2010-3-30

노트 제목 2010-03-27

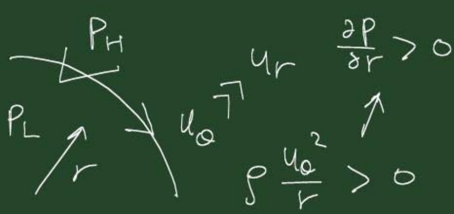
⊙ Euler-n eq.

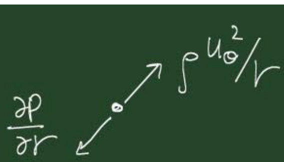
In a cylindrical coordinate,

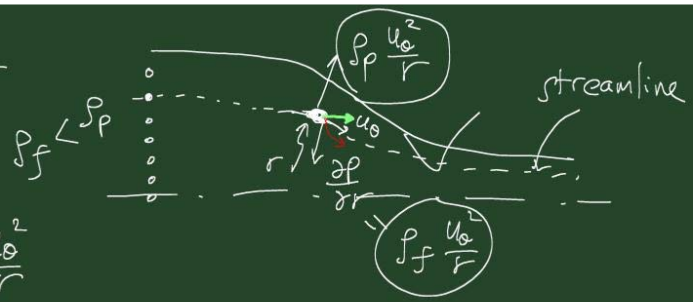
$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial r} + \cancelto{0}{\mu}\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + \rho f_r$$

In many cases, the press. grad. term is balanced with $\rho u_\theta^2/r$; i.e.,

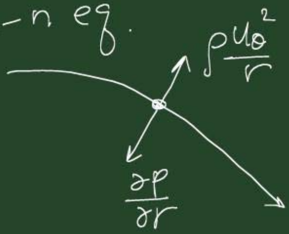
$$\rho\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}$$







Euler-n eq.



$$\rho\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r} \;:\; \text{Euler-n eq.}$$

PART II : Ideal Fluid Flow

neglecting viscous / compressible effects $\longrightarrow$ mathematical simplification

$\nabla \cdot \underline{u} = 0$

$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\,\underline{u} = -\frac{1}{\rho}\nabla p + \underline{f}$ Euler equation

1st order PDE in space

$\rightarrow$ only one b.c. per boundary

$\Rightarrow$ $\boxed{\underline{u} \cdot \underline{n} = \underline{U} \cdot \underline{n}}$ on solid boundaries

$\hookrightarrow$ velocity of the body

$\hookrightarrow$ Surface of the body must be a streamline.

If the flow of an ideal fluid ($\nu = 0$) about a body originates in an irrotational flow ($\underline{\omega} = 0$) such as a uniform flow, then the flow will remain irrotational even near the body (Kelvin Th.).

$\rightarrow$ $\underline{\omega} \equiv 0 \longrightarrow \nabla \times \underline{u} = 0$

irrotational $\rightarrow$ $\boxed{\underline{u} = \nabla \phi}$ ϕ : velocity potential

"potential flow"

$\nabla \cdot \underline{u} = 0 \rightarrow \boxed{\nabla^2 \phi = 0} \Leftarrow$ linear eq !

superposition principle

solve this eq. to get ϕ $\rightarrow$ get $\underline{u} = \nabla \phi \rightarrow p$?

Use Bernoulli eq.

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi - G = F(t)$$

$\rightarrow$ simple algebraic eq. for p.

Ch. 4 Two-dimensional potential flows

stream function, vel. potential $\longrightarrow$ "complex potential"

- stream function : a ft. which automatically satisfies the continuity eq.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

ψ : stream function
valid for all 2D flows.

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2\psi$$

$\mu = 0$ $\quad$ $\underline{\omega} = 0$

For potential flow (inviscid irrotational flow),

$$\boxed{\nabla^2\psi = 0} \rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

solve this eq. to get ψ $\rightarrow$ get u & v $\rightarrow$ get p from Bernoulli eq.

- The flow lines which correspond to ψ = const are the streamlines of the flow field.

