Ch. 5 Three-dimensional potential flows

노트 제목 2010-04-22

(axisymmetric flows)

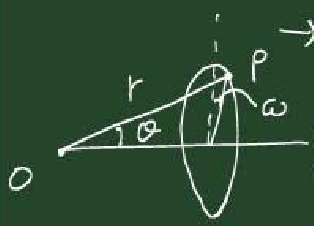
ψ does not exist. (r, θ) coord.

- Velocity potential

$$\nabla \times \underline{u} = 0 \rightarrow \underline{u} = \nabla \phi \rightarrow u_r = \frac{\partial \phi}{\partial r}, \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\nabla \cdot \underline{u} = 0 \rightarrow \nabla^2 \phi = 0$$

$$\rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$



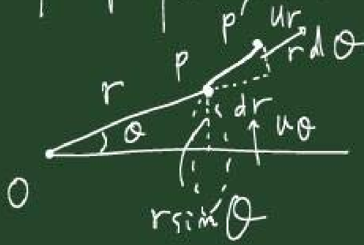
$(u_\omega = 0)$

- Stokes stream ft.

$$\nabla \cdot \underline{u} = 0 \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) = 0$$

$$\rightarrow u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}.$$

ψ physical meaning

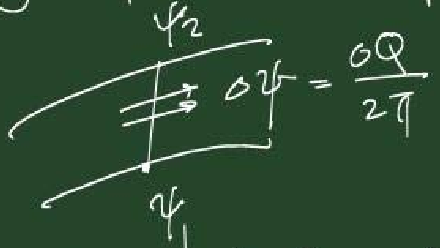


$$\underset{(p-p')}{dQ} = 2\pi r \sin\theta \left(u_r r d\theta - u_\theta dr\right)$$

$$= 2\pi \cancel{r} \cancel{\sin\theta} \left(\frac{1}{\cancel{r^2} \cancel{\sin\theta}}\frac{\partial \psi}{\partial \theta} \cdot \cancel{r} d\theta + \frac{1}{\cancel{r}\cancel{\sin\theta}}\frac{\partial \psi}{\partial r} dr\right)$$

$$= 2\pi \left(\frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial r} dr\right) = 2\pi d\psi$$

$\therefore$ $2\pi d\psi$ corresponds to the volume of fluid crossing the surface of revolution which is formed by rotating the position vector pp' around the reference axis.



- Solution of the potential equation $\nabla^2\phi = 0$

Separation of variables $\underline{\phi(r,\theta) = R(r)\,T(\theta)}$

$$\rightarrow \frac{T}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0$$

$$\rightarrow \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{ft. } r} = \underbrace{-\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)}_{\text{ft. } \theta} = \text{const} = l(l+1)$$

$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$: Euler-Cauchy eq.

$$\rightarrow R_l(r) = A_l r^l + B_l r^{-(l+1)}$$

$-\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = l(l+1)$: Legendre eq.

Let $x = \cos\theta$

$$\rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right] + l(l+1)T = 0$$

$$\rightarrow T_l(\theta) = \boxed{C_l P_l(\cos\theta)} + D_l Q_l(\cos\theta)$$

$\uparrow$ 1st kind $\qquad\qquad$ $\uparrow$ 2nd kind

for all ℓ, $Q_\ell(\pm 1) \to \infty$

$P_\ell(\pm 1)$ diverges unless ℓ is an integer.

$$\therefore \phi_\ell(r,\theta) = (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$$

$$\therefore \boxed{\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$

Legendre ft. $P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

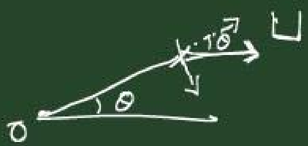
$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, ...

We use this sol. for some specific potential flows.

- Uniform flow

$B_\ell = 0$ and $A_\ell = \begin{cases} 0 & \text{for } \ell \neq 1 \\ U & \text{for } \ell = 1 \end{cases}$ $\qquad P_1(\cos\theta) = \cos\theta$

$\rightarrow \phi(r,\theta) = Ur\cos\theta$



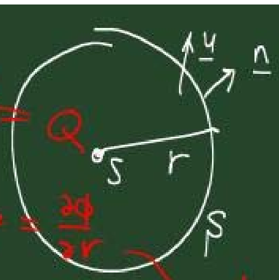
$$\begin{cases} u_r = \dfrac{\partial\phi}{\partial r} = U\cos\theta = \dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta} \\ u_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -U\sin\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial r} \end{cases}$$

$$\psi = \frac{1}{2}Ur^2\sin\theta$$

- Source and sink

$A_\ell = 0$ and $B_\ell = \begin{cases} 0 & \text{for } \ell \neq 0 \\ B_0 & \text{for } \ell = 0 \end{cases}$, $\quad P_0(\cos\theta) = 1$

$$\rightarrow \phi(r,\theta) = \frac{B_0}{r}$$



$$u_r 4\pi r^2 = Q$$

$$\downarrow$$

$$u_r = \frac{Q}{4\pi r^2} = \frac{\partial\phi}{\partial r} \rightarrow \phi = -\frac{Q}{4\pi r}$$

$$\begin{cases} u_r = -\dfrac{B_0}{r^2} \\ u_\theta = 0 \end{cases}$$

$$Q = \int_S \underline{u}\cdot\underline{n}\, dS = -\frac{B_0}{r^2}\cdot 4\pi r^2 = -4\pi B_0$$

