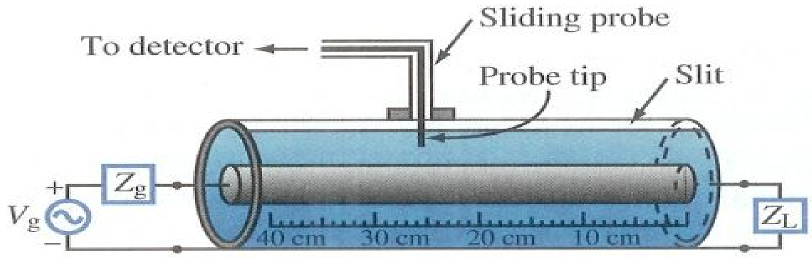
(b) $Z_L = 0$ (short circuit) : $\Gamma = -1$



(c) $Z_L = \infty$ (open circuit) : $\Gamma = 1$

| SWR | $l_{\min}(\lambda)$ | $\Gamma$ | $Z_L(\Omega)$ | Voltage on line |
|---|---|---|---|---|
| 1 | | 0 | 100 |  |
| ∞ | 0 & $\frac{1}{2}$ | $1\angle 180°$ | 0 |  |
| ∞ | $\frac{1}{4}$ | $1\angle 0°$ | ∞ |  |

Determination SWR, $|\Gamma|$, $\theta_\Gamma$ and $Z_L$ by a slotted-line probe:



Measurements of $|V_{\max}|$, $|V_{\min}|$, and $l_{\min}$

$\Rightarrow$ Determine $S$ and $|\Gamma|$ from $S = \dfrac{|V_{\max}|}{|V_{\min}|}$ (8-90) , $|\Gamma| = \dfrac{S-1}{S+1}$ (8-91)

$\theta_\Gamma$ from $l_{\min} = \left(\dfrac{\theta_\Gamma}{\pi} + 1\right)\dfrac{\lambda}{4}$ (7)

$\Gamma$ and $Z_L$ from $\Gamma = |\Gamma| e^{j\theta_\Gamma} = \dfrac{Z_L - Z_o}{Z_L + Z_o}$ (8-88)

*(e.g. 8-5)* For a lossless terminated transmission line,

given $Z_0 = R_o = 50\,(\Omega)$, $S = 3$, $l_{\min} = 5$ (cm), volt. min. dist.= 20 (cm)

a) $\Gamma = ?$ and b) $Z_L = ?$

a) $\lambda = 2 \times 0.2 = 0.4$ , $|\Gamma| = (3-1)/(3+1) = 0.5$

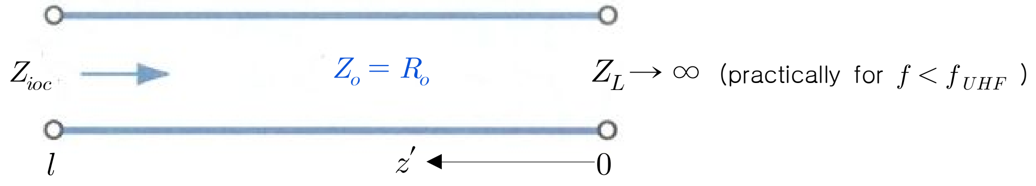
$$l_{\min} = \left(\frac{\theta_\Gamma}{\pi} + 1\right)\frac{\lambda}{4} \quad \Rightarrow \quad \theta_\Gamma = \pi(4 l_{\min}/\lambda - 1) = \pi(4 \times 0.05/0.4 - 1) = -\pi/2$$

$$\therefore \quad \Gamma = |\Gamma| e^{j\theta_\Gamma} = 0.5\, e^{-j\pi/2} = -j0.5$$

b) $\Gamma = \dfrac{Z_L - Z_o}{Z_L + Z_o} \quad \Rightarrow \quad -j0.5 = \dfrac{Z_L - 50}{Z_L + 50} \quad \Rightarrow \quad Z_L = 30 - j40\ (\Omega)$

