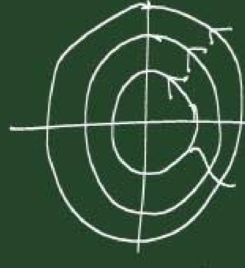
노트 제목 2010-04-06

- Vortex flow $F(z) = -ic \ln z$

$$F(z) = -ic \ln re^{i\theta} = c\theta - ic \ln r$$

$$\phi = c\theta, \quad \psi = -c \ln r$$



$$W(z) = \frac{dF}{dz} = -ic \cdot \frac{1}{z} = -i\frac{c}{r}e^{-i\theta}$$

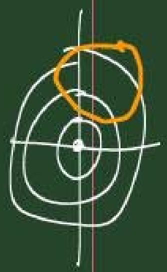
$$u_r = 0, \quad \boxed{u_\theta = \frac{c}{r}} \qquad c > 0 \rightarrow u_\theta > 0 \quad \text{positive vortex}$$

strength → circulation

$$\Gamma = \oint \underline{u} \cdot d\underline{l} = \int_0^{2\pi} u_\theta r d\theta = 2\pi c$$

$$\rightarrow F(z) = -i\frac{\Gamma}{2\pi} \ln z \qquad \left(F(z) = -i\frac{\Gamma}{2\pi} \ln(z - z_0) \right)$$

$z = 0$: singular pt.



that is, the line vortex is located at $z = 0$

For any closed contour which does not include the singularity, the circulation is zero and the flow is irrotational (free vortex).

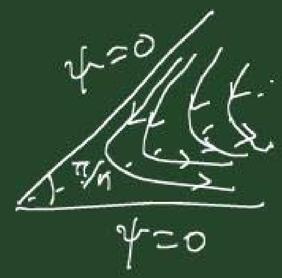
(cf. solid-body rotation)

- Flow in a sector $F(z) = Uz^n$

The flows in sharp bends or sectors are represented by complex potentials which are proportional to z^n.

$$F(z) = Uz^n = Ur^n e^{in\theta}$$

$$= \underbrace{Ur^n \cos n\theta}_{\phi} + i\,\underbrace{Ur^n \sin n\theta}_{\psi}$$



$\psi = 0$: $\sin n\theta = 0 \rightarrow \theta = 0, \frac{\pi}{n}$

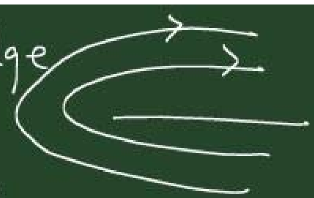
$$W(z) = nUz^{n-1} = nUr^{n-1}e^{i(n-1)\theta}$$

$$= (\underbrace{nUr^{n-1}\cos n\theta}_{= u_r} + i\underbrace{nUr^{n-1}\sin n\theta}_{= -u_\theta})\, e^{-i\theta}$$

$n=1$: uniform flow

$n=2$: right-angled corner

$n = \frac{1}{2}$: flow around a sharp edge



$$F(z) = Uz^{\frac{1}{2}} = Ur^{\frac{1}{2}}e^{i\frac{\theta}{2}}$$

$$= \underbrace{(Ur^{\frac{1}{2}}\cos\tfrac{\theta}{2})}_{\phi} + i\,\underbrace{Ur^{\frac{1}{2}}\sin\tfrac{\theta}{2}}_{\psi}$$

$\psi = 0$

$\rightarrow \theta = 0,\ 2\pi$

$$W(z) = \frac{U}{2z^{\frac{1}{2}}} = \frac{U}{2r^{\frac{1}{2}}}e^{-i\frac{\theta}{2}}$$

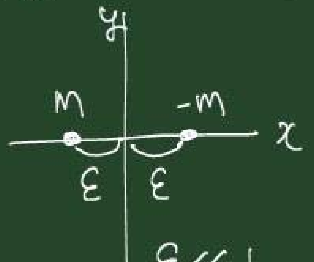
$$= \frac{U}{2r^{\frac{1}{2}}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)e^{-i\theta}$$

u_r $\rightarrow -u_\theta$

$r = 0$ (corner): singular pt.