$\psi(x,y)$

pf. $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$

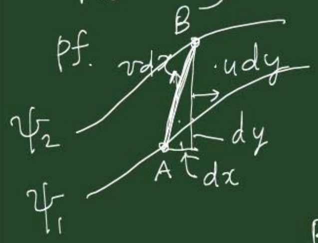
$= -v\,dx + u\,dy = 0$ for const ψ

$\rightarrow \left.\frac{dy}{dx}\right|_{\psi} = \frac{v}{u}$: streamline for const ψ

(streamline : $\frac{dx}{u} = \frac{dy}{v} = ds$)

- The difference of the stream ft's values between two streamlines gives the volume of fluid which is flowing between these two streamlines.

pf.



total vol. of fluid flowing bet. ψ_1 and ψ_2 per unit time

$$Q = \int_A^B u\,dy - \int_A^B v\,dx$$

$$\int_A^B \left(d\psi = -v\,dx + u\,dy \right)$$

$$\rightarrow \psi_2 - \psi_1 = \int_A^B -v\,dx + \int_A^B u\,dy = Q$$

- The streamlines ($\psi = \text{const}$) and the (equi)potential lines ($\phi = \text{const}$) are orthogonal to each other.

pf. $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u\,dx + v\,dy = 0$ for $\phi = \text{const}$.

$$\rightarrow \left.\frac{dy}{dx}\right|_{\phi} = -\frac{u}{v}$$

$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v\,dx + u\,dy = 0$ for $\psi = \text{const}$.

$$\rightarrow \left.\frac{dy}{dx}\right|_{\psi} = \frac{v}{u}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\phi} \cdot \left.\frac{dy}{dx}\right|_{\psi} = \left(-\frac{u}{v}\right)\left(\frac{v}{u}\right) = -1$$

$$\Rightarrow (\psi = \text{const}) \perp (\phi = \text{const}).$$

⓪ Complex potential and complex velocity

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned}\right) \text{ Cauchy - Riemann eq}$$

$F(z) \equiv \phi(x, y) + i\psi(x, y)$: complex potential

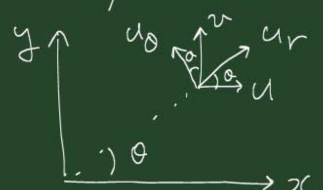
$z = x + iy$

$$\boxed{W(z) = \frac{dF}{dz}} = \frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv \text{ : Complex velocity}$$

$$= \frac{\partial F}{\partial (iy)} = \frac{1}{i}\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} = u - iv$$

$$W\bar{W} = (u - iv)(u + iv) = u^2 + v^2$$

In cylindrical coord.



$$u = u_r\cos\theta - u_\theta\sin\theta$$

$$v = u_r\sin\theta + u_\theta\cos\theta$$

$$\underline{W = u - iv}$$

$$= (u_r\cos\theta - u_\theta\sin\theta) - i(u_r\sin\theta + u_\theta\cos\theta)$$

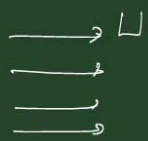
$$= u_r e^{-i\theta} - i u_\theta e^{-i\theta}$$

$$\therefore \underline{W = (u_r - iu_\theta)e^{-i\theta}} \text{ in cyl. coor.}$$

① uniform flow : $F(z) = cz$

$$W(z) = \frac{dF}{dz} = c = u - iv \Rightarrow \begin{aligned} u &= c = U \\ v &= 0 \end{aligned}$$

$$\therefore \boxed{F(z) = Uz}$$



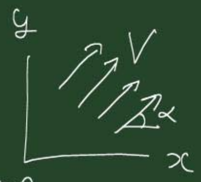
$F(z) = -iVz \rightarrow W(z) = \frac{dF}{dz} = -iV = u - iv$

$\rightarrow u = 0, \quad v = V$

$\underline{F(z) = Ve^{-i\alpha}z} \rightarrow W(z) = \frac{dF}{dz} = Ve^{-i\alpha}$

$= V\cos\alpha - iV\sin\alpha$

$= u - iv$



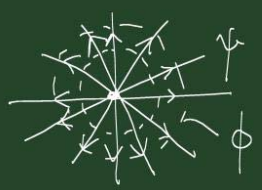
$u = V\cos\alpha, \quad v = V\sin\alpha$

② Source/sink : $F(z) = c\ln z \quad (z = re^{i\theta})$

$= c\ln re^{i\theta}$

$= c\ln r + ic\theta = \phi + i\psi$

$\phi = c\ln r, \quad \psi = c\theta$



direction?

$W(z) = \frac{dF}{dz} = \frac{c}{z} = \frac{c}{r}e^{-i\theta}$

$= (u_r - iu_\theta)e^{-i\theta}$

$\rightarrow u_r = \frac{c}{r}, \quad u_\theta = 0$

$c > 0$ source

$c < 0$ sink

$r = 0$; singular pt.

strength $m \equiv \int u_r dl = \int_0^{2\pi} u_r r d\theta = 2\pi c$

$\rightarrow \boxed{F(z) = \frac{m}{2\pi}\ln z}$

$F(z) = \frac{m}{2\pi}\ln(z - z_0)$

source located at $z = z_0$