4) Characteristics in lossless finite lines $(\alpha=0,\ X_o=0\ ;\ \gamma=j\beta,\ Z_o=R_o)$

a) Open-circuited line $(Z_L\to\infty\ ,\ \Gamma=1,\ S\to\infty)$



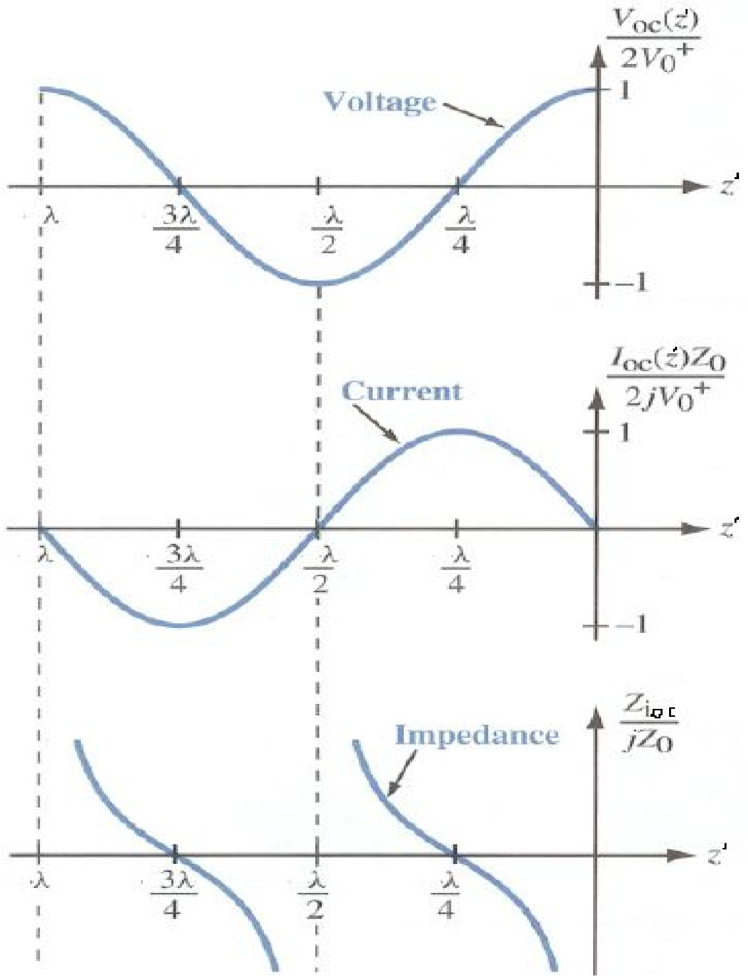
(8-62)⋆ $\Rightarrow$ $V_{oc}(z')=V_o^+(e^{j\beta z'}+e^{-j\beta z'})=2V_o^+\cos\beta z'$ (8-62)⋆$_{oc}$

(8-63)⋆ $\Rightarrow$ $I_{oc}(z')=\dfrac{V_o^+}{R_o}(e^{j\beta z'}-e^{-j\beta z'})=\dfrac{2jV_o^+}{R_o}\sin\beta z'$ (8-63)⋆$_{oc}$

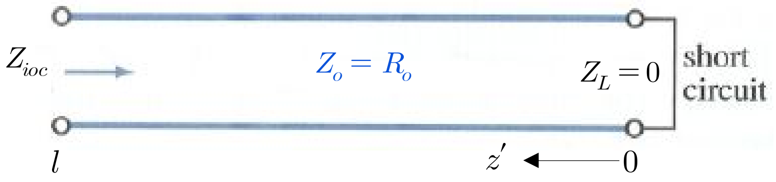
(8-79) $\Rightarrow$ $Z_{ioc}=\dfrac{V_{oc}(l)}{I_{oc}(l)}=jX_{ioc}=\dfrac{-jR_o}{\tan\beta l}=-jR_o\cot\beta l$ (purely reactive) (8-80)