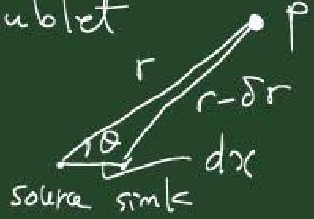
$$\therefore \phi(r,\theta) = -\frac{Q}{4\pi r}$$

$Q > 0$ source

$Q < 0$ sink

$$\rightarrow \psi = -\frac{Q}{4\pi}(1+\cos\theta)$$

- Doublet



$$\delta r \simeq \delta x \cos\theta$$

$$\phi(r,\theta) = -\frac{Q}{4\pi r} + \frac{Q}{4\pi(r-\delta r)}$$

$$= -\frac{Q}{4\pi}\left(\frac{1}{r} - \frac{1}{r-\delta r}\right) \simeq +\frac{Q\,\delta r}{4\pi r^2}$$

$$\simeq \frac{Q}{4\pi r^2}\,\delta x\cos\theta = \frac{Q\,\delta x}{4\pi r^2}\cos\theta$$

$$= \frac{\mu}{4\pi r^2}\cos\theta$$

($Q\delta x \rightarrow$ finite as $\delta x \rightarrow 0$)

$$u_r = \frac{\partial\phi}{\partial r} = -\frac{\mu}{2\pi r^3}\cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

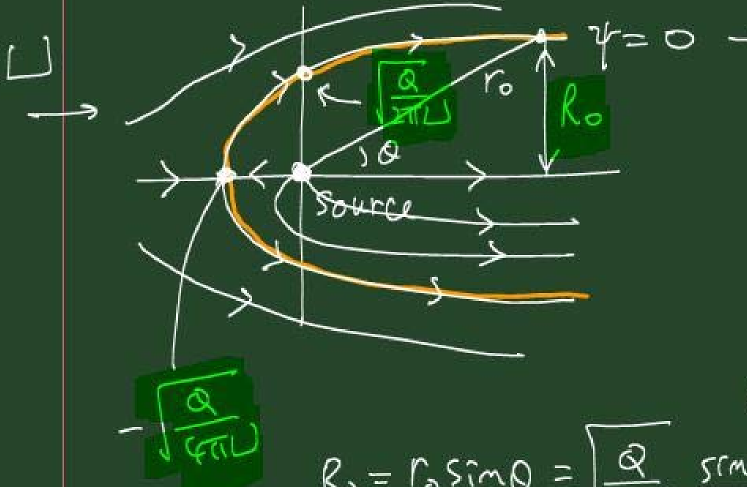
$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{-\mu}{4\pi r^3}\sin\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$$\rightarrow \psi = -\frac{\mu}{4\pi r}\sin^2\theta$$

• Flow near a blunt nose : uniform flow + source

$$\psi = \frac{1}{2}Ur^2\sin^2\theta - \frac{Q}{4\pi}(1+\cos\theta)$$

$\left(1+\cos\theta = 2\cos^2\frac{\theta}{2}, \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \right.$



$\psi = 0 \rightarrow$

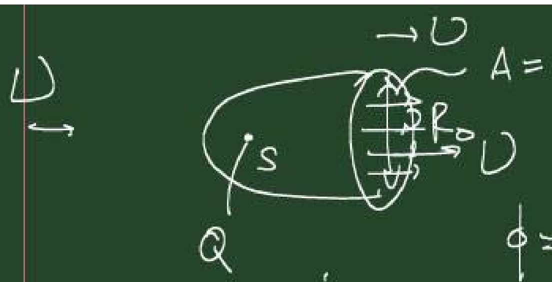
$$r_0 = \sqrt{\frac{Q}{4\pi U}\frac{1}{\sin^2\frac{\theta}{2}}}$$

$\theta = 0 :\ r_0 \rightarrow \infty$

$\theta = \frac{\pi}{2} :\ r_0 = \sqrt{\frac{Q}{2\pi U}}$

$\theta = \pi :\ r_0 = \sqrt{\frac{Q}{4\pi U}}$

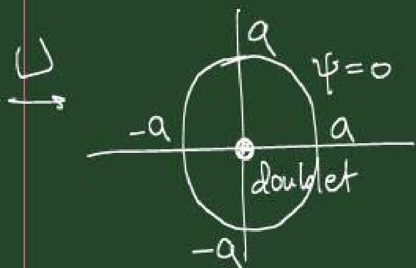
$$R_0 = r_0\sin\theta = \sqrt{\frac{Q}{4\pi U}}\,\frac{\sin\theta}{\sin\frac{\theta}{2}} \rightarrow \text{as } \theta \rightarrow 0,\quad R_0 = \sqrt{Q/\pi U}$$



$A = \pi R_0^2$ $\qquad Q = U \cdot \pi R_0^2$

$$\phi = Ur\cos\theta - \frac{Q}{4\pi r}$$

• Flow around a sphere : uniform flow + doublet



$$\psi = \frac{1}{2}Ur^2\sin^2\theta - \frac{\mu}{4\pi r}\sin^2\theta$$

$$\psi = 0 :\ 0 = \frac{1}{2}Ur_0^2\cancel{\sin^2\theta} - \frac{\mu}{4\pi r_0}\cancel{\sin^2\theta}$$

$$\rightarrow r_0 = \left(\frac{\mu}{2\pi U}\right)^{\frac{1}{3}}$$

Let $\mu = 2\pi U a^3 \rightarrow r_0 = a$: radius of sphere.

Then, $\psi = \frac{1}{2} U \left(r^2 - \frac{a^3}{r} \right) \sin^2 \theta$

$$\phi = U \left(r + \frac{1}{2} \frac{a^3}{r^2} \right) \cos \theta .$$