For $F(z) = Uz^n$, $\frac{\pi}{n} > \pi \rightarrow n < 1 \rightarrow$ corner is singular

- Flow due to a doublet — superposition



$\varepsilon \ll 1$

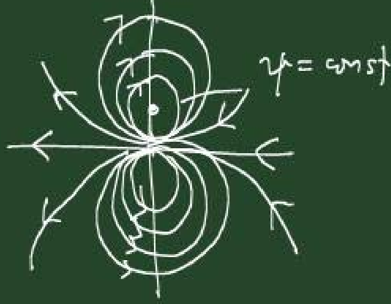
$m\varepsilon$: finite

$$F(z) = \frac{m}{2\pi}\ln(z+\varepsilon) - \frac{m}{2\pi}\ln(z-\varepsilon)$$
$$= \frac{m}{2\pi}\ln\frac{z+\varepsilon}{z-\varepsilon} = \frac{m}{2\pi}\ln\frac{1+\varepsilon/z}{1-\varepsilon/z}$$
$$= \frac{m}{2\pi}\ln\left[1+2\frac{\varepsilon}{z}\left(1+\frac{\varepsilon}{z}+\cdots\right)\right]$$
$$\simeq \frac{m}{2\pi}\ln\left(1+2\frac{\varepsilon}{z}\right)$$
$$\simeq \frac{m}{2\pi}\cdot\frac{2\varepsilon}{z} = \frac{m\varepsilon}{\pi z} = \frac{\mu}{z}$$

$$F(z) = \frac{\mu}{z} = \frac{\mu}{x+iy} = \mu\cdot\frac{x-iy}{x^2+y^2}$$

$$\psi = -\mu\frac{y}{x^2+y^2}$$

$$\rightarrow x^2+y^2+\frac{\mu}{\psi}y = 0$$
$$\rightarrow x^2+\left(y+\frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2$$



$$W(z) = -\frac{\mu}{z^2} = -\frac{\mu}{r^2}e^{-i2\theta}$$
$$= -\frac{\mu}{r^2}(\cos\theta - i\sin\theta)e^{-i\theta}$$

$$\rightarrow u_r = -\frac{\mu}{r^2}\cos\theta,\quad u_\theta = -\frac{\mu}{r^2}\sin\theta$$

Singularity at $r=0$: doublet

$$\left(F(z) = \frac{\mu}{z-z_0}\right)$$

- Circular cylinder without circulation

$F(z) = Uz + \frac{\mu}{z}$ ← uniform + doublet

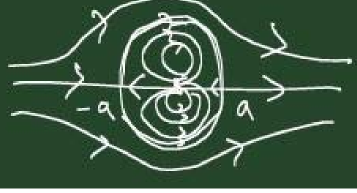
For a certain choice of μ, the circle $r=a$ becomes a streamline.

On $r=a$, $z = ae^{i\theta}$

$\rightarrow F(z) = Uae^{i\theta} + \frac{\mu}{a}e^{-i\theta}$

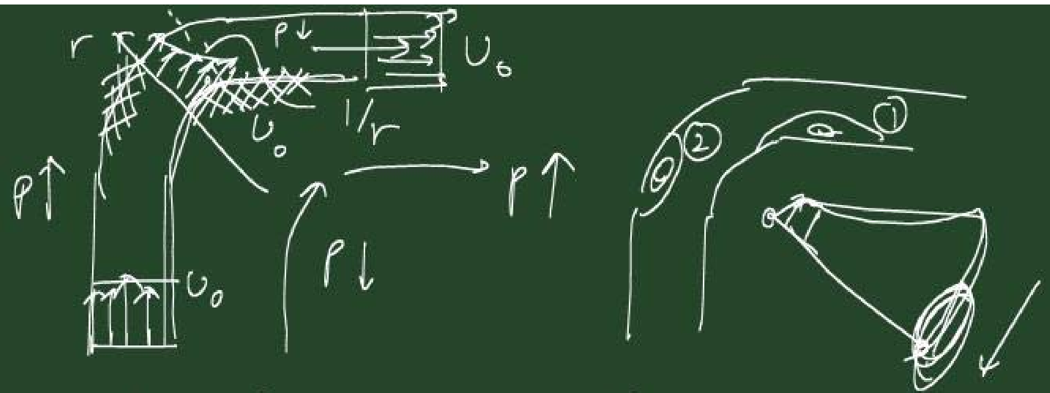
$= (Ua + \frac{\mu}{a})\cos\theta + i(Ua - \frac{\mu}{a})\sin\theta$