For very shot line $(\beta l=2\pi l/\lambda\ll 1)$,

$Z_{ioc}=jX_{ioc}\cong\dfrac{-jR_o}{\beta l}=-j\dfrac{\sqrt{L/C}}{\omega\sqrt{LC}l}=-j\dfrac{1}{\omega Cl}$ : capacitively reactive (8-81)



### b) Short-circuited line ($Z_L = 0$, $\Gamma = -1$, $S \to \infty$)



(8-62)* $\Rightarrow$ $V_{oc}(z') = V_o^+(e^{j\beta z'} - e^{-j\beta z'}) = 2jV_o^+\sin\beta z'$ (8-62)*$_{sc}$

(8-63)* $\Rightarrow$ $I_{oc}(z') = \dfrac{V_o^+}{R_o}(e^{j\beta z'} + e^{-j\beta z'}) = \dfrac{2V_o^+}{R_o}\cos\beta z'$ (8-63)*$_{sc}$

(8-79) $\Rightarrow$ $Z_{isc} = \dfrac{V_{sc}(l)}{I_{sc}(l)} = jX_{isc} = jR_o\tan\beta l = jR_o\tan\left(\dfrac{2\pi l}{\lambda}\right)$ (8-82)

$\Rightarrow$ purely reactive (inductive or capacitive depending on $\tan\beta l$)

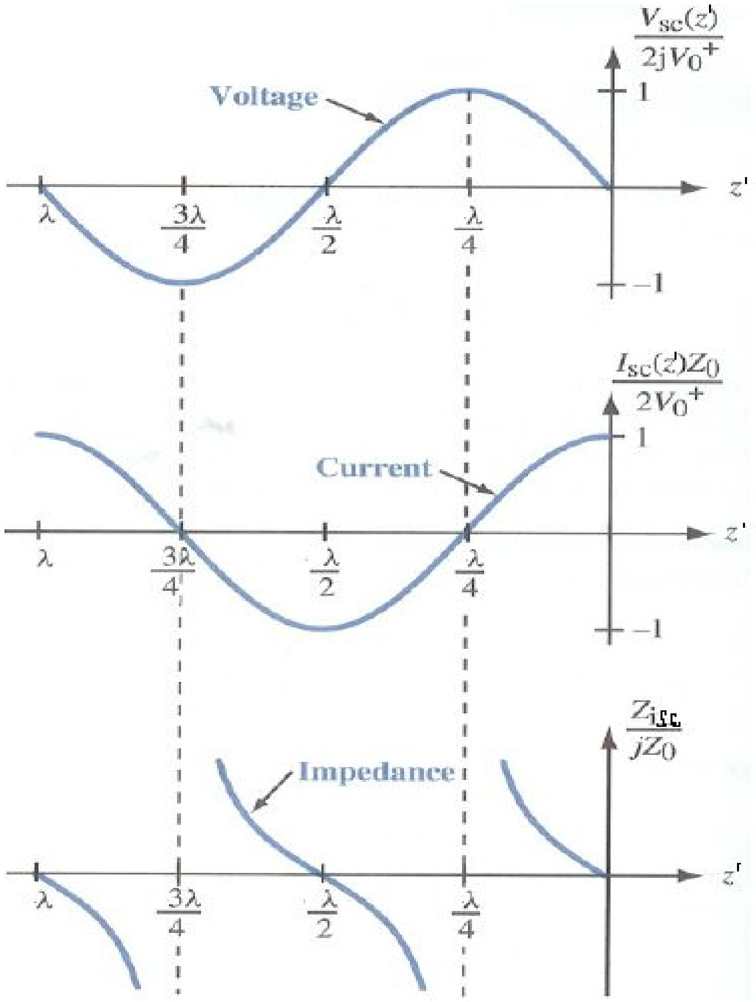
$\Rightarrow$ Proper choice of $l$ of s.c. line can substitute for inductors and capacitors.

For very shot line ($\beta l = 2\pi l/\lambda \ll 1$),

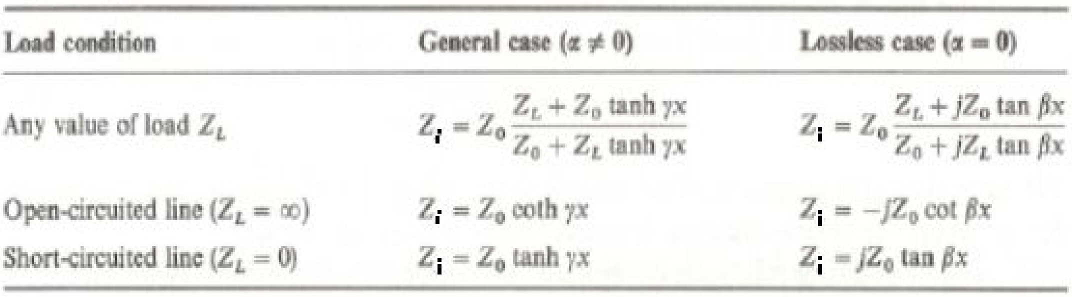
$Z_{isc} = jX_{isc} \cong jR_o\beta l = j\sqrt{L/C}\,\omega\sqrt{LC}\,l = j\omega L\,l$ : inductively reactive (8-83)

For $\beta l = \pi/2$, $i.e.$, $l = \lambda/4$ , $Z_{isc} \to \infty$

$\Rightarrow$ A s.c. quarter-wavelength line is effectively an o.c. line.



**Input impedance of terminated transmission line**[†]



| Load condition | General case ($\alpha \neq 0$) | Lossless case ($\alpha = 0$) |
|---|---|---|
| Any value of load $Z_L$ | $Z_i = Z_0 \dfrac{Z_L + Z_0 \tanh \gamma x}{Z_0 + Z_L \tanh \gamma x}$ | $Z_i = Z_0 \dfrac{Z_L + jZ_0 \tan \beta x}{Z_0 + jZ_L \tan \beta x}$ |
| Open-circuited line ($Z_L = \infty$) | $Z_i = Z_0 \coth \gamma x$ | $Z_i = -jZ_0 \cot \beta x$ |
| Short-circuited line ($Z_L = 0$) | $Z_i = Z_0 \tanh \gamma x$ | $Z_i = jZ_0 \tan \beta x$ |

[†] $\gamma = \alpha + j\beta$, where $\alpha$ = attenuation constant in nepers per meter, $\beta = 2\pi/\lambda$ = phase constant in radians per meter, and $\lambda$ = wavelength.

### c) Half-wavelength lossless line $(l = n\lambda/2)$

For $l = n\lambda/2$ $(n = 1, 2, 3, .....)$, $\tan\beta l = \tan\left(\dfrac{2\pi l}{\lambda}\right) = \tan n\pi = 0$ in (8-79):

$$Z_i = Z_L \quad \text{for} \quad l = n\lambda/2 \quad (n = 1, 2, 3, .....) \qquad (10)$$

$\Rightarrow$ A half-wave lossless line transfers a load impedance to the generator end without change.

$\Rightarrow$ The generator induce the same $V$ and $I$ across the load as when the line does not exist there.