$\psi = (Ua - \frac{\mu}{a})\sin\theta = 0 \rightarrow \mu = Ua^2$



$\therefore F(z) = Uz + \frac{Ua^2}{z}$

$= U(z + \frac{a^2}{z})$



- Circular cylinder with circulation

$F(z) = U(z + \frac{a^2}{z}) + \frac{i\Gamma}{2\pi}\ln z + c$

↑ to make $\psi = 0$ on the cylinder surface.

$z = ae^{i\theta}$

$F(z) = U(ae^{i\theta} + ae^{-i\theta}) + i\frac{\Gamma}{2\pi}\ln ae^{i\theta} + c$

$= 2Ua\cos\theta - \frac{\Gamma}{2\pi}\theta + i\frac{\Gamma}{2\pi}\ln a + c$

imaginary part $\psi \equiv 0 = i\frac{\Gamma}{2\pi}\ln a + c$

$\rightarrow c = -i\frac{\Gamma}{2\pi}\ln a$

$\therefore F(z) = U\left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln\frac{z}{a}$

$W(z) = \frac{dF}{dz} = U\left(1 - \frac{a^2}{z^2}\right) + i\frac{\Gamma}{2\pi}\cdot\frac{1}{z}$

$= U\left(1 - \frac{a^2}{r^2}e^{-i2\theta}\right) + i\frac{\Gamma}{2\pi r}e^{-i\theta}$

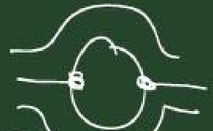
$= \left[\underbrace{U\left(1-\frac{a^2}{r^2}\right)\cos\theta}_{u_r} + i\underbrace{\left\{U\left(1+\frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right\}}_{-u_\theta}\right]e^{-i\theta}$

on $r=a$, $u_r = 0$ good! (and should be!)

$u_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi a}$

stagnation pt.: $u_r = u_\theta = 0$

$\rightarrow u_\theta = 0$: $\boxed{\sin\theta_s = -\frac{\Gamma}{4\pi U a}}$



For $\Gamma = 0$: $\sin\theta_s = 0 \rightarrow \theta_s = 0, \pi$

For $\Gamma \neq 0$:

① $0 < \frac{\Gamma}{4\pi U a} < 1$, $\sin\theta_s < 0$

② $\frac{\Gamma}{4\pi U a} = 1$, $\sin\theta_s = -1$, $\theta_s = \frac{3\pi}{2}$

③ $\frac{\Gamma}{4\pi U a} > 1$, $\sin\theta_s < -1$ X

$\rightarrow$ stag. pt. is not on the cylinder surface.

(r_s, θ_s) $\quad u_r = U\left(1-\frac{a^2}{r_s^2}\right)\cos\theta_s = 0 \rightarrow \cos\theta_s = 0 \rightarrow \theta_s = \cancel{\pi/2}\ \ 3\pi/2$

$$-u_\theta = U\left(1+\frac{a^2}{r_s^2}\right)\underbrace{\sin\theta_s}_{=\,-1} + \frac{\Gamma}{2\pi r_s} = 0$$

$\downarrow$

$$r_s^2 - \frac{\Gamma}{2\pi U} r_s + a^2 = 0$$

$$\begin{cases} \dfrac{r_s}{a} = \dfrac{\Gamma}{4\pi U a}\left[1 \pm \sqrt{1-\left(\dfrac{4\pi U a}{\Gamma}\right)^2}\right] \quad \because r_s > a \\ \theta_s = 3\pi/2 \end{cases}$$

vel. sym. wrt y → no drag

asym. 〃 x → lift $\xrightarrow{\text{Bernoulli}}$ positive lift.