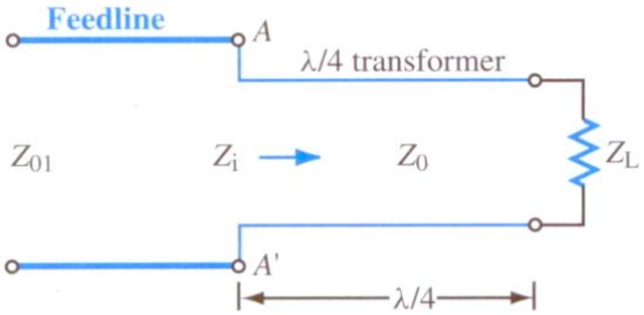
### d) Quater-wavelength lossless line $(l = \lambda/4 + n\lambda/2)$

For $l = \lambda/4 + n\lambda/2$ $(n = 0, 1, 2, 3, .....)$,

$$\tan\beta l = \tan\left(\frac{2\pi l}{\lambda}\right) = \tan(\pi/2 + n\pi) \rightarrow \infty$$ in (8-79):

$$Z_i = \frac{Z_o^2}{Z_L} \quad \text{for} \quad l = \lambda/4 + n\lambda/2 \quad (n = 0, 1, 2, 3, .....) \qquad (8\text{-}111)$$

$\Rightarrow$ Quater-wave transformer to eliminate reflections at the load terminal.



If $Z_i = Z_{01}$ , no reflections at the terminal AA'.

By (8-111), $Z_{01} = Z_o^2/Z_L \quad \Rightarrow \quad Z_o = \sqrt{Z_{01}Z_L}$ (11)

Therefore, if a quarter-wave lossless line having a characteristic impedance of $Z_o = \sqrt{Z_{01}Z_L}$ is inserted between the feedline and the load, there are no reflections at the terminal and all the incident power is transferred into the load.

e) Determination of $Z_o$ and $\gamma$ by input impedance measurements

From (8-78) $Z_i(l) = Z_o \dfrac{Z_L + Z_o \tanh\gamma l}{Z_o + Z_L \tanh\gamma l}$,

$Z_{ioc} = Z_o \coth\gamma l$ for $Z_L \to \infty$ and $Z_{isc} = Z_o \tanh\gamma l$ for $Z_L = 0$ (8-84a, b)

$$\Rightarrow \quad Z_o = \sqrt{Z_{ioc} Z_{isc}} \quad (\Omega) \tag{8-85}$$

$$\gamma = \frac{1}{l}\tanh^{-1}\sqrt{\frac{Z_{isc}}{Z_{ioc}}} \quad (\mathrm{m}^{-1}) \tag{8-86}$$

d) Power flow along the transmission lines

Time-average power flow along the line by analogy with (7-79),

$$P(z') = \mathscr{P}_{av}(z') = \frac{1}{2} Re[V\ I^*] \tag{12}$$

For lossless lines ($\gamma = j\beta$),

$$V(z') = V_o^+ (e^{j\beta z'} + \Gamma e^{-j\beta z'}) \tag{8-62)*}$$

$$I(z') = \frac{V_o^+}{Z_o}(e^{j\beta z'} - \Gamma e^{-j\beta z'}) \tag{8-63)*}$$

At the load ($z' = 0$), the incident and reflected waves are

$$V_i(0) = V_o^+, \qquad I_i(0) = V_o^+ / Z_o \tag{13}$$

$$V_r(0) = \Gamma V_o^+, \qquad I_i(0) = -\Gamma V_o^+ / Z_o \tag{14}$$

(13), (14) in (12) :

$$P_i = \frac{1}{2} Re[V_o^+ V_o^{+*} / Z_o] = \frac{|V_o^+|^2}{2Z_o} \tag{12)$_i$}$$

$$P_r = \frac{1}{2} Re[\Gamma V_o^+ (-\Gamma^* V_o^{+*} / Z_o)] = -|\Gamma|^2 \frac{|V_o^+|^2}{2Z_o} \tag{12)$_r$}$$

Net average power delivered to the load :

$$P = P_i + P_r = \frac{|V_o^+|^2}{2Z_o}(1 - |\Gamma|^2) \qquad (cf)\ (7\text{-}104) \tag{15}$$

## Homework Set 4

1) P.8-5 2) P.8-7 3) P.8-9

4) P.8-11 5) P.8-13 6) P.